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Pareto-Optimal Low-Thrust Lunar Transfers with Resonant Encounters

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Contents

- Problem statement
- General scheme of low-thrust lunar transfer
- Transfer design algorithm
- Results for the fuel-optimal transfer
- Conclusions

Problem statement

Given:
- Near-Earth orbit (GTO or LEO)
- Halo orbit around the Earth-Moon L1
- Small spacecraft equipped with a low-thrust engine

Find:
- Transfer trajectories from the near-Earth orbit to the halo orbit exploiting the resonant encounters with the Moon

Take into account:
- Radiation belts
- Shadow regions
- Lunisolar perturbations
- Earth oblateness

Three Principal Transfer Stages

- First stage: orbit raising by the continuous tangential thrust until the perigee is above the radiation belts
- Second stage: time-optimal (or fuel-optimal) orbit raising and reorientation until the lunar encounters become efficient enough
- Third stage: a series of resonant encounters followed by entering into the stable manifold of the halo orbit

Shirobokov, M., Trofimov, S. “Parametric Analysis of Low-Thrust Lunar Transfers with Resonant Encounters,” Advances in the Astronautical Sciences, 2016, Vol. 158, pp. 579–603.

Transfer Design Algorithm

Point in a parking orbit + λ_S, $\Delta\Omega$

First stage

Tangential thrust
+
Atmospheric drag
Earth oblateness (J2)
Lunisolar perturbations
Earth's shadow

$r_\pi = r_{rad}$

Orbital elements at the beginning of the second stage

Point in a libration point orbit + Resonance sequence

Third stage

Perigee active arcs that ensure resonant encounters

CR3BP

$\Delta v_k > \Delta v_{max}$
$r_\pi < r_{rad}$

Orbital elements at the end of the second stage

Second stage

Time-optimal or fuel-optimal control
+
Earth oblateness (J2)
Lunisolar perturbations
Earth's shadow

Resonant orbits

- Resonance $l{:}m$ – l revolutions around the Earth while the Moon makes m revolutions

$$a_{res} = (1-\mu)^{1/3}(m/l)^{2/3}$$

- Constraints:

$$m \leq 3$$
$$l/m \leq 4$$

- Chains of resonances:

$$l_1 : m_1 \rightarrow l_2 : m_2 \rightarrow l_3 : m_3 \rightarrow \ldots$$

Example of 5:2 → 3:1 chain

$\varphi = 0.2$

$\Delta v_0 = 0.78 \text{ m/s}$

$5:2 \to 3:1$

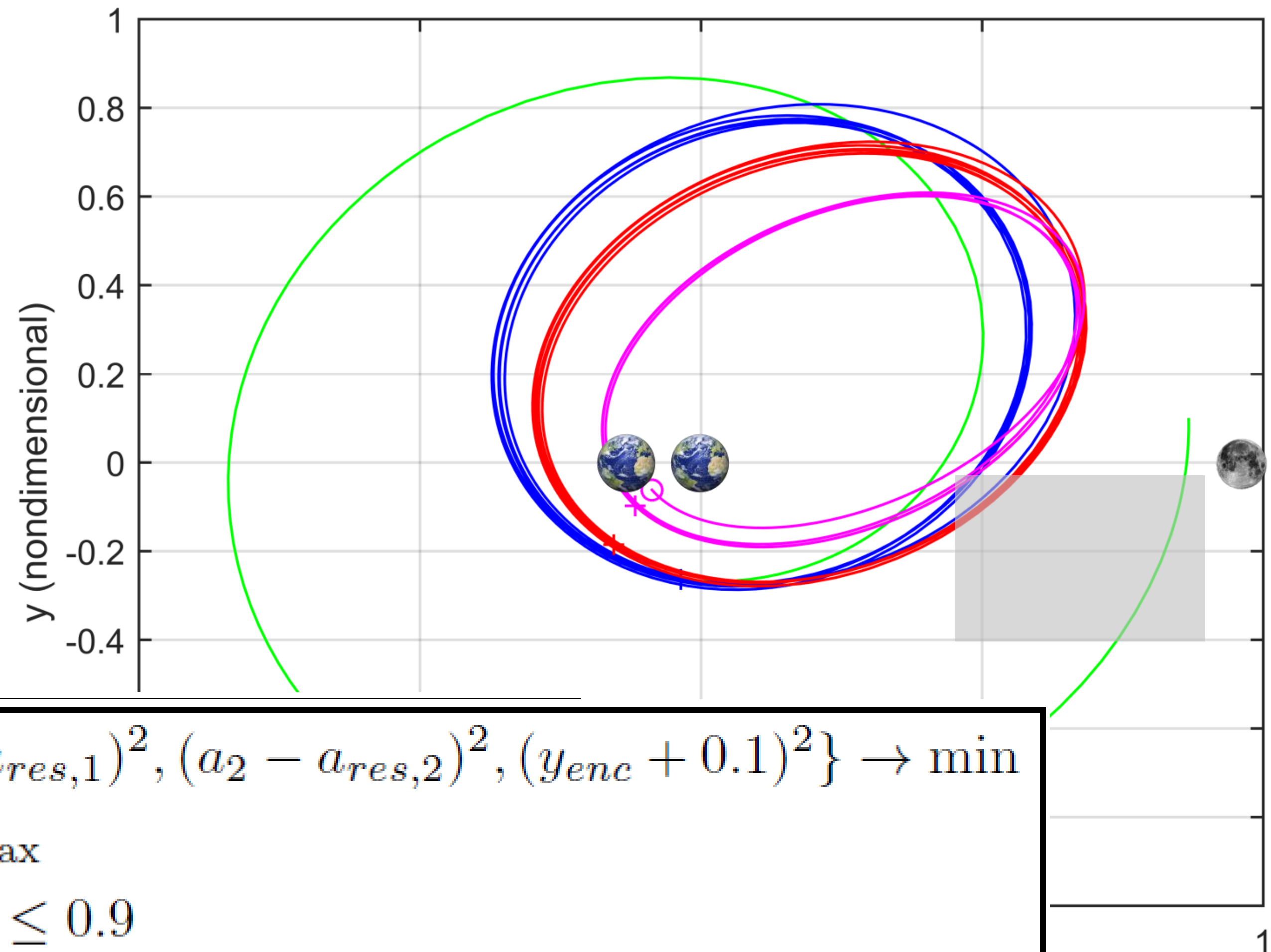
$\Delta v_1 = 47.5 \text{ m/s}$

$\Delta v_2 = 0 \text{ m/s}$

$r_\pi = 41,127 \text{ km}$

r_α

TO



$$
\begin{cases}
\max\{(a_1 - a_{res,1})^2, (a_2 - a_{res,2})^2, (y_{enc} + 0.1)^2\} \to \min \\
|\Delta \mathbf{v}| \leq \Delta v_{\max} \\
0.5 \leq x_{enc} \leq 0.9 \\
-0.3 \leq y_{enc} \leq -0.005
\end{cases}
$$

Results for the time-optimal case

	$\lambda_{S0} = 0°$	$\lambda_{S0} = 90°$	$\lambda_{S0} = 180°$	$\lambda_{S0} = 270°$
$\Delta\Omega_0 = 270°$	$T_{\min} = 252.79$ days $T_{op} = 136.75$ days $T_{sh} = 1.68$ hours $\Delta M/M_0 = 0.1952$ $\varphi = 0.5$ $3:1 \to 2:1$	$T_{\min} = 256.65$ days $T_{op} = 142.73$ days $T_{sh} = 1.65$ hours $\Delta M/M_0 = 0.2038$ $\varphi = 0.5$ $3:1 \to 2:1$	$T_{\min} = 249.92$ days $T_{op} = 133.70$ days $T_{sh} = 0.74$ hours $\Delta M/M_0 = 0.1909$ $\varphi = 0.5$ $3:1 \to 2:1$	$T_{\min} = 241.68$ days $T_{op} = 126.65$ days $T_{sh} = 1.67$ hours $\Delta M/M_0 = 0.1808$ $\varphi = 0.5$ $3:1 \to 2:1$
$\Delta\Omega_0 = 180°$	$T_{\min} = 260.77$ days $T_{op} = 137.13$ days $T_{sh} = 2.19$ hours $\Delta M/M_0 = 0.1958$ $\varphi = 0.6$ $8:3 \to 2:1$	$T_{\min} = 252.95$ days $T_{op} = 138.82$ days $T_{sh} = 0.95$ hours $\Delta M/M_0 = 0.1982$ $\varphi = 0.5$ $3:1 \to 2:1$	$T_{\min} = 251.72$ days $T_{op} = 136.69$ days $T_{sh} = 0.45$ hours $\Delta M/M_0 = 0.1951$ $\varphi = 0.5$ $3:1 \to 2:1$	$T_{\min} = 259.00$ days $T_{op} = 137.25$ days $T_{sh} = 2.43$ hours $\Delta M/M_0 = 0.1959$ $\varphi = 0.6$ $8:3 \to 2:1$
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Spacecraft and engine characteristics

- Spacecraft mass: 300 kg
- Ideally regulated thruster of limited power
- Maximal power: $P_{\max} = 1350\mathrm{W}$
- Efficiency: $\eta = 45\%$
- Fuel-optimal control gives minimum to the integral of the squared thrust acceleration:

$$\frac{1}{m(t_2)} - \frac{1}{m(t_1)} = \frac{1}{2\eta P_{\max}} \int_{t_1}^{t_2} u^2 \, dt$$

The optimal control problem

$$\frac{1}{2}\int_{t_1}^{t_2} |\mathbf{u}|^2\,dt \to \min \text{ subject to } \begin{cases} \dot{\mathbf{r}} = \mathbf{v} \\ \dot{\mathbf{v}} = -\mathbf{r}/r^3 + \mathbf{u} \\ \mathbf{x}(t_1) = \mathbf{x}_1 \\ \mathbf{x}(t_2) = \mathbf{x}_2(t_2) \end{cases}$$

$$H(\mathbf{u}) = -\frac{1}{2}|\mathbf{u}|^2 + \boldsymbol{\lambda}_{\mathbf{r}}^T \mathbf{v} + \boldsymbol{\lambda}_{\mathbf{v}}^T \left(-\frac{\mathbf{r}}{r^3}\right) + \boldsymbol{\lambda}_{\mathbf{v}}^T \mathbf{u} \qquad \mathbf{u}_{opt} = \boldsymbol{\lambda}_{\mathbf{v}}$$

$$\dot{\mathbf{r}} = \mathbf{v} \qquad \dot{\boldsymbol{\lambda}}_{\mathbf{r}} = \left(\frac{I_{3\times 3}}{r^3} + \frac{3\mathbf{r}\mathbf{r}^T}{r^5}\right)\boldsymbol{\lambda}_{\mathbf{v}}$$

$$\dot{\mathbf{v}} = -\frac{\mathbf{r}}{r^3} + \boldsymbol{\lambda}_{\mathbf{v}} \qquad \dot{\boldsymbol{\lambda}}_{\mathbf{v}} = -\boldsymbol{\lambda}_{\mathbf{r}}$$

Find $\boldsymbol{\lambda}_{\mathbf{r}}(t_1)$, $\boldsymbol{\lambda}_{\mathbf{v}}(t_1)$ such that $\mathbf{x}(t_2) - \mathbf{x}_2(t_2) = 0$

Differential continuation technique

Consider a problem $\mathbf{f}(\mathbf{z}) = 0$ and the initial guess $\mathbf{z}_0$

Let $\mathbf{f}(\mathbf{z}_0) = \mathbf{b}$, consider a curve $\mathbf{z}(\tau),\, \tau \in [0,1]$ such that

$$\mathbf{f}(\mathbf{z}(\tau)) = \mathbf{b}(1-\tau) \qquad \mathbf{z}(0) = \mathbf{z}_0$$

Differentiating by τ gives the initial-value problem

$$\frac{d\mathbf{z}}{d\tau} = -\left(\frac{\partial \mathbf{f}}{\partial \mathbf{z}}\right)^{-1} \cdot \mathbf{b}$$

$$\mathbf{z}(0) = \mathbf{z}_0$$

Continuation over the gravitational parameter

Consider $\mu(\tau) = \mu_0 + (1 - \mu_0)\tau$ where μ_0 is the initial value of the gravitational parameter such that

1. the transfer angular distances are the same for

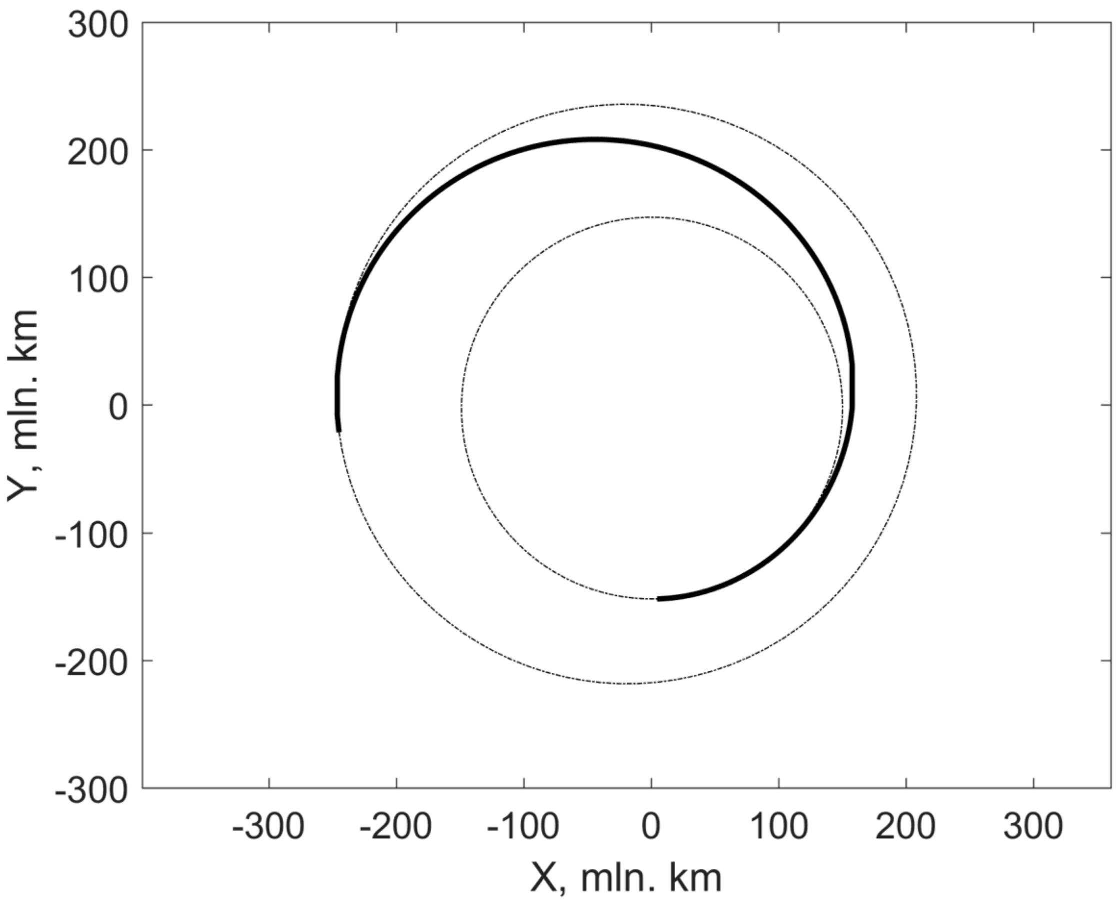
$$\tau = 0 \quad \text{and} \quad \tau = 1$$

2. the case $\tau = 0$ corresponds to the passive motion

Then $\mu_0 = \dfrac{(M_2 + 2\pi N - M_1)^2 a^3}{T^2}$

V. G. Petukhov, "Optimization of interplanetary trajectories for spacecraft with ideally regulated engines using the continuation method," Cosmic Research, Vol. 46, No. 3, 2008, pp. 219–232.

Example of continuation: Earth-Mars transfer



$$N = 0$$

$$\tau = 1.00$$

Continuation method for solving the optimal control problem

Consider $\mathbf{z} = [\boldsymbol{\lambda}_{\mathbf{r}}(t_1), \boldsymbol{\lambda}_{\mathbf{v}}(t_1)]$ and a function

$$\boldsymbol{\varphi}(\mathbf{z}, \mu, \mathbf{v}(t_1)) := \mathbf{x}(t_2) - \mathbf{x}_2(t_2)$$

Consider $\mathbf{z}_0$ such that $\boldsymbol{\varphi}(\mathbf{z}_0, \mu_0, \sqrt{\mu_0}\mathbf{v}_1) = \mathbf{b}$

Consider a curve $\mathbf{z}(\tau)$, $\tau \in [0, 1]$ such that

$$\mathbf{g}(\mathbf{z}(\tau), \tau) := \boldsymbol{\varphi}(\mathbf{z}(\tau), \mu(\tau), \sqrt{\mu(\tau)}\mathbf{v}_1) = \mathbf{b}(1 - \tau)$$

Then

$$\frac{d\mathbf{z}}{d\tau} = -\left(\frac{\partial \mathbf{g}}{\partial \mathbf{z}}\right)^{-1} \left(\frac{\partial \mathbf{g}}{\partial \tau} + \mathbf{b}\right), \quad \mathbf{z}(0) = \mathbf{z}_0$$

Considered cases

	$\lambda_{S0} = 0°$	$\lambda_{S0} = 90°$	$\lambda_{S0} = 180°$	$\lambda_{S0} = 270°$
$\Delta\Omega_0 = 270°$	$T_{\min} = 252.79$ days $T_{op} = 136.75$ days $T_{sh} = 1.68$ hours $\Delta M/M_0 = 0.1952$ $\varphi = 0.5$ $3:1 \to 2:1$	$T_{\min} = 256.65$ days $T_{op} = 142.73$ days $T_{sh} = 1.65$ hours $\Delta M/M_0 = 0.2038$ $\varphi = 0.5$ $3:1 \to 2:1$	$T_{\min} = 249.92$ days $T_{op} = 133.70$ days $T_{sh} = 0.74$ hours $\Delta M/M_0 = 0.1909$ $\varphi = 0.5$ $3:1 \to 2:1$	$T_{\min} = 241.68$ days $T_{op} = 126.65$ days $T_{sh} = 1.67$ hours $\Delta M/M_0 = 0.1808$ $\varphi = 0.5$ $3:1 \to 2:1$
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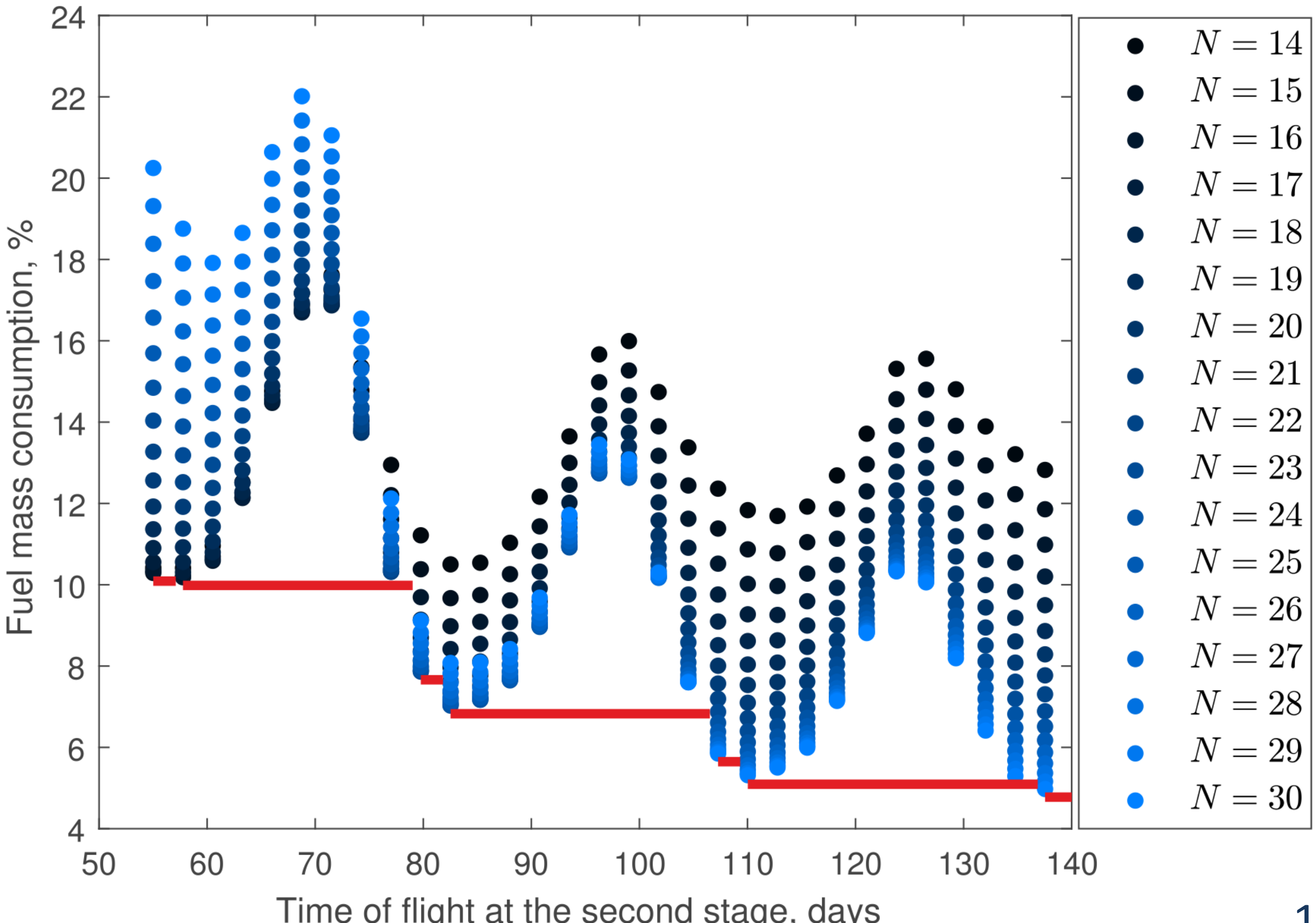
Fuel mass consumption vs TOF
$\lambda_{S0} = 270°$, $\Delta\Omega_0 = 270°$



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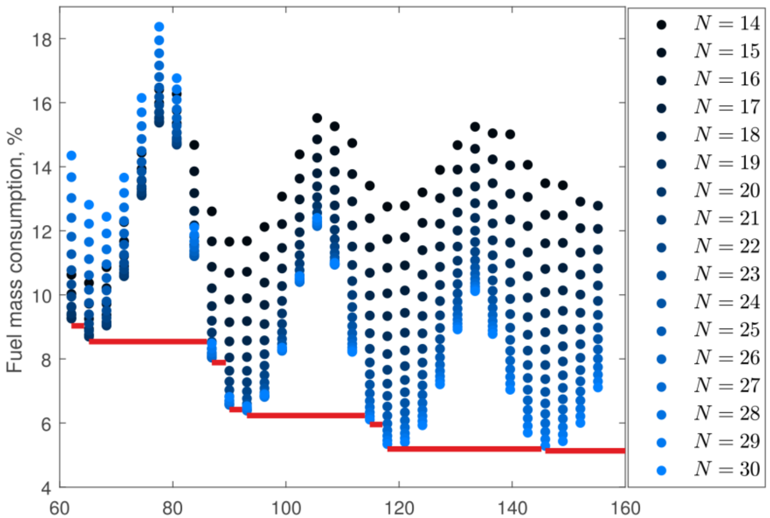
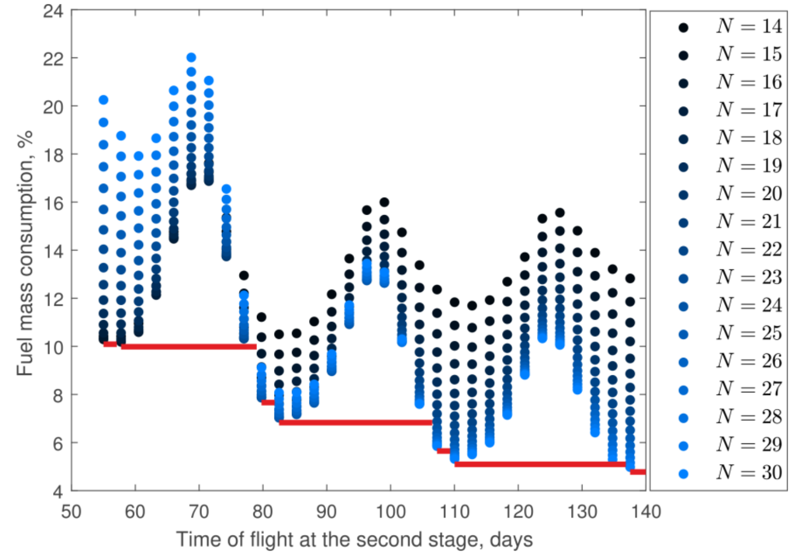
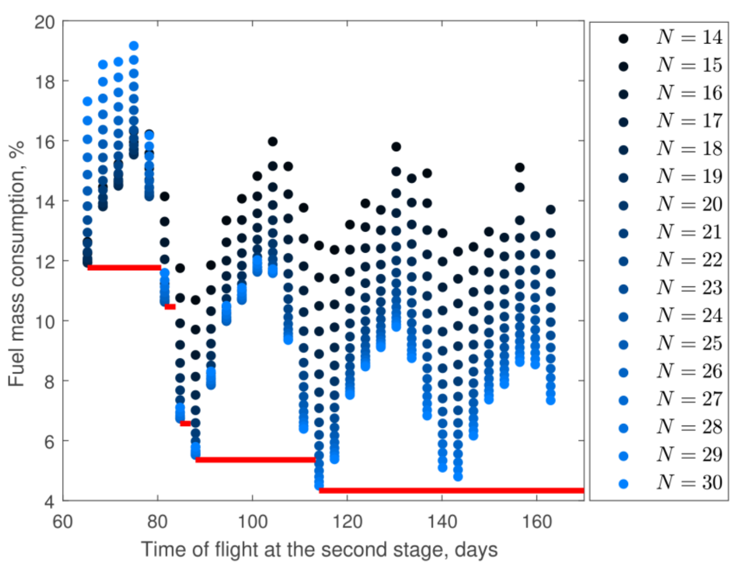
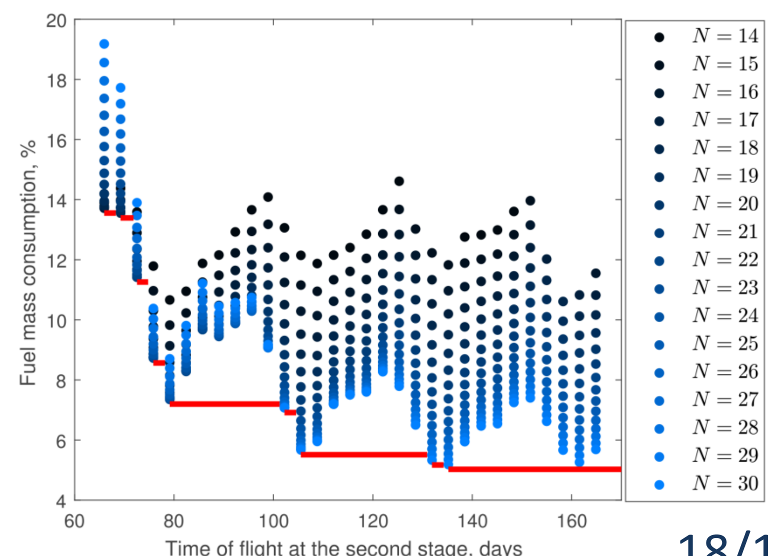
Fuel mass consumption vs TOF









Conclusions

- We propose a simple and reliable scheme for designing a transfer between a near-Earth orbit and a halo orbit around the Earth-Moon L1 point.
- The transfer is divided into three stages: fast orbit raising by the tangential thrust, time-optimal or fuel-optimal orbit raising and reorientation, and a series of resonant encounters that ends on the stable manifold of a halo orbit.
- The analysis of the Pareto fronts shows that increasing the time of flight by 1.5-2.5 times saves 7-23 kg of fuel for the spacecraft with the initial total mass of 300 kg.
- Preliminary results indicate that savings strongly depend on the date and, especially, the time of start. The more realistic thrust model (with constant exhaust velocity) will correct our estimations and decrease the savings.

This work is fully supported by the Russian Science Foundation grant 14-11-00621

First Stage: Leaving the Radiation Belts

$$\frac{d\text{œ}}{dt} = \mathbf{Ma}$$

$$\frac{dL}{dt} = \frac{\xi^2}{h^3 \mu_E} + \frac{h\eta}{\xi} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathbf{a}$$

$$\frac{dM}{dt} = -\frac{F_T}{v_{ex}}$$

Equations are integrated until the perigee distance reaches 30,000 km

$$\mathbf{M} = \frac{h}{\xi} \begin{bmatrix} 0 & h & 0 \\ \xi \sin L & (\xi + 1)\cos L + e_x & -e_y \eta \\ -\xi \cos L & (\xi + 1)\sin L + e_y & e_x \eta \\ 0 & 0 & \varphi \cos L \\ 0 & 0 & \varphi \sin L \end{bmatrix}$$

$$\text{œ} = [h, e_x, e_y, i_x, i_y]^T$$
$$\xi = 1 + e_x \cos L + e_y \sin L$$
$$\eta = i_x \sin L - i_y \cos L$$
$$\varphi = (1 + i_x^2 + i_y^2)/2$$

$$\mathbf{a} = [a_r, a_s, a_w]^T$$

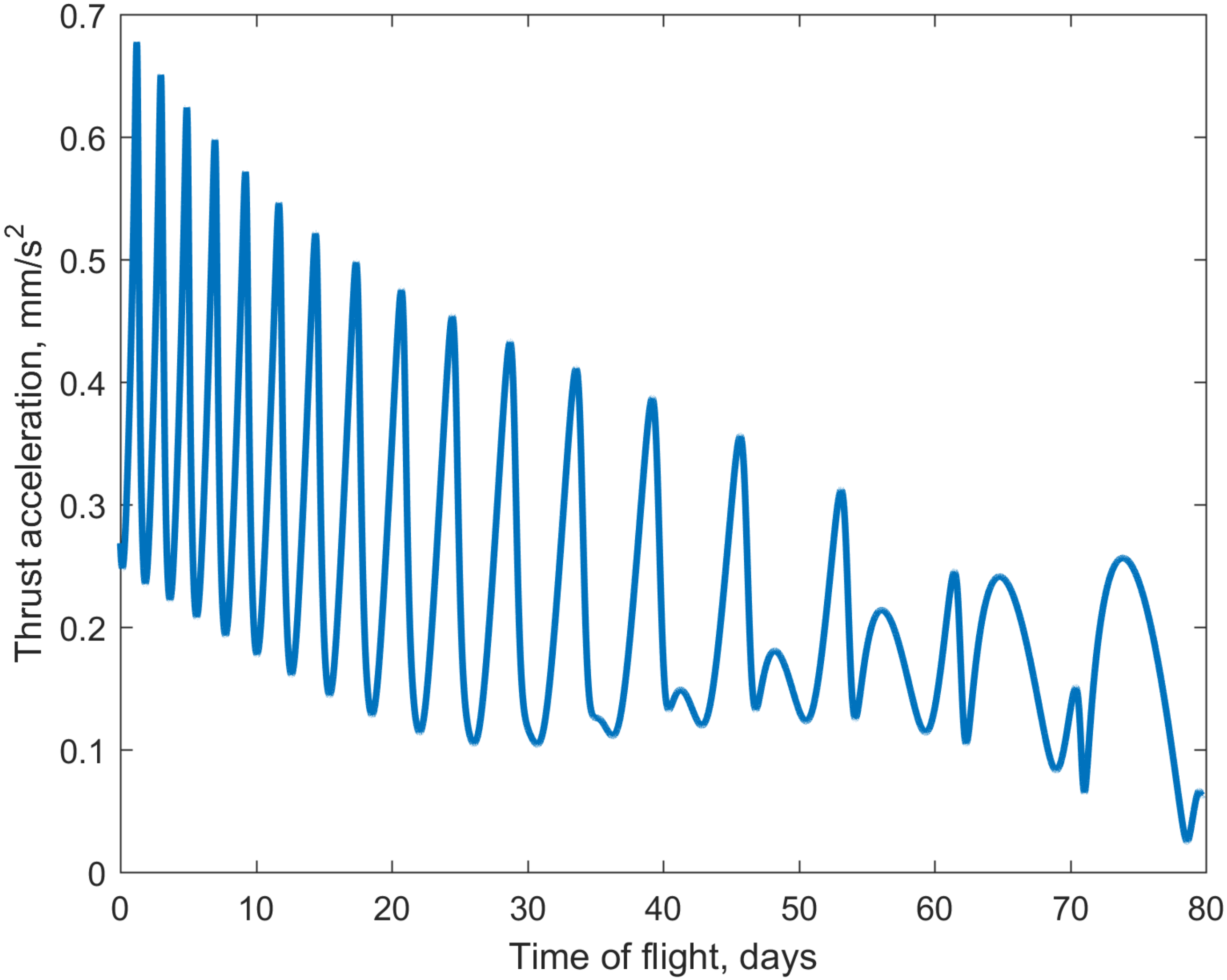
$$\mathbf{a} = \mathbf{a}_T + \mathbf{a}_A + \mathbf{a}_{J2} + \mathbf{a}_S + \mathbf{a}_M$$

$$\mathbf{a}_T = \begin{cases} \frac{F_{\max}}{M} \frac{\mathbf{v}}{|\mathbf{v}|} & \text{out of the Earth's shadow} \\ 0 & \text{in the Earth's shadow} \end{cases}$$

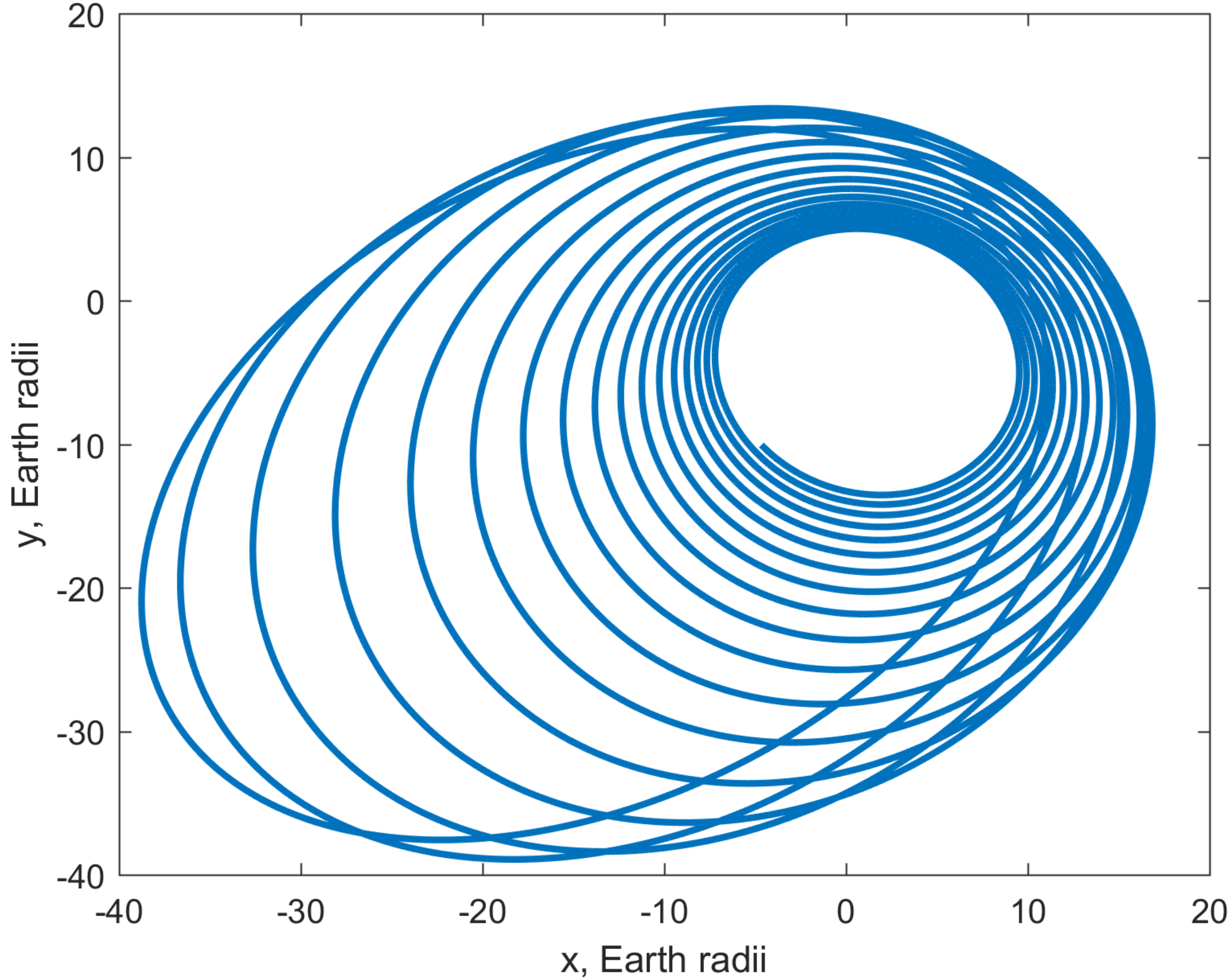
Efficient Resonance Sequences

φ	$l:m$ seq.	r_π	r_α	a	e
0.0	7:2 → 3:1	0.123	0.740	0.431	0.715
0.0	11:3 → 7:2 → 3:1	0.119	0.722	0.421	0.718
0.1	10:3 → 3:1	0.169	0.745	0.457	0.630
0.2	3:1 → 5:2	0.111	0.781	0.446	0.751
0.2	7:2 → 3:1 → 5:2	0.112	0.757	0.434	0.743
0.2	11:3 → 7:2 → 3:1 → 5:2	0.100	0.736	0.418	0.760
0.2	4:1 → 11:3 → 7:2 → 3:1 → 5:2	0.091	0.710	0.400	0.773
0.4	3:1 → 5:2	0.150	0.790	0.470	0.681
0.4	7:2 → 3:1 → 5:2	0.125	0.742	0.433	0.712
0.4	7:2 → 10:3 → 5:2	0.156	0.716	0.436	0.643
0.5	3:1 → 2:1	0.151	0.802	0.476	0.684
0.5	7:2 → 3:1 → 2:1	0.114	0.757	0.436	0.738
0.6	3:1 → 2:1	0.179	0.774	0.476	0.625
0.6	8:3 → 2:1	0.209	0.733	0.471	0.557
0.6	7:2 → 3:1 → 2:1	0.141	0.725	0.433	0.675
0.6	10:3 → 3:1 → 2:1	0.162	0.756	0.459	0.647
0.6	11:3 → 7:2 → 3:1 → 2:1	0.132	0.705	0.418	0.685
0.7	3:1 → 2:1	0.158	0.792	0.475	0.667
0.7	11:3 → 7:2 → 3:1 → 2:1	0.110	0.723	0.416	0.737
0.8	8:3 → 6:3	0.217	0.788	0.503	0.567
0.8	8:3 → 7:3	0.253	0.772	0.512	0.506
0.8	3:1 → 8:3 → 6:3	0.145	0.772	0.459	0.684
0.8	10:3 → 3:1 → 7:3	0.193	0.714	0.453	0.575
0.8	3:1 → 8:3 → 7:3	0.213	0.748	0.481	0.557
0.8	10:3 → 3:1 → 8:3 → 7:3	0.191	0.713	0.452	0.578

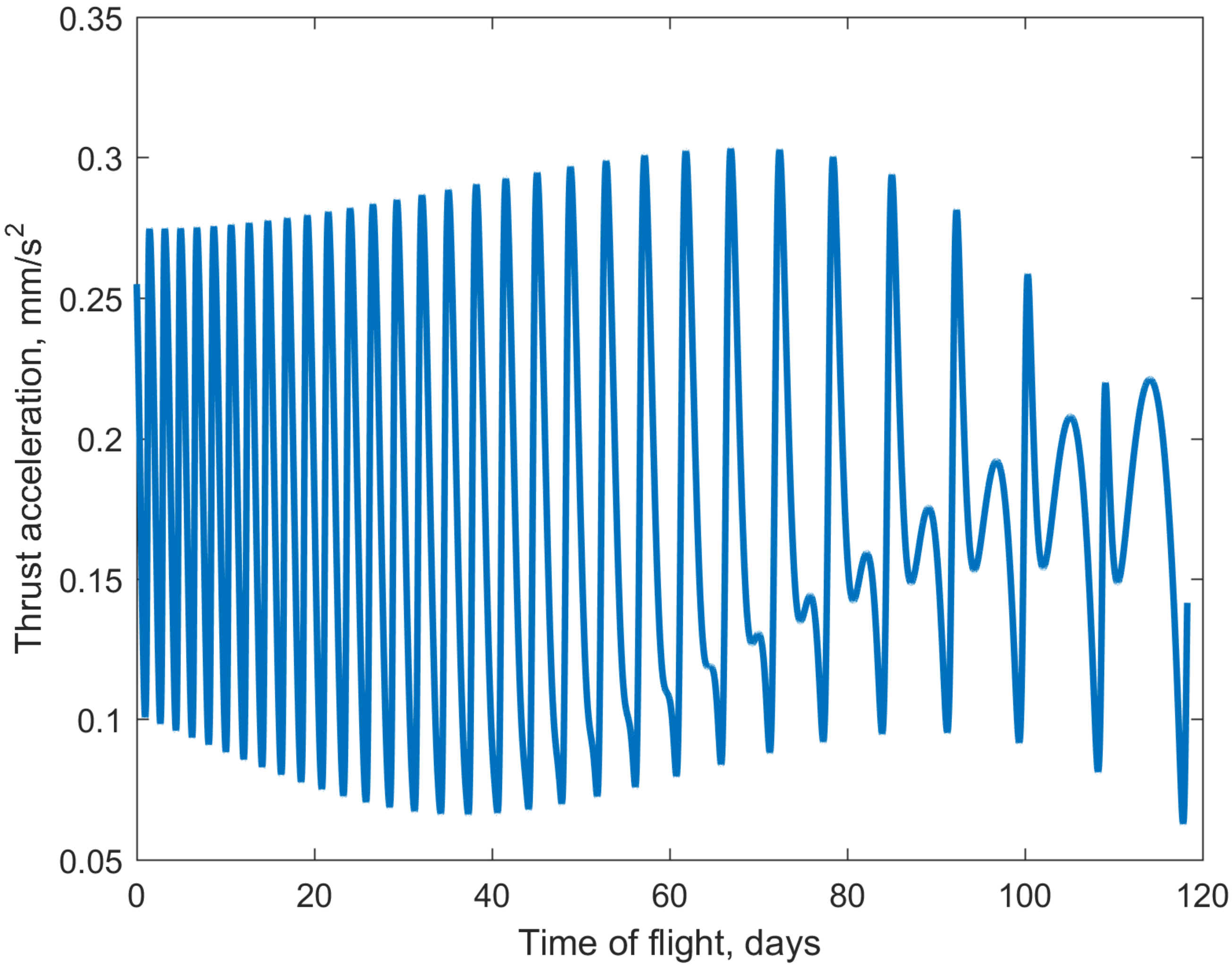
Thrust acceleration history (2^{nd} stage) $N = 17$, $TOF = 1.45*TOF_{min}$



Trajectory example (2nd stage) N = 17, TOF = 1.45*TOF_{min}



Thrust acceleration history (2^{nd} stage) $N = 30$, $TOF = 2.15*TOF_{min}$



Trajectory example (2^{nd} stage) $N = 30$, $TOF = 2.15*TOF_{min}$