A Comparison of Three Main Nonlinear Control Methods

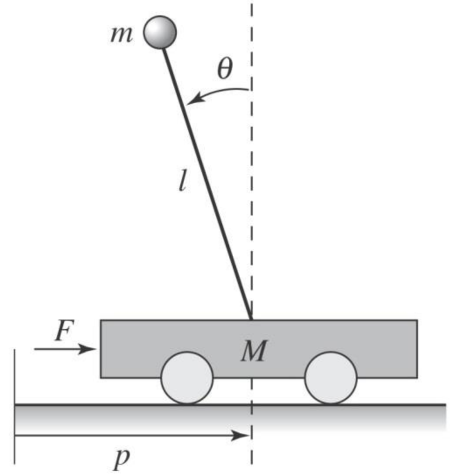
Author: Aleksey I. Shestopyorov
Organization: Keldysh Institute of Applied Mathematics
Cell: +7 915 458 17 34
E-mail: alex.shestoperov@yandex.ru

Contents

Mathematical Model of Mechanical Systems

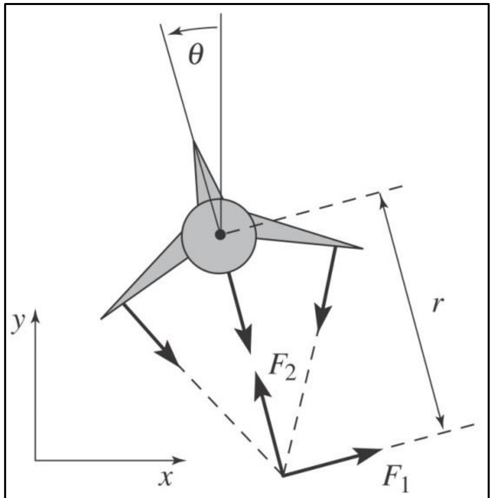
- We will consider mechanical systems.
- Describing the behavior of mechanical systems we use basic **laws of motion** such as Newton's laws.
- Mathematical language - **differential equations**.

Segway



Cart–pendulum system

Harrier “jump jet”



Simplified model

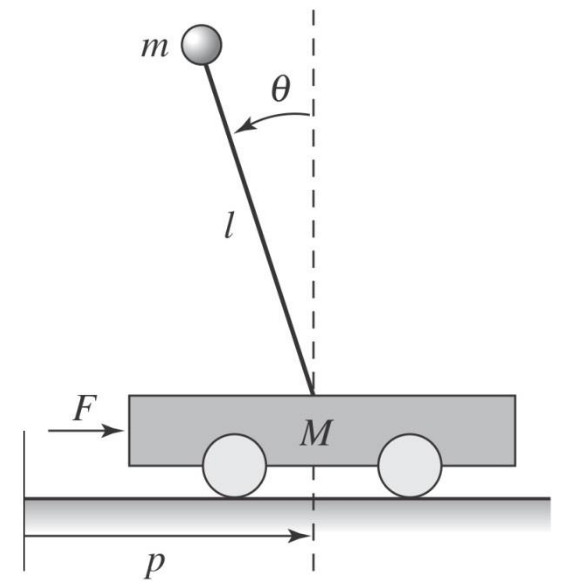
Equations of Motion

- General form of equations of motion:
  $\dot{x}_i = f_i(t, x_1, \dots, x_n, u_1, \dots, u_m), i = 1,2, \dots, n.$
- $x_i, i = 1,2, \dots, n$ – **state of the system**, parameters, which totally describe the system's behavior.
- In the following we suppose that $n = 2$.
- For Segway the states of its mathematical

model are

  – the angle between the rod and vertical;

  – the position of the cart.



Cart–pendulum system

- $u_i, i = 1,2, \dots, m$ – **control forces**, which

allows us to change the states of the system in a desired way.

- For Segway the control force F is produced by the cart motor.

Autonomous Systems and Equilibrium Point

For simplicity consider **the autonomous system** without control force.

- Vector notation: $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \boldsymbol{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$.
- **The autonomous system**: $\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases} \rightarrow \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$.
- In autonomous system f_i doesn't depend on time t.

An equilibrium point $\boldsymbol{x}_{eq} = \begin{pmatrix} x_{1eq} \\ x_{2eq} \end{pmatrix}$ is a state of a system such that if once $\boldsymbol{x} = \boldsymbol{x}_{eq}$ the system remains at this point for all future moments.

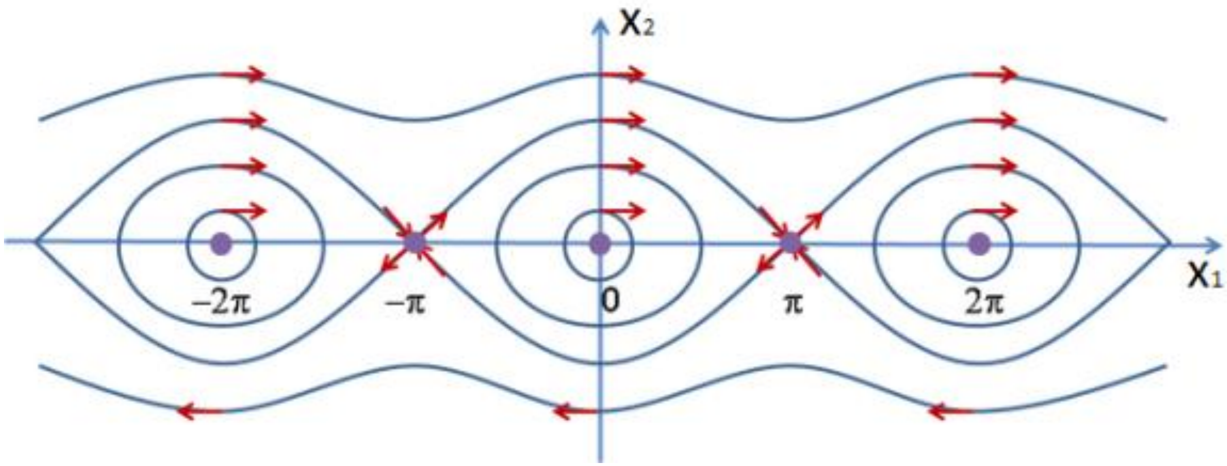
At equilibrium point $\boldsymbol{x}_{eq}$ the following conditions must be hold:

$$\begin{cases} f_1(x_1, x_2) = 0 \\ f_2(x_1, x_2) = 0 \end{cases}$$

Further we suppose that $\boldsymbol{x}_{eq} = \begin{pmatrix} x_{1eq} \\ x_{2eq} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Phase Space and Phase Portrait

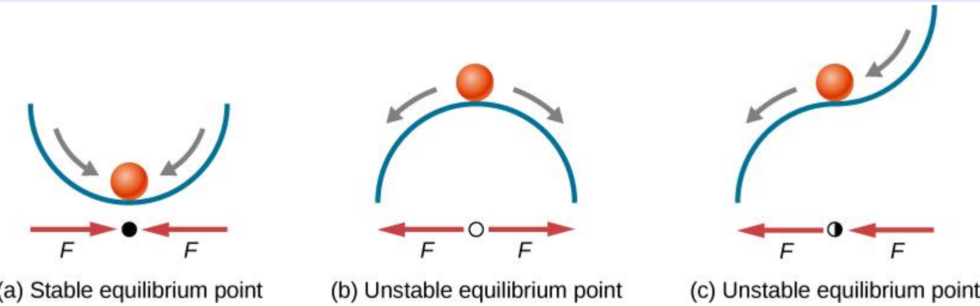
- Geometrically all admissible states of the system can be represented on so called **phase plane** (x_1, x_2).
- Let $\boldsymbol{x}(t)$ be a **solution** of this equation. A curve $\boldsymbol{x}(t)$ on a phase plane is called **trajectory**.
- Trajectories corresponding to various initial conditions form a **phase portrait** of the system.



Undamped pendulum

Types of Stability

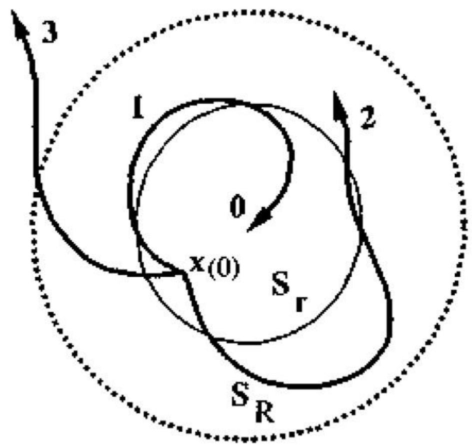
- The idea of definition: the system trajectory will be arbitrarily close to the zero state if it starts sufficiently close to it.



- An equilibrium point $\boldsymbol{x}_{eq} = \boldsymbol{0}$ is **stable** if for any $R > 0$ there exist $r > 0$ such that if $x_1^2(0) + x_2^2(0) < r$ then $x_1^2(t) + x_2^2(t) < R$ for all $t \geq 0$.

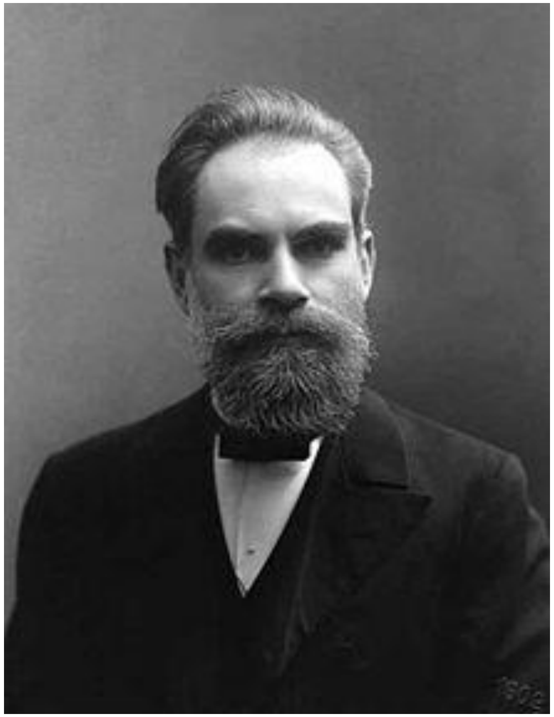
Otherwise the equilibrium point is **unstable**.

- An equilibrium point $\boldsymbol{x}_{eq} = \boldsymbol{0}$ is **asymptotically stable** if it is stable and if in addition there exists some $r > 0$ such that $x_1^2(0) + x_2^2(0) < r$ implies that $\boldsymbol{x}(t) \rightarrow \boldsymbol{0}$ as $t \rightarrow \infty$.



Curves: 1- asymptotically stable, 2- stable, 3-unstable.

Lyapunov Direct Method



- **Aleksandr Mikhailovich Lyapunov**

(June 6, 1857- November 3, 1918**)**

is a great Russian mathematician.

- The philosophy of **Lyapunov's direct method**:

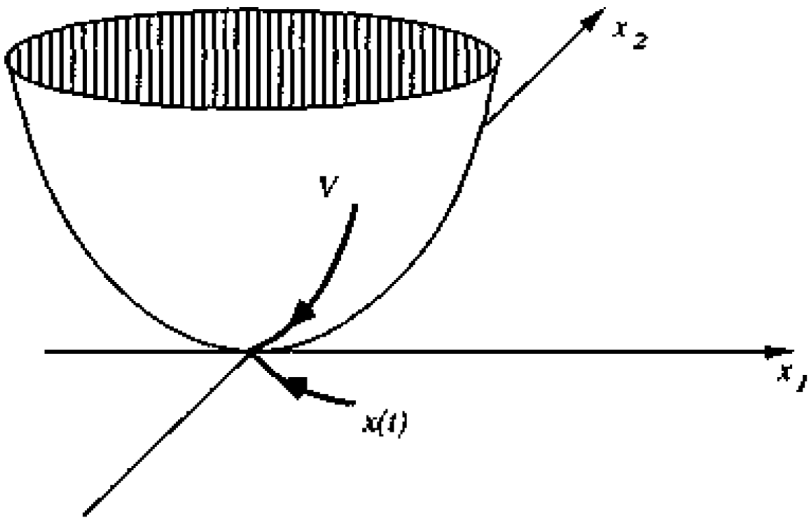
if the total energy of a mechanical system is continuously dissipated, then the system must eventually settle down to an equilibrium point.

To examine the stability of the equilibrium points Lyapunov supposed to use «energy-like» scalar functions called **Lyapunov functions**.

Lyapunov Stability Theorem

Theorem. If in some neighborhood of the origin, there exists a scalar Lyapunov function $V(x_1, x_2)$ such that

1. $V(x_1, x_2) > 0$ and $V(0,0) = 0$;
2. $\dot{V}(x_1, x_2) \leq 0$ along any trajectory of the system, then equilibrium point $\boldsymbol{x}_{eq} = \mathbf{0}$ is stable.

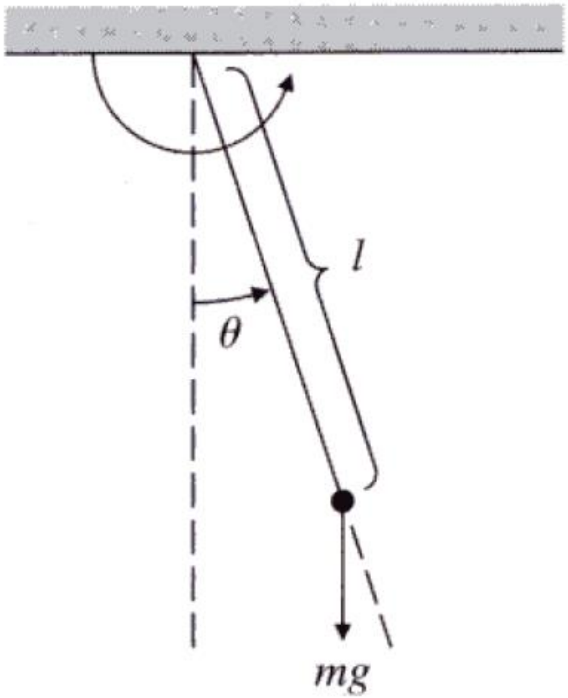


Besides if

2'. $\dot{V}(x_1, x_2) < 0$ along any trajectory of the system when $x_1 \neq 0$ and $x_2 \neq 0$, then equilibrium point $\boldsymbol{x}_{eq} = \mathbf{0}$ is asymptotically stable.

This theorem gives local sufficient stability conditions.

Mathematical Pendulum



The equation of motion:

$$ml^2\ddot{\theta} = -lmg\sin\theta - b\dot{\theta},$$

where $b\dot{\theta}$ – is a damping force in a joint,

θ is an angle between the rod and the vertical,

m – mass of the ball, l – length of rod.

State of the system: $x_1 = \theta, x_2 = \dot{\theta}$.

Rewrite the equation of motion in standard form

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{g}{l}\sin x_1 - \dfrac{b}{ml^2}x_2 \end{cases}$$

At equilibrium point: $\begin{cases} f_1(x_1, x_2) = x_2 = 0 \\ f_2(x_1, x_2) = -\frac{g}{l}\sin x_1 - \frac{b}{ml^2}x_2 = 0 \end{cases}$

Determination of Stability Type

So: $x_{1eq} = \theta = 0, x_{2eq} = \dot{\theta} = 0$.

Suppose that $b = 0$. No damping. Then

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{g}{l}\sin x_1 \end{cases}$$

Consider the function $V(x_1, x_2) = \frac{1}{2}x_2^2 + \frac{g}{l}(1 - \cos x_1)$.

Then: $\dot{V} = x_2\dot{x}_2 + \frac{g}{l}x_2 \sin x_1 = x_2\left(-\frac{g}{l}\sin x_1 + \frac{g}{l}\sin x_1\right) = 0$.

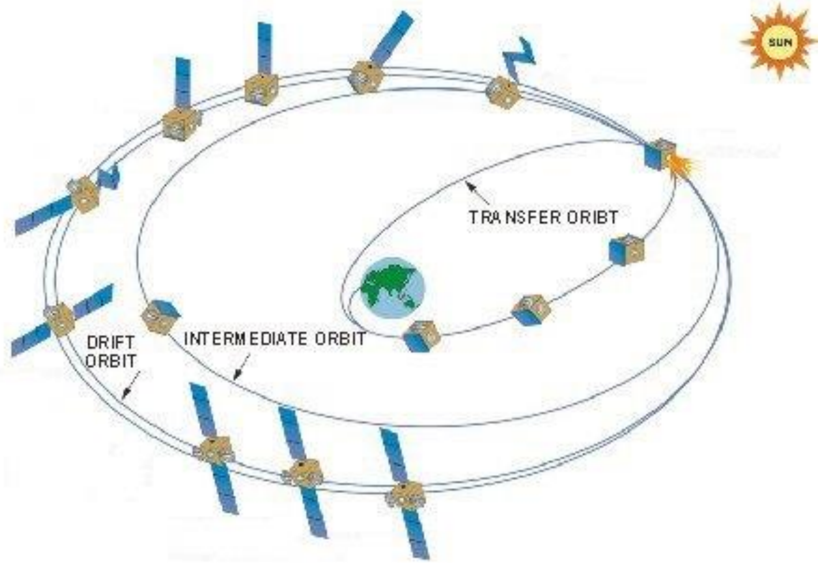
Thus, $\boldsymbol{x}_{eq} = \mathbf{0}$ is stable.

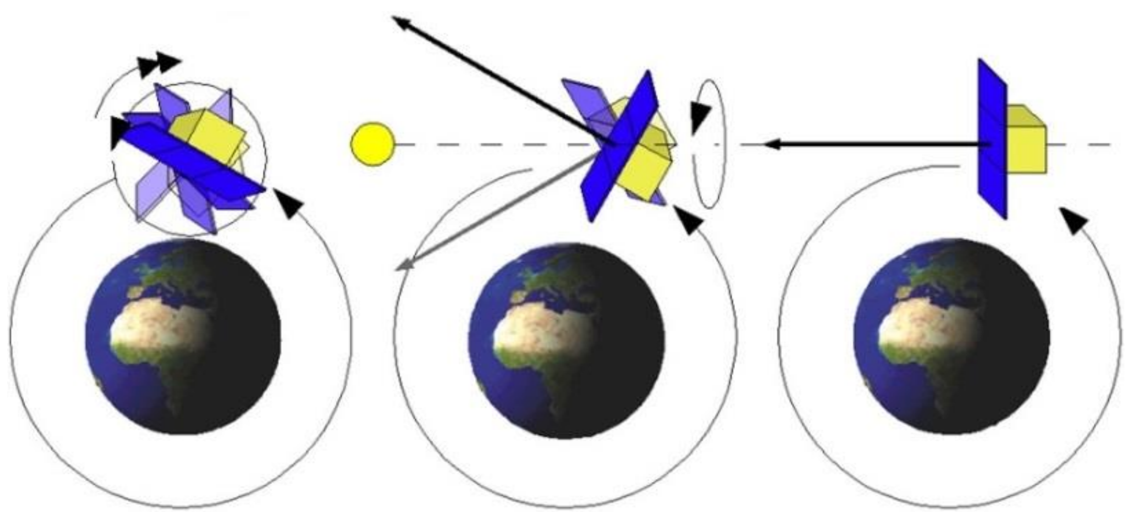
When there is damping force in the system $(b \neq 0)$ the equilibrium point $\boldsymbol{x}_{eq} = \mathbf{0}$ is asymptotically stable (using Barbashin-Krasovskii theorem).

Satellite Motion

- Orbital motion around planets or asteroids, interplanetary flights.
- Attitude motion around satellite center of mass.
- In real missions one has to control both types of motion simultaneously.

It’s a challenging task!





Feedback Control

- In general case, control $\boldsymbol{u}$ is a function of time t and state $\boldsymbol{x}$:
  $\boldsymbol{u} = \boldsymbol{g}(t, \boldsymbol{x})$
- Control is called **feedback control** if it is possible to use the values of state $\boldsymbol{x}$ while constructing function $\boldsymbol{g}$.
- When we substitute the proposed control law $\boldsymbol{u} = \boldsymbol{g}(t, \boldsymbol{x})$ in equations of motion we obtain so called **closed-loop system** :
  $\dot{\boldsymbol{x}} = \boldsymbol{f}\big(\boldsymbol{x}, \boldsymbol{g}(t, \boldsymbol{x})\big)$.
- Its behavior we have to investigate!

Types of Control Problems

During the control missions (in particular, space missions) the most important control goals are:

- Stabilize the system in the neighborhood of certain equilibrium point

Example: stabilize the satellite in desired angular position

- Neglect the influence of bounded disturbance forces on the system's motion

Example: stabilize the satellite taking into account the Earth oblateness.

We will discuss this concepts below using simple example: the motion of mathematical pendulum.

- Find optimal system's motion in some sense

Example: during the interplanetary transfer one wants to waste as less as possible time and fuel to realize the desired maneuver.

Stabilization Problem

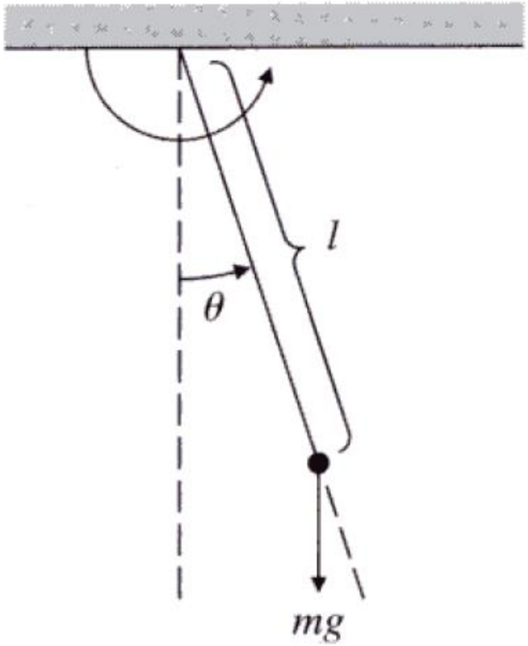
- Space missions that can be viewed as stabilization problems:
  - attitude rotation maneuvers
  - orbital transfers
  - vibration suppression in satellite flexible appendages
  - spacecraft formation maneuvering
  - spacecraft for formation keeping
- **Stabilization problem**: ensure asymptotic stability of a certain state $\boldsymbol{x}_f = const.$
- Note, that $\boldsymbol{x}_f$ is usually an equilibrium point of system!
- Further we will consider the stabilization problem of undamped pendulum near its equilibrium point $\boldsymbol{x}_{eq} = \boldsymbol{0}$.

Method of Control Lyapunov Functions

Suppose that we can use a force u to control the motion of pendulum.

Suppose that $b = 0$. No damping. Then

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{g}{l}\sin x_1 + u \end{cases}$$



State of the system: $x_1 = \theta, x_2 = \dot{\theta}$.

Equilibrium point: $x_{1eq} = \theta = 0, x_{2eq} = \dot{\theta} = 0$ or

In vector notation: $\boldsymbol{x}_{eq} = \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0}$

Without damping: $\boldsymbol{x}_{eq} = \mathbf{0}$ is stable.

But we can create «damping» by control force and ensure asymptotic stability of the equilibrium point $\boldsymbol{x}_{eq} = \mathbf{0}$!

Control Lyapunov Functions

- Goal: to obtain a control law such that the equilibrium point $\boldsymbol{x}_{eq} = \boldsymbol{0}$ of closed-loop system will be asymptotically stable.
- The stability analysis of closed-loop system is carried out using the Lyapunov direct method.
- Consider the feedback control law $u = -kx_2$ and the Lyapunov function

$$V(x_1, x_2) = \frac{1}{2}x_2^2 + \frac{g}{l}(1 - \cos x_1).$$

- When we concern with control problems, Lyapunov functions are called **Control Lyapunov Functions** (CLF).
- The equations of motion of closed-loop system are as follows

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{g}{l}\sin x_1 - \dfrac{k}{ml^2}x_2 \end{cases}$$

- The stability analysis of closed-loop system is carried out using Barbashin-Krasovskii theorem (one of the Lyapunov direct method theorems).

Sliding Mode Control

- Robust control methods aim to achieve asymptotic stability of satellite motion in the presence of bounded disturbance forces.
- Let the pendulum's motion be perturbed by force of wind $f_{wind}(t)$ and we don't know its explicit form.
- The equation of motion of undamped pendulum is as follows:

$ml^2\ddot{\theta} = -lmg\sin\theta + f_{wind}(t) + u$, where $f_{wind}(t)$ is an **unknown**.

- Suppose that $|f_{wind}(t)| < a$, where a – known constant.
- State of the system: $x_1 = \theta, x_2 = \dot{\theta}$.
- Goal: to stabilize the pendulum at the equilibrium point $\boldsymbol{x}_{eq} = \mathbf{0}$ in spite of **unknown** force of wind $f_{wind}(t)$.

Sliding Mode Control - Algorithm

1. Define so called **sliding surface** $s(x_1, x_2) = 0$ in a phase plane.

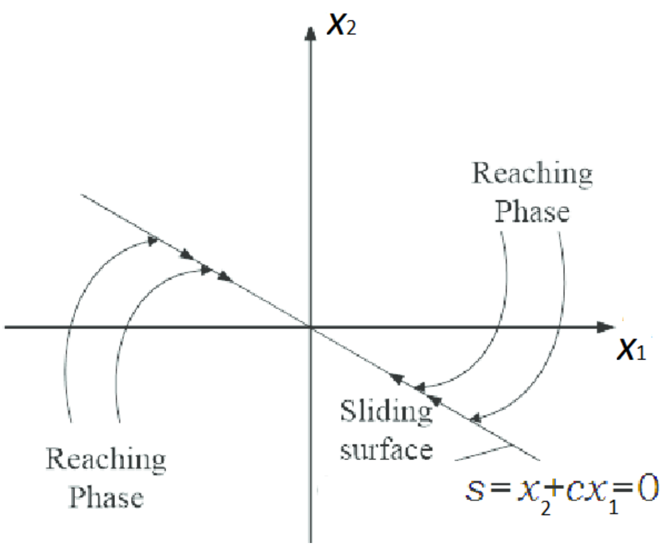
For pendulum: $s(x_1, x_2) = x_2 + cx_1 = \dot{\theta} + c\theta = 0$, where $c > 0$.

Then $\theta(t) = c_0 e^{-ct} \to 0$.

Thus, staying on the surface $f_{wind}(t)$ has no influence on pendulum's motion and we reach the origin asymptotically!

2. Reach sliding surface in a finite time due to control forces.

Idea: using control force $u(t, x)$ we have to minimize the squared «distance» s^2 to the surface.



The sliding condition: $\frac{1}{2}\frac{d}{dt}s^2 \leq -\sigma|s|$.

Using the method of Control Lyapunov Functions we obtain control law:
$u = lmg \sin\theta - ml^2 c\dot{\theta} - \left(\sigma + ml^2\right)\mathrm{sign}(s)$.

Optimal Control

- **Optimal control theory** deals with the problem of finding a control law for a given system such that a certain optimality criterion is achieved.
- **Optimality criterion** is a certain function with depends on object's trajectory and values of control forces.
- Examples of criterions which have to usually minimized during space missions:
  - the tracking error
  - fuel consumption
  - the mission time
- Note, that some control laws can optimize the control efforts and stabilize the satellite motion simultaneously. Examples:
  - Linear Quadratic Regulator (LQR)
  - State-Dependent Riccati Equation (SDRE) Control

Comparison of Control Laws

Control laws	Can this law stabilize the system near equilibrium point?	Is this law optimal in some sense?	Can this law prevent influence of external perturbations?
Method of Control Lyapunov Functions	Yes	No	No
Sliding mode control	Yes	No	Yes
SDRE control	Yes, in certain circumstances	Yes	No

Thank for Your Attention!