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THE VISHIK–LYUSTERNIK METHOD IN GENERAL ELLIPTIC BOUNDARY VALUE PROBLEMS WITH SMALL PARAMETER

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Abstract

L. Volevich[1]. The Vishik–Lysternik Method in General Elliptic Boundary Value Problems with Small Parameter.

This paper gives a survey on the concept of ellipticity with small parameter for general elliptic boundary value problems with operator and boundary conditions depending polynomially on a small parameter. We combine the methods of the theory of general elliptic boundary value problems with the Vishik–Lyusternik method of exponential boundary layer. The main result includes necessary and sufficient conditions for the existence of an a priori estimate of the problem uniform with respect to the parameter. These conditions are formulated in terms of interior and boundary symbols of the problem with parameter introduced in this paper.

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1 Introduction

On a manifold M with the smooth boundary ∂M the equation is considered

$$A(x,D,\varepsilon)u(x)=f(x)\quad x\in M, \tag{1-1}$$

where

$$A(x,D,\varepsilon):=\varepsilon^{2m-2\mu}A_{2m}(x,D)$$
$$+\varepsilon^{2m-2\mu-1}A_{2m-1}(x,D)+\cdots+A_{2\mu}(x,D). \tag{1-2}$$

Here $A_{2m-j},\quad j=0,\dots,2m-2\mu$ is an operator of order $2m-j$ with principal part A^0_{2m-j}.

The boundary conditions are of the form

$$B_j(x',D,\varepsilon)u(x')=g_j(x'),\quad x'\in\partial M,\quad j=1,\dots,m, \tag{1-3}$$

where

$$B_j(x',D,\varepsilon):=\varepsilon^{b_j-\beta_j}B_{b_j}(x',D)$$
$$+\varepsilon^{b_j-\beta_j-1}B_{b_j-1}(x',D)+\cdots+B_{\beta_j}(x',D), \tag{1-4}$$

and $B_{b_j-k},\quad k=0,\dots,b_j-\beta_j$ is an operator of order b_j-k with principal part $B^0_{b_j-k}$. We shall suppose that for a fixed $\varepsilon>0$ the problem (1.1), (1.3) is a standard elliptic problem (i. e. the operator $A_{2m}(x,D)$ is elliptic and operators $\{B_{b_j}(x',D),j=1,\dots,m\}$ are connected with $A_{2m}(x,D)$ by the standard Shapiro–Lopatinskii condition).

If we replace operators A_{2m-j} by A^0_{2m-j} in (1.2) and, respectively, B_{b_j-k} by $B^0_{b_j-k}$ in (1.4), we obtain the principal parts $A^0(x,D,\varepsilon)$ and $B^0_j(x,D,\varepsilon)$ of operators (1.2), (1.4). If we assign ε the weight -1 and ξ the weight 1, then the symbols of these operators become homogeneous functions:

$$A(x,\rho\xi,\rho^{-1}\varepsilon)=\rho^{2\mu}A(x,\xi,\varepsilon),\quad B_j(x,\rho\xi,\rho^{-1}\varepsilon)=\rho^{\beta_j}B_j(x,\xi,\varepsilon). \tag{1-5}$$

The main problem is to discribe necessary and sufficient conditions on symbols of operator in (1.1) and boundary operators in (1.3), which guarantee

(A) A priori estimate of the problem uniform with respect to $\varepsilon\searrow 0$.

(B) Existence of formal asymptotic solution (FAS) of (1.1), (1.3).

(C) Justification of FAS (in other words, when FAS is the expansion of the real solutions in powers of the small parameter).

We shall start from (B) and in informal way present the Vishik–Lyusternik method, which will suggest the conditions on inner and boundary symbols of the problem. Then we shall mainly study (A).

Although problems of type (1.1), (1.3) for high-order elliptic equations with small parameter in higher derivatives arise in mathematical physics (mainly in the fluid dynamics and in the elasticity) the profound theory of such problems begun from the remarkable paper of Vishik-Lyusternik [**?**], where the basic idea of exponential boundary layer was developped and the so-called the Vishik–Lyusternik method was introduced. The main achievement of this method is the possibility to calculate corrections near the boundary by solving ODE problems in the direction normal to the boundary.

This approach with great success was used in applications. There is a lot of applied papers, where the boundary layer method of Vishik–Lyusternik is used to write down the asymptotics for concrete boundary value problems.

As for purely mathematical papers devoted to this problem, there are not many of them. The so-called general elliptic theory in mid fiftieth (when the paper of Vishik–Lyusternik was written) was not so popular as it became a decade later. Vishik and Lyusternik restricted themselves to the Dirichlet problem for strongly elliptic equations. The generalization of their results to general boundary value problems was discussed by Frank in series of papers starting from [**?**] and Nazarov [**?**] and later. The presentation below has common points with these works.

The goal of my lecture is to selebrate Vishik's anniversary by presenting the small parameter theory as a part of general elliptic theory.

2 Formal asymptotic solution of the problem (1.1), (1.3)

The traditional localization of elliptic problems makes possible to "glue" the FAS on M from local FAS on $\mathbb{R}^n$ and on $\mathbb{R}^n_+$.

2.1. Formal asymptotic solution on $\mathbb{R}^n$.

The construction is absolutely traditional. We seek FAS in the form

$$U(x,\varepsilon)=\sum_{k=0}^{\infty}\varepsilon^k u_k(x). \tag{2-1}$$

Substituting (2.1) in equation (1.1) and equating the terms with the same power of ε we obtain recurrent relations

$$A_{2\mu}u_0=f, \quad A_{2\mu}u_1=-A_{2\mu+1}u_0, \tag{2-2}$$

and for an arbitrary $k > 1$ we obtain

$$A_{2\mu}u_k = -A_{2\mu+1}u_{k-1} - \cdots - A_{2m}u_{k-2m+2\mu}, \tag{2-3}$$

where we formally set $u_{k-l} = 0$ for $l > k$.

Equations (2.2), (2.3) show that our recurrent system can be solved if the operator, say,

$$A_{2\mu}(x, D) : H^r(M) \to H^{r-2\mu}(M) \tag{2-4}$$

has a bounded inverse for some r. It means that $A_{2\mu}(x, D)$ is elliptic and, in principal, some additional conditions on the lower terms of $A_{2\mu}$ are satisfied.

To justify FAS we need some (weak) estimate from below of the operator $A(x, D, \varepsilon)$ providing unicity in $\mathbb{R}^n$.

Note, that if the right-hand side f belongs to a space $H^s(M)$ the series (2.1) according to (2.2), (2.3) belongs, in general, to $H^{-\infty}(M)$. However, if $f \in C^\infty$, then according to the hypoellipticity of the elliptic operator $A_{2\mu}(x, D)$ the FAS (6) also belongs to $A_{2\mu}(x, D)$.

2.2. Formal asymptotic solution in the half-space. Boundary layer method.

We shall consider the problem (1.1), (1.3) in the half-space

$$\mathbb{R}^n_+ := \{x = (x', x_n), \quad x' \in \mathbb{R}^{n-1}, \quad x_n \geq 0\}.$$

We shall use the indexing of boundary operators (1.3) such that

$$\beta_1 \leq \beta_2 \leq \cdots \leq \beta_m.$$

In addition, we make a very important assumption:

$$\beta_\mu < \beta_{\mu+1}. \tag{2-5}$$

We seek the solution of the problem (1.1), (1.3) in the form

$$U(x, \varepsilon) + V(x', \frac{x_n}{\varepsilon}, \varepsilon), \tag{2-6}$$

where the first term is the so-called exterior expansion (2.1) and the second term is the interior expansion (boundary layer) of the form

$$V(x', \frac{x_n}{\varepsilon}, \varepsilon) = \sum_{l=0}^{\infty} \varepsilon^{l_0+l} v_l(x', \frac{x_n}{\varepsilon}). \tag{2-7}$$

The integer l_0 will be indicated below.

For the exterior expansion we obtain equations (2.2), (2.3), which we rewrite in the form

$$A_{2\mu}(x,D)u_k(x) = \mathcal{F}(x,u_0,\ldots,u_{k-1}). \tag{2-8}$$

These equations will be supplimented with μ boundary conditions

$$B_{\beta_j}(x',D)u_k(x',0) = \mathcal{G}_j(x',u_0,\ldots,u_{k-1}), \quad j=1,\ldots,\mu. \tag{2-9}$$

It is natural to suppose that equation (2.8) and boundary conditions (2.9) are connected by means of the Shapiro–Lopatinskii condition.

The interior expansion we shall search as solution of the equation

$$A(x,D,\varepsilon)V(x',\frac{x_n}{\varepsilon},\varepsilon) = 0,$$

In this equation we change variable x_n by $t = x_n/\varepsilon$. After the change the equation can be rewritten in the form

$$\sum_{l=0}^{\infty} \varepsilon^{l_0+l-2m}(A(\varepsilon t,x',\varepsilon D',D_n)v_l)(x',t) = 0.$$

Expanding $A(\varepsilon t,x',\varepsilon D',D_n)$ in powers of ε and equating the terms corresponding to the same power of ε we obtain ordinary differential equations with respect to t (parametrized by $x'\in\mathbb{R}^{n-1}$)

$$A(0,x',0,D_n,1))v_l(x',t) = \mathcal{F}'(x',t,v_0,\ldots,v_{l-1}). \tag{2-10}$$

These equations will be supplimented by $m-\mu$ boundary conditions

$$B_j(x',0,D_n,1)v_l(x',0) = \mathcal{G}'_j(x',v_0,\ldots,v_{l-1}), \quad j=\mu+1,\ldots,m. \tag{2-11}$$

It is natural to suppose that ODE problem (15), (16) is uniquelly solvable.

Now we see that the construction of FAS is reduced to the definition of the right-hand sides in (2.9) and (2,11). In this process the important role plays the choice of the parameter l_0:

$$l_0 = \beta_\mu + 1.$$

First of all note that

$$B_j(x',D,\varepsilon)U(x,\varepsilon)|_{x_n=0} = \sum_{l=0}^{\infty}\varepsilon^l[(B_{\beta_j}(x',D)u_l)(x',0)$$

$$+\sum_{s=1}^{b_j-\beta_j}\varepsilon^s(B_{\beta_j+s}(x',D)u_{l-s})(x',0)] \tag{2-12}$$

and

$$B_j(x',D,\varepsilon)V(x',\frac{x_n}{\varepsilon},\varepsilon)|_{x_n=0}=\sum_{l=0}^{\infty}\varepsilon^{l+1+\beta_\mu-\beta_j}[(B_j(x',0,D_n,1)v_l(x',0)$$

$$+\sum_{s=1}\varepsilon^s B_{js}(x',D)v_{l-s}(x',0)]. \tag{2-13}$$

Now we substitute expressions (2.12) and (2.13) in the boundary conditions. For $j\le\mu$ the second sum is $O(\varepsilon)$ and the first sum gives the relations

$$B_{\beta_j}(x',D)u_0(x',0)=g_j(x'),\quad j=1,\dots,\mu. \tag{2-14}$$

To deal with the boundary conditions with $j>\mu$ we suppose, that

$$\beta_j=\beta_\mu+1,\quad j=\mu+1,\dots,\nu$$

,

$$\beta_j>\beta_\mu+1,\quad j=\nu+1,\dots,m.$$

. The sum (2.12) is $O(1)$ and (2.13) for $j\le\nu$ is also $O(1)$ and gives

$$B_j(x',0,D_n,1)v_0(x',0)=g_j-B_{\beta_j}(D)u_0,\quad j=\mu+1,\dots,\nu. \tag{2-15}$$

For $j>\nu$ expression (2.13) contains negative powers of ε. Equating to zero the coefficient of the greatest negative power we obtain

$$B_j(x',0,D_n,1)v_0(x',0)=0,\quad j=\nu+1,\dots,m. \tag{2-16}$$

Solving equation (2.8) (corresponding to the case $k=0$) with boundary conditions (2.9) we obtain $u_0(x)$. Substituting $u_0(x)$ in (2.15) we obtain the full set of boundary conditions for $v_0(x)$. Solving equation (2.10) with these boundary conditions we obtain $v_0(x)$.

This process can be recurrently repeated.

3 Small parameter-elliptic boundary value problems

Fix a point $x^0\in M$ and consider the interior symbol

$$A(\xi,\varepsilon)=A^0(x^0,\xi,\varepsilon) \tag{3-1}$$

at this point. If $x^0 \in \partial M$ we define the symbols

$$B_j(\xi, \varepsilon) = B_j^0(x^0, \xi, \varepsilon), \quad j = 1, \ldots, m. \tag{3-2}$$

We introduce a local coordinate system $x = (x', t)$ such that ∂M is given by the equation $\{t = 0\}$. In the traditional theory of elliptic problems the ODE problem on the half-line $\mathbb{R}_+ = \{t > 0\}$

$$A(\xi', D_t, \varepsilon)v(t) = 0 \quad t > 0, \tag{3-3}$$

$$B_j(\xi', D_t, \varepsilon)v(t)|_{t=0} = \phi_j, \quad j = 1, \ldots, m, \tag{3-4}$$

$$v(t) \to 0 \quad t \to +\infty$$

is called the boundary symbol of the problem (1.1), (1.3), here ξ' is the variable dual to x'. The invertibility of this symbol for $\xi' \neq 0$ is the Shapiro–Lopatinskii condition. In the case of problems with small parameter the analogs of ellipticity condition for (3.1) and the Shapiro–Lopatinskii condition for (3.3), (3.4) are more complicated.

3.1. Condition on the interior symbol. Symbol $A(\xi, \varepsilon)$ is called *small parameter-elliptic* if

$$|A(\xi, \varepsilon)| \geq C|\xi|^{2\mu}(1 + \varepsilon|\xi|)^{2m-2\mu}. \tag{3-5}$$

This condition comes from the paper of Vishik–Lyusternik. It is not difficult to show that inequality (3.5) is equivalent to the following conditions:

(i) $A^0_{2m}(\xi)$ is elliptic, i. e. $A^0_{2m}(\xi) \neq 0, \quad \xi' \neq 0;$

(ii) $A^0_{2\mu}(\xi)$ is elliptic; i. e. $A^0_{2\mu}(\xi) \neq 0, \quad \xi' \neq 0;$

(iii) $A(\xi', \varepsilon) \neq 0, \quad |\xi'| > 0, \quad \varepsilon \geq 0.$

As another equivalent definition of the small parameter-ellipticity we can take the estimate from above for $G(x, \xi, \varepsilon) := A^{-1}(x, \xi, \varepsilon)$:

$$|G(x, \xi, \varepsilon)| \leq C|\xi|^{-2\mu}(1 + \varepsilon|\xi|)^{-2m+2\mu}$$

.

3.2. Small parameter-ellipticity and (weak) parameter-ellipticity with large parameter

We set $\lambda = 1/\varepsilon$ and denote by $\tilde{A}(x, D, \lambda)$ and $\tilde{B}_j(x, D, \lambda)$ operators (1.2), (1.4) multiplied by $\lambda^{2m-2\mu}$, and, respectively, by $\lambda^{b_j - \beta_j}$. Replacing in (1), (3) operators A and B_j by $\tilde{A}$ and, respectively, by $\tilde{B}_j$ we obtain a problem with a "large" parameter. The theory of it is parallel to the theory of (1), (3). The principal symbol of $\tilde{A}$ at the point x^0 is of the form

$$\tilde{A}(\xi, \lambda) = A^0_{2m}(\xi) + \lambda A^0_{2m-1}(\xi) + \cdots + \lambda^{2m-2\mu} A^0_{2\mu}(\xi).$$

For this symbol the small parameter–ellipticity condition leads to the inequality

$$|\tilde{A}(\xi,\lambda)| > \text{const}\,|\xi|^{2\mu}(|\lambda|+|\xi|)^{2m-2\mu}. \tag{3-6}$$

In the papers of Denk-Mennicken-Volevich [?], [?] it is called *weak parameter– ellipticity condition.* This condition is a generalization of the Agmon– Agranovich–Vishik parameter-ellipticity condition corresponding to the case $\mu = 0$. The theory of weak parameter-elliptic problems is based on the boundary layer method.

Note that the weak parameter–ellipticity condition also arise when one studies parabolic operators which are not resolved with respect to the highest time derivative. In this case the inequality must be satisfied for λ belonging to a lower half-plane of the complex plane. This analogy shows that the boundary layer method can be also used in such problems.

3.3. Newton's polygon and parameter-ellipticity conditions

Inequality (3.6) is connected with the Newton polygon of the symbol $\tilde{A}$ and makes it possible to use in the context some ideas of the Newton polygon method.

Consider the polynomial

$$A(\xi,\lambda) = \sum_{\alpha,k} a_\alpha \xi^\alpha \lambda^k. \tag{3-7}$$

Let $N(A)$ be the convex hull in $\mathbb{R}^2$ of

$$\{(|\alpha|,k) : a_{\alpha k} \neq 0,\ (0,0),(|\alpha|,0),(0,k)\}.$$

The polygon $N(A)$ is called Newton's polygon of polynomial (3.7). In the case of polynomial $\tilde{A}(\xi,\lambda)$ satisfying (3.6) (note, that this estimate is two-sided) the Newton polygon of $\tilde{A}$ is a trapezoid and has the shape indicated in Figure 1

Inequality (3.6) can be rewritten in the form

$$|\tilde{A}(\xi,\lambda)| \geq C \sum_{(i,k)\in N(\tilde{A})\cap\mathbb{Z}^2} |\xi|^i|\lambda|^k$$

3.4. Roots of small parameter-elliptic symbols

In the study of the boundary value problems (1.1), (1.3) an important role play the zeros of the algebraic equation

$$A(\xi',\tau,\varepsilon) = 0, \tag{3-8}$$

belonging to the half-plane $\mathbb{C}_+$ of the complex plane. It will be convinient to rewrite equation (3.8) in the equivalent form

$$\tilde{A}(\xi',\tau,1/\varepsilon) = 0, \tag{3-9}$$

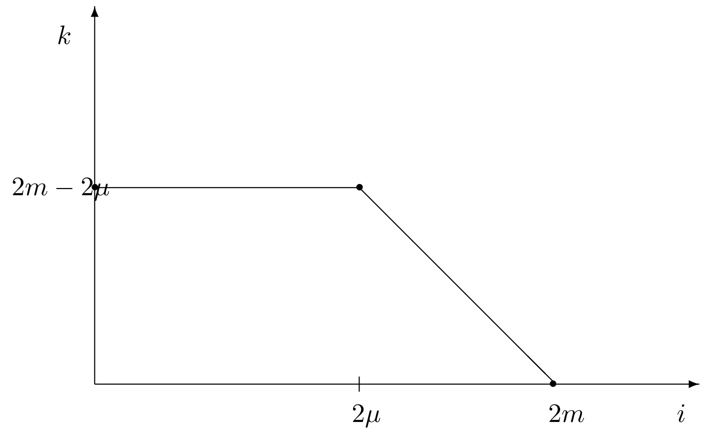

Fig. 1. The Newton polygon.

and denote its zeros as $\tau_j(\xi', 1/\varepsilon)$, $j = 1, \ldots, 2m$, they depend continuously on $(\xi', 1/\varepsilon)$. Let $m^{\pm}(\xi', 1/\varepsilon)$ be the number of zeros belonging to $\mathbb{C}_{\pm}$. According to (3.5) or (3.6) equations (3.8), (3.9) have no real zeros τ, so $m^{\pm}(\xi', 1/\varepsilon)$ is independent of $(\xi', 1/\varepsilon)$, and is denoted by $m^{\pm}$. From the continuity of zeros as $\varepsilon \to \infty$ follow that $m^{\pm}$ coincide with the the corresponding numbers for the equation $P^0_{2m}(\xi', \tau) = 0$.

Denote by $\mu^{\pm}$ the number of roots of $P^0_{2\mu}(\xi', \tau) = 0$ belonging to $\mathbb{C}_{\pm}$.

Following Vishik and Lyusternik we introduce the polynomial

$$Q(\tau) := \tau^{-2\mu} A(0, \tau, 1).$$

If we pose $\xi' = 0, \varepsilon = 1$ in (3.5) and divide both sides of the inequality by by $\tau^{2\mu}$ we obtain

$$Q(\tau) > C(1 + |\tau|)^{2m-2\mu}.$$

Therefore $Q(\tau)$ has no real zeros. Denote by $q^{\pm}$ the number of roots of $Q(\tau) = 0$ belonging to $\mathbb{C}_{\pm}$.

Proposition. Following relations take place

$$m^+ = \mu^+ + q^+, \quad m^- = \mu^- + q^-. \tag{3-10}$$

The idea of the proof is following. As was established by Vishik and Lyusternik (for details see [**?**]) zeros of (3.8) can be splitted in the two groups:

$$\{\tau_j(\xi', 1/\varepsilon), \quad j = 1, \dots, 2\mu\} \bigcup \{\tau_j(\xi', 1/\varepsilon), \quad j = 2\mu + 1, \dots, 2m\}. \tag{3-11}$$

The first set in (3.11) consists of the roots (with regard to multiplicities) uniformly bounded for small ε, and as $\varepsilon \to 0$ it tends to the set

$$\{\tau_j^0(\xi'), \quad P_{2\mu}(\xi', \tau_j^0(\xi') = 0, \quad j = 1, \dots, 2\mu\}.$$

More exactly, for a fixed ξ', small $\delta > 0$, and $\varepsilon < \varepsilon(\delta, \xi')$ in the disk of radius δ surrounding a root $\tau_j^0(\xi')$ of the multiplicity p_j there are exactly p_j roots $\tau_j(\xi', 1/\varepsilon)$. The roots in the second set (3.11) are $O(1/\varepsilon)$ for $\varepsilon \to 0$ and

$$\varepsilon\tau_j(\xi', \varepsilon) \to \nu_j, \quad j = 2\mu + 1, \dots, 2m,$$

where ν_j are the roots of $Q(\tau) = 0$. More exactly, for a fixed ξ', small $\delta > 0$, and $\varepsilon < \varepsilon(\delta, \xi')$ in the disk of radius δ surrounding a root ν_j of the multiplicity q_j there are exactly q_j roots $\varepsilon\tau_j(\xi', 1/\varepsilon)$. Since polynomials $A(\xi, \varepsilon), A_{2m}^0(\xi, \varepsilon), A_{2\mu}^0(\xi, \varepsilon)$ and $Q(\tau)$ have no real roots, we come to relations (3.10).

3.5. Properly small parameter-elliptic symbols The small parameter-elliptic polynomial $A(\xi, \varepsilon)$ is called *properly small parameter-elliptic*, if

$$m^+ = \mu^- = m, \quad \mu^+ = \mu^- = \mu \tag{3-12}$$

Note that relations (3.12) are satisfied automatically in the case $n > 2$, and only in the case $n = 2$ it is an additional condition.

Comparing (3.10) and (3.12) we obtain that in the case of properly small parameter-elliptic polynomial

$$q^+ = q^- = m - \mu. \tag{3-13}$$

It is the so-called condition of regular degeneration of Lyusternik and Vishik.

Remark. The existence of the two groups of roots with different behaviour with respect to the parameter is the main difference of small parameter ellipticity (or weak parameter ellipticity) from the standard ellipticity or parameter–ellipticity. Namely, one of this group leads to the boundary layer type solutions.

3.6. Conditions on the boundary symbol

Now we can formulate the analog of the Shapiro–Lopatinskii condition for the small parameter–elliptic operators.

Condition I. For every $\xi' \in \mathbb{R}^{n-1}\backslash\{0\}$ and $\varepsilon \in [0,\infty)$ the ordinary differential equation on the half-line

$$A(\xi', D_t, \varepsilon)v(t) = 0 \quad t > 0, \tag{3-14}$$

$$B_j(\xi', D_t, \varepsilon)v(t)|_{t=0} = \phi_j, \quad j = 1, \ldots, m, \tag{3-15}$$

$$v(t) \to 0 \quad t \to +\infty$$

has a unique solution for arbitrary $(\phi_1, \ldots, \phi_m) \in \mathbb{C}^m$.

Condition II. For every $\xi' \in \mathbb{R}^{n-1}\backslash\{0\}$ the ordinary differential equation on the half-line

$$A_{2\mu}(\xi', D_t)v(t) = 0 \quad t > 0, \tag{3-16}$$

$$B_{\beta_j}(\xi', D_t)v(t)|_{t=0} = \phi_j, \quad j = 1, \ldots, \mu, \tag{3-17}$$

$$v(t) \to 0 \quad t \to +\infty$$

has a unique solution for arbitrary $(\phi_1, \ldots, \phi_\mu) \in \mathbb{C}^\mu$.

Condition III. The ordinary differential equation on the half-line

$$A(0, D_t, 1)v(t) = 0 \quad t > 0, \tag{3-18}$$

$$B_j(0, D_t, 1)v(t)|_{t=0} = \phi_j, \quad j = \mu + 1, \ldots, m, \tag{3-19}$$

$$v(t) \to 0 \quad t \to +\infty$$

has a unique solution for arbitrary $(\phi_{\mu+1}, \ldots, \phi_m) \in \mathbb{C}^{m-\mu}$.

Denote by $v_j(t, \xi', \varepsilon), \quad j = 1, \ldots, m$ the fundamental system of solutions of the problem (3.14), (3.15), i. e.

$$A(\xi', D_t, \varepsilon)v_j(t) = 0 \quad t > 0, \tag{3-20}$$

$$B_k(\xi', D_t, \varepsilon)v_j(t)|_{t=0} = \delta_{kj}, \quad k = 1, \ldots, m, \tag{3-21}$$

and

$$v_j(t) \to 0 \quad t \to +\infty$$

The existence of such solutions follows from Condition I. Since the solutions of (3.15), (3.16) decay exponentially as $t \to \infty$, we obtain

$$\int_0^{+\infty} |D_t^l v_j(t, \xi', \varepsilon)|^2 dt) := h_{lj}^2(\xi', \varepsilon) < \infty. \tag{3-22}$$

In the traditional cases (ellipticity or parameter– ellipticity) the right hand side can be easily obtained from the homogeneity. In our case it is a rather difficult task leading to cumbersome expressions.

Main Lemma. Let $v_j(t), \quad j=1,\dots,m$ are solutions of (3.20) and (3.21). Then integrals (3.22) converge for $l=0,1,\dots$ and the right-hand sides $h_{lj}(\xi',\varepsilon)$ are not greater than constant times

$$|\xi'|^{l-\beta_j-1/2}(1+\varepsilon|\xi'|)^{\beta_j-b_j}, \quad j\le\mu, l\le\beta_{\mu+1};$$

$$\varepsilon^{\beta_{\mu+1}-l+1/2}|\xi'|^{\beta_{\mu+1}-\beta_j}(1+\varepsilon|\xi'|)^{l-\beta_{\mu+1}+\beta_j-b_j-1/2}, \quad j\le\mu, l>\beta_{\mu+1};$$

$$\varepsilon^{\beta_j-b_\mu}|\xi'|^{l-\beta_\mu-1/2}(1+\varepsilon|\xi'|)^{b_\mu-b_j}, \quad j>\mu, l\le b_\mu;$$

$$\varepsilon^{\beta_j-l+1/2}(1+\varepsilon|\xi'|)^{l-b_j-1/2}, \quad j>\mu, l>b_\mu.$$

The rough idea of the proof of the Lemma is following. As in the case of FAS we seek the solution of (3.20), (3.21) in the form

$$v_j(t,\xi',\varepsilon)=U_j(t,\xi',\varepsilon)+V_j(\frac{t}{\varepsilon},\xi',\varepsilon), \tag{3-23}$$

where

$$U_j(t,\xi',\varepsilon)=\sum_{p=1}^{\mu}\phi_{pj}(\xi',\varepsilon)\exp(i\tau_p^+(\xi',\varepsilon)t)$$

and

$$V_j(\frac{t}{\varepsilon},\xi',\varepsilon)=\sum_{p=\mu+1}^{m}\phi_{pj}(\xi',\varepsilon)\exp(i(\varepsilon\tau_p^+(\xi',\varepsilon))t/\varepsilon).$$

The function U_j is a solution of some perturbation of the problem (3.14), (3.15) in condition II, and V_j is a solution of some perturbation of the problem (3.18), (3.19) in condition III. The perturbation argument leads to special form of the unknown coefficients ϕ_{pj}.

3.7. Weakly parameter–elliptic problems with small parameter.

Replacing $1/\varepsilon$ by λ and $A,B_1,\dots,B_m$ by $\tilde A,\tilde B_1,\dots,\tilde B_m$ we obtain a problem with large parameter, see [**?**]–[**?**]. In this case conditions (I), (II), (III) can be trivially rewritten.

Consider the corresponding system of fundamental solutions of the ODE problen on the half-line

$$\tilde A(\xi',D_t,\lambda)\tilde v_j(t)=0 \quad t>0, \tag{3-24}$$

$$\tilde B_k(\xi',D_t,\lambda)\tilde v_j(t)|_{t=0}=\delta_{kj}, \quad k=1,\dots,m, \tag{3-25}$$

and

$$\tilde v_j(t)\to 0 \quad t\to+\infty.$$

It folllows from the unicity of solutions of the ODE problems under consideration, that

$$\tilde{v}_j(t,\xi',\lambda) = \lambda^{\beta_j - b_j} v_j(t,\xi',1/\lambda),$$

and according to the Main Lemma the integrals

$$\left(\int_0^{+\infty} |D_t^l \tilde{v}_j(t,\xi',\lambda)|^2 dt\right)^{1/2}$$

are not greater than constant times

$$|\xi'|^{l-\beta_j-1/2}(|\lambda|+|\xi'|)^{\beta_j-b_j}, \quad j \le \mu, l \le \beta_{\mu+1};$$

$$|\xi'|^{\beta_{\mu+1}-\beta_j}(|\lambda|+|\xi'|)^{l-\beta_{\mu+1}+\beta_j-b_j-1/2}, \quad j \le \mu, l > \beta_{\mu+1};$$

$$|\xi'|^{l-\beta_\mu-1/2}(|\lambda|+|\xi'|)^{b_\mu-b_j}, \quad j > \mu, l \le b_\mu;$$

$$(|\lambda|+|\xi'|)^{l-b_j-1/2}, \quad j > \mu, l > b_\mu.$$

In the case of boundary operators independent of λ (i. e. $b_j = \beta_j, j = 1,\dots,m$) we come to the results of [**?**].

4. A priori estimates for small parameter–elliptic problems.

All the estimates are based on the Main Lemma and follow the plan of [**?**]-[**?**]. This technique also allows in the case of variable cvoefficients to construct left and right parametrices and (under additional conditions on the lower terms) to construct the inverse operator.

4.1 A priori estimates on a manifold without boundary.

The small parameter–ellipticity condition sugggests the choice of the functional space $H^{r,s}(M)$. In the case $M = \mathbb{R}^n$ this space is defined as the space of $u \in H^{-\infty}$ with the norm

$$||u||_{r,s} := ||(1+|D|^2)^{s/2}(1+|\varepsilon D|^2)^{(r-s)/2} \tag{3-26}$$

uniformly bounded for $\varepsilon \le \varepsilon_0$. The standard localization technique permits to define these spaces on a smooth manifold M.

Theorem. For a symbol $A(x,\xi,\varepsilon)$ with smooth coefficients following conditions are equivalent

(I) $A(x^0,\xi,\varepsilon)$ for each $x^0 \in M$ satisfy the small parameter–ellipticity condition.

(II) For arbitrary real r,s and large enough R the estimate

$$||u,M||_{r,s} \le C(||A(x,D,\varepsilon)u||_{r-2m,s-2\mu} + ||u,M||_{(-R)}) \tag{3-27}$$

holds with constant independent of ε.

4.2. A priori estimate in $\mathbb{R}^n_+$.

Denote by $H^{r,s}(\mathbb{R}^n_+)$ the space of restrictions to $\mathbb{R}^n_+$ of the elements from $H^{r,s}(\mathbb{R}^n)$. We shall consider only the case of *positive integer* r and $r \geq s$.

In the case $r > 1/2$ the elements $u(x) \in H^{r,s}(\mathbb{R}^n_+)$ have traces $u(x',0)$ belonging to the space $H^{r-1/2,s-1/2}(\mathbb{R}^{n-1})$ with norm (see [**?**]– [**?**]).

$$||g, \mathbb{R}^{n-1}||_{r-1/2,s-1/2} := ||\Xi_{r-1/2,s-1/2}(D',\varepsilon)g, \mathbb{R}^{n-1}||,$$

where

$$\Xi_{r-1/2,s-1/2}(\xi',\varepsilon) = \begin{cases} (1+|\xi'|)^{s-\frac{1}{2}}(1+\varepsilon|\xi'|)^{r-s}, & s > 1/2, \\ \varepsilon^{\frac{1}{2}-s}(1+\varepsilon|\xi'|)^{r-\frac{1}{2}}, & s \leq 1/2 \end{cases} \tag{3-28}$$

It is also useful to note that according to the form of operators (1.2), (1.4)

$$A(D,\varepsilon)H^{r,s}(\mathbb{R}^n_+) \subset H^{r-2m,s-2\mu}(\mathbb{R}^n_+),$$

$$B_j(D,\varepsilon)H^{r,s}(\mathbb{R}^n_+) \subset H^{r-b_j,s-\beta_j}(\mathbb{R}^n_+), \quad j = 1, \ldots, m.$$

We now can correspond to our problem the continuous operator

$$\{A(D,\varepsilon), B_1(D,\varepsilon), \ldots, B_m(D,\varepsilon)\} : H^{r,s}(\mathbb{R}^n_+)$$

$$\to H^{r-2m,s-2\mu}(\mathbb{R}^n_+) \times \prod_{j=1}^{m} H^{r-b_j-1/2,s-\beta_j-1/2}(\mathbb{R}^{n-1}), \tag{3-29}$$

whose norm is uniformly bounded with respect to ε.

Main Theorem. Suppose that the symbol $A(\xi,\varepsilon)$ is properly small parameter–elliptic and inequality $\beta_\mu < \beta_{\mu+1}$ holds. Then following conditions are equivalent.

(A) Conditions (I), (II), (III) for the boundary symbol.

(B) Condition (I) and estimates of the Main Lemma for fundamental system of solutions (3.24), (3.25).

(C) For natural $r > b_m + 1/2$ and s satisfying

$$\beta_\mu + 1/2 \leq s < \beta_{\mu+1} + 1/2$$

the estimate

$$||u, \mathbb{R}^n_+||_{r,s} \leq C(||A(D,\varepsilon)u, \mathbb{R}^n_+||_{r-2m,s-2\mu} + ||u, \mathbb{R}^n_+||$$

$$+ \sum_{j=1}^{m} ||u, \mathbb{R}^{n-1}||_{r-b_j-1/2,s-\beta_j-1/2}) \tag{3-30}$$

holds with the constant independent of ε.

The main analytical part of the proof of the theorem is the estimate of the solution in $\mathbb{R}^n_+$ of the homogeneous problem

$$A(D,\varepsilon)v(x) = 0 \quad x_n > 0, \tag{3-31}$$

$$B_k(D,\varepsilon)v(x)|_{x_n=0} = g_k(x'), \quad k = 1,\dots,m, \tag{3-32}$$

which directly follows from the Main Lemma. The reduction of the non-homogeneous case to homogeneous is following the standard lines.

Now consider the special case of the Dirichlet problem:

$$B_j(D,\varepsilon) = (D_n)^{j-1}, \quad b_j = \beta_j = j-1.$$

In this case $r > m - 1/2$ and $\mu - 1/2 \le s < \mu - 1/2$, so we can take $s = \mu$. The estimate (3.30) takes form

$$||u,\mathbb{R}^n_+||_{r,\mu} \le C(||A(D,\varepsilon)u,\mathbb{R}^n_+||_{r-2m,-\mu} + ||u,\mathbb{R}^n_+||$$

$$+\sum_{j=1}^{\mu} ||(1+|D'|)^{\mu-j-1/2}(1+\varepsilon|D'|)^{r-\mu}(D_n)^{j-1}u(.,0),\mathbb{R}^{n-1}||$$

$$+\sum_{j=\mu+1}^{m} \varepsilon^{j-\mu-1/2}||(1+\varepsilon|D'|)^{r-j-1/2}(D_n)^{j-1}u(.,0),\mathbb{R}^{n-1}||).$$

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