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Sharp Estimates for the Number of Degrees of Freedom for the Damped-Driven 2-D Navier-Stokes Equations

A. A. Ilyin[1] and E. S. Titi[2]

[1] Keldysh Institute of Applied Mathematics, Russian Academy of Sciences, Miusskaya Sq. 4, 125047 Moscow, Russia
e-mail: ilyin@spp.keldysh.ru

[2] Department of Mathematics and Department of Mechanical and Aerospace Engineering, University of California, Irvine, CA 92697
e-mail: etiti@math.uci.edu
Also: Department of Computer Science and Applied Mathematics, Weizmann Institute of Science, P.O. Box 26, Rehovot, 76100, Israel
e-mail: edriss.titi@weizmann.ac.il

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Summary. We derive upper bounds for the number of asymptotic degrees (determining modes and nodes) of freedom for the two-dimensional Navier-Stokes system and Navier-Stokes system with damping. In the first case we obtain the previously known estimates in an explicit form, which are larger than the fractal dimension of the global attractor. However, for the Navier-Stokes system with damping, our estimates for the number of the determining modes and nodes are comparable to the sharp estimates for the fractal dimension of the global attractor. Our investigation of the damped-driven 2-D Navier-Stokes system is inspired by the Stommel-Charney barotropic model of ocean circulation where the damping represents the Rayleigh friction. We remark that our results equally apply to the viscous Stommel-Charney model.

Key words. Determining modes and nodes, fractal dimension, Stommel-Charney model, Navier-Stokes equations, barotropic ocean circulation model

MSC numbers. 35Q30, 76D05, 76U05, 76E20, 86A05, 86A10

1. Introduction

In this paper we derive estimates for the number of determining modes, nodes, and other determining projections for the two-dimensional Navier-Stokes system (2.1) and for the two-dimensional damped-driven Navier-Stokes system (3.1). The latter system is inspired by the viscous Stommel-Charney barotropic ocean circulation model [4], [35], [37]:

$$\partial_t u + \sum_{i=1}^{2} u^i \partial_i u + \mathbf{k}l \times u \ = \ -\mu u + \nu \Delta\, u - \nabla\, p + f,$$
$$\operatorname{div} u \ = \ 0, \tag{1.1}$$

where the damping μu represents the so-called Rayleigh friction term in the ocean circulation model. In recent years there has been some analytical study of the Stommel-Charney model from the dynamical systems point of view (see, for instance, [3], [5], [19], [20], [23], [27], [36], [41]). For the sake of clarity in our mathematical presentation and in order to make a straightforward comparison to the physical literature about the 2-D turbulence, we focus here on the Navier-Stokes system (2.1) and the damped-driven Navier-Stokes system (3.1). It worth stressing, however, that our results concerning the system (3.1), especially about determining modes, equally applies to the barotropic model (1.1), which we report in a forthcoming paper.

The history of determining modes goes back to [15], where it was shown that for a sufficiently large m the first m Fourier modes are determining for the 2-D Navier-Stokes system. This means, by definition, that two solutions converge as $t \to \infty$ provided that their m-dimensional projections converge. The Fourier projections have subsequently been generalized to projections onto an m-dimensional grid in the x-space (determining nodes) [16], m-dimensional averages on finite volumes [29] (determining finite volume elements), and general determining projections and functionals [8], [9] (see also [7]). For the Navier-Stokes equations, the best-to-date estimates for the number of explicitly identified determining modes, nodes, and projections in terms of the physical parameters can be found in [30], [8], and [9] (see also [13]). On the other hand, the reader is referred to [14] for improved estimates, which are comparable to the dimension of the global attractor, regarding no explicitly identified generic determining projections. Similar results are obtained in [17] and [18] for no explicitly identified generic determining nodes.

Our paper is organized as follows. In Section 2 we obtain previously known estimates reported in [8], [9] and [30] for the number of determining modes, nodes, and other determining projections for the two-dimensional space-periodic Navier-Stokes system that are linear with respect to the Grashof number. We use, however, the scalar vorticity formulation, which makes it possible to give all the estimates and constants in an explicit form. Furthermore, the dependence on the aspect ratio of the periodic domain is explicitly singled out. For the Dirichlet boundary conditions we obtain an explicit estimate for the number of determining modes that is quadratic with respect to the Grashof number (see, for example, [21], [31]). It is worth mentioning that these best-known estimates for the number of explicit determining modes and nodes of the 2-D Navier-Stokes equations without damping are still much larger than the dimension of the global attractor.

In Section 3 we consider the Navier-Stokes system with damping, subject to periodic boundary conditions and stress-free boundary conditions, and obtain estimates for the number of determining modes and nodes that are of the same order as the sharp estimates for the fractal dimension of the global attractor [27]. These remarkable estimates are extensive, that is, depend linearly on the area of the spatial domain, which is consistent with physical intuition. We remark, again, that such an observation is not known to exist in the case of the 2-D Navier-Stokes equations without damping.

Finally, in the Appendix in Section 5 we prove some auxiliary inequalities, namely, we derive sharp constant in the Agmon inequality on the two-dimensional torus and prove a variant of the embedding theorem reported in [30].

Since we are interested in obtaining explicit bounds about the constants involved in our asymptotic estimates for the numbers of degrees of freedom, we focus in this paper on the notions of determining modes and nodes. However, it is worth stressing that our asymptotic estimates, in terms of the physical parameters, are valid for other determining functionals and projections, as has been demonstrated in [7], [8], [9], with constants that may vary depending on the underlying chosen determining functionals.

2. Determining Modes and Nodes for Two-Dimensional Navier-Stokes Equations

Dirichlet Boundary Conditions

We consider in this section the two-dimensional Navier-Stokes system,

$$\begin{aligned} \partial_t u + \sum_{i=1}^{2} u^i \partial_i u \;&=\; \nu\Delta\, u - \nabla\, p + f, \\ \operatorname{div} u \;&=\; 0, \quad u(0) = u_0, \end{aligned} \tag{2.1}$$

where u is the velocity vector field satisfying Dirichlet boundary conditions $u|_{\partial\Omega} = 0$, p is the pressure, and $\nu > 0$ is the kinematic viscosity. The right-hand side $f = f(x, t)$ is given, and the domain Ω is an arbitrary open connected set in $\mathbb{R}^2$ with finite measure $|\Omega| < \infty$.

We use the standard notation and facts from the theory of Navier-Stokes equations (see, for instance, [10], [32], [39], [40]) and denote by P the Helmholtz-Leray orthogonal projection in $L_2(\Omega)^2$ onto the Hilbert space H, which is the closure in $L_2(\Omega)^2$ of the set of smooth solenoidal vector functions with compact supports in Ω. Applying P to the first equation in (2.1), we obtain

$$\partial_t u + B(u, u) + \nu A u = f, \quad u(0) = u_0, \tag{2.2}$$

where $A = -P\Delta$ is the Stokes operator, $B(u, v) = P\left(\sum_{i=1}^{2} u^i \partial_i v\right)$ is the nonlinear term, and $f = Pf \in H$.

Next, we denote by $\{\lambda_j\}_{j=1}^{\infty}$, $0 < \lambda_1 \le \lambda_2 \le \cdots$ and $\{w_j\}_{j=1}^{\infty}$ the eigenvalues and the corresponding eigenfunctions of the Stokes operator A: $A\, w_j = \lambda_j w_j$. The asymptotic behavior $\lambda_k \sim \frac{4\pi k}{|\Omega|}$ as $k \to \infty$ was established in [34], while in this work we use the following explicit nonasymptotic lower bounds for the eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ (see [24]):

$$\sum_{j=1}^{m} \lambda_j \ge \frac{\pi m^2}{|\Omega|}, \quad \text{and consequently,} \quad \lambda_m \ge \frac{\pi m}{|\Omega|},\ m \ge 1; \quad \lambda_1 \ge \frac{2\pi}{|\Omega|}. \tag{2.3}$$

The Hilbert space $V = D(A^{1/2})$ is the space $H_0^1(\Omega)^2 \cap H$ with norm

$$\|u\|_{D(A^{1/2})} = \|\nabla\, u\| = \|\operatorname{rot} u\|,$$

where $\|\cdot\| = \|\cdot\|_{L_2(\Omega)}$. The nonlinear operator $B(v, v)$ satisfies the well-known inequalities (see, for instance, [10], [32], [39], [40])

$$|(B(v,v),u)| = |(B(v,u),v)| \le c_1\|v\|\|\nabla v\|\|\nabla u\|,$$
$$|(B(u,v),w)| \le c_2\|u\|^{1/2}\|\nabla u\|^{1/2}\|\nabla v\|\|w\|^{1/2}\|\nabla w\|^{1/2}, \tag{2.4}$$

where it was shown in [6] that

$$c_1 =: c_b \le \left(\frac{8}{27\pi}\right)^{1/2}, \qquad c_2 \le \sqrt{2}c_b \le \left(\frac{16}{27\pi}\right)^{1/2}. \tag{2.5}$$

Let u and v be the solutions of the Navier-Stokes equations

$$\partial_t u + B(u,u) + \nu Au = f, \quad u(0) = u_0,$$
$$\partial_t v + B(v,v) + \nu Av = g, \quad v(0) = v_0, \tag{2.6}$$

where $f, g \in L_\infty(0,\infty; H)$.

We denote by P_m the L_2-orthogonal projection onto the space $\mathrm{Span}\{w_1, \dots, w_m\}$, and we set $Q_m = I - P_m$.

Definition 2.1. We call a set of modes $\{w_j\}_{j=1}^m$ determining (see [12], [15]) if

$$\lim_{t\to\infty} \|u(t) - v(t)\| = 0, \tag{2.7}$$

as long as

$$\lim_{t\to\infty} \|f(t) - g(t)\| = 0 \quad \text{and} \quad \lim_{t\to\infty} \|P_m(u(t) - v(t))\| = 0.$$

Accordingly, a set of points $\{x^i\}_{i=1}^N \subset \Omega$ is called a set of determining nodes (see [12], [16]) if (2.7) holds as long as

$$\lim_{t\to\infty} \|f(t) - g(t)\| = 0 \quad \text{and} \quad \lim_{t\to\infty} \eta(u(t) - v(t)) = 0,$$

where $\eta(w) = \max_{j=1,\dots,N} |w(x^j)|$.

We further suppose that

$$\limsup_{t\to\infty} \|f(t)\| =: \mathbf{f} < \infty. \tag{2.8}$$

Subtracting in (2.6) the second equation from the first and setting $w(t) = u(t) - v(t)$ and $h(t) = f(t) - g(t)$, we obtain

$$\partial_t w + \nu Aw + B(u,w) + B(w,u) - B(w,w) = h(t).$$

We write $w = p + q$, where $p = P_m w$, $q = Q_m w$, and take the scalar product with q:

$$\frac{1}{2}\partial_t\|q\|^2 + \nu\|\nabla q\|^2 + b(q,u,q) = (h,q) - b(u,p,q) - b(p,u,q)$$
$$+ b(p,p,q) + b(q,p,q),$$

where $b(u,v,w) = (B(u,v),w)$.

A variant of the Gronwall lemma [12], [13], [30] is essential in the estimates below.

Lemma 2.1. *Suppose that $\alpha(t)$ and $\beta(t)$ are locally integrable functions on $(0, \infty)$ satisfying for some $T > 0$ the following conditions:*

$$\liminf_{t\to\infty} \frac{1}{T}\int_t^{t+T} \alpha(\tau)\, d\tau = \gamma,$$

$$\limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} \alpha^-(\tau)\, d\tau = \Gamma,$$

$$\lim_{t\to\infty} \frac{1}{T}\int_t^{t+T} \beta^+(\tau)\, d\tau = 0,$$

where $\gamma > 0$, $\Gamma < \infty$ and $\alpha^- = \max\{-\alpha, 0\}$, $\beta^+ = \max\{\beta, 0\}$. If $\xi(t) \geq 0$, an absolutely continuous function, satisfies

$$\xi' + \alpha\xi \leq \beta \quad \text{on} \quad (0, \infty),$$

then $\xi(t) \to 0$ as $t \to \infty$.

All the terms on the right-hand side containing h or p can be absorbed in the function $\beta(t)$ in Lemma 2.1. For example, using the second inequality in (2.4), we obtain

$$\begin{aligned}|b(u, p, q)| &\leq c_2\|u\|^{1/2}\|\nabla u\|^{1/2}\|\nabla p\|\|q\|^{1/2}\|\nabla q\|^{1/2}\\ &\leq c_2(\lambda_m/\lambda_1)^{1/2}\|p\|\|\nabla u\|\|\nabla q\| \leq (c_2/2)(\lambda_m/\lambda_1)^{1/2}\|p\|(\|\nabla u\|^2 + \|\nabla q\|^2).\end{aligned}$$

Our claim follows since $\|p(t)\| \to 0$ and in view of (2.9),

$$\begin{aligned}&\frac{1}{T}\int_t^{t+T} \|p(\tau)\|(\|\nabla u(\tau)\|^2 + \|\nabla q(\tau)\|^2)\, d\tau\\ &\quad\leq \max_{\tau\in[t,t+T]} \|p(\tau)\| \frac{1}{T}\int_t^{t+T} (\|\nabla u(\tau)\|^2 + \|\nabla q(\tau)\|^2)\, d\tau \to 0 \quad \text{as} \quad t \to \infty.\end{aligned}$$

The remaining terms can be treated in exactly the same way. Therefore we obtain

$$\partial_t\|q\|^2 + 2\nu\|\nabla q\|^2 + 2b(q, u, q) \leq \beta(t),$$

where $\beta(t) \to 0$ as $t \to \infty$.

Using the first inequality in (2.4), we have

$$2|b(q, u, q)| \leq \nu\|\nabla q\|^2 + c_b^2\nu^{-1}\|q\|^2\|\nabla u\|^2,$$

and then using the Poincaré inequality $\lambda_{m+1}\|q\|^2 \leq \|\nabla q\|^2$, we obtain

$$\partial_t\|q\|^2 + \alpha(t)\|q\|^2 \leq \beta(t), \quad \text{where} \quad \alpha(t) = \nu\lambda_{m+1} - \nu^{-1}c_b^2\|\nabla u(t)\|^2.$$

Since by the well-known estimates for the Navier-Stokes system (see, for instance, [10], [40]),

$$\limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} \|\nabla u(\tau)\|^2\, d\tau \leq \frac{\mathbf{f}^2}{T\nu^3\lambda_1^2} + \frac{\mathbf{f}^2}{\nu^2\lambda_1}, \tag{2.9}$$

it follows that α satisfies the conditions of Lemma 2.1 provided that T is sufficiently large and

$$\lambda_{m+1} > \frac{c_b^2 \mathbf{f}^2}{\nu^4 \lambda_1}. \tag{2.10}$$

In view of (2.3) and (2.4), (2.5), we have proved the following theorem.

Theorem 2.1. *The first m eigenfunctions of the Stokes operator are determining for the two-dimensional Navier-Stokes system with Dirichlet boundary conditions if*

$$m + 1 > \frac{4}{27\pi^3} G^2, \quad \textit{where} \quad G = \frac{\mathbf{f}|\Omega|}{\nu^2}. \tag{2.11}$$

Remark 2.1. The above theorem without explicit value of the constant has been mentioned as a remark in [21] and was also proved in [12] and [31].

Periodic Boundary Conditions: Determining Modes

We now consider the Navier-Stokes system (2.1) with space-periodic boundary conditions $x \in \Omega = [0, L_1] \times [0, L_2]$. We set $L_2 = L$ and $L_1 = L/\gamma$. Without loss of generality we assume that $\gamma \le 1$. As before, $|\Omega|$ denotes the measure of the periodic domain Ω: $|\Omega| = L_1 L_2 = L^2/\gamma$.

We further assume that u, p, and f have mean value zero over the torus. Applying the rot (curl) operator to the first equation in (2.1), we obtain the well-known vorticity equation

$$\partial_t \varphi + J(\Delta^{-1}\varphi, \varphi) - \nu \Delta \varphi = \operatorname{rot} f, \tag{2.12}$$

where $\operatorname{rot} u = \varphi$, $u = \nabla^\perp \Delta^{-1}\varphi$, $J(a, b) = \partial_1 a \partial_2 b - \partial_2 a \partial_1 b = \nabla^\perp a \cdot \nabla b$, $\nabla^\perp \psi = \mathbf{k} \times \nabla \psi = (-\partial_2 \psi, \partial_1 \psi)$, and $\mathbf{k}$ is the vertical unit vector.

We now recall that the spectrum $\{\lambda_j\}_{j=1}^\infty$ of the Stokes operator with periodic boundary conditions coincides with that of the negative scalar Laplacian

$$-\Delta \varphi_j = \lambda_j \varphi_j,$$

and the corresponding eigenfunctions are as follows:

$$A\, w_j = \lambda_j w_j, \qquad w_j = \lambda_j^{-1/2} \nabla^\perp \varphi_j = \lambda_j^{-1/2}(-\partial_2 \varphi_j, \partial_1 \varphi_j).$$

Therefore the modes $\{w_1, \dots, w_m\}$ are determining for the Navier-Stokes system (2.1) with periodic boundary conditions if the modes $\{\varphi_1, \dots, \varphi_m\}$ are determining for equation (2.12).

Similarly to (2.6) we write

$$\begin{aligned} \partial_t \varphi + J(\Delta^{-1}\varphi, \varphi) - \nu \Delta \varphi &= \operatorname{rot} f(t), \\ \partial_t \psi + J(\Delta^{-1}\psi, \psi) - \nu \Delta \psi &= \operatorname{rot} g(t). \end{aligned} \tag{2.13}$$

Setting $\omega = \varphi - \psi$ and $H(t) = \operatorname{rot} f(t) - \operatorname{rot} g(t)$, we obtain for ω the equation

$$\partial_t \omega - \nu \Delta \omega + J(\Delta^{-1}\varphi, \omega) + J(\Delta^{-1}\omega, \varphi) - J(\Delta^{-1}\omega, \omega) = H. \tag{2.14}$$

As before, we write $\omega = p + q$, where $p = P_m\omega$ and $q_m = Q_m\omega$ and where P_m is the orthogonal projection $P_m : L_2(\Omega) \to \mathrm{Span}(\varphi_1, \dots, \varphi_m)$. Taking the scalar product with q, we obtain

$$\begin{aligned}\frac{1}{2}\partial_t\|q\|^2 + \nu\|\nabla q\|^2 + (J(\Delta^{-1}q,\varphi), q) \\ &= (H, q) - (J(\Delta^{-1}\varphi, p), q) - (J(\Delta^{-1}p,\varphi), q) \\ &\quad + (J(\Delta^{-1}p, p), q) + (J(\Delta^{-1}q, p), q) \\ &=: \beta(t).\end{aligned}$$

As before, all the terms on the right-hand side containing p can be absorbed in $\beta(t)$ in Lemma 2.1. Next we have

$$\begin{aligned}|(J(\Delta^{-1}q,\varphi), q)| &\le \int |\nabla\Delta^{-1}q||\nabla\varphi|\,|q|\,dx \le \|\nabla\Delta^{-1}q\|_{L_4}\|\nabla\varphi\|\|q\|_{L_4} \\ &\le c_{\mathrm{L}}(\gamma)^2\|\nabla\Delta^{-1}q\|^{1/2}\|q\|\|\nabla q\|^{1/2}\|\nabla\varphi\| \\ &\le \lambda_m^{-1/2}c_{\mathrm{L}}(\gamma)^2\|q\|\|\nabla\varphi\|\|\nabla q\|,\end{aligned}$$

where we used the Ladyzhenskaya inequality (see, for instance, [10], [32], [39])

$$\|\varphi\|_{L_4} \le c_{\mathrm{L}}(\gamma)\|\varphi\|^{1/2}\|\nabla\varphi\|^{1/2}, \quad \|\nabla\varphi\|_{L_4} \le c_{\mathrm{L}}(\gamma)\|\nabla\varphi\|^{1/2}\|\Delta\varphi\|^{1/2}.$$

We arrive at the same estimate if we use the integral identity

$$(J(f, g), h) = (J(h, f), g)$$

and the Agmon inequality (see Theorem 5.1)

$$\|\varphi\|_\infty \le c_{\mathrm{AT}}(\gamma)\|\varphi\|^{1/2}\|\Delta\varphi\|^{1/2}.$$

In fact,

$$\begin{aligned}|(J(\Delta^{-1}q,\varphi), q)| &= |(J(\varphi, q), \Delta^{-1}q)| \le \|\Delta^{-1}q\|_\infty\|\nabla\varphi\|\|\nabla q\| \\ &\le c_{\mathrm{AT}}(\gamma)\|\Delta^{-1}q\|^{1/2}\|q\|^{1/2}\|\nabla q\|\|\nabla\varphi\| \\ &\le \lambda_m^{-1/2}c_{\mathrm{AT}}(\gamma)\|q\|\|\nabla\varphi\|\|\nabla q\|,\end{aligned}$$

which gives

$$|(J(\Delta^{-1}q,\varphi), q)| \le \lambda_m^{-1/2}c_{\mathrm{J}}\|q\|\|\nabla\varphi\|\|\nabla q\|, \quad \text{where} \quad c_{\mathrm{J}} = \min(c_{\mathrm{L}}(\gamma)^2, c_{\mathrm{AT}}(\gamma)).$$

As before we obtain the differential inequality

$$\partial_t\|q\|^2 + \alpha(t)\|q\|^2 \le 2\beta(t), \quad \text{where} \quad \alpha(t) = \nu\lambda_{m+1} - \nu^{-1}\lambda_{m+1}^{-1}c_{\mathrm{J}}^2\|\nabla\varphi(t)\|^2.$$

It follows from the well-known a priori estimate on the time average of the H^2-norm of a solution u (see, for instance, [2], [10], [30], [40])

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\|\nabla\varphi(\tau)\|^2\,d\tau &= \limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\|A\,u(\tau)\|^2\,d\tau \\ &\le \frac{\mathbf{f}^2}{T\nu^3\lambda_1} + \frac{\mathbf{f}^2}{\nu^2}\end{aligned} \tag{2.15}$$

that α satisfies conditions of Lemma 2.1 provided that T is sufficiently large and

$$\lambda_{m+1}^2 > \frac{c_{\mathrm{J}}^2 \mathbf{f}^2}{\nu^4}. \tag{2.16}$$

It was shown in [27], [26] that $c_{\mathrm{L}}(\gamma) \le (6/(\gamma\pi))^{1/4}$. In the Appendix (Section 5) we will show that $c_{\mathrm{AT}}(\gamma) \le 1/\sqrt{\gamma\pi}$. Hence we can take $c_{\mathrm{J}} = 1/\sqrt{\gamma\pi}$. Furthermore, for $\gamma = 1$, $\lambda_m \ge (\lambda_1/4)m$, where $\lambda_1 = 4\pi^2 L^{-2}$. We obtain the following theorem.

Theorem 2.2. *The first m eigenfunctions of the Stokes operator are determining for the two-dimensional Navier-Stokes system with periodic boundary conditions if*

$$\lambda_{m+1} > \left(\frac{1}{\gamma\pi}\right)^{1/2} \frac{\mathbf{f}}{\nu^2}. \tag{2.17}$$

For a square torus ($\gamma = 1$), this condition is satisfied if

$$m + 1 > \frac{1}{\pi^{3/2}}\, G, \quad \textit{where} \quad G = \frac{\mathbf{f} L^2}{\nu^2}. \tag{2.18}$$

Remark 2.2. The first eigenvalues $\lambda_1, \lambda_2, \ldots$ of the Laplacian on the periodic domain $\Omega = [0, L/\gamma] \times [0, L]$ are of order γ^2 when $\gamma \ll 1$. It was shown in [26] (see Proposition 4.1) that if $m \ge 2/\gamma$, then

$$\lambda_m \ge \frac{m\gamma}{8} \cdot \frac{4\pi^2}{L^2} = \frac{\pi^2}{2} \cdot \frac{m}{|\Omega|}. \tag{2.19}$$

Therefore condition (2.17) is satisfied if

$$m + 1 > \frac{2}{\gamma} + \frac{2}{\pi^2} \left(\frac{1}{\gamma\pi}\right)^{1/2} \frac{\mathbf{f}|\Omega|}{\nu^2}. \tag{2.20}$$

Determining Nodes

Suppose that the periodic domain Ω is divided into N equal squares with side of length l and for each square there is a point x^j, $j = 1, \ldots, N$ chosen arbitrarily in it. For the two solutions u and v of the Navier-Stokes equations we assume that

$$\eta(w(t)) = \max_{j=1,\ldots,N} |w(t, x^j)| \to 0 \quad \text{as} \quad t \to \infty, \quad \text{where} \quad w = u - v.$$

We take the scalar product of (2.14) and ω:

$$\partial_t \|\omega\|^2 + 2\nu \|\nabla \omega\|^2 = \beta(t) - 2(J(\Delta^{-1}\omega, \varphi), \omega), \quad \text{where} \quad \beta(t) = 2(H, \omega).$$

For the nonlinear term using inequalities (5.14), (5.17) and Young's inequality, we have

$$\begin{aligned} 2|(J(\Delta^{-1}\omega, \varphi), \omega)| &\le 2\|\nabla \Delta^{-1}\omega\|_\infty \|\nabla \varphi\| \|\omega\| \\ &\le 2c_{\mathrm{AT}}(\gamma) \|\nabla \Delta^{-1}\omega\|^{1/2} \|\nabla \omega\|^{1/2} \|\omega\| \|\nabla \varphi\| \\ &\le 2c_{\mathrm{AT}}(\gamma)(4|\Omega|)^{1/4} \eta^{1/2}(w) \|\nabla \omega\|^{1/2} \|\omega\| \|\nabla \varphi\| \\ &\quad + 2c_{\mathrm{AT}}(\gamma) 68^{1/4} l \|\nabla \omega\| \|\omega\| \|\nabla \varphi\| \\ &\le \beta_1(t) + c_{\mathrm{AT}}(\gamma)^2 68^{1/2} l^2 \nu^{-1} \|\omega\|^2 \|\nabla \varphi\|^2 + \nu \|\nabla \omega\|^2. \end{aligned}$$

Hence,

$$\partial_t \|\omega\|^2 + \nu \|\nabla \omega\|^2 \le \beta_1(t) + c_{\rm AT}(\gamma)^2 68^{1/2} l^2 \nu^{-1} \|\omega\|^2 \|\nabla \varphi\|^2.$$

Using inequality (5.16) in the form $\|\nabla \omega\|^2 \ge 68^{-1/2} l^{-2} \|\omega\|^2 - \beta_2(t)$ to bound from below the second term on the left-hand side, we obtain

$$\partial_t \|\omega\|^2 + \alpha(t) \|\omega\|^2 \le \beta(t),$$

where

$$\alpha(t) = 68^{-1/2} \nu l^{-2} - c_{\rm AT}(\gamma)^2 68^{1/2} \nu^{-1} l^2 \|\nabla \varphi\|^2.$$

Taking into account (2.15), we see that if the number of squares $N = |\Omega|/l^2$ is sufficiently large (or, equivalently, the typical distance l between the nodes is sufficiently small), then α satisfies conditions of Lemma 2.1 and the corresponding N nodes are determining. We obtain the following result.

Theorem 2.3. *If*

$$N > \left(\frac{68}{\gamma\pi}\right)^{1/2} \frac{\mathbf{f}|\Omega|}{\nu^2} = \left(\frac{68}{\gamma^3\pi}\right)^{1/2} \frac{\mathbf{f}L^2}{\nu^2} \tag{2.21}$$

equal squares tile $\Omega = [0, L/\gamma] \times [0, L]$*, then N nodes* (*chosen arbitrarily, one in each square*) *are determining for the space-periodic Navier-Stokes system in* Ω.

Remark 2.3. The estimates for determining modes and nodes of Theorems 2.2 and 2.3 were obtained for $\gamma = 1$ in [30].

3. Determining Modes and Nodes for Damped Navier-Stokes Equations

In this section we consider the damped-driven Navier-Stokes system's important applications in geophysical hydrodynamics [11], [35]:

$$\begin{aligned} \partial_t u + \sum_{i=1}^{2} u^i \partial_i u &= \nu \Delta u - \mu u - \nabla p + f, \\ \operatorname{div} u &= 0, \quad u(0) = u_0. \end{aligned} \tag{3.1}$$

Periodic Boundary Conditions

We first consider this system on the torus $x \in \Omega = [0, L/\gamma] \times [0, L]$, with space-periodic boundary conditions. The right-hand side, $f = f(t)$, satisfies the condition

$$\limsup_{t\to\infty} \|\operatorname{rot} f(t)\|_\infty =: \mathbf{F}_\infty < \infty. \tag{3.2}$$

Lemma 3.1. *The following bound holds for* $u(t)$*:*

$$\limsup_{t\to\infty} \|\operatorname{rot} u(t)\|_\infty \le \frac{\mathbf{F}_\infty}{\mu}. \tag{3.3}$$

Proof. We use the vorticity formulation of the system (3.1)

$$\partial_t \varphi + J(\Delta^{-1}\varphi, \varphi) - \nu \Delta \varphi + \mu\varphi = \operatorname{rot} f, \tag{3.4}$$

and take the scalar product with φ^{2k-1}, where $k \geq 1$ is integer, and use the identity $(J(\psi, \varphi), \varphi^{2k-1}) = (2k)^{-1} \int J(\psi, \varphi^{2k})\, dx = (2k)^{-1} \int \operatorname{div}(\varphi^{2k} \nabla^{\perp}\psi)\, dx = 0$. We obtain

$$\begin{aligned}\|\varphi\|_{L_{2k}}^{2k-1} \partial_t \|\varphi\|_{L_{2k}} + (2k-1)\nu \int |\nabla \varphi|^2 \varphi^{2k-2}\, dx + \mu \|\varphi\|_{L_{2k}}^{2k} &= (\operatorname{rot} f(t), \varphi^{2k-1}) \\ &\leq \|\operatorname{rot} f(t)\|_{L_{2k}} \|\varphi\|_{L_{2k}}^{2k-1}.\end{aligned}$$

Hence, by Gronwall's inequality,

$$\|\varphi(t+\tau)\|_{L_{2k}} \leq \|\varphi(\tau)\|_{L_{2k}} e^{-\mu t} + \mu^{-1} \sup_{s \in [\tau, \infty)} \|\operatorname{rot} f(s)\|_{L_{2k}} (1 - e^{-\mu t}),$$

and passing to the limit as $k \to \infty$, we find

$$\|\varphi(t+\tau)\|_{\infty} \leq \|\varphi(\tau)\|_{\infty} e^{-\mu t} + \mu^{-1} \sup_{s \in [\tau, \infty)} \|\operatorname{rot} f(s)\|_{\infty} (1 - e^{-\mu t}).$$

Now, we let $t \to \infty$ to obtain

$$\limsup_{t \to \infty} \|\varphi(t)\|_{\infty} \leq \frac{\mathbf{F}_{\infty}}{\mu}. \qquad \square$$

We consider the systems (3.1) with right-hand sides f and g such that

$$\lim_{t \to \infty} \|\operatorname{rot}(f(t) - g(t))\|_{\infty} = 0.$$

Similarly to (2.13) for $\omega = \varphi - \psi$, we obtain the equation

$$\partial_t \omega - \nu \Delta \omega + \mu\omega + J(\Delta^{-1}\varphi, \omega) + J(\Delta^{-1}\omega, \varphi) - J(\Delta^{-1}\omega, \omega) = H, \tag{3.5}$$

where $H(t) = \operatorname{rot} f(t) - \operatorname{rot} g(t)$.

Determining Modes

We take the scalar product of (3.5) with $q = Q_m \omega$:

$$\begin{aligned}\partial_t \|q\|^2 + 2\nu \|\nabla q\|^2 + 2\mu \|q\|^2 &\leq 2\beta(t) + 2|(J(\Delta^{-1}q, \varphi), q)| \\ &\leq 2\beta(t) + 2\|\nabla q\| \|\nabla \Delta^{-1} q\| \|\varphi\|_{\infty} \\ &\leq 2\beta(t) + \nu \|\nabla q\|^2 + \nu^{-1} \lambda_{m+1}^{-1} \|q\|^2 \|\varphi\|_{\infty}^2,\end{aligned}$$

where we used Young's and Poincaré inequalities. Dropping the μ-term on the left-hand side and again using the Poincaré inequality, we obtain

$$\partial_t \|q\|^2 + \|q\|^2 (\nu \lambda_{m+1} - \nu^{-1} \lambda_{m+1}^{-1} \|\varphi(t)\|_{\infty}^2) \leq 2\beta(t).$$

By estimate (3.3) and Lemma 2.1, the first m modes are determining provided

$$\lambda_{m+1} \geq \frac{\mathbf{F}_\infty}{\mu\nu}. \tag{3.6}$$

Using (2.19) we see that this condition is satisfied if

$$m+1 > \max\left\{\frac{2}{\gamma},\ \frac{2}{\pi^2}\frac{\mathbf{F}_\infty|\Omega|}{\mu\nu}\right\}. \tag{3.7}$$

For a square torus $\lambda_m \geq \lambda_1/4$, $\lambda_1 = 4\pi^2/L^2$, and hence condition (3.6) is satisfied if

$$m+1 \geq \frac{1}{\pi^2}\frac{\mathbf{F}_\infty L^2}{\mu\nu}. \tag{3.8}$$

Determining Nodes

Suppose that our periodic domain Ω is divided into N equal squares Q_j with side of length l, $j = 1, \dots, N$, and we chose arbitrarily a point $x^j \in Q_j$ for each $j = 1, \dots, N$.

For $u \in H^2_{\mathrm{per}}(\Omega)$, we set

$$\eta(u) = \max_{j=1,\dots,N} |u(x^j)|. \tag{3.9}$$

Suppose that $\eta(w(t)) \to 0$ as $t \to \infty$ for $w(t) = u(t) - v(t)$, where u and w are two solutions of (3.1).

We take the scalar product of (3.5) with ω:

$$\frac{1}{2}\partial_t\|\omega\|^2 + \nu\|\nabla\omega\|^2 + \mu\|\omega\|^2 = (H,\omega) - (J(\Delta^{-1}\omega,\varphi),\omega).$$

We estimate the nonlinear term by means of inequality (5.17):

$$\begin{aligned}|(J(\Delta^{-1}\omega,\varphi),\omega)| &= |(J(\omega,\Delta^{-1}\omega),\varphi)| \leq \|\nabla\omega\|\|\nabla\Delta^{-1}\omega\|\|\varphi\|_\infty \\ &\leq 2|\Omega|^{1/2}\eta(w)\|\nabla\omega\|\|\varphi\|_\infty + \sqrt{68}|\Omega|N^{-1}\|\nabla\omega\|^2\|\varphi\|_\infty.\end{aligned}$$

As a result we obtain

$$\frac{1}{2}\partial_t\|\omega\|^2 + \alpha(t)\|\nabla\omega\|^2 \leq \beta(t),$$

where

$$\alpha(t) = \nu - \sqrt{68}|\Omega|N^{-1}\|\varphi(t)\|_\infty, \qquad \beta(t) = (H(t),\omega) + 2|\Omega|^{1/2}\eta(w(t))\|\nabla\omega\|\|\varphi(t)\|_\infty.$$

As before, $\beta(t) \to 0$ as $t \to \infty$, while in view of (3.3) $\alpha(t) \geq \text{const} > 0$ for all t large enough provided that $N > \sqrt{68}\,\mathbf{F}_\infty|\Omega|/(\mu\nu)$.

We combine the results obtained above in the following theorem.

Theorem 3.1. *The first m modes of the Stokes operator are determining for the space-periodic Navier-Stokes system with damping* (3.1) *if*

$$m + 1 > \frac{2}{\gamma} + \frac{2}{\pi^2}\frac{\mathbf{F}_\infty|\Omega|}{\mu\nu}. \tag{3.10}$$

If Ω *is tiled by* N *equal squares, then any collection of nodes* (*one in each square*) *is determining if*

$$N > \sqrt{68}\,\frac{\mathbf{F}_\infty|\Omega|}{\mu\nu}. \tag{3.11}$$

Remark 3.1. We observe that estimates for the number of the determining modes and nodes (3.7) and (3.11) depend linearly on the measure of the periodic domain $|\Omega|$ and depend on the aspect ratio γ of the torus only via $|\Omega| = L^2/\gamma$. Furthermore, the characteristic microscopic length l of the lattice of the determining nodes satisfies the following Ω-independent estimate from above:

$$l < 68^{-1/4}\left(\frac{\mu\nu}{\mathbf{F}_\infty}\right)^{1/2}. \tag{3.12}$$

Remark 3.2. It was shown in [27] that the fractal dimension of the global attractor of the autonomous system (3.1) on the torus $[0, L]^2$ satisfies the estimate

$$\dim_F \mathcal{A} \le \left(\frac{6}{\pi^3}\right)^{1/2}\frac{\|\operatorname{rot} f\|L}{\mu\nu}. \tag{3.13}$$

It was also shown that for the Kolmogorov forcing of the form

$$f = f_s = \begin{cases} f_1 = A(\mu,\nu)\sin s\dfrac{2\pi x_2}{L}, \\ f_2 = 0, \end{cases} \tag{3.14}$$

one has a lower bound

$$\dim_F \mathcal{A} \ge \text{const}\frac{\|\operatorname{rot} f\|L}{\mu\nu} \tag{3.15}$$

in the limit $\nu \to 0^+$ (accordingly, $s = (\mu L^2/\nu)^{1/2} \to \infty$).

Since $\|\operatorname{rot} f\| \le \|\operatorname{rot} f\|_\infty L$, it follows that

$$\dim_F \mathcal{A} \le \left(\frac{6}{\pi^3}\right)^{1/2}\frac{\|\operatorname{rot} f\|_\infty L^2}{\mu\nu}, \tag{3.16}$$

and since $\|\operatorname{rot} f_s\| = (L/\sqrt{2})\|\operatorname{rot} f_s\|_\infty$, it follows from (3.15) that the estimate (3.16) is also sharp with respect to the dimensionless number $\frac{\|\operatorname{rot} f\|_\infty L^2}{\mu\nu}$.

Hence the bounds (3.8) and (3.11) for the number of determining modes and nodes for the damped Navier-Stokes system are of the same order as the fractal dimension of the global attractor.

We point out that in general and for the Navier-Stokes equations there is a gap between the number of the determining modes and nodes and the dimension of the global attractor. However, the works [14], [22], and [18] indicate that one can perturb the points or the projections to obtain the number of nodes and the rank of the projections comparable with the dimension of the global attractor. Here, however, we show that there is no need for perturbation and that the usual projections P_m and any choice of points will do. It will therefore be interesting to understand the role of the damping term here in terms of the generic results of [14], [22], [18], which rely heavily on the Mãné embedding theorem.

Stress-Free Boundary Conditions

Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain with C^2 boundary. Let n be the outward unit normal vector. We consider the system (3.1) supplemented with the so-called stress-free boundary conditions

$$u \cdot n|_{\partial\Omega} = 0, \qquad \operatorname{rot} u|_{\partial\Omega} = 0. \tag{3.17}$$

Then any smooth vector field u, $\operatorname{div} u = 0$, satisfying (3.17) has a unique single-valued stream function ψ, $u = \nabla^{\perp}\psi$ with $\psi|_{\partial\Omega} = 0$ and $\Delta\,\psi|_{\partial\Omega} = \operatorname{rot} u|_{\partial\Omega} = 0$. Therefore, the vorticity formulation for the system (3.1), (3.17) is the equation (3.4) with zero boundary condition both for φ and ψ:

$$\begin{aligned} \partial_t\varphi + J(\psi,\varphi) \;-\; \nu\Delta\,\varphi + \mu\varphi &= \operatorname{rot} f, \\ \Delta\,\psi \;&=\; \varphi, \\ \varphi|_{\partial\Omega} \;&=\; \psi|_{\partial\Omega} = 0. \end{aligned} \tag{3.18}$$

For the Stokes eigenvalue problem with boundary conditions (3.17)

$$\begin{aligned} -\Delta\, w_k + \nabla\, p_k = \lambda_k w_k, \quad \operatorname{div} w_k = &= 0, \\ w_k \cdot n|_{\partial\Omega} = 0, \quad \operatorname{rot} w_k|_{\partial\Omega} &= 0, \end{aligned} \tag{3.19}$$

we have (as in the case of periodic boundary conditions) that $\{\lambda_k\}_{k=1}^{\infty}$ are the eigenvalues of the scalar Dirichlet problem $-\Delta\,\varphi_k = \lambda_k\varphi_k$, $\varphi_k|_{\partial\Omega} = 0$ and $w_k = \lambda_k^{-1/2}\nabla^{\perp}\varphi_k$. Hence we can use the Li-Yau lower bound [33] for the eigenvalues λ_k

$$\lambda_k \geq \frac{2\pi k}{|\Omega|}. \tag{3.20}$$

Lemma 3.1 and the subsequent argument for estimates of the determining modes still hold, and we obtain that condition (3.6) is sufficient for the first m modes to be determining. In view of (3.20), we obtain the following result.

Theorem 3.2. *The first m modes of the Stokes operator are determining for the Navier-Stokes system with damping* (3.1) *with stress-free boundary conditions* (3.17) *if*

$$m + 1 > \frac{1}{2\pi}\frac{\mathbf{F}_{\infty}|\Omega|}{\mu\nu}.$$

Remark 3.3. As in the space-periodic case, this estimate agrees with the estimate for the fractal dimension of the attractor [27] but, unlike the latter, does not involve constants depending on the smoothness and shape of the boundary.

Remark 3.4. A similar result holds for the determining nodes and other determining functionals and projections (see [8], [9]) if we use extension operators mapping Sobolev spaces defined on Ω to the spaces defined on corresponding periodic rectangular domain containing Ω. In this case, however, the estimate involves a constant depending on the smoothness and shape of the boundary.

4. Concluding Remarks

The damped-driven Navier-Stokes system has important geophysical applications and is essentially the viscous Charney-Stommel model of ocean circulation. In this paper we provide explicit bounds for the number of determining degrees of freedom for this system and study the effect of the damping/drag/friction term $-\mu u$.

For the situation without damping, i.e. $\mu = 0$, we recover the previously known estimates for the number of explicitly identified determining modes and nodes. These estimates are of the order G^2 for the case of no-slip Dirichlet boundary conditions, and of the order G for the case of periodic boundary conditions, where G is the Grashof number. Unlike the previous results, all the constants involved in our estimates are given in an explicit form. We also observe that the case without damping these estimates is much larger than the estimates for the fractal dimension of the global attractor (see, however, [14], [17], and [18] for comparable estimates to the dimension of the global attractor for the number of none explicitly identified determining modes and nodes).

We observe that the addition of the damping/drag/friction term $-\mu u$, i.e. $\mu > 0$, drastically changes the situation. Here we are able to derive in the space-periodic case upper bounds for the number of determining modes and nodes that are of the order $\|\operatorname{rot} f\|_\infty |\Omega|/(\mu\nu)$. A remarkable fact is that these estimates depend linearly on the area $|\Omega|$ of the periodic rectangular domain and agree with the sharp estimate of the fractal dimension of the global attractor for this system, which was obtained previously by the authors in [27].

5. Appendix. Proof of Auxiliary Inequalities

The embedding of the Sobolev space $H^l(M)$ with norm $\|u\|^2_{H^l} = \|u\|^2 + \|(-\Delta)^{l/2}u\|^2$ into the space of bounded continuous functions $C(M)$, where $\dim M = n$ and $l > n/2$, can be written as a multiplicative inequality

$$\|u\|_\infty \le c_M(l)\|u\|^\theta \|(-\Delta)^{l/2}u\|^{1-\theta}, \quad \text{where} \quad \theta = (2l - n)/2l. \tag{5.1}$$

Inequalities of this type are sometimes called the Agmon inequalities (see [1]). The best constant $c_M(l)$ for $M = \mathbb{R}$ in this inequality was found in [38] (the results of [38] can easily be generalized to the case when $M = \mathbb{R}^n$). Sharp constants in inequalities for periodic functions and functions defined on the sphere were found in [25]. Following [25], we consider below the case of a two-dimensional torus.

The constant in inequality (5.1) on a two-dimensional torus clearly depends only on the aspect ratio γ of the torus. We first consider the case of a square torus $\gamma = 1$, and then without loss of generality we assume that $\Omega = \mathrm{T}^2 = [0, 2\pi]^2$.

We consider the negative Laplacian $-\Delta$ in $H = L_2(\mathrm{T}^2) \cap \{\varphi,\ \int \varphi\, dx = 0\}$ and order its eigenvalues according to magnitude and multiplicity:

$$1 = \lambda_1 \le \lambda_2 \le \cdots, \qquad \{\lambda_j,\ j = 1, \ldots\} = \{k^2 = k_1^2 + k_2^2,\ k = (k_1, k_2) \in \mathbb{Z}_0^2\}, \tag{5.2}$$

where $\mathbb{Z}_0^2 = \mathbb{Z}^2 \backslash \{0\}$. The corresponding basis of orthonormal eigenfunctions $w_j(x)$, $-\Delta\, w_j = \lambda_j w_j$, is the basis of trigonometric functions

$$\bigcup_{j\in\mathbb{N}} w_j(x) = \bigcup_{k\in\mathbb{Z}_+^2} \left\{ (\sqrt{2}\pi)^{-1} \sin kx, \quad (\sqrt{2}\pi)^{-1} \cos kx \right\},$$

$$\mathbb{Z}_+^2 = \{k \in \mathbb{Z}_0^2, \quad k_1 \ge 0,\ k_2 \ge 0\} \cup \{k \in \mathbb{Z}_0^2, \quad k_1 \ge 1,\ k_2 \le 0\}. \tag{5.3}$$

Similarly to (5.2), we write

$$1 = \Lambda_1 \le \Lambda_2 \le \ldots, \qquad \{\Lambda_j,\ j = 1, \ldots\} = \{k^2,\ k \in \mathbb{Z}_+^2\}, \tag{5.4}$$

and observe that

$$\bigcup_{j=1}^{\infty} \{\lambda_j\} = \bigcup_{l=1}^{\infty} \{\Lambda_l, \Lambda_l\}. \tag{5.5}$$

Hence, for $j \ge 1$, we have $\Lambda_j = \lambda_{2j} = \lambda_{2j-1}$ and, corresponding to each $\Lambda = \Lambda_j$, there are two eigenfunctions $u_j(x) = (\sqrt{2}\pi)^{-1} \sin kx$ and $v_j(x) = (\sqrt{2}\pi)^{-1} \cos kx$ for some uniquely defined $k_j = (k_1(j), k_2(j))$ with $k_j^2 = \Lambda_j$. We obviously have

$$u_j(x)^2 + v_j(x)^2 = \frac{1}{2\pi^2}. \tag{5.6}$$

Theorem 5.1. *The sharp constant c_{AT} in the inequality*

$$\|\varphi\|_\infty \ \le\ c_{\mathrm{AT}} \|\varphi\|^{1/2} \|\Delta\, \varphi\|^{1/2}, \quad \varphi \in H \cap H^2_{\mathrm{per}}(\mathrm{T}^2), \tag{5.7}$$

is given by

$$c_{\mathrm{AT}}^2 = \frac{1}{\pi^2} \sup_{\mu>0} \mu \sum_{n=1}^{\infty} \frac{1}{\mu^2 + \Lambda_n^2}, \tag{5.8}$$

and, in particular,

$$c_{\mathrm{AT}}^2 \ <\ \frac{1}{\pi}. \tag{5.9}$$

Proof. Writing φ in terms of the Fourier series $\varphi(x)=\sum_{n=1}^{\infty} c_n w_n(x)$, for an arbitrary point x_0 and a positive parameter ν, we have

$$\varphi(x_0)^2 \;=\; \left(\sum_{n=1}^{\infty} c_n w_n(x_0)\right)^2 \le \sum_{n=1}^{\infty} \frac{w_n(x_0)^2}{1+\nu\lambda_n^2}\;\sum_{n=1}^{\infty} c_n^2(1+\nu\lambda_n^2)$$

$$= \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{1+\nu\Lambda_n^2}\cdot\big(\|\varphi\|^2+\nu\|\Delta\varphi\|^2\big), \tag{5.10}$$

where we used (5.5), (5.6). Since the right-hand side of (5.10) is independent of x_0, it follows that

$$\|\varphi\|_\infty^2 \;\le\; \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{1+\nu\Lambda_n^2}\cdot\big(\|\varphi\|^2+\nu\|\Delta\varphi\|^2\big). \tag{5.11}$$

Let x_0 be fixed. Then there is equality in (5.10), (5.11) if and only if

$$c_n = (1+\nu\lambda_n^2)^{-1} w_n(x_0),$$

that is, if

$$\varphi(x) = \sum_{n=1}^{\infty}\frac{w_n(x)w_n(x_0)}{1+\nu\lambda_n^2} = \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{\cos(k_n(x-x_0))}{1+\nu\Lambda_n^2}. \tag{5.12}$$

We now set $\nu=\nu_*=\|\varphi\|^2/\|\Delta\varphi\|^2$. Then $\|\varphi\|^2+\nu_*\|\Delta\varphi\|^2 = 2\nu_*^{1/2}\|\varphi\|\|\Delta\varphi\|$, and therefore

$$\|\varphi\|_\infty^2 \;\le\; \frac{1}{\pi^2}\,\nu_*^{1/2}\sum_{n=1}^{\infty}\frac{1}{1+\nu_*\Lambda_n^2}\cdot\|\varphi\|\|\Delta\varphi\| \le \frac{1}{\pi^2}\,\sup_{\nu>0}\,\nu^{1/2}\sum_{n=1}^{\infty}\frac{1}{1+\nu\Lambda_n^2}\cdot\|\varphi\|\|\Delta\varphi\|,$$

which shows (with $\nu=\mu^{-2}$) that c_{AT}^2 is less than or equal to the right-hand side of (5.8).

Suppose now that the supremum of the function $H(\nu)=\nu^{1/2}\sum_{n=1}^{\infty}(1+\nu\Lambda_n^2)^{-1}$ is attained at a finite point ν_*, $0<\nu_*<\infty$. Then,

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{1+\nu_*\Lambda_n^2} \;=\; \nu_*\sum_{n=1}^{\infty}\frac{\Lambda_n^2}{(1+\nu_*\Lambda_n^2)^2},$$

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{1+\nu_*\Lambda_n^2} \;=\; \sum_{n=1}^{\infty}\frac{1}{(1+\nu_*\Lambda_n^2)^2}. \tag{5.13}$$

In fact, the first equality follows from $H'(\nu_*)=0$. Summing the the first and the second equalities, we obtain a valid identity; hence, the second equality also holds.

Next we set $\nu=\nu_*$ in (5.12) and $x_0=0$. Then for the corresponding $\varphi=\varphi_*$, we have

$$\|\varphi_*\|^2 = \sum_{n=1}^{\infty}\frac{1}{(1+\nu_*\Lambda_n^2)^2},\qquad \|\Delta\varphi_*\|^2 = \sum_{n=1}^{\infty}\frac{\Lambda_n^2}{(1+\nu_*\Lambda_n^2)^2}.$$

Then it follows from (5.13) that $\|\varphi_*\|^2/\|\Delta\varphi_*\|^2=\nu_*$. Hence,

$$\|\varphi_*\|_\infty^2 = \frac{1}{\pi^2}H(\nu_*)\|\varphi_*\|\|\Delta\varphi_*\|,$$

which proves the theorem in the case when $0 < \nu_* < \infty$.

Suppose now that the supremum is attained as $\nu \to 0$ (observe that $H(\nu) \to 0$ as $\nu \to \infty$). We consider inequality (5.7) on the finite-dimensional space $T_N = \mathrm{Span}\{\sin kx,\ \cos kx\}, k^2 \le N$. The corresponding sharp constant $c_{\mathrm{AT}}(N)$ is given by the formula

$$c_{\mathrm{AT}}(N)^2 = \frac{1}{\pi^2}\max_{\nu>0} H_N(\nu), \qquad H_N(\nu) = \nu^{1/2}\sum_{\Lambda_n \le N}\frac{1}{1+\nu\lambda_n^2}.$$

The maximum is attained since $H_N(0) = 0$ and $H_N(\nu) \to 0$ as $\nu \to \infty$. Hence there exists an extremal function $\varphi_*^N \in T_N$. Since the spaces T_N are dense in the Sobolev space $H^2_{\mathrm{per}}(\mathrm{T}^2)$, it follows that

$$c_{\mathrm{AT}} = \lim_{N\to\infty} c_{\mathrm{AT}}(N), \quad \text{and} \quad c_{\mathrm{AT}}^2 = \frac{1}{\pi^2}\sup_{\nu>0}\nu^{1/2}\sum_{n=1}^{\infty}\frac{1}{1+\nu\Lambda_n^2}.$$

It remains to prove (5.9). Using the lower bound $\Lambda_n \ge n/2$ (see [27]), we have

$$\begin{aligned}\nu^{1/2}\sum_{n=1}^{\infty}\frac{1}{1+\nu\Lambda_n^2} &= \frac{1}{\mu}\sum_{n=1}^{\infty}\frac{1}{1+(\Lambda_n/\mu)^2} < \frac{1}{\mu}\sum_{n=1}^{\infty}\frac{1}{1+(n/(2\mu))^2}\\ &< \int_0^\infty f(x)\,dx = \pi,\end{aligned}$$

where $f(x) = 1/(1+(x/2)^2)$ is monotone decreasing and the third term in the above formula is the Riemann sum with step $1/\mu$ for the corresponding integral. □

Remark 5.1. In the vector case $u \in H^2_{\mathrm{per}}(\mathrm{T}^2)^2$, we have the same constant in the corresponding inequality

$$\|u\|_\infty \le c_{\mathrm{AT}}\|u\|^{1/2}\|\Delta u\|^{1/2}. \tag{5.14}$$

In fact, for $u = \{u^1, u^2\}$ we have

$$\begin{aligned}\|u\|_\infty^2 &\le \|u^1\|_\infty^2 + \|u^2\|_\infty^2 \le c_{\mathrm{AT}}^2(\|u^1\|\|\Delta u^1\| + \|u^2\|\|\Delta u^2\|)\\ &\le \frac{c_{\mathrm{AT}}^2}{2}(\varepsilon(\|u^1\|^2+\|u^2\|^2) + \varepsilon^{-1}(\|\Delta u^1\|^2 + \|\Delta u^2\|^2))\\ &= \frac{c_{\mathrm{AT}}^2}{2}(\varepsilon\|u\|^2 + \varepsilon^{-1}\|\Delta u\|^2),\end{aligned}$$

and minimizing with respect to ε, we obtain inequality (5.14).

Corollary 5.1. *The constant $c_{\mathrm{AT}}(\gamma)$ on the torus $\Omega = [0, L/\gamma]\times[0, L]$, $\gamma \le 1$, satisfies the estimate $c_{\mathrm{AT}}(\gamma) \le c_{\mathrm{AT}}/\sqrt{\gamma} \le 1/\sqrt{\gamma\pi}$.*

Proof. Given a function $\varphi \in H^2_{\mathrm{per}}(\Omega)$, we consider the function $\widetilde{\varphi}, \widetilde{\varphi}(x_1, x_2) = \varphi(x_1, \gamma x_2)$. Then $\widetilde{\varphi} \in H^2_{\mathrm{per}}(\widetilde{\Omega})$, where $\widetilde{\Omega} = [0, L/\gamma]^2$ is a square-shaped periodic domain so that

$\|\widetilde{\varphi}\|_{L_\infty(\widetilde{\Omega})} \le c_{\rm AT}\|\widetilde{\varphi}\|^{1/2}_{L_2(\widetilde{\Omega})}\|\Delta\widetilde{\varphi}\|^{1/2}_{L_2(\widetilde{\Omega})}$. Since $\|\varphi\|_{L_\infty(\Omega)} = \|\widetilde{\varphi}\|_{L_\infty(\widetilde{\Omega})}$, $\gamma^{-1}\|\varphi\|^2_{L_2(\Omega)} = \|\widetilde{\varphi}\|^2_{L_2(\widetilde{\Omega})}$, the corollary will be proved once we show that $\|\Delta\widetilde{\varphi}\|^2_{L_2(\widetilde{\Omega})} \le \gamma^{-1}\|\Delta\varphi\|^2_{L_2(\Omega)}$. We have

$$\begin{aligned}\|\Delta\widetilde{\varphi}\|^2_{L_2(\widetilde{\Omega})} &= \gamma^{-1}\|\varphi_{x_1x_1}(x_1,x_2)+\gamma^2\varphi_{x_2x_2}(x_1,x_2)\|^2_{L_2(\Omega)}\\ &\le \gamma^{-1}\|\varphi_{x_1x_1}(x_1,x_2)+\varphi_{x_2x_2}(x_1,x_2)\|^2_{L_2(\Omega)} = \gamma^{-1}\|\Delta\varphi\|^2_{L_2(\Omega)},\end{aligned}$$

since $\int_\Omega \varphi_{x_1x_1}\varphi_{x_2x_2}\,dx_1\,dx_2 = \|\varphi_{x_1x_2}\|^2_{L_2(\Omega)} \ge 0$ and $\gamma \le 1$. □

We now prove the remaining two inequalities used for estimates of the number of the determining nodes.

Lemma 5.1 (see [12], [30]). *Let $\Omega = [0, L_1]\times[0, L_2]$ be divided into N equal squares Q_j with side l and let $x^j \in Q_j$ for $j = 1,\dots,N$. Then for $u \in H^2_{\rm per}(\Omega)$ the following inequalities hold:*

$$\|u\|^2 \le 4l^2N\,\eta^2(u) + 68l^4\|\Delta u\|^2, \tag{5.15}$$

$$\|\nabla u\|^2 \le 2\cdot 68^{-1/2}N\eta^2(u) + 68^{1/2}l^2\|\Delta u\|^2, \tag{5.16}$$

where $\eta(u) = \max_{j=1,\dots,N}|u(x^j)|$.

Proof. We consider the scalar case and prove the first inequality. Let $u \in H^2(Q)$, where $Q = [0,l]^2$. For any two points $\mathbf{x} = (x,y)$ and $\mathbf{x}^0 = (x_0,y_0)$ in Q, we have

$$u(\mathbf{x}) - u(\mathbf{x}^0) = \int_{x_0}^{x} u_x(\xi,y)\,d\xi + \int_{y_0}^{y} u_y(x_0,\eta)\,d\eta.$$

Hence,

$$(u(\mathbf{x}) - u(\mathbf{x}^0))^2 \le 2l\int_0^l u_x(\xi,y)^2\,d\xi + 2l\int_0^l u_y(x_0,\eta)^2\,d\eta,$$

which after integration over Q with respect to x, y gives that

$$\|u - u(\mathbf{x}^0)\|^2_{L_2(Q)} \le 2l^2\|u_x\|^2_{L_2(Q)} + 2l^3\int_0^l u_y(x_0,\eta)^2\,d\eta.$$

For the second term on the right we have

$$u_y(x_0,\eta)^2 \le u_y(x,\eta)^2 + 2\int_0^l |u_y(\xi,\eta)||u_{yx}(\xi,\eta)|\,d\xi,$$

and hence, integrating with respect to x and η over Q, we find

$$\begin{aligned} l\int_0^l u_y(x_0,\eta)^2\,d\eta &\le \|u_y\|^2_{L_2(Q)} + 2l\int_0^l\int_0^l |u_y(\xi,\eta)||u_{yx}(\xi,\eta)|\,d\xi\,d\eta\\ &\le 2\|u_y\|^2_{L_2(Q)} + l^2\|u_{xy}\|^2_{L_2(Q)}.\end{aligned}$$

Therefore,

$$\|u - u(\mathbf{x}^0)\|^2_{L_2(Q)} \le 4l^2 \|\nabla u\|^2_{L_2(Q)} + 2l^4 \|u_{xy}\|^2_{L_2(Q)}.$$

Temporarily denoting the right-hand side by K and using Young's inequality, we have

$$\begin{aligned}\|u\|^2_{L_2(Q)} &\le K + 2u(\mathbf{x}^0) \int_Q u(x, y)\, dx\, dy - l^2 u(\mathbf{x}^0)^2 \\ &\le K + 2u(\mathbf{x}^0)\, l \|u\|_{L_2(Q)} - l^2 u(\mathbf{x}^0)^2 \le K + l^2 u(\mathbf{x}^0)^2 + \frac{1}{2}\|u\|^2_{L_2(Q)}.\end{aligned}$$

Hence,

$$\|u\|^2_{L_2(Q)} - 2l^2 u(\mathbf{x}^0)^2 \le 8l^2 \|\nabla u\|^2_{L_2(Q)} + 4l^4 \|u_{xy}\|^2_{L_2(Q)}.$$

We now divide $\Omega = [0, L_1] \times [0, L_2]$ into N equal squares of side $l = (|\Omega|/N)^{1/2}$ and choose a point $\mathbf{x}^j$ in each square Q_j, $j = 1, \dots, N$. Summing over j, we obtain

$$\|u\|^2 - 2l^2 \sum_{j=1}^{N} u(\mathbf{x}^j)^2 \le 8l^2 \|\nabla u\|^2 + 2l^4 \|\Delta u\|^2,$$

where we used the periodic boundary conditions so that $\|u_{xy}\|^2 = \int u_{xx} u_{yy}\, dx\, dy \le \frac{1}{2}\|\Delta u\|^2$. Next we use the interpolation inequality

$$\|\nabla u\|^2 \le \|u\| \|\Delta u\| \le \frac{1}{16l^2}\|u\|^2 + 4l^2 \|\Delta u\|^2,$$

and finally obtain

$$\|u\|^2 \le 4l^2 \sum_{j=1}^{N} u(\mathbf{x}^j)^2 + 68l^4 \|\Delta u\|^2 \le 4l^2 N\, \eta^2(u) + 68l^4 \|\Delta u\|^2,$$

which proves inequality (5.15) for the scalar case. For the vector case we apply the above inequality for each component and add up the results.

Finally, if $u \in H^2_{\text{per}}(\Omega)$, $\operatorname{div} u = 0$, and $\operatorname{rot} u = \omega$, then taking into account that $\|u\| = \|\nabla \Delta^{-1} \omega\|$ and $\|\nabla \omega\| = \|\Delta u\|$ we can write the previous inequality in the form

$$\|\nabla \Delta^{-1} \omega\|^2 \le 4l^2 N \eta^2(u) + 68l^4 \|\nabla \omega\|^2 = 4|\Omega| \eta^2(u) + 68|\Omega|^2 N^{-2} \|\nabla \omega\|^2. \quad (5.17)$$

For the proof of (5.16) we have

$$\begin{aligned}\|\nabla u\|^2 &\le \|u\| \|\Delta u\| \le \varepsilon \|u\|^2 + (4\varepsilon)^{-1} \|\Delta u\|^2 \\ &\le 4Nl^2 \varepsilon \eta^2(u) + (68l^4 \varepsilon + (4\varepsilon)^{-1}) \|\Delta u\|^2,\end{aligned}$$

which gives (5.16) by setting $\varepsilon^{-1} = 2 \cdot 68^{1/2} l^2$. □

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