Extremal problems for vector potentials and their applications to the asymptotics of Hermite-Pade approximants.

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Program of the talk

- Extremal problems for vector potentials with matrix of interaction

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- Rational approximants of analytic functions with branch points. (Definitions, Pade aproximants)

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- Extremal problems for vector potentials with matrix of interaction
- Rational approximants of analytic functions with branch points. (Definitions, Pade aproximants)
- Asymptotics of Hermite-Pade approximants for a vector of multivalued functions.

Scalar case

- Logarithmic potential and energy

$$V^{\mu}(z) = \int \log \frac{1}{|z-t|}\, d\mu(t), \quad I(\mu) = \int V^{\mu}(x)\, d\mu(x)$$

of a measure μ, $\quad S(\mu) \subset \Delta, \quad \Delta$ is a compact in $\mathbb{C}$,

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- Equilibrium of the potential

$$V^{\lambda}(x) = \mathrm{const}, \qquad x \in \Delta.$$

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- A.Gonchar, E.Rakhmanov (1982).

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- $\vec{W}^{\vec{\lambda}} = (W_1^{\vec{\lambda}}, \ldots, W_p^{\vec{\lambda}})$ is called the *vector potential of the vector valued measure* $\vec{\lambda}$ *with respect to the interaction matrix D*.

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- Scalar case : $(|\lambda_1| + |\lambda_2| = 1)$,
  $D := \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad (\kappa_1 = \kappa_2)$.

Definition of HP-approximants

- Given $\vec{f} = (f_1, \ldots, f_p)$ a vector of p.s.

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- The *HP-rational approximants* (of type II)

$$\pi_{\vec{n}} = \left(\frac{Q_{\vec{n},1}}{P_{\vec{n}}}, \dots, \frac{Q_{\vec{n},p}}{P_{\vec{n}}} \right)$$

for the vector $\vec{f}$ and multi-index $\vec{n} \in \mathbb{N}^p$ are defined by

$$\deg P_{\vec{n}} \leq |\vec{n}| = n_1 + \cdots + n_p$$

$$f_j(z) P_{\vec{n}}(z) - Q^{(j)}_{\vec{n}}(z) =: R^{(j)}_{\vec{n}}(z) = \left(\frac{1}{z^{n_j+1}} \right), \qquad z \to \infty.$$

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- C. Hermite, *Sur la fonction exponentielle*, C.R. Acad. Sci. Paris **77** (1873), 18–24; 74–79; 226–233.

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- Poles of Rational Approximants $\rightarrow$ Singularities of the function.
- Goal is to determine the limiting distribution of the zeros of the common denominator $P_{\vec{n}}$:

$$\nu_{P_{\vec{n}}} = \frac{1}{n}\sum_{k=1}^{pn}\delta(z - z_{k,\vec{n}}) \xrightarrow{*} \lambda\,? \qquad n \rightarrow \infty.$$

Class of functions

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- Domain of convergence of R.A. $\leftrightarrow$ Domain of holomorphicity
  Limiting positions of the poles of R.A. $\leftrightarrow$ Cuts making functions singlevalued

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  - $$\begin{cases} V^\lambda = \text{const.} \\ \dfrac{\partial V^\lambda}{\partial n_+} = \dfrac{\partial V^\lambda}{\partial n_-} \end{cases}, \qquad z \in \Delta \text{ a.e.}$$

Statement of problem

- H-P approximants

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$$f_j(z)P_n(z) - Q_n^{(j)}(z) =: R_n^{(j)}(z) = \left(\frac{1}{z^{n+1}}\right), \quad z \to \infty, \quad j = 1, 2.$$

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- The limiting distribution of the zeros of the common denominator $P_{\vec{n}}$:

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Example 1.

- $A_j := \{a_j, b_j\}, \quad j = 1, 2.$

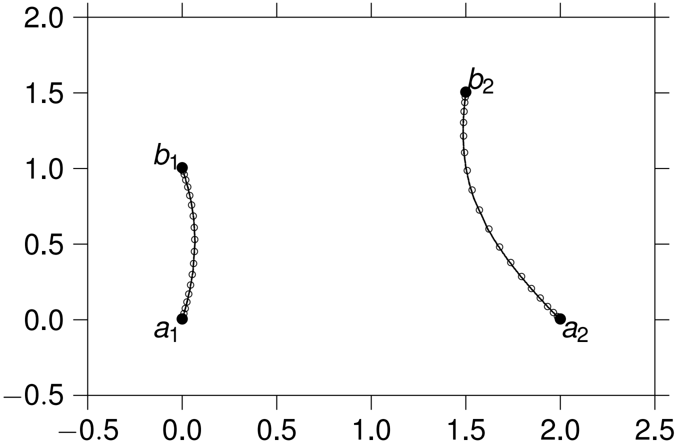

Figure: $a_1 = 0$, $b_1 = i$, $a_2 = 2$, $b_2 = (3 + 3i)/2$; zeros of $P_{20,20}$

Example 2.

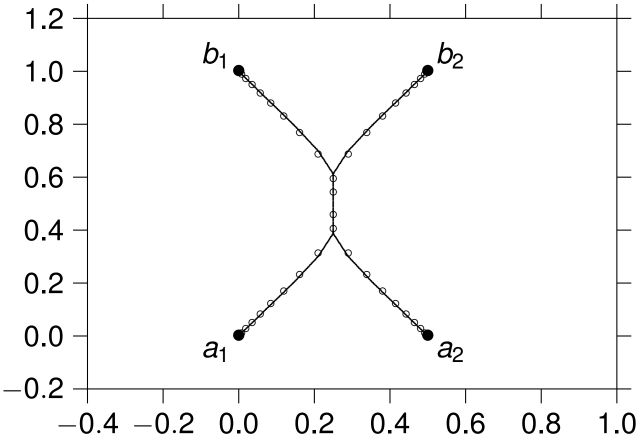

Figure: $a_1 = 0$, $b_1 = i$, $a_2 = 1/2$, $b_2 = 1/2 + i$; zeros of $P_{20,20}$ indicated by $\circ$

Example 3.

- new phenomena

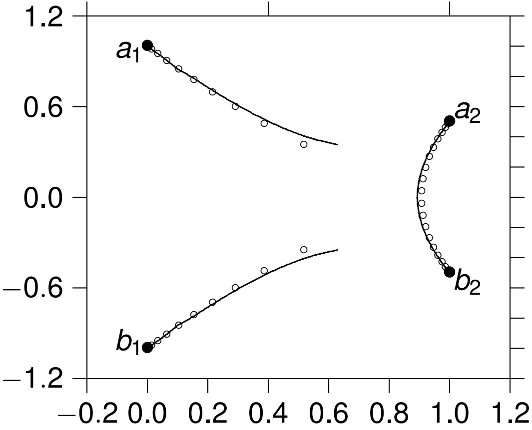

Figure: $a_1 = i$, $b_1 = -1$, $a_2 = 1 + i/2$, $b_2 = 1 - i/2$

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- Goal : (λ, μ_1, μ_2)

Theorem

a) $\exists \Delta_1, \Delta_2$ which make f_1, f_2 holomorphic

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b) There exists a triple of measures $(\lambda_1, \tilde{\lambda}_2, \mu_1)$ with supports

$$S(\lambda_1) \subset \Delta_1, \quad S(\tilde{\lambda}_2) \subset \widetilde{\Delta}_2 := \Delta_2 \setminus \Delta_{1,2}, \quad S(\mu_1) \subset E,$$

and with the relations on their total mass

$$\begin{cases} |\lambda_1| + |\tilde{\lambda}_2| = 2 \\ |\lambda_1| - |\mu_1| = 1. \end{cases}$$

Theorem

c) This triple of measures possesses the following equilibrium relations with some constants κ_1 and $\tilde{\kappa}_2$:

$$
{}_1 := 2V^{\lambda_1} + V^{\tilde{\lambda}_2} - V^{\mu_1} \begin{cases} = \kappa_1, & \text{on } S(\lambda_1), \\ \geq \kappa_1, & \text{on } \Delta_1, \end{cases}
$$

$$
U_2 := V^{\lambda_1} + 2V^{\tilde{\lambda}_2} + V^{\mu_1} \begin{cases} = \tilde{\kappa}_2, & \text{on } S(\tilde{\lambda}_2), \\ \geq \tilde{\kappa}_1, & \text{on } \widetilde{\Delta}_2, \end{cases}
$$

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U_3 := -V^{\lambda_1} + V^{\tilde{\lambda}_2} + 2V^{\mu_1} \begin{cases} = \tilde{\kappa}_2 - \kappa_1, & \text{on } S(\mu_1), \\ \geq \tilde{\kappa}_2 - \kappa_1, & \text{on } E. \end{cases}
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Theorem

d) The supports of the measures possess the following symmetry relations

$$\begin{cases} \dfrac{\partial U_1}{\partial n_+} = \dfrac{\partial U_1}{\partial n_-}, & \text{on } S(\lambda_1), \\ \dfrac{\partial U_2}{\partial n_+} = \dfrac{\partial U_2}{\partial n_-}, & \text{on } S(\tilde{\lambda}_2), \\ \dfrac{\partial U_3}{\partial n_+} = \dfrac{\partial U_3}{\partial n_-}, & \text{on } S(\mu_1). \end{cases}$$

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$$\lambda := \lambda_1 + \tilde{\lambda}_2 = \lambda_2 + \tilde{\lambda}_1, \quad S(\lambda) \subset \Delta_0 := \Delta_1 \cup \Delta_2,$$

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IV. The measure λ is the weak limit of the poles of the approximants of the functions, and the measures μ_1 and μ_2 are the weak limits of the extra interpolation points.