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We compute the spectrum of the Feigenbaum period-doubling operator in the space of bounded analytical functions in an ellipse. The spectral properties of the period-doubling operator in this space are not the same as in the space of even analytical functions. In particular, it was found that the dimension of the unstable manifold is not one (Feigenbaum's conjecture), but three. We analyze several articles devoted to this problem and compare different approaches and algorithms.

Вычисляется спектр оператора удвоения Фейгеннбаума в пространстве ограниченных аналитических функций в эллипсе. Спектральные свойства оператора удвоения в этом пространстве отличаются от свойств оператора в пространстве четных аналитических функций. В частности, было установлено, что размерность неустойчивого многообразия не один (гипотеза Фейгеннбаума), а три. Рассматриваются различные работы, посвященные этой задаче, и проводится сравнение различных методов и алгоритмов.

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§ 1. Introduction

Since the publication of the seminal works by Feigenbaum [11, 12], hundreds of studies were devoted to this very interesting and still expanding field of research. The author of this study was familiar for many years with the subject, but in the most general terms. One of the subjects that the author specializes in is the development of efficient numerical and symbolic algorithms for solving various mathematical problems. In the course of testing one of such algorithms for solution of functional equations, the Feigenbaum universality equation appeared to be a very convenient model problem. The fundamental constants associated with this equation are computed to more than a thousand decimal places, which gives a perfect opportunity for tuning various settings of the algorithm.

So it was without any expectations to find anything new that the author performed the tests, which gave very satisfactory results pertaining to the algorithm. However, it was unsettling that some of the results were in disagreement with the well known and long established facts such as the Feigenbaum conjecture and the spectral properties of the doubling or universality operator.

Thanks to the popularity of this field of mathematical and physical sciences and to the Internet, most papers on the subject are readily available. The present paper is a comparative study of the spectral properties of the doubling operator and a review of several works dealing with the original problem. It is also an attempt to reconcile apparent contradictions and to trace their origin.

Let us recapitulate briefly the setting of the problem. It deals with the mapping of an interval onto itself $f\colon [a, b] \to [a, b]$, where f is a generic unimodal function. Here unimodal means having only one extremum (maximum) on the interval $[a, b]$ (one-hump map), and generic means that the function is smooth and the extremum is quadratic. The function f depends on one parameter. The iterations of such maps can display an infinite cascade of period doubling bifurcations as parameter changes. The bifurcations occur when a stable solution x_n to the equation $x = f^{(n)}(x)$, $n = 2^{k-1}$, $k \in \mathbb{N}$, loses stability, and two new stable cycles are born, i.e., two solution $x_{2n}^{\pm}$ to the equation $x = f^{(2n)}(x)$. It was shown in [11, 12] with the help of renormalization involving rescaling and stretching of the iterated maps that, as the period of cycles tends to infinity, the sequence of bifurcations displays a universal character independent of the initial function $f(x)$. Asymptotically, the cascade of bifurcations possesses self-similarity with the universal con-

stants $\delta \approx 4.6$ in parameter space, and $\alpha \approx -2.5$ in the phase space (on the interval). These universal constants can be found from the period-doubling (or universality) equation:

$$g(x) = T(g)(x) = g(g(g(1)x))/g(1), \quad x \in [-1, 1]. \tag{1}$$

Here the function $g(x)$ is the result of an infinite number of renormalizations of iterations of the original mapping $f(x)$, and so it totally forgets its prehistory. The constant $\alpha = 1/g(1)$; and the constant δ is determined from the spectrum of the operator $dT(g)$, i.e., Fréchet derivative of the operator T on the solution $g(x)$ to the universality equation (1). This solution cannot be found by iterations of the operator T, since this operator is hyperbolic. We recall that the Feigenbaum conjecture (in its modern form) states that all the eigenvalues of the operator $dT(g)$ except one lie within the unit circle, the unstable eigenvalue being δ. So the unstable manifold at the fixed point $g(x)$ is one-dimensional.

A few remarks on the preceding paragraph. First, the equation (1) is not a unique form of universality equation. In various papers, there are used other forms of universality operator (see Sect. 2, 3). They possess the common solution $g(x)$, which is an even analytical function in the neighborhood of the interval $[-1, 1]$. But these universality operators do not have the same spectrum, and have different eigenfunctions for the same eigenvalues. We will stick to the equation (1) as canonical in this paper, and mark the differences as they appear. Second, note the absence of the normalization condition $g(0) = 1$ usually imposed on the solution in the definition (1). The reason will be given in Sect. 4. Third, the renormalization used in [11, 12] preserves the nature of the extremum of the original function f. So the limit function must have the same type of extremum, and so the equation (1) must possess different solutions. This fact, of course, is well known, and only mentioned here to avoid misunderstanding. We will deal primarily with the function $g(x)$ having quadratic extremum, and discuss other solutions in Sect. 4. The most important remark here is this: the spectrum of an operator $dT(g)$ depends strongly on the functional space where the operator is acting. For example, in the space $L_2[-1, 1]$ of complex-valued functions integrable with square, the spectrum of the operator $dT(g)$ is continuous and complex (Sect. 2).

Feigenbaum never mentioned specifically the functional space for the operator T. In the framework of papers [11, 12] it is hardly to be expected. But then, the function $g(x)$ had to be found, and the most natural space for this is the space $\mathcal{E}$ of even bounded analytical functions, since the function $g(x)$ must be smooth and even by construction. Again, it was not made

explicit, but the numerical algorithm described in [12, page 693] clearly uses discretization in the space $\mathcal{E}$ (or rather in its subset, see Sect. 4). Since the finite dimensional approximation to the operator $dT(g)$ is obtained as a by-product of the Newton iterations scheme used for numerical solution of (1), the spectrum found (numerically) in [12] corresponds to even eigenfunctions. Hence, the Feigenbaum conjecture (Sect. 4).

In Sections 2, 3, we compare the spectrum of the operator $dT(g)$ and its various representations in different functional spaces. We also discuss the most common mistakes made in various papers and monographs in the analysis of the spectral problem for the universality operator. Some mistakes are obvious as such, and some are the result of misquoting or the wrong assumptions and peer pressure.

In Section 4, we solve numerically the spectral problem for the universality operator in the Banach space $\mathcal{F}$ of bounded analytical functions in an ellipse with the focal points ± 1, with the supremum norm, continuous on the closure of the ellipse. Let us give a few reasons for this choice. First, computer experiments revealed that the function $g(x)$ belongs to this space. A rigorous proof is still to be found, despite some computer assisted efforts [14, 15]. But, as "mathematics is an experimental science" (V.I. Arnold), we will consider this fact established. The second reason is the fact that the functional space $\mathcal{F}$ admits extremely good discretization with Chebyshev polynomials as a basis. The coefficients of expansions of the functions in $\mathcal{F}$ in Chebyshev series decrease exponentially [18]. This is why we choose an ellipse and not a disk (see [8, page 1264]). Finally, and this is a purely physical argument open for discussion: there is no reason to restrict the space $\mathcal{F}$ to the space $\mathcal{E}$ of even analytical functions. The function $g(x)$ forgets its prehistory and is even, but all the pre-limit functions subject to renormalizations used in [11, 12] still keep some information about the original function $f(x)$, which is unimodal, and so the perturbations of the function $g(x)$ need not be even. It is a part of universality that we need not impose some symmetry on the function f (as in the logistic map) in order to obtain the function g.

§ 2. Explicit spectrum

The notion of universality in dynamical systems can now be found in almost every monograph remotely concerned with chaotic dynamics. An excellent exposition of the Feigenbaum universality can be found in the book [16, Chap. 7] aimed at physical scientists and engineers. The book also illustrates how physical intuition fails when simple mathematics is neglected. We will use this book as a typical example.

The universality equation in [16] is given in the form of rescaling equation (7.2.39) [16, page 491]:

$$g(x) = \alpha g(g(x/\alpha)) = T(g)(x), \quad g(0) = 1. \tag{2}$$

Since the normalizing condition $g(0) = 1$ is included in the definition, it immediately follows that $\alpha = 1/g(1)$, and this equation is identical with (1). From the previous exposition in [16], it also follows that the authors consider general maps, i.e., unimodal and generic in the sense of Sect. 1, and so they implicitly operate in the space $\mathcal{F}$.

To investigate the stability of the fixed point $g(x)$, the authors compute the linearized period-doubling operator introducing a perturbation $g(x) + \varepsilon h(x)$ and, linearizing, obtain the linear operator (Gâteaux derivative) in the form

$$L(g)h(x) = \alpha \left(g'(g(x/\alpha))h(x/\alpha) + h(g(x/\alpha)) \right). \tag{3}$$

Then the authors refer to Feigenbaum [11, 12] and claim (Feigenbaum conjecture) that the spectrum of the operator $L(g)$ has a single unstable (i.e., lying outside the unit circle) eigenvalue $\delta \approx 4.669$ [16, page 492]. Unfortunately, both this claim and the linearized equation (3) itself are not true. So it is not clear how much of the following physical argument in [16] will survive.

Let us compute the linearized period-doubling operator proceeding exactly as described in [16, page 491], but keeping in mind that $\alpha = 1/g(1)$, i.e., that α depends on the function g.

Proposition 1. *The formal Gâteaux derivative of the operator T defined in (1) is given by the formula*

$$dT(g)h(x) = L(g)h(x) + \alpha \left(g'(g(x/\alpha))g'(x/\alpha)x - \alpha g(g(x/\alpha)) \right) h(1). \tag{4}$$

It seems that this easily verified formula (4) for the operator $dT(g)$ was never computed. An (almost) correct formula for the derivative was found in [22], but for another form of the universality operator (Sect. 3). The formula (4) is applicable in any functional space where the Gâteaux derivative of the operator T coincides with the Fréchet derivative. It is certainly the case in the space $\mathcal{F}$.

Proposition 2. *The operator T is compact ([22, page 16]) in the space $\mathcal{F}$, and the operator $dT(g)$ has the following spectrum*

$$S = [\lambda_1, \lambda_2, \ldots] = [\alpha^2, \delta, \alpha, \frac{1}{\alpha}, \frac{1}{\alpha^2}, \lambda_6, \frac{1}{\alpha^3}, \lambda_8, \frac{1}{\alpha^4}, \frac{1}{\alpha^5}, \ldots], \tag{5}$$

where $|\lambda_i| > |\lambda_j|$, $i < j$.

We will compute numerically the eigenvalues and the corresponding eigenfunctions of the operator $dT(g)$ in Section 4. They coincide with S in (5). But now we note that all the eigenvalues in S where α is present are found explicitly together with the corresponding eigenfunctions.

Proposition 3. *Let k be any complex number except 1. Then $\lambda = \alpha^{1-k}$ is an eigenvalue of the formal spectral problem $dT(g)h = \lambda h$ with the eigenfunction*

$$h(x) = g(x) - xg'(x) - g^k(x) + x^k g'(x). \tag{6}$$

In addition, α^2 is the eigenvalue with the eigenfunction

$$h(x) = g(x) - xg'(x). \tag{7}$$

Proof. If we differentiate the equation (2) and put there $x = 0$, we obtain the identity $g'(1) = \alpha$. We use this, along with the equation (2) and its derivative, for simplifying substitutions. We observe that if we put $x = 0$ into the formal spectral problem $dT(g)h(x) = \lambda h(x)$, then we derive the identity $(\alpha^2 - \lambda)h(0) = 0$. Hence, for analytical functions h, either $\lambda = \alpha^2$, or $h(0) = 0$. The rest of the proof is a simple, although very bulky, symbolic calculation better made on a computer. ∎

The spectral problem is formal until we specify the functional space we are working with. In the space $\mathcal{F}$, obviously, $k = 0, 2, 3, \ldots$. So we have found explicitly 7 out of the first 10 eigenvalues of the operator $dT(g)$, and at least two of them lie outside of the unit circle. This result is easily verified analytically (and numerically, Sect. 4) and is in direct contradiction with the Feigenbaum conjecture. So let us trace the origin of this apparent paradox. But before we turn to the original paper [12], where we hope to find an answer, we need the spectrum of the operator $L(g)$ for comparison.

Proposition 4. *The spectrum of the operator $L(g)$ in the space $\mathcal{F}$ is*

$$\tilde{S} = [\delta, \alpha, 1, \frac{1}{\alpha}, \frac{1}{\alpha^2}, \lambda_6, \frac{1}{\alpha^3}, \lambda_8, \frac{1}{\alpha^4}, \ldots], \tag{8}$$

where λ_i, $i = 6, 8, \ldots$ are the same as in (5). The eigenvalues α^{1-k}, $k = 0, 1, \ldots$ in $\tilde{S}$ have the eigenfunctions

$$h(x) = g^k(x) - x^k g'(x). \tag{9}$$

Proof. Numerically, it is demonstrated in Sect. 4. The part of the spectrum where α is involved is found explicitly. ∎

We will return in Section 4 to all the spectral problems discussed in this and the following Section.

§ 3. The problem with the spectrum

Now we turn to the paper [12], which is referenced in almost every publication dealing with the Feigenbaum universality. To avoid confusion, we will keep our notation $\alpha = 1/g(1) < 0$, which is common now (Feigenbaum used $\alpha = -1/g(1) > 0$, [12, page 675], [13, page 73]), and translate the corresponding formulas when needed.

Feigenbaum used a different form of the universality equation from what we use (1). It is given in the abstract of [12] as

$$g(x) = \alpha g(g(-x/\alpha)) = T_2(g)(x). \tag{10}$$

The normalizing condition $g(0) = 1$ is given later on in [12].

Strangely, in the abstract of [12], Feigenbaum gives the linear operator $\mathcal{L}$, which coincides with (3) on the function $g(x)$, since $g(x)$ is even. The correct formula (assuming $\alpha = $ const) should be

$$L_2(g)h(x) = \alpha \left(g'(g(-x/\alpha))h(-x/\alpha) + h(g(-x/\alpha)) \right), \tag{11}$$

and the corresponding operator $dT_2(g)$ (correct Fréchet derivative) is

$$dT_2(g)h(x) = L_2(g)h(x) - \alpha \left(g'(g(-x/\alpha))g'(-x/\alpha)x + \alpha g(g(-x/\alpha)) \right) h(1). \tag{12}$$

Note, that the formula (11) for the derivative of (10) was found in [11, page 47, formula (42)]. The formulas (11) and (12) can be simplified using the fact that g is even and g' is odd. But this should be done after and not before the computation of the derivative of the operator. In addition, the function h in these formulas need not be even, so no simplifications there.

The spectral properties of the operators $L(g)$ and $L_2(g)$, and, respectively, of the operators $dT(g)$ and $dT_2(g)$ are different in the space $\mathcal{F}$. In the space $\mathcal{E}$, each pair of operators possesses identical spectrum (Sect. 4).

Later on in [12], Feigenbaum uses the operator $L(g)$ as the derivative of the operator T_2 on $g(x)$, but periodically switches to $L_2(g)$ (see [12, page 677, formula (28); page 682, 685].

It is also not exactly clear, what Feigenbaum meant by his conjecture. First, in the abstract of the paper [12]: "$\mathcal{L}$ possesses a unique eigenvalue in excess of 1." Then (we quote from [12, page 687] using our notation and correcting a misprint): "The spectrum of the operator $dT(g)$ is δ and $\alpha^{1-\rho}$, $\rho \geq 1$, and, moreover, the spectrum is complete."

We used here $dT(g)$ rather than $L(g)$, since here it was clearly meant the derivative of the operator T.

The part about the spectrum being complete was refuted numerically in many works, since other eigenvalues were found (Sect. 4). In Proposition 2, they are λ_6, λ_8, etc.

After numerical investigation of the spectral problem in [12], Feigenbaum states his conjecture in the form [12, page 694]: "δ *is the solitary eigenvalue of $dT(g)$ greater than 1.*"

Note that all these conjectures imply that 1 is an eigenvalue, and so they contradict the conjecture in its modern interpretation. Although this difficulty is fixed by the normalization $g(0) = 1$, which simply means that we choose one solution from the family of solutions, still, this eigenvalue is the product of a wrong assumption. If the derivative $dT(g)$ was computed correctly, the eigenvalue 1 would not appear (Sect. 4).

To complicate matters even more, Feigenbaum actually found the eigenvalue α, since $\rho = 0$ perfectly fits the citation above, with the analytical eigenfunction $1 - g'(x)$ [12, page 686]. This eigenvalue is not in excess of 1, since $\alpha < 0$, but α lies outside of the unit circle.

So let us draw a line here and try to explain these paradoxes.

First, Feigenbaum used the wrong linearization $L(g)$ instead of $L_2(g)$ of the universality operator. In addition, both these linearizations are wrong, since they assume $\alpha = $ const independent of the fixed point $g(x)$. This assumption is later rejected in [12, page 693, formula (80)], when the variation of α is used together with the variation of $g(x)$. The analysis of the spectrum is performed in some unspecified functional space, which is clearly not a space of even functions, since some of the eigenfunctions (9) are not even.

The second misunderstanding in [12] compounding the first is the use of numerically obtained data in the same context as analytically obtained eigenvalues and eigenfunctions. These are two different sets of objects, since the numerical algorithm described in [12, page 693] operates in a subset of the space $\mathcal{E}$ (see Sect. 4). To unite the numerical and analytical data, we need the space $\mathcal{F}$ and correctly linearized operator $dT(g)$.

In the afterword to the paper [12], Feigenbaum states that his spectral conjecture was verified by Collet et al. We have no access to that paper (then in draft), but in the subsequent publications of the same authors, the space of even functions was postulated [6, page 211, 212], [14, page 427], [15, page 521].

Now we consider how the spectral problem for the universality operator was treated in several frequently cited papers and in some books.

In the study [7, page 4], the authors use the same notation as in this paper, but consider the problem in a broader space of functions mapping the interval $[-1, 1]$ onto itself, i.e., the functions are not necessarily even. The four assumptions, M1-M4, all agree with our conclusions so far, but then the authors wrongly compute the derivative of the operator (1) as $L(g)$ (3) and proceed with the analysis. In particular, Lemma 1 in [7] coincides with Proposition 4 here, so the following assumptions M5, M6 [7, page 5] can be considered as either true or wrong depending on what operator is taken for the derivative of T. On the other hand, the authors found the eigenvalue 1, so the solution g to the equation $g = T(g)$ is either degenerate, or belongs to a one parameter family of solutions (implicit function theorem). Both facts are not true (Sect. 4).

In the paper [8], Eckmann gave some substantiation to the choice of the space of even analytical functions, where $g(x)$ "is supposed to lie", [8, page 1264]. His space is similar to the space $\mathcal{E}$, except it is defined on a disk, not an ellipse. However, the properties P1-P3 (including the Feigenbaum conjecture) hold there only with an additional stipulation (see Sect. 4).

Feigenbaum renormalizations preserve the property of the function f being symmetric with respect to its hump, so the choice of the functional space of even functions is justified for such maps (logistic map, for example). But the Feigenbaum universality is now understood in a broader sense (see [16, Chap. 7]), meaning the functions f need not be symmetric. This confusion of notions leads to many erroneous statements on the dimension of the unstable manifold at $g(x)$. For example, in the paper [20, page 425], the author refers to Lanford's computer-assisted proof, but explains his results in a general space of analytic functions; in the book [19], analytical unimodal maps are considered, so Proposition 2 in [19, page 191] and its corollaries are not true; in the book [1], the Feigenbaum universality is explained on a typical example $f(x) = Ax \exp(-x)$ [1, page 338, 339], but the doubling operator J, identical to T (1), is defined on even functions with some restrictions [1, page 340], and the Feigenbaum conjecture is formulated in an unspecified functional space.

Although we are not concerned with proofs of the Feigenbaum conjecture in this paper, some of the works on the subject deserve a special attention, since they apparently disagree with our results stated above.

Lanford is reputed to have given the first of the computer-assisted proofs of the Feigenbaum conjecture. His proof seemingly contradicts our conclusions, but only if the results are taken out of the context. In the paper [14], he introduces the space $\mathcal{M}$ of continuously differentiable even mappings ψ of the interval $[-1, 1]$ into itself such that $\psi(0) = 1$ (among other things) [14,

page 427]. But the condition $\psi(0) = 1$ makes $\mathcal{M}$ a set, not a space, since functions cannot be added or multiplied by a constant in $\mathcal{M}$. Further [14, page 428], he introduces a Banach space $\mathcal{B}$ of bounded even analytic functions on a disk equipped with the supremum norm, and its subspace $\mathcal{B}_0$ of functions vanishing to second order at 0. Theorem 3 on hyperbolicity of $dT(g)$ [14, page 428] is formulated in the space $\mathcal{B}_0$, where it is not true, since the functions in this space do not satisfy the universality equation. It is, probably, a misprint, since Theorem 3 is true in the set (or an affine space) $\mathcal{B}_1 = \mathcal{B}_0 + 1$ (see Sect. 4). Further [14, page 429], Lanford introduces the expansion $\psi(x) = 1 - x^2 h(x^2)$ corresponding to the set $\mathcal{B}_1$, which was used in many papers implicitly.

In the paper [9], where another computer assisted proof of the Feigenbaum conjectures is given, the word "even" is not mentioned even once. However, even functions are implied [9, Theorem 2.2]. It is also the case in [10, page 396] and many other papers.

To the best of the author's knowledge, there is a unique paper [22] where the correct formula for the derivative of the doubling operator was found (but for the wrong operator). The authors consider generic unimodal maps as defined in Sect. 1 [22, page 14] (we quote the Russian edition), and the Feigenbaum conjecture is formulated in its modern form without reference to even maps. The doubling operator is defined [22, page 13] as

$$T_3(g)(x) = -ag(g(x/a)), \ a = -\frac{g(0)}{g(g(0))}, \ g(0) = \text{const}, \quad x \in [-1, 1]. \quad (13)$$

Here we substitute a for α to avoid confusion. If $g(0) = 1$, then $a = -\alpha$. The authors compute the correct derivative [22, page 16], but for the operator

$$T_4(g)(x) = -ag(g(-x/a)), \quad (14)$$

which is not the same as (13) for analytic functions. The analysis of spectral properties of the operator $dT_4(g)$ in [22] is very similar to that in the present paper, although it is more difficult due to a more complicated form of the doubling operator. The authors have found the eigenvalue 1, and $-a = \alpha \approx -2.5$, as well as other powers of α, except α^2. These results contradict the Feigenbaum conjecture stated earlier in [22]. So the authors have tried to dismiss unwanted eigenvalues on the following grounds [22, page 17]. First, they are not relevant to the universality, since they are linked to coordinate transformations (i.e., not to the parameter space). Not many people would subscribe to this point of view today, since α is considered now on a par with δ as a universal constant. For example, in [16, page 488], it is explained that both α and α^2 play a part in the rescaling of periodic solutions. The

second argument the authors use to conform to the Feigenbaum conjecture is (a) – the eigenvalue 1 is eliminated by the condition $g(0) = \mathrm{const}$; and (b) – the eigenvalue α is eliminated by the condition $g'(0) = 0$. The condition (a) means that we choose one solution from a family, so the eigenvalue 1 is simply ignored; and the condition (b) was not imposed in the statement of the problem, and anyway, it follows from the universality equation, i.e., $g'(0)(\alpha - 1) = 0$ follows from (1), and similarly for (14). The property $g'(0) = 0$ of the solution $g(x)$ to (1) or (14) is a result of an infinite number of renormalizations. But perturbations of the solution need not conform to this restriction. In addition, this projection does not explain what to do with other powers of α present in the spectrum in both spaces $\mathcal{F}$ and $\mathcal{E}$ (Sect. 4). Further, the authors give incorrect form of the doubling operator [22, page 19, formula 4.1] with $\alpha = g(1)$, but this is clearly a misprint.

In some papers, the derivative of the operator (1) is computed incorrectly, but then never used; so the mistake is not revealed [5, page L713S]. And the use of the space of even functions can only be deduced by a dedicated reader. In [5, page L713S], it was only indicated as "Lanford's expansion" of the function $g(x)$.

We conclude this survey with two works devoted to precise computation of the Feigenbaum constants.

In the paper [3], Briggs uses the same notation and the same operator (1) as we used in this paper [3, formula (5)]. Numerical algorithm is similar to that used by Feigenbaum and the most authors [3, page 437], i.e., it operates in the subset of the space of even functions. The Feigenbaum conjecture is formulated for historical reference; then the wrong "local linearization of T" [3, formula (8)] is obtained by "simple calculation". In fact, this local linearization coincides with that of Feigenbaum in [11, page 47, formula (42)], where it is found for the positive $\alpha = -1/g(1)$. Fortunately, this "local linearization" was never used in [3].

In his PhD thesis [4], Briggs uses the same notation as in [3] (see [4, formula (1.5)]). But then, the derivative DT_g of the operator T is upgraded to include the dependence of α on the solution $g(x)$ [4, page 5]. This new formula for the derivative DT_g is remarkably similar to that found in [22, page 16] for the different operator (14). Then, [4, page 12], the "local linearization of T" is found again by "simple calculation" as in [3], but this time with the correct sign of α. Apparently, Briggs is familiar with eigenvalues which do not comply with the Feigenbaum conjecture, but he explains them as "extra eigenvalues" introduced by a finite-dimensional approximation [4, page 22]. Briggs recommends to select the good eigenvalues, which are readily

identified, and discard the bad ones.

In the next Section, we will not follow this advice.

§ 4. Numerical analysis of the spectral problems

In this Section, we compute the spectrum for all spectral problems mentioned in previous Sections in different functional spaces. We will also use different algorithms including that described by Feigenbaum in [12, page 693], which was used (with various modifications) by Lanford [14], Briggs [3], and many other researches.

First, we describe an algorithm based on the use of Chebyshev polynomials as a basis in the space $\mathcal{F}$.

The solution $g(x)$ to the equation (1) is approximated by the polynomial

$$g(x) = \sum_{i=1}^{n} g(x_i) p_{ni}(x), \quad x_i = \cos \frac{(2i-1)\pi}{2n}, \quad i = 1, \ldots, n, \qquad (15)$$

where x_i are Chebyshev roots, and

$$p_{ni}(x) = \frac{T_n(x)}{(x - x_i) T_n'(x_i)}, \quad i = 1, \ldots, n \qquad (16)$$

are Chebyshev fundamental polynomials of Lagrange interpolation. We will use the notation $g(x)$ both for the analytic solution to (1) and for its polynomial approximation (15) (and others), but this will not lead to confusion.

The equation (1) is rewritten as $\Phi(g) = g - T(g) = 0$, and the solution is found by the Newton iterations

$$g_{k+1} = g_k - A_k^{-1} \Phi(g_k), \quad k = 0, 1, \ldots,$$

where g_k is the k-th approximation to the solution g; $A_k = d\Phi(g_k)$ is the Jacobian matrix at g_k. The iterations are done until the polynomial $g_{k+1} - g_k$ (evaluated at the nodes x_i) is zero in the sup norm within the round-off error.

After the final iteration, we have found the polynomial solution g represented by the values $\{g(x_i), i = 1, \ldots, n\}$, and the matrix $A = A(g)$ on the solution. The matrix $I - A$ is an approximation to the derivative $dT(g)$ (4), where I is the unit matrix. Then we compute the spectrum of the matrix $I - A$ by standard linear algebra subroutines.

In the course of these computations, we need to evaluate the polynomial $g(x)$ at the points that are not Chebyshev roots. This can be done very efficiently, if the function $g(x)$ is expanded in Chebyshev series

$$g(x) = a_0/2 + \sum_{k=1}^{n-1} a_k T_k(x), \quad T_k(x) = \cos(k \arccos x), \qquad (17)$$

where $T_k(x)$ are Chebyshev polynomials. The coefficients a_k are found by the discrete Fourier-Chebyshev transform. This operation is stable and does not accumulate the round-off errors [18]. The evaluation of g is done with the series (17) using the recurrence relations for Chebyshev polynomials. These operations are also stable [18].

The Fourier-Chebyshev transform also provides a built in precision control, since the coefficients $\{a_k, k = 1, \ldots, n\}$ must decrease exponentially. This can be seen on a plot of $\log(1/|a_k|)$ versus k, $k = 1, \ldots, n$.

The elements of Jacobian matrix A are approximated by finite differences. It needs not be done with high precision for Newton iterations to converge quadratically. Only after the final iteration this matrix needs to be evaluated with maximal precision, since it is used for the approximation of the spectral problem.

We have also used an alternative way to approximate the operator $dT(g)$ (4). If the doubling operator T (1) is applied to a polynomial p of an order m, then $T(p)$ is a polynomial of the order m^2. Since $T(p)$ can be restored by its values at $m^2 + 1$ points, the same is true for the derivative $dT(p)$. So if we take the dimension n of the projection such that $n \geq m^2 + 1$, then we can compute the operator $dT(p)$ exactly, i.e., in the same sense as Gauss quadratures are exact on polynomials up to a certain order.

The finite difference approximations are proved to be faster, but slightly less accurate.

This algorithm takes about as many lines in a computer language as it took to describe it. For general analytic functions in $\mathcal{F}$, it is also one of the most efficient. It follows from the approximative properties of the Chebyshev series (17) and asymptotically optimal distribution of Chebyshev nodes (see [18]). However, for the same accuracy of the solution g, this algorithm takes about 4 times more memory and 8 times more CPU time than the algorithms that use the symmetry of the solution $g(x)$. This is probably why it was never used before. Recently, Chebyshev series representation of $g(x)$ was used in [17], but on the interval $[0, 1]$, i.e., for even functions.

We are not about to break any records in the number of digits of the universal constants. The original plan was to test the algorithm, so we fix the number of nodes on the interval $[-1, 1]$ as $n = 32$, and we fix the floating point arithmetic at 64 digits. The software for such computations is available as an open source (see [2]). The chosen precision is equivalent to working with infinite number of digits, since the round-off errors can be neglected in comparison with the errors of the approximation. We should mention that all computations were verified with different settings (more/less digits/nodes,

and different linear algebra routines for solution of the spectral problems), but they gave similar results and not reported here.

The constant $\alpha = 1/g(1)$ is found with the accuracy 0.5×10^{-22}, which is a small number in comparison with the last coefficient ($\approx 0.2454065396 \times 10^{-13}$) at x^{30} in the Taylor expansion of $g(x)$. The reason for this is the value of the last Chebyshev coefficient at $T_{30}(x)$ in the expansion (17), which is $0.4571053006 \times 10^{-22}$. The constant δ is found with 22 correct decimal places. We stress that no normalization needs to be imposed on the solution $g(x)$. The Newton iterations converge quadratically, provided a good initial approximation is taken, and the solution is found uniquely in the space $\mathcal{F}$.

In Table 1, we cite the first 11 eigenvalues of the operator $dT(g)$ computed as described above. They correspond to the spectrum S in Proposition 2 to the indicated accuracy, which was estimated by comparison with the values of α and δ found in [3]. We cite here only 10 decimal places and can send the computed values on demand.

Table 1. First 11 eigenvalues of the spectrum S.

$\lambda_1 =$	6.264547831	α^2	0.7×10^{-21}
$\lambda_2 =$	4.669201609	δ	0.2×10^{-21}
$\lambda_3 =$	-2.502907875	α	0.2×10^{-21}
$\lambda_4 =$	-0.399535280	α^{-1}	0.5×10^{-21}
$\lambda_5 =$	0.159628440	α^{-2}	0.4×10^{-18}
$\lambda_6 =$	-0.123652712		
$\lambda_7 =$	-0.063777193	α^{-3}	0.3×10^{-18}
$\lambda_8 =$	-0.057307021		
$\lambda_9 =$	0.025481238	α^{-4}	0.1×10^{-12}
$\lambda_{10} =$	-0.010180653	α^{-5}	0.9×10^{-17}
$\lambda_{11} =$	-0.010145805		

The eigenvalues that correspond to the powers of α in Table 1 have the eigenfunctions given in Proposition 3 (after the normalization). The eigenvalues $\lambda_2 = \delta$, λ_6, λ_8, λ_{11}, etc., that are not related to the powers of α (at least not in an obvious way), have even eigenfunctions.

The spectrum of the operator $L(g)$ (3), that was frequently mistaken for the derivative of the operator T, is given in Proposition 4. To verify it numerically, we only need to fix the value of α in the program, after $g(x)$ and α are already found. The eigenvalue 1 indicates that there is a one-parameter family of solutions. We will find the family below for another problem.

The spectrum of the operator $dT_2(g)$ (12) in the space $\mathcal{F}$ is

$$S_2 = [\alpha^2, \delta, -\alpha, -\frac{1}{\alpha}, \frac{1}{\alpha^2}, \lambda_6, -\frac{1}{\alpha^3}, \lambda_8, \frac{1}{\alpha^4}, -\frac{1}{\alpha^5}, \ldots],$$

i.e., α in S (5) is replaced with $-\alpha$. The same is true for the operator $L_2(g)$ (11), i.e., it is the spectrum $\tilde{S}$ of the operator $L(g)$ (3) with the substitution $\alpha \to -\alpha$. However, only even eigenfunctions in Propositions 3, 4 are preserved, i.e., (7) for α^2, and (6), (9) for odd k. For even k, we could not find explicit formulas.

Now we compute the spectrum of the operators used in [22], i.e., $T_4(g)$ (14), and $T_3(g)$ (13). We recall that in [22], the doubling operator was defined as $T_3(g)$, but the derivative was computed for the operator $T_4(g)$. Since both operators have 1 as an eigenvalue, the Newton iterations do not converge (i.e., there is a family of solutions). So we used $g(x)$ found earlier, which satisfies all universality equations. The spectrum of the operator $dT_4(g)$ is $\tilde{S}$ (8), i.e., it coincides with the spectrum of the operator $L(g)$. However, the explicit eigenfunctions corresponding to $\lambda = \alpha^{1-k}$, $k = 0, 2, 3, \ldots$ are the same as for the operator $dT(g)$ in Proposition 3; and the eigenvalue 1 has the eigenfunction $g(x) - xg'(x)$ (as was found in [22]). The eigenvalue α^2 is missing in these problems.

It turns out that the spectra of $T_4(g)$ and $T_3(g)$ stand in the same relationship as the spectra of $T(g)$ and $T_2(g)$, i.e., the spectrum of $dT_3(g)$ is obtained from the spectrum of $dT_4(g)$ by the substitution $\alpha \to -\alpha$, and half of the explicit eigenfunctions (for even k) could not be recovered.

The one-parameter family of solutions to the equations (2) and (14) can be found if we take $g(0)$ as a parameter on the family and fix it in the procedure. Numerical solutions that we found correspond to the family $\tilde{g}(x) = \mu g(x/\mu)$, $\mu \in \mathbb{R}$. The value α and the spectrum are preserved on the family; however, only one explicit eigenfunction $\tilde{g}(x) - x\tilde{g}'(x)$ is left for the eigenvalue 1 in each problem.

Thus, the equation $y(x) = \beta y(y(x/\beta))$ has a family of solutions only for a discrete set of values β. One of them is $\beta = \alpha \approx -2.5$, another is $\beta = 1$ for $y(x) \equiv 1$. For each family, there is a solution $y_0(x)$ on the family for which $\beta = 1/y_0(1)$. For $\beta = \alpha$, $y_0(x) = g(x)$ (the solution to the equation (1)); for $\beta = 1$, $y_0(x) \equiv 1$, and the family itself is $y(x) = \text{const}$ with the spectrum $1, 0, 0, \ldots$ Since the spectrum is preserved on each family of solutions, the families cannot intersect.

Other families can be found for different types of extrema of the unimodal solution to the equation (1), i.e., $g(x) - 1 = O(x^{2k})$, $k = 2, 3, \ldots$ We found

the solution for $k = 2$ (with $n = 70$)

$$g(x) = 1 - 1.834107907\, x^4 + 0.012962226\, x^8 + 0.311901736\, x^{12} \\ - 0.062014622\, x^{16} - 0.037539249\, x^{20} + 0.017665496\, x^{24} + \ldots \tag{18}$$

for which $1/g(1) = \alpha_2 \approx -1.690302971$ with the accuracy 0.1×10^{-19}, as compared to [3]. The constant $\delta_2 \approx 7.284686217$ is found with 19 correct decimal places. The spectrum of the operator dT on the solution (18) is

$$[\gamma_1, \gamma_2, \ldots] = [\alpha_2^4, \delta_2, \alpha_2^3, \alpha_2^2, \alpha_2, \frac{1}{\alpha_2}, \frac{1}{\alpha_2^2}, \gamma_8, \gamma_9, \frac{1}{\alpha_2^3}, \ldots],$$

where $|\gamma_i| > |\gamma_j|$, $i < j$, and $\gamma_8 \approx 0.291838408$, $\gamma_9 \approx -0.255664558$. Proposition 3 holds here formally except for the eigenfunction (7), which corresponds now to the eigenvalue α_2^4.

Now we turn to different functional spaces, and, to make it more demonstrative, we will use a different quasi-numerical algorithm.

We will consider various Taylor expansions of the solution $g(x)$ to the equation (1). The coefficients of these series are found exactly (symbolically) in rational arithmetic by the symbolic values of the polynomial $g(x)$ at the chosen rational nodes. Symbolic approach avoids the floating point arithmetic at this very crucial stage, since the corresponding linear systems of equations are very ill-conditioned. In this way, we obtain an analog of the Fourier-Chebyshev transform for arbitrary distributed rational nodes. Thus, the floating point arithmetic is only used for the evaluation of polynomials.

Let us verify the Feigenbaum conjecture for the equation (1) using this algorithm and the Lanford's expansion $g(x) = 1 - x^2 y(x^2)$. As it was mentioned, this substitution is frequently used for the numerical solution of the equation (1) (including the paper [12, page 693]).

The polynomial $g(x)$ is expanded in the Taylor series

$$g(x) = 1 + a_1 x^2 + a_2 x^4 + \ldots + a_m x^{2m}, \tag{19}$$

where m is the dimension of the approximation and the number of nodes taken on the interval $[0, 1]$. This set of functions is not a space, but a subset in the space $\mathcal{E}$ of even analytical functions.

We take $m = 15$, and choose the nodes $x_i = i/m$, $i = 1, 2, \ldots, m$. Then we solve symbolically the linear system $\{g(x_i) = g_i\}$, $i = 1, 2, \ldots, m$ with respect to the coefficients a_k, $k = 1, 2, \ldots, m$ of the Taylor expansion (19). Then we evaluate this exact solution as needed in floating point arithmetic on different sets of values $g(x_i)$, $i = 1, 2, \ldots, m$. The Newton iterations are done as described above for Chebyshev nodes, and the spectral problem is

solved after the final iteration for the obtained matrix $I - A$. Thus we find the spectrum

$$S_3 = [\delta, \frac{1}{\alpha^2}, \lambda_6, \lambda_8, \frac{1}{\alpha^4}, \ldots],$$

i.e., we recovered the Feigenbaum conjecture. Only those eigenvalues of the spectrum S (5) are left in S_3 that correspond to even eigenfunctions (except for α^2). The constant δ is found with 19 correct decimal places, although the power of the polynomial solutions is the same, i.e., 30. This is due to the poor choice of the nodes, compensated only by the rational arithmetic.

In many papers, only the space of even analytical functions is defined, and the Lanford's expansion (19) is not stipulated (see, for example, [8, page 1264]). This makes the Feigenbaum conjecture not true. To demonstrate this, we take the expansion

$$g(x) = a_0 + a_1 x^2 + a_2 x^4 + \ldots + a_m x^{2m}, \tag{20}$$

and proceed as described above, but for $x_i = i/m$, $i = 0, 1, 2, \ldots, m$. We obtain the spectrum S_3 plus the missing eigenvalue $\alpha^2 \approx 6.26$.

In both cases (19) and (20), the same solution $g(x)$ is obtained, and the Newton iterations converge quadratically (as the spectrum indicates they should).

Now we demonstrate that even functions are not necessary for the Feigenbaum conjecture to be fulfilled.

First, we take the expansion

$$g(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m \tag{21}$$

on the interval $[-1, 1]$. We take the Chebyshev nodes and approximate them as rational numbers (with small denominators). Then we solve symbolically the linear system $\{g(x_i) = g_i\}$, $i = 0, 1, 2, \ldots, m$ and proceed as described above for the quasi-numerical algorithm. This is another projection of the space $\mathcal{F}$ on a finite dimensional one. As expected, we duplicated the results obtained with the Chebyshev approximation (15) and obtained the spectrum S (5).

Now we fix $a_0 = 1$ in the expansion (21), decrease the dimension by 1, and repeat the process. This will kill the eigenvalue α^2. If we keep a_0 arbitrary and fix the coefficient $a_1 = 0$, then we kill the eigenvalue α, but α^2 is still present. Finally, if we fix both $a_0 = 1$ and $a_1 = 0$ and repeat the process, then we kill both eigenvalues α^2 and α and recover the Feigenbaum conjecture.

§ 5. Conclusion

In the paper [6], the condition P3), i.e., the functions being even, is imposed "mostly for convenience; it simplifies matters and is satisfied by the ψ's we are able to analyze in detail" [6, page 211, 212]. A rhetorical question is: how much our convenience and ability to analyze something in detail are related to the physical relevance of such an analysis? We do not pretend to know the answer to this question with respect to the Feigenbaum universality. However, in other problems, for example, bifurcations of periodic solutions in a dynamical system, there is no reason to restrict the analysis to symmetric functions if the solutions in question possess the symmetry. On the contrary, the loss of the symmetry is one of the possible bifurcations (see [21]).

These considerations lead us to believe that the Feigenbaum conjecture, as it is usually understood, is a numerical artifact. Although at this stage, it is just a personal opinion.

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