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ВЫЧИСЛИТЕЛЬНЫЙ КОД SPIDER. РЕШЕНИЕ ЗАДАЧИ РАВНОВЕСИЯ ПЛАЗМЫ С АНИЗОТРОПНЫМ ДАВЛЕНИЕМ И ВРАЩЕНИЕМ В ТОКАМАКЕ

В МГД теории плазмы в токамаках давление плазмы обычно считается изотропным. Однако нагрев плазмы с помощью инжекции нейтрального пучка и ВЧ нагрев могут приводить к сильной анизотропии плазменных параметров и вращению плазмы. Теория МГД равновесия с учётом инерции плазмы и анизотропии давления начала разрабатываться уже достаточно давно, но до последнего времени она не была последовательно применена в вычислительных кодах для инженерных расчётов равновесия и эволюции плазмы в токамаке.

В данной статье представлена версия кода SPIDER, предназначенная для численного моделирования равновесия плазмы токамака с учётом тороидального вращения плазмы и анизотропии давления; даётся подробный вывод уравнения равновесия осесимметричной плазмы в самом общем виде (с произвольным вращением и анизотропным давлением); излагается метод расчета равновесий с анизотропным давлением при заданном вращательном преобразовании. Также приводятся примеры расчетов и обсуждаются полученные результаты.

Ключевые слова: токамак, плазма, МГД равновесие, анизотропное давление

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THE SPIDER CODE. SOLUTION OF TOKAMAK PLASMA EQUILIBRIUM PROBLEM WITH ANISOTROPIC PRESSURE AND ROTATION

In the MHD tokamak plasma theory plasma pressure is usually assumed to be isotropic. However, plasma heating by neutral beam injection and RF heating can lead to a strong anisotropy of plasma parameters and rotation of the plasma. The development of MHD equilibrium theory with the plasma inertia and the anisotropic pressure began a long time ago, but until now it has not been consistently applied in computational codes for engineering calculations of the plasma equilibrium and evolution in tokamak.

This paper is devoted to description of the version of the SPIDER code for numerical simulation of the tokamak plasma equilibrium with the toroidal rotation and pressure anisotropy; detailed derivation of the axisymmetric plasma equilibrium in the most general form (with arbitrary rotation and anisotropic pressure) and a method of calculation of the equilibrium with anisotropic pressure and prescribed rotational transform are given. Examples of calculations and discussion of the results are also presented.

Key words: tokamak, plasma, MHD equilibrium, anisotropic pressure

1. Introduction

In the MHD tokamak plasma theory plasma pressure is usually assumed to be isotropic. However, in some cases of practical interest, such as plasma heating by neutral beam injection and RF heating, the rotation of the plasma and a strong anisotropy of some plasma parameters can occur [1]. The development of MHD equilibrium theory with the plasma inertia and the anisotropic pressure began to a long time ago [2-5], but until now it has not been consistently applied in computational codes for engineering calculations of the plasma equilibrium and evolution in tokamak.

This paper is devoted to the description of the version of the SPIDER code [6-7] intended for numerical simulation of the tokamak plasma equilibrium with the toroidal rotation and pressure anisotropy.

In the section 2 a detailed derivation of the axisymmetric plasma equilibrium formulation in the most general form (with arbitrary rotation and anisotropic pressure) is given. The equilibrium equation with the toroidal rotation and the static plasma equilibrium equation with anisotropic pressure follow from the general equation. The method of calculating of the fixed boundary plasma equilibrium with anisotropic pressure and prescribed rotational transform $q(\psi)$ is also presented.

The section 3 is devoted to the investigation of the influence of toroidal rotation on the position of the free boundary plasma with pedestal profiles corresponding to the basic equilibrium of the ITER Scenario 4.

In the section 4 examples of the ITER Scenario 2 plasma equilibrium with prescribed rotational transform $q(\psi)$ and varying anisotropy of the plasma pressure are presented.

The part 5 is devoted to the discussion of the obtained results.

2. General formulation of the tokamak plasma equilibrium problem with anisotropic pressure and rotation

2.1 Ideal MHD equations and some of its consequences

The dynamics of a perfectly conducting fluid is described by the equations of motion, energy (in terms of internal energy per unit of mass for adiabatic flows) and magnetic field frozen into fluid:

$$\rho \frac{d\vec{v}}{dt} = -\mathrm{div}\left(\hat{\pi} + \hat{T}\right), \tag{1}$$

$$\rho\frac{d\varepsilon}{dt} = -\hat{\pi}\circ\circ\nabla\vec{v}\,, \tag{2}$$

$$\rho\frac{d}{dt}\left(\frac{\vec{B}}{\rho}\right) = \left(\vec{B},\nabla\right)\vec{v}\,. \tag{3}$$

where $\rho, \vec{v}, \varepsilon$ are the density, velocity, internal energy, $\vec{B}$ is a magnetic field vector. The equations include the Maxwell stress tensor $\hat{T} = \frac{B^2}{2}\hat{I} - \widehat{BB}$ and the pressure tensor:

$$\hat{\pi} = p_\perp \hat{I} + \sigma_\parallel \widehat{BB}\,, \qquad \sigma_\parallel = \frac{p_\parallel - p_\perp}{B^2}\,. \tag{4}$$

The continuity equation

$$\frac{d\rho}{dt} = -\rho\,\mathrm{div}(\vec{v}) \tag{5}$$

and the condition of the solenoidality of the magnetic field

$$\mathrm{div}\,\vec{B} = 0 \tag{6}$$

close the set of equations (1) - (3).

Rewriting the equation (3) in the form

$$\rho\frac{d}{dt}\left(\frac{B^2}{2\rho}\right) = -\hat{T}\circ\circ\nabla\vec{v}\,, \tag{3a}$$

and combining the equations (1) and (2), we obtain the following equation for the total energy per unit mass $E = \frac{v^2}{2} + \varepsilon + \frac{1}{\rho}\frac{B^2}{2}$:

$$\rho\frac{dE}{dt} = -\mathrm{div}\left(\hat{\pi}\circ\vec{v} + \hat{T}\circ\vec{v}\right). \tag{7}$$

Using the explicit form of the tensor (4), it can be shown that

$$\hat{\pi}\circ\circ\nabla\vec{v} = p_\perp \mathrm{div}\,\vec{v} + \sigma_\parallel\left(\vec{B},\left(\vec{B},\nabla\right)\vec{v}\right).$$

On the other hand, from the equations (3) and (5) for the Lagrangian time derivative of the magnetic field squared we have:

$$\frac{d}{dt}\left(\frac{B^2}{2}\right) = B^2\mathrm{div}\,\vec{v} + \left(\vec{B},\left(\vec{B},\nabla\right)\vec{v}\right).$$

With the use of the last expression for the term $\left(\vec{B},\left(\vec{B},\nabla\right)\vec{v}\right)$ the energy equation in the adiabatic case can be written as follows [5]:

$$\rho\frac{d\varepsilon}{dt} = \frac{p_\parallel}{\rho}\frac{d\rho}{dt} - \frac{\sigma_\parallel}{2}\frac{dB^2}{dt}\,. \tag{8}$$

Thus, we can assume that $\varepsilon = \varepsilon\left(V = 1/\rho, B^2, \psi\right)$ is a function of three parameters, with ψ being a label of magnetic surface frozen into the flow, so in a liquid particle the following relation holds :

$$d\varepsilon = -p_\parallel dV - \sigma_\parallel V d\left(\frac{B^2}{2}\right).$$

Let us introduce the function $h=\varepsilon+p_{\|}V$ as an analogue of the enthalpy. Then

$$dh = Vdp_{\|} - \sigma_{\|}Vd\left(\frac{B^2}{2}\right) ,$$

or in the whole space

$$\nabla h = V\nabla p_{\|} - \sigma_{\|}V\nabla\left(\frac{B^2}{2}\right) + \frac{\partial\varepsilon}{\partial\psi}\nabla\psi . \qquad (9)$$

Divergence of the tensor $\hat{\pi}$ can be written as follows:

$$\mathrm{div}\hat{\pi} = \left(\mathrm{rot}\,\sigma_{\|}\vec{B}\right)\times\vec{B} - \sigma_{\|}\nabla\left(\frac{B^2}{2}\right) + \nabla p_{\|} , \qquad (10)$$

or, using (9), in terms of the thermodynamic parameters:

$$\mathrm{div}\hat{\pi} = \left(\mathrm{rot}\,\sigma_{\|}\vec{B}\right)\times\vec{B} + \rho\nabla h - \rho\frac{\partial\varepsilon}{\partial\psi}\nabla\psi .$$

These relations will be used later to derive the axisymmetric equilibrium equation in the most general form.

Bernoulli integral. Let us take the scalar products of the equations (1) and (3) with $\frac{\vec{B}}{\rho}$ and $\vec{v}$ respectively. Summing the obtained equations gives the following relations under the assumption that $(\nabla\psi, B) = 0$:

$$\rho\frac{d}{dt}\left(\frac{\vec{v},\vec{B}}{\rho}\right) = -\frac{1}{\rho}\left(\vec{B}, \mathrm{div}\,\hat{\pi}\right) + \left(\vec{B}, \nabla\left(\frac{\vec{v}^2}{2}\right)\right), \qquad (11)$$

or

$$\rho\frac{d}{dt}\left(\frac{\vec{v},\vec{B}}{\rho}\right) = \mathrm{div}\left(\vec{B}\left(\frac{v^2}{2} - h\right)\right), \qquad (12)$$

or in the integral form:

$$\frac{d}{dt}\left(\int_V (\vec{v},\vec{B})dV\right) = \int_{\partial V}\left(\frac{v^2}{2} - h\right)(\vec{B}, d\vec{S}), \qquad (13)$$

where V is the Lagrangian volume, and ∂V is its boundary.

2.2 Axisymmetric plasma equilibrium with rotation

The axisymmetric magnetic field consists of the poloidal and toroidal components:

$$\vec{B}_p = \nabla\psi\times\nabla\varphi \ , \quad \vec{B}_\varphi = F\nabla\varphi \ .$$

Plasma equilibrium equation with a scalar pressure taking into account inertial forces can be written as:

$$\rho(\vec{v},\nabla)\vec{v} = \mathrm{rot}\,\vec{B}\times\vec{B} - \nabla p . \qquad (14)$$

In the steady state when

$$(\vec{v}_p, \nabla\psi) = 0, \quad \mathrm{div}(\rho\vec{v}_p) = 0,$$

the poloidal components of the velocity and magnetic field are related as follows:

$$\rho\vec{v}_p = \mathrm{rot}(\chi\nabla\varphi) = \nabla\chi \times \nabla\varphi,$$
$$\rho\vec{v}_p = \chi'\vec{B}_p, \quad \chi = \chi(\psi).$$

It means that the velocity vector $\vec{v}$ lies on the magnetic surfaces $\psi = const$.

The toroidal component of the velocity is expressed through the angular velocity ω:

$$\vec{v}_\varphi = \omega R^2 \nabla\varphi.$$

Assuming the ideal conductivity of plasma we can write:

$$\vec{E} = -\vec{v} \times \vec{B}, \quad \vec{E}_\varphi = -\vec{v}_p \times \vec{B}_p = 0,$$
$$\vec{E} = \vec{E}_p = -\Omega\nabla\psi,$$

i.e. the electric field $\vec{E}$ is orthogonal to magnetic surfaces. The function Ω is the flux quantity in the steady state:

$$\mathrm{rot}\,\vec{E}_p = 0 \;\rightarrow\; \nabla\Omega \times \nabla\psi = 0 \;\rightarrow\; \Omega(\psi).$$

Using an explicit representation of the poloidal magnetic field through χ, ρ, ω, F

$$\vec{E}_p = -\vec{v}_p \times \vec{B}_\varphi - \vec{v}_\varphi \times \vec{B}_p = -\frac{\chi'}{\rho}\vec{B}_p \times \vec{B}_\varphi - \vec{v}_\varphi \times \vec{B}_p =$$
$$= \vec{B}_p \times \left(\vec{v}_\varphi - \frac{\chi'}{\rho}\vec{B}_\varphi\right) = \vec{B}_p \times \left(\omega R^2 - \frac{F\chi'}{\rho}\right)\nabla\varphi = -\left(\omega - \frac{F\chi'}{\rho R^2}\right)\nabla\psi,$$

we obtain

$$\Omega(\psi) = \omega - \frac{F\chi'}{\rho R^2}, \tag{15}$$

$$\vec{v} = \frac{\chi'}{\rho}\vec{B} + \Omega R^2 \nabla\varphi. \tag{16}$$

In the case of the steady state axisymmetric flow the Bernoulli integral reads:

$$h + \frac{v^2}{2} - \omega\Omega R^2 = H(\psi). \tag{17}$$

Let us denote the vorticity $\vec{W} = \mathrm{rot}\,\vec{v}$ and the electric current density $\vec{j} = \mathrm{rot}\,\vec{B}$. Then the steady state equilibrium equation (14) becomes:

$$\rho\nabla\left(\frac{v^2}{2}\right) + \rho\vec{W} \times \vec{v} = \vec{j} \times \vec{B} - \nabla p. \tag{18}$$

Substituting the expression $\rho\vec{v}$ from (16) to (18), we get:

$$(\chi'\vec{W} - \vec{j}) \times \vec{B} + \rho\Omega R^2(\vec{W} \times \nabla\varphi) = -\nabla p - \rho\nabla\left(\frac{v^2}{2}\right), \tag{19}$$

and using $\vec{W}_p = \nabla(\omega R^2)\times\nabla\varphi$ in (19) arrives to:

$$\vec{I}\times\vec{B} + \rho\omega R^2\nabla\Omega = -\nabla p - \rho\nabla\left(\frac{v^2}{2} - \omega\Omega R^2\right), \tag{20}$$

where $\vec{I} = \chi'\vec{W} - \vec{j}$.

The toroidal part of the equation (20) gives $\vec{I}_p\times\vec{B}_p = 0$, from which it follows

$$\begin{aligned} &F - \chi'\omega R^2 = \Lambda(\psi), \\ &\vec{I}_p = -\nabla\Lambda\times\nabla\varphi + \omega R^2\chi''\vec{B}_p, \end{aligned} \tag{21}$$

i.e. the vector $\vec{I}$ lies on magnetic surfaces. The poloidal part of (20) is:

$$\vec{I}_p\times\vec{B}_\varphi + \vec{I}_\varphi\times\vec{B}_p = -\rho\omega R^2\nabla\Omega - \nabla p - \rho\nabla\left(\frac{v^2}{2} - \omega\Omega R^2\right). \tag{22}$$

Let us expand the two terms in the left-hand side of (22):

$$\vec{I}_\varphi\times\vec{B}_p = \left(\chi'\vec{W}_\varphi - \vec{j}_\varphi\right)\times\vec{B}_p = \left(\left(\chi' rot\left(\frac{\chi'}{\rho}\nabla\psi\times\nabla\varphi\right) - rot(\nabla\psi\times\nabla\varphi)\right),\nabla\varphi\right)\nabla\psi,$$

$$\vec{I}_p\times\vec{B}_\varphi = -F(\nabla\Lambda\times\nabla\varphi)\times\nabla\varphi + F\omega R^2\chi''(\nabla\psi\times\nabla\varphi)\times\nabla\varphi = \frac{F}{R^2}\nabla\Lambda - F\omega\chi''\nabla\psi.$$

Now the equations (22) can be rewritten as:

$$\begin{aligned} &\chi'\left(rot\left(\frac{\chi'}{\rho}\nabla\psi\times\nabla\varphi\right),\nabla\varphi\right)\nabla\psi - (rot(\nabla\psi\times\nabla\varphi),\nabla\varphi)\nabla\psi = \\ &= -\frac{F}{R^2}\nabla\Lambda + F\omega\chi''\nabla\psi - \rho\omega R^2\nabla\Omega + \rho\nabla\left(\omega\Omega R^2 - \frac{v^2}{2}\right) - \nabla p. \end{aligned} \tag{23}$$

If the entropy S is introduced in the usual way as $\nabla h = V\nabla p + T\nabla S$ through the enthalpy h and the temperature T, and it is the flux function $S = S(\psi)$, then in the view of (17) and (9) the equation (23) takes the form:

$$\begin{aligned} &\chi'\left(\mathrm{rot}\left(\frac{\chi'}{\rho}\nabla\psi\times\nabla\varphi\right),\nabla\varphi\right) - (\mathrm{rot}(\nabla\psi\times\nabla\varphi),\nabla\varphi) = \\ &= -\frac{F}{R^2}\Lambda' + F\omega\chi'' - \rho\omega R^2\Omega' - \rho H' + \rho T S'. \end{aligned} \tag{24}$$

If alternatively the density is the flux function $\rho = \rho(\psi)$, then in the view of $\nabla p = \rho\nabla\left(\frac{p}{\rho}\right) - \frac{p\nabla\rho}{\rho}$ the last two terms in the right hand side of (23) are transformed to:

$$\rho\nabla\left(\omega\Omega R^2 - \frac{v^2}{2} - \frac{p}{\rho}\right) + \frac{p\nabla\rho}{\rho}, \qquad \frac{v^2}{2} + \frac{p}{\rho} - \omega\Omega R^2 = \tilde{H}(\psi),$$

and the last two terms in the right-hand side of (24) convert to

$$-\rho\tilde{H}' + \frac{p\rho'}{\rho}.$$

2.3 Axisymmetric plasma equilibrium with anisotropic pressure

Recalling the expression (10) for the pressure tensor divergence, we can rewrite the static ($\vec{v}=0$) equilibrium equation with the anisotropic pressure in the form:

$$\operatorname{div}\hat{\pi} = \left(\operatorname{rot}\sigma_{\parallel}\vec{B}\right)\times\vec{B} - \sigma_{\parallel}\nabla\left(\frac{B^2}{2}\right) + \nabla p_{\parallel} = \vec{j}\times\vec{B}, \tag{25}$$

where $\hat{\pi} = p_{\perp}\hat{I} + \sigma_{\parallel}\hat{BB}$, $\sigma_{\parallel} = \dfrac{p_{\parallel}-p_{\perp}}{B^2}$,

or

$$\vec{k}\times\vec{B} = \nabla p_{\parallel} - \sigma_{\parallel}\nabla\left(\frac{B^2}{2}\right),$$

where $\vec{k} = \operatorname{rot}(\sigma\vec{B})$, $\sigma = 1-\sigma_{\parallel}$. The projection of (25) on the toroidal direction $\nabla\varphi$ gives $F\sigma = \Lambda(\psi)$.

Poloidal part of (25) gives

$$\nabla\psi\left(\operatorname{rot}(\sigma\nabla\psi\times\nabla\varphi),\nabla\varphi\right) - \nabla\Lambda\frac{F}{R^2} - \rho\nabla h + \rho\frac{\partial\varepsilon}{\partial\psi}\nabla\psi = 0. \tag{26}$$

Thus we have:

$$h\left(\rho, B^2, \psi\right) = H(\psi), \tag{27}$$

and the equilibrium equation (26) can be written as:

$$\left(\operatorname{rot}(\sigma\nabla\psi\times\nabla\varphi),\nabla\varphi\right) - \Lambda'\frac{F}{R^2} - \rho H' + \rho\frac{\partial\varepsilon}{\partial\psi} = 0. \tag{28}$$

Note that from (27) $p_{\parallel}$ can be expressed as a function of two variables $\left(B^2,\psi\right)$.

Then (28) can be rewritten as

$$\left(\operatorname{rot}(\sigma\nabla\psi\times\nabla\varphi),\nabla\varphi\right) - \Lambda'\frac{F}{R^2} - \frac{\partial p_{\parallel}\left(\psi, B^2\right)}{\partial\psi} = 0. \tag{29}$$

Taking into account $p_{\parallel} = p_{\parallel}\left(\psi, B^2\right)$ in the equation (25) we obtain:

$$\frac{\partial p_{\parallel}}{\partial B^2} = \frac{\sigma_{\parallel}}{2}. \tag{29a}$$

Averaging of (29) over the volume dV_{ψ} between the magnetic surfaces $\psi = const$ gives:

$$dK_{\varphi}d\psi - d\Phi d\Lambda = \left\langle\frac{\partial p_{\parallel}}{\partial\psi}\right\rangle_{V\psi} dV_{\psi}d\psi, \tag{30}$$

or

$$dK_{\varphi} + qd\Lambda = \left\langle\frac{\partial p_{\parallel}}{\partial\psi}\right\rangle_{V_{\psi}} dV_{\psi}, \tag{31}$$

where $K_{\varphi} = \int\limits_{S_{\psi}} k_{\varphi}dS$, S_{ψ} - area inside the level line $\psi = const$ in the toroidal cross-section.

Let us obtain the expression for the averaged current density parallel to the magnetic field, which may be needed for derivation of the magnetic field diffusion equation averaged over the magnetic surfaces:

1) Starting with the expression

$$\left(\vec{j},\vec{B}\right)=\left(\vec{j}_\varphi,\vec{B}_\varphi\right)+\left(\vec{j}_p,\vec{B}_p\right)=F\left(\vec{j}_\varphi,\nabla\varphi\right)+\left(\vec{j}_p,\vec{B}_p\right), \tag{32}$$

and integrating it over the volume dV_F between the poloidal current surfaces $F=const$, we obtain:

$$dV\left\langle\left(\vec{j},\vec{B}\right)\right\rangle_{V_F}=FdI_F-I_F dF\,,$$

where $I_F=\int\limits_{S_F}\vec{j}_\varphi dS$, S_F - area inside the level line $F=const$ in the toroidal cross-section.

2) From the expression

$$\left(\vec{k},\vec{B}\right)=\left(\mathrm{rot}\left(\sigma\vec{B}\right),\vec{B}\right)=\left(\sigma\vec{B},\mathrm{rot}\vec{B}\right)=\sigma\left(\vec{j},\vec{B}\right) \tag{33}$$

we get $\left(\vec{k},\sigma\vec{B}\right)=\sigma^2\left(\vec{j},\vec{B}\right)=\left(\vec{k}_\varphi,\sigma\vec{B}_\varphi\right)+\left(\vec{k}_p,\sigma\vec{B}_p\right)=\Lambda\left(\vec{k}_\varphi,\nabla\varphi\right)+\left(\vec{k}_p,\sigma\vec{B}_p\right)$, and integration over the volume between the magnetic surfaces leads to

$$\Lambda dK_\varphi-K_\varphi d\Lambda=\left\langle\left(\vec{k},\sigma\vec{B}\right)\right\rangle_{V_\psi}dV_\psi\,. \tag{34}$$

Currents and fluxes. The toroidal current I through the magnetic surface $\psi=const$ cross-section poloidal flux of the magnetic field are related as follows [8]:

$$I=\alpha_{22}\frac{d\psi}{dV_\psi}\,, \tag{35}$$

$$\alpha_{22}=\oint\limits_{\psi=const}\frac{\left|\nabla V_\psi\right|}{R}dl\,, \tag{36}$$

where V_ψ is the volume inside the magnetic surface. In general case of equilibrium with anisotropic pressure, when the magnetic and current surfaces are not the same, the analogous relation between the poloidal current F and the toroidal flux Φ_F can be written through the surface $F=const$ but not the magnetic surface:

$$F=\alpha_{33}\frac{d\Phi_F}{dV_F}\,, \tag{37}$$

$$\alpha_{33}=\left\langle\frac{1}{R^2}\right\rangle_{V_F}^{-1}, \tag{38}$$

where V_F is the volume inside the current surface $F=const$. In addition to that, for the fluxes of the vector $\vec{k}$ we have:

$$K_\varphi=C_{22}\frac{d\psi}{dV_\psi}\,, \tag{39}$$

$$C_{22}=\oint\limits_{\psi=const}\sigma\frac{\left|\nabla V_\psi\right|}{R}dl\,, \tag{40}$$

where V_ψ is the volume bounded inside the surface $\psi = const$,

$$\Lambda = C_{33}\frac{d\Phi}{dV_\psi} \ , \tag{41}$$

$$C_{33} = \left\langle \frac{1}{\sigma R^2} \right\rangle_{V_\psi}^{-1} \ . \tag{42}$$

Flux conserving equilibrium equation with anisotropic pressure. Let us specify how the equilibrium equation (29) can be solved with the prescribed functions $q(\psi)$, $\frac{\partial p_\parallel}{\partial \psi}(\psi, B^2)$ and the total poloidal flux in the plasma $\delta\psi = \psi_m - \psi_b$ (see the section 4).

Averaged equilibrium equation (31) with the relations (39) - (42) can be rewritten as:

$$\frac{d}{dV_\psi}\left(C_{22}\frac{d\psi}{dV_\psi}\right) - q\frac{d}{dV_\psi}\left(C_{33}q\frac{d\psi}{dV_\psi}\right) = \left\langle \frac{\partial p_\parallel}{\partial \psi} \right\rangle_{V_\psi} \ . \tag{43}$$

To solve the equilibrium problem the iterative procedure similar to the procedure in [7] is used. At each iteration the one-dimensional equation (43) is solved. From its solution we can determine the quantity

$$\Lambda\Lambda' = -qC_{33}\frac{d}{dV_\psi}\left(C_{33}q\frac{d\psi}{dV_\psi}\right). \tag{44}$$

Then the two-dimensional equation (29) with $\Lambda\Lambda'$ from (44) and prescribed $\frac{\partial p_\parallel}{\partial \psi}(\psi, B^2)$ is solved.

Remark. It is easy to show that the equilibrium equation with anisotropic pressure and plasma rotation can be wrirren as follows:

$$\chi'\left(\mathrm{rot}\left(\frac{\chi'}{\rho}\nabla\psi \times \nabla\varphi\right), \nabla\varphi\right) - \left(rot(\sigma\nabla\psi \times \nabla\varphi), \nabla\varphi\right) = -\frac{F}{R^2}\Lambda' + F\omega\chi'' - \rho\omega R^2\Omega' - \rho H' + \rho T S' \ . \tag{45}$$

3. Influence of the toroidal rotation on the plasma equilibrium for the basic ITER scenario

In the case of purely toroidal rotation the equilibrium equation (23) significantly simplifies and takes the form [4]:

$$-\frac{1}{R^2}\left(\Delta^*\psi + FF'\right) = \frac{\partial p(\psi, R)}{\partial \psi} \ . \tag{46}$$

Poloidal current F and ω depend on the poloidal flux function ψ only, i.e. current and magnetic surfaces coincide.

If we assume that the density ρ is constant on magnetic surfaces, i.e. $\rho = \rho(\psi)$, the expression for the pressure p can be written as follows:

$$p(\psi, R) = p_0(\psi) + \frac{\rho(\psi)\omega^2 R^2}{2}. \tag{47}$$

From this it follows:

$$\frac{\partial p}{\partial \psi} = p_0' + \frac{\left(\rho\omega^2\right)' R^2}{2}. \tag{48}$$

To study the influence of toroidal rotation on the equilibrium, the following series of calculations with different values of the rotation velocity v were performed for the equilibrium configuration of the ITER tokamak (Fig.1):

- the basic equilibrium configuration "Scenario 4" from the ITER database with the value $\beta_{ax} = 2\mu_0 p_0 / B_0^2 = 6.2\%$ was chosen;
- the normalized poloidal flux $\overline{\psi} = (\psi - \psi_b)/(\psi_a - \psi_b)$ varies from 1 at the magnetic axis to 0 at the plasma boundary;
- the profile $\left(\rho\omega^2\right)' = A_0\left(1-(1-\overline{\psi})^{\alpha_1}\right)^{\alpha_2}$ with $\alpha_1 = 2$, $\alpha_2 = 0.5$ was prescribed;
- the maximal velocity $v_{\varphi_{\max}} = \omega_{\max} R_0$ changes within the range of 100-300 км/с, $\rho_{\max} = 1.5 \cdot 10^{-10} \frac{g}{cm^3}$, which corresponds to $n = 10^{14} cm^{-3}$ for a hydrogen plasma; value $\mu_0 \rho_{\max} v_{\varphi\max}^2 / B_0^2 = 0.0075\%$ for $v_{\varphi_{\max}}$ =100 km/sec.

Fig.2 shows the profiles of the main flux functions in plasma. For the series of calculations with different values of toroidal rotation velocity the Table 1 and Fig.3 show the dependence of the radial position of the magnetic axis and the X-point - $R_m(m)$ and $R_x(m)$ respectively - on the magnitude of the rotation velocity.

Table 1 - Radial coordinates of the magnetic axis and the X-point in the equilibria with free boundary and plasma toroidal rotation, depending on the magnitude of the rotation velocity

$v(km/\sec)$	0	100	200	300
$R_m(m)$	6.690	6.6920	6.696	6.7056
$R_x(m)$	5.085	5.086	5.088	5.0908

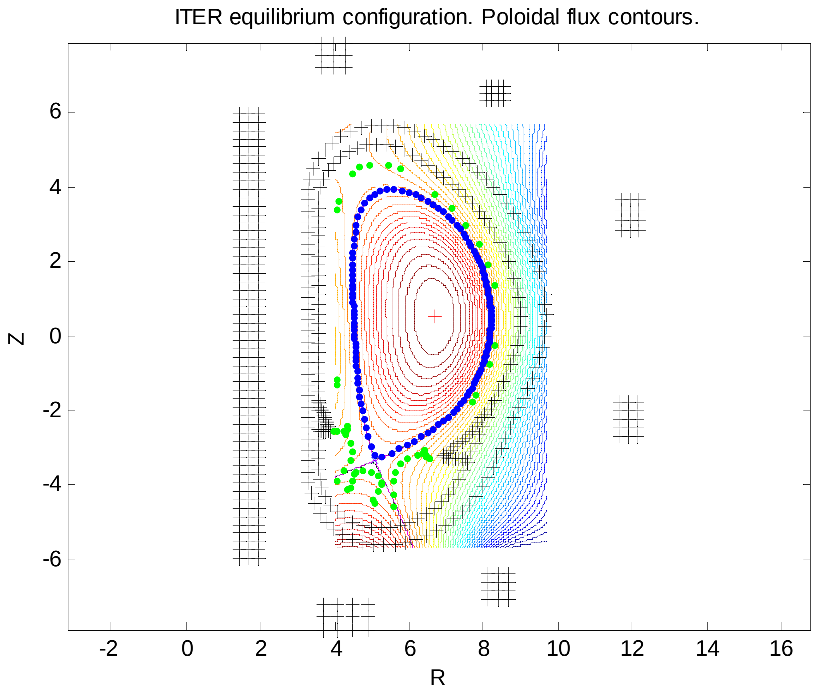

Figure 1 - Basic free boundary plasma equilibrium ($\bar{v} = 0$). ITER Scenario 4

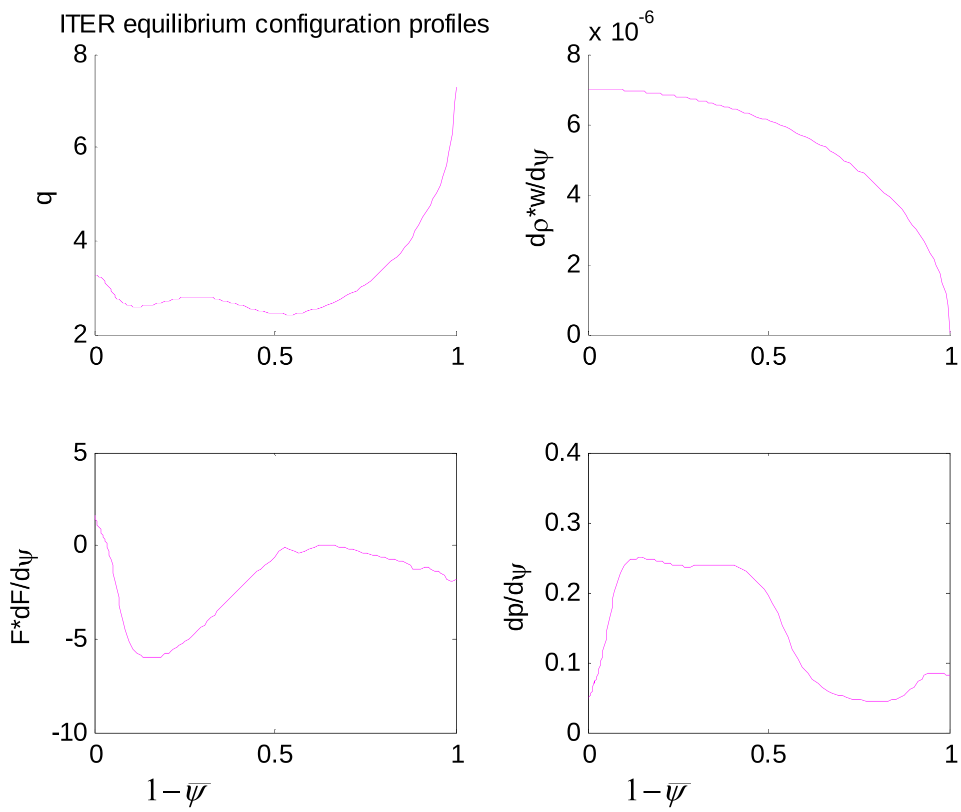

Figure 2 - Profiles of the flux functions in plasma for Fig.1 basic equilibrium configuration versus normalized poloidal flux. The input flux functions for the equilibrium - p', FF', $(\rho\omega^2)'$

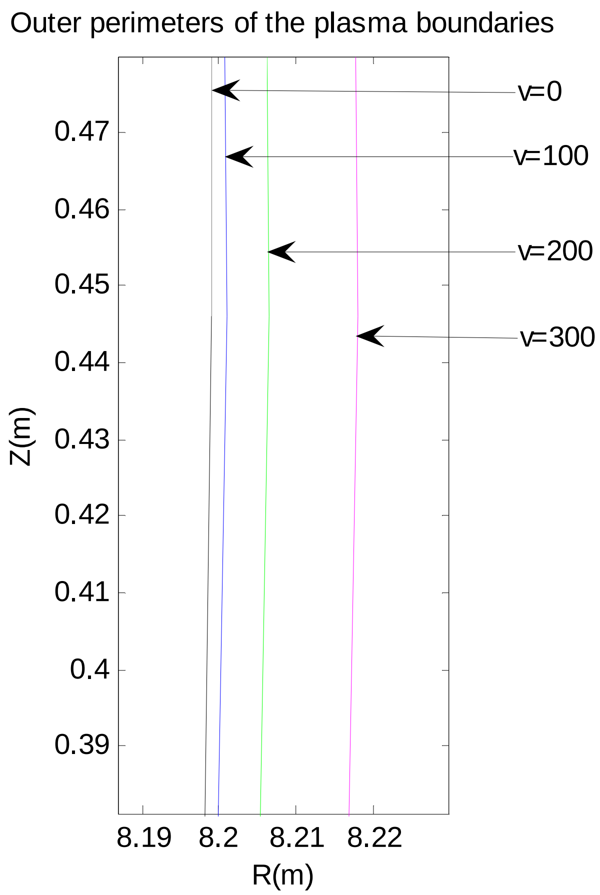

Figure 3 - The position of the equilibrium plasma boundary at the outer side of the torus depending on the magnitude of the toroidal rotation velocity (numbers correspond to the maximal rotation velocities in km/sec)

4. Calculation of the axisymmetric plasma equilibria with the anisotropic pressure due to high fraction of energetic particles

Equation (29a) can be conveniently converted to the form [9]:

$$\frac{\partial}{\partial B}\left(\frac{p_{\parallel}}{B}\right) = -\frac{p_{\perp}}{B^2}\,. \tag{49}$$

Suppose that $p_{\perp} = p_{\perp}(\psi)$ is a function of ψ. Then the integration of (49) gives:

$$p_{\parallel}(\psi, B) = p_{\perp}(\psi) + f(\psi)\cdot B\,. \tag{50}$$

Without loss of generality, the expression (50) can be rewritten as:

$$p_{\parallel} = p_{\perp}(\overline{\psi}) + f_0(\overline{\psi})\frac{B}{B_{\max}}\,, \tag{51}$$

where normalized poloidal flux of the magnetic field $\overline{\psi} = (\psi - \psi_b)/(\psi_a - \psi_b)$ varies from 1 on the magnetic axis to 0 at the plasma boundary. The calculations were performed with the prescribed profiles of $p_{\|}(\overline{\psi}, B)$ satisfying (51). The initial equilibrium was taken from the ITER database for the Scenario 2 with plasma current $I_{pl} = 15MA$ and assuming the perpendicular pressure to coincide with the initial equilibrium total pressure $p_{\perp} = p_0$. Fig.4 shows the profiles of the main flux functions in plasma. In Fig.5 the computational grid adaptive to the magnetic surfaces $\psi = const$ of the base equilibrium configuration is plotted. The following parameters were prescribed:

- the safety factor q and the total poloidal flux in the plasma from the initial equilibrium which correspond to a flux conserving series;
- $f_0(\overline{\psi}) = A_0\left(1 - (1 - \overline{\psi})^{\alpha_1}\right)^{\alpha_2}$;
- $p_{\perp}(\overline{\psi}) = p_0(\overline{\psi})$;
- $A_0 = 2p_0(1)$, $\alpha_1 = 4$, $\alpha_2 = 0.25$.

The calculations show that the qualitative changes in the equilibrium take place for high longitudinal beta: when the value of the longitudinal beta is about three times the value of the perpendicular beta the shift of the magnetic axis is quite large: $R_m = 6.568$ m as compared to the shift in the initial equilibrium $R_{m_0} = 6.418$ m under strong pressure anisotropy. The level lines $\psi = const$ (magnetic surfaces) strongly deviate from the level curves $\frac{\partial p_{\|}}{\partial \psi} = const$ (Fig.6). In addition, the current surfaces $F = const$ deviate significantly from the magnetic flux surfaces (Fig.7). For the equilibrium with $A_0 = p_0(1)$ (the longitudinal beta being two times lower) the discrepancy between the current and flux surfaces is still significant (Fig.8).

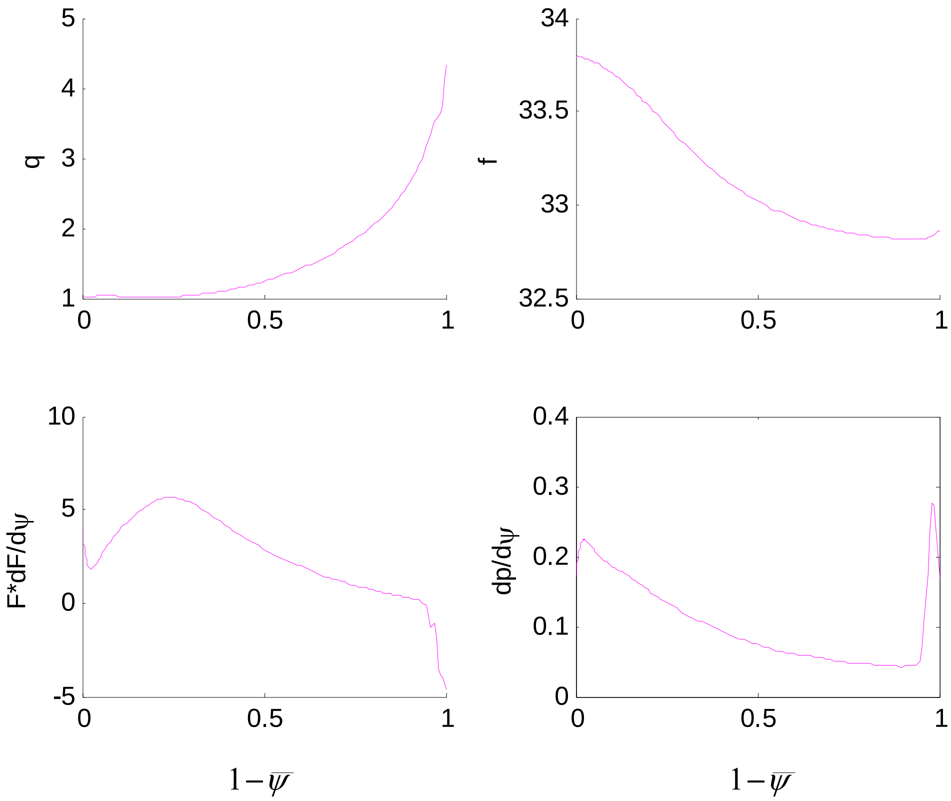

Figure 4 - Profiles of major flux functions in the plasma versus the normalized poloidal flux for the basic equilibrium configuration of Figure 5. The input parameters of the problem - p', FF'

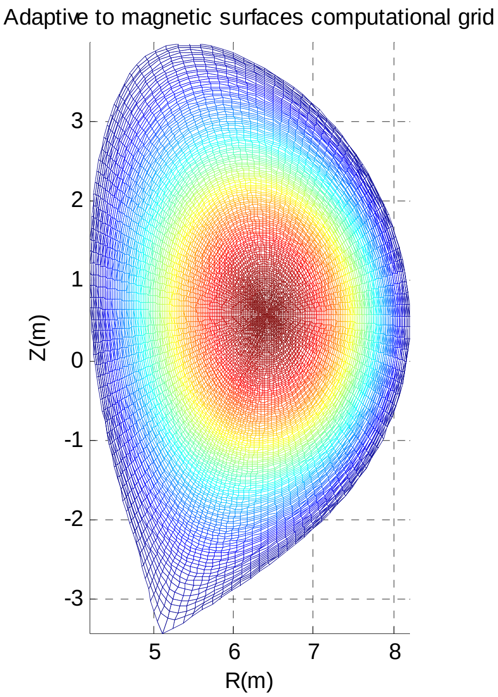

Figure 5 – The computational grid adaptive to the magnetic surfaces $\psi = const$ *of the basic ITER Scenario 2 equilibrium configuration*

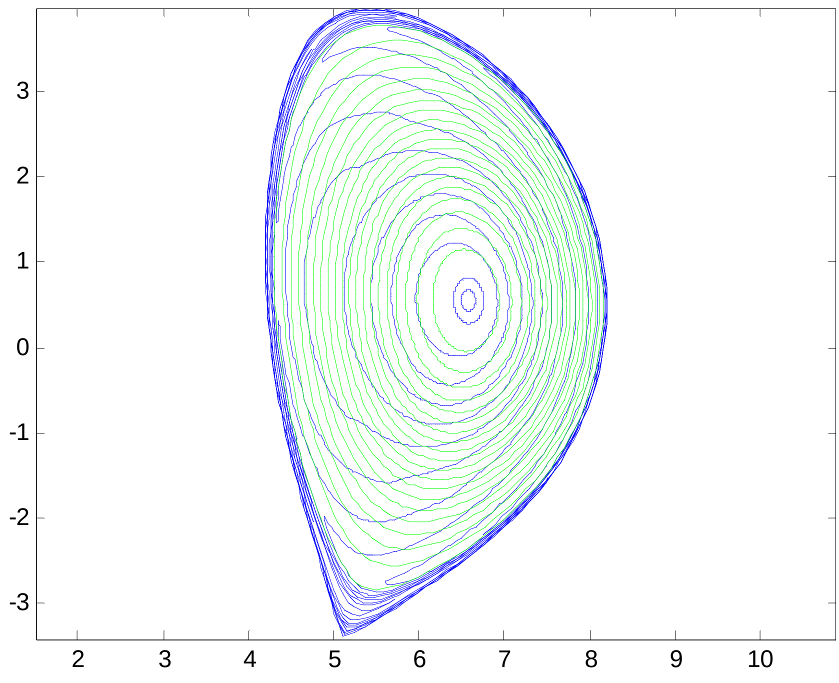

Figure 6 - Equilibrium with large longitudinal beta. Level lines $\psi = const$ *(magnetic surfaces) – green, level lines* $\frac{\partial p_{\parallel}}{\partial \psi} = const$ *– blue*

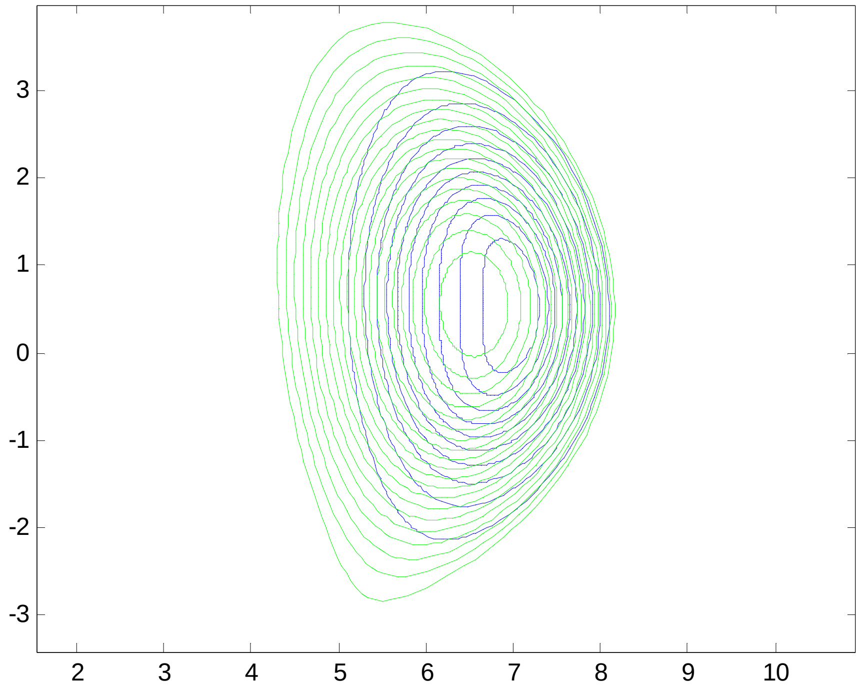

Figure 7 - Equilibrium with the large longitudinal beta. Level lines $\psi = const$ *(magnetic surfaces) – green, level lines* $F = const$ *(current surfaces) – blue*

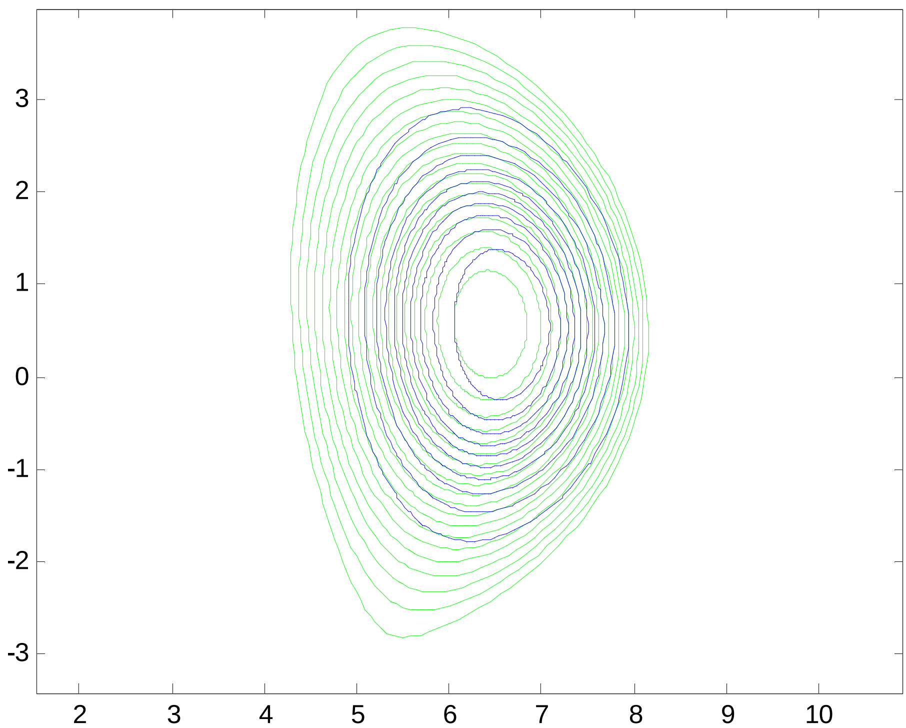

Figure 8 - Equilibrium with the lower longitudinal beta. Level lines $\psi = const$ *(magnetic surfaces) – green; level lines* $F = const$ *(current surfaces) – blue*

5. Conclusions

The calculations of plasma equilibrium with toroidal rotation confirm a weak influence of the rotation on the free plasma boundary position [10]: for predicted ITER plasma rotation speeds - below 100 km/s ($\mu_0\rho_{\max}v^2_{\varphi\max}/B_0^2 = 0.0075\%$) - the shift of the plasma boundary at the outer side of the torus is less than a few millimeters. Fig.3 shows that the effect increases proportionally to the square of the speed, while the position of the magnetic axis and the X-point change slightly (see Tab.1).

For the flux conserving equilibria with the anisotropic pressure and large fraction of energetic particles (beta of the energetic particles comparable with the total beta in plasma) we can see significant deviation of the current surfaces from the magnetic surfaces. This fact suggests some restrictions on the choice of a realistic equilibrium model with anisotropic pressure in ITER.

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References

1. Fasoli A. et al. - Nucl.Fusion, 2007, v.47, p.128.
2. Grad H. - Phys.Fluids, 1967, v.10, p.137.
3. L.E.Zakharov, V.D.Shafranov - Voprosy teorii plasmy. Atomizdat, 1982, v.11, p.118.
4. E.K.Maschke and H.Perrin - Plasma Phys.1980, v.22, p.579.
5. R.Iacono A.Bondeson et al. - Phys.Fluids 1990 B vol.2 No.6 p.1794.
6. A.A.Ivanov, R.R.Khayrutdinov, S.Yu.Medvedev, Yu.Yu.Poshekhonov - The SPIDER code-axisymmetric fixed boundary plasma equilibrium solver, KIAM preprint No7, 2006.
   URL: http://www.keldysh.ru/papers/2006/source/prep2006_07.doc
7. A.A.Ivanov, R.R.Khayrutdinov, S.Yu.Medvedev, Yu.Yu.Poshekhonov - The SPIDER code – solution of direct and inverse problems for free boundary tokamak plasma equilibrium, KIAM preprint No39, Moscow, 2009.
   URL: http://www.keldysh.ru/papers/2009/prep39/prep2009_39.pdf
8. L.M.Degtyarev, V.V.Drozdov, S.Yu.Medvedev - Numerical modeling of toroidal plasma equilibrium and stability (in Russian). Keldysh Institute of Applied Mathematics, Moscow 1989.
9. V.D.Pustovitov - AIP Conf. Proc. 2012, v.1478, p.50.
10. V.D.Pustovitov - Plasma Physics Reports, 2003, v.29, p.105. (Translated from Fizika Plazmy, Vol. 29, No. 2, 2003, pp. 124–130).

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