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A SOLUTION OF THE BLASIUS PROBLEM

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The classical Blasius boundary layer problem in its simplest statement consists in finding an initial value for the function satisfying the Blasius ODE on semi-infinite interval such that a certain condition at infinity be satisfied. Despite an apparent simplicity of the problem and more than a century of effort of numerous scientists, this elusive constant is determined at present numerically and not much better than it was done by Töpfer in 1912. Here we find this (Blasius) constant rigourously in closed form as a convergent series of rational numbers. Asymptotic behaviour, and lower and upper bounds for the partial sums of the series are also given.

В.П. Варин. Решение проблемы Блазиуса. Препринт Института прикладной математики им. М.В. Келдыша РАН, Москва, 2013.

Классическая проблема Блазиуса о погранслое в ее простейшей формулировке состоит в определении начального значения функции, удовлетворяющей дифференциальному уравнению Блазиуса на полубесконечном интервале, таким образом, что выполняется некоторое условие на бесконечности. Несмотря на кажущуюся простоту этой задачи и более чем вековую историю ее исследования многими учеными, эта константа к настоящему времени вычисляется численными методами и не намного лучше, чем это было сделано Топфером в 1912 г. Здесь эта константа (Блазиуса) находится строго и в явном виде как сходящийся ряд рациональных чисел. Также приводится асимптотика частичных сумм этого ряда и их нижняя и верхняя оценка.

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§ 1. Introduction

The classical Blasius problem is, historically, the first and the simplest boundary layer problem [1]. For more than a century, it attracted attention of numerous scientists, among which we can name such prominent figures as Weyl and von Neumann.

We recapitulate the problem briefly, keeping in mind that information on this problem is widely available.

The Blasius BVP on semi-infinite interval $[0, \infty]$ takes the form

$$2\, f_{xxx} + f\, f_{xx} = 0, \quad f(0) = f_x(0) = 0,\ f_{xx}(0) = s,\ f_x(\infty) = 1, \tag{1}$$

where f_x denotes the derivative of $f(x)$ with respect to x.

In its simplest statement, the problem consists in finding the initial value $s = f_{xx}(0)$ such that the condition $f_x(\infty) = 1$ at infinity be satisfied. We will call the value s the *Blasius constant*.

There are numerous modifications of this problem that can be found in literature. They are either generalizations of this problem or equivalent to (1) with a linear change of variables.

Nowadays the interest to this problem seems only to increase, and the evidence for this is the burst of activity around this problem at the turn of its centenary in 2007. If one types "Blasius problem" in Google search engine, hundreds of references appear along with attached PDF files of relevant (and sometimes irrelevant) papers.

In our opinion, there are several reasons for such an undiminished interest. One of them is the need to test some new computational techniques implemented on powerful computers. This was the prime motivation for the author of this paper. Another one, obviously, is the fact that this problem is far from being solved. On the contrary, it seems that attempts to solve this problem have made a full circle after a century of efforts.

In 1912, Töpfer [2] used a Runge-Kutta integrator and obtained virtually by hand several digits of the Blasius constant. Töpfer's idea was to exploit the symmetry of the problem, namely, that it is invariant under the change of variables

$$x \to t/\alpha, \quad f(t) \to \alpha\, g(t),$$

where α is an arbitrary constant. Hence we can take $s = 1$ in (1), then integrate this IVP as far as we deem reasonable, then take

$$s \approx h^{-3/2}, \quad h = f_x(x_{max}),$$

where x_{max} approximates infinity. Thus, taking $x_{max} = 200$, and using ordinary double float arithmetic, we obtain in a matter of seconds

$$s \approx 0.33205733621, \tag{2}$$

where we retained only the digits that coincide with more accurate results in [3, 4].

A century later, this approach is still the most effective and gives the best results. But even the best results obtained on powerful computers today are only twice (or may be thrice) better than the result obtained by Töpfer. For example, in the paper [3], Boyd gives the Blasius constant with 16 decimal places that all "are believed correct". Recently [4], this result was improved with the solution of (almost) exactly the same IVP by the same method.

However, this purely numerical approach was never considered satisfactory. First, this is not a solution but a numerical approximation to the solution. Second, it is unknown how good an approximation it is, i.e., there are no rigorous error estimates. And third, it is not feasible to improve the accuracy of computations significantly for practical as well as theoretical reasons.

There are other approaches to this problem, too numerous to recite them here (see [3, 4]), among which the most promising seem to be the series acceleration techniques such as Padé approximations and sequence transformations. These techniques were used for summation of the Blasius power series for the function $f(x)$ at the origin outside its radius of convergence.

So far, there is no proof that these methods give a convergent process; and, numerically, they are proved to be less effective than the Töpfer's approach: they give less accurate results also without rigorous error estimates.

The Crocco transformation [5] (see Sect. 2) reduces the Blasius problem (1) to solution of the Crocco equation of the second order on a finite interval. Until recently [6], the Crocco equation received much less attention than the original problem (1) on semi-infinite interval. In the paper [7], power series for the Crocco solution were used to compute the Blasius constant about as accurately as it was done by Töpfer. In addition, it was established experimentally in [7] that Padé approximations in this case diverge.

There are also numerous reductions of the Crocco equation to the planar vector field, which are all equivalent to the first such reduction attributed by Weyl to von Neumann (see historical review in [6]). In other words, the problem is reduced to solution of a BVP for one differential equation of the first order, but on a semi-infinite interval again. This approach, as yet, is no more successful in finding the Blasius constant as the others.

To round up this by necessity short review, we remark that there is, actually, a convergent process that gives solution to the Blasius problem. As it was proved by Weyl, this problem can be solved by iteration of an integral equation for the function $f_{xx}(x)$ [8]. However, this approach is totally impractical.

In this paper we examine the Crocco equation and the properties of its solution as an analytical function (Sect. 2). The infinity in the problem (1) is shifted by Crocco transformation to the end of the interval $[0, 1]$. We will demonstrate that there are no other singularities of the solution in the unit circle. This fact partially explains why Padé approximations do work in the case of Blasius series for the original problem, and why they do not work in the Crocco equation.

At the end of Sect. 2, we use formal power series transformations and find, although heuristically, an explicit expression for the Blasius constant as a sum of a series of rational numbers. This series follows from the Crocco equation almost trivially, and it seems a pure chance that this solution was missed before.

In Sect. 3, we find a very simple recursive formula for the coefficients of the series that represents the Blasius constant. This is a key part of the paper, where we prove the convergence of the series and find some of its asymptotic properties. In particular, some analytical properties of the Crocco solution at the singularity are derived from the local properties of its expansion at the origin.

In Sect. 4, we compute the Blasius constant numerically using analytical properties of the (inverted) Crocco solution. We demonstrate that computational error can be made exponentially small.

Finally, we use the structure of the obtained convergent series for the Blasius constant and propose a new sequence transformation that relies precisely on the slow convergence rate that we found empirically. This allowed us to find the Blasius constant with guaranteed 16 decimal places.

§ 2. Analytical properties of the Crocco solution

Integrating Eq. (1), we obtain the identity

$$f_{xx}(x) = s\,\exp\left(-\frac{1}{2}\int_0^x f(t)dt\right), \tag{3}$$

and, since $s > 0$, $f_x(0) = 0$, $f_{xx}(x) > 0$, it is clear that $f_x(x)$ is a monotonically increasing function that tends to 1 as $x \to \infty$. In other words, f_x is a diffeomorphism of the intervals $[0, \infty) \to [0, 1)$.

Thus, using Crocco's idea, we make the change of dependant and independent variables in Eq. (1) respectively as

$$y = f_{xx}, \quad t = f_x,$$

and obtain the *Crocco equation*

$$2\,y(t)\,y''(t) + t = 0. \tag{4}$$

The boundary conditions for the Blasius BVP (1) translate to the boundary conditions for the Crocco equation as

$$y(0) = s, \quad y'(0) = 0, \quad y(1) = 0, \tag{5}$$

where the boundary condition $y(1) = 0$ follows form (3).

This set of boundary conditions seems overdetermined, but it is not, since the constant s is unknown.

We will call the function $y(t)$ satisfying the Crocco equation (4) and the boundary conditions (5) the *Crocco solution*, although, apart from its boundary conditions, we know a priori little about this function.

First, it is clear from Cauchy theorem that the Crocco solution is a holomorphic function at the origin. Thus we obtain

$$y(t) = s - \frac{1}{12\,s}\,t^3 - \frac{1}{720\,s^3}\,t^6 - \frac{1}{17280\,s^5}\,t^9 + O(t^{12}). \tag{6}$$

It seems very tempting to develop the series (6) further, then truncate it and put there $t = 1$ and $y(1) = 0$, then solve this polynomial equation for s. However, this approach gives a very poor accuracy for apparently no reason.

The reason for this failure, however, is a very nasty singularity of the Crocco solution at $t = 1$ (see Sect. 3). It is, in fact, a movable singularity of the Crocco equation that depends on the value of the Blasius constant s. The nature of this singularity, at present, is not completely elucidated, since there is no explicit asymptotic expansion for $y(t)$ at $t = 1$. An asymptotic equivalence of the Crocco solution at $t = 1$ to a logarithmic function was found in [6, Prop. 7.1]. In particular, it was found that $y'(t) \to -\infty$ as $t \to 1$. In this paper, we will find this property of the function $y(t)$ by different means.

Thus it is useless to try and use Padé approximants for accelerating the series (6). Rational approximations will try to shoot right through the singularity, which is impossible, since it is manifestly not a pole.

The series (6) contains only the powers of t^3, that we will prove shortly. But to use a sparse series is uneconomical, so we (keeping notation) make the change of variable $t \to x^{1/3}$. The Crocco equation takes the form

$$18\,x\,y(x)\,y''(x) + 12\,y(x)\,y'(x) + 1 = 0, \tag{7}$$

with the same boundary conditions (5) minus $y'(0) = 0$.

Eq. (7) is singular at the origin, but the Crocco solution is a regular function there. We prove this by substituting a formal power series into Eq. (7). After some rearrangement, we obtain the solution

$$y(x) = s - \sum_{n=1}^{\infty} \frac{a_n}{s^{2n-1}}\, x^n, \tag{8}$$

where

$$a_1 = 1/12, \quad n\,(3\,n-1)\,a_n = \sum_{j=1}^{n-1} j\,(3\,j-1)\,a_j\,a_{n-j},\ n > 1. \tag{9}$$

The first 6 coefficients a_n are

$$\frac{1}{12}, \frac{1}{720}, \frac{1}{17280}, \frac{1}{304128}, \frac{2099}{9580032000}, \frac{31453}{1954326528000}.$$

The substitution $x \to t^3$ shows that the series (8) has the same coefficients as the series (6), and thus both series are regular at the origin.

The properties of the Crocco solution are completely determined by the set of (obviously) positive rational numbers $\{a_n,\ n \in \mathbb{N}\}$ that satisfy the recurrence relation (9), and the Blasius constant s, which remains undetermined.

The Töpfer's idea, however, still holds, and we can eliminate the Blasius constant from the Crocco solution (8). Namely, Eq. (7) is invariant under the change of variables

$$x \to s^2\,x, \quad y(x) \to s\,y(x), \tag{10}$$

and the new solution is obtained if we put $s = 1$ into the series (8).

The singularity $x = 1$ is shifted to the location $x = 1/s^2$, which remains to be found. But let us assume for the moment that we know the Blasius constant s exactly. Then the following theorem holds.

Theorem 1. *The Crocco solution* (8) *is a holomorphic function inside the unit circle. The point $x = 1$ is a unique singularity on the boundary $|x| = 1$.*

Proof. By Cauchy theorem (applied to Eq. (4)), the function $y(x)$ is holomorphic at the origin. Let R be the radius of convergence of the series (8). The point $x = 1$ cannot be regular for the Crocco solution, since the contrary statement immediately leads to a contradiction. We substitute a series

$$y(x) = b_m\,(x-1)^m + b_{m+1}\,(x-1)^{m+1} + \dots$$

into Eq. (7) and find that the constant term cannot be cancelled with any constants $b_m, b_{m+1}, \dots$ and $m \geq 1$.

Note that we already know this fact in a different form [6, Prop. 7.1].

Thus $R \leq 1$. Suppose that $R = R_0 < 1$. Then there is a singularity somewhere on the circle $|x| = R_0$. But since the coefficients of the series (8) are of constant sign, it follows from the Pringsheim theorem [9, page 133] that the point $x = R_0$ is singular. This is impossible, since by the binomial identity, $y'(x) < 0$ for at least $x \in [0, R_0]$. In other words, the function $y(x)$ monotonically decreases from s to $y(R_0)$, which is positive by definition of the Blasius constant. Hence, again by Cauchy theorem, the point $x = R_0$ is regular.

Thus we proved that $R = 1$, and the point $x = 1$ is singular. It remains to prove that there are no other singularities on the boundary $|x| = 1$.

This again follows from the coefficients a_n being positive. The series (8) converges everywhere on the boundary $|x| = 1$, since the series $s - y(x)$ with $x = 1$ is a majorant for any $x\colon |x| = 1$. For the same reason, $y(\exp(i\,\varphi)) \neq 0$ if $\varphi \neq 0 \bmod 2\,\pi$, and, again by Cauchy theorem, all these points are regular. End of proof (**EoP**).

A similar result was obtained in [10] with more elaborate means.

Theorem 1 partially explains why Padé approximations for the original Blasius problem seem to work well, i.e., it is possible to perform a summation of the divergent Blasius series up to $x = \infty$ (although it is not yet proved). In our opinion, this is because the three singularities of the solution $f(x)$ to (1) (see [3] and related references there) are spurious, i.e., they do not affect the constant s, since they are removed by the Crocco transformation.

Now we can express the Blasius constant through the radius of convergence of the series (8). By Cauchy-Hadamard theorem, we have

$$s = \lim_{n\to\infty} \sup a_n^{\frac{1}{2n-1}},$$

where "sup" can be omitted, and by the ratio test

$$s^2 = \lim_{n\to\infty} \frac{a_{n+1}}{a_n},$$

provided the last limit exists (which it does). For example, taking $n = 300$, the last formula gives $s \approx 0.33091290$, where $a_{300} \approx 0.120444961 \times 10^{-292}$, or, by Cauchy-Hadamard formula, $s \approx 0.32432937$.

So, unless we find asymptotic behavior of the sequence $\{a_n\}$ as $n \to \infty$, this approach is impractical due to purely technical, i.e., numerical, problems, which are formidable (underflow, cancellation of significant digits, etc). It would be ideal to find a closed expression $a_n = A(n)$ for this sequence, but this seems hopeless, since it would mean explicit integration of the Crocco equation (although with a new special function).

Now we return to the idea that we criticized earlier, namely, an attempt to find the Blasius constant as a solution to a polynomial equation obtained from the series (8).

Here we need to perform two operations successively. First, we approximate the equation by truncating the series (8) and introducing an unknown error. Next, we solve the polynomial equation numerically by an iterative process. As it was mentioned, this approach does not work.

Our approach is to unite these two operations, i.e., to construct the equation and its solution simultaneously. We need to solve the equation

$$s = \sum_{n=1}^{\infty} \frac{a_n}{s^{2\,n-1}}$$

with respect to s. It is clear that the solution exists and unique, since each side of this equation monotonically increases (decreases) with s.

Let us write a more general equation with additional parameters

$$v = \sum_{n=1}^{\infty} \frac{a_n}{s^{2\,n-1}}\, x^n. \tag{11}$$

We can formally solve Eq. (11) with respect to $x = x(v)$ considering s as a parameter. This can be done uniquely in the form of a formal power series

$$x = \frac{s\,v}{a_1} - \frac{a_2\,v^2}{a_1^3} - \frac{\left(a_1\,a_3 - 2\,a_2^2\right)\,v^3}{a_1^5\,s} - \frac{\left(5\,a_2^3 - 5\,a_1\,a_2\,a_3 + a_1^2\,a_4\right)\,v^4}{a_1^7\,s^2} + \ldots \tag{12}$$

It is easily seen that the powers of s decrease from 1 by 1 in the series (12). Now we recall that, incidentally, $x = 1$, and $v = s$. Thus, dividing (12) by s^2, we obtain the series

$$\frac{1}{s^2} = \frac{1}{a_1} - \frac{a_2}{a_1^3} - \frac{a_1\,a_3 - 2\,a_2^2}{a_1^5} + \ldots = \sum_{k=1}^{\infty} b_k, \tag{13}$$

where each coefficient b_n is uniquely determined by the coefficients $\{a_k,\, k = 1, \ldots n\}$ for every $n \in \mathbb{N}$.

There is no indication so far that these manipulations have any sense. The validity of these operations depends upon the radius of convergence of the series (12) being no less than s, and the convergence of this series at $v = s$. And the series itself depends upon the value that we want to find, so there seems to be no clear cut way to prove the formula (13) directly.

To the extent of our knowledge, there are no general theorems that give bounds for the radius of convergence of an inverse series. There are, however, several general algorithms for inversion of the series. The most famous (but not the most efficient) is the Lagrange formula for coefficients of the inverse series. The inversion can also be done with the Bell polynomials [11], and with nested derivatives [12].

Using these techniques, it is possible to calculate a few tens of coefficients b_n and verify experimentally that this approach does work. The first 7 coefficients b_n are

$$12,\ -\frac{12}{5},-\frac{6}{25},\ -\frac{27}{275},-\frac{2484}{48125},\ -\frac{255807}{8181250},\ -\frac{65309031}{3149781250}.$$

We computed the first 100 coefficients b_n and obtained by the formula (13), $s \approx 0.33200308$, which is clearly better than what we get with the radius of convergence.

Apart from a rigorous proof, there are two major problems that we need to overcome before this approach can be made effective. The first one is how to compute thousands (or millions) of coefficients b_n if necessary. The second is how to derive asymptotic properties of these coefficients. The general techniques for inversion of the series are just too general to tackle either of these problems.

§ 3. The inversion of the Crocco solution

Man muss immer umkehren.
C.G.J. Jacobi

Following this maxim, we are going to invert the Crocco solution here rather than the series (11). It is, obviously, one and the same thing expressed differently.

We consider the Crocco equation (7) where the change of variables (10), or rescaling, is made. Thus, the Crocco solution $y(x)$ is a holomorphic function inside the circle of radius $r = 1/s^2$. In addition, $y(0) = 1$, $y(r) = 0$, and $y(x)$ monotonically decreases from 1 to 0 on the interval $[0, r]$.

We take the function $u(x) = 1 - y(x)$ as a new dependant variable. This function has, obviously, the coefficients a_n (9) in its power series expansion in variable x. The function $u(x)$ is invertible on the interval $[0, r]$, and the value u can be taken as a new independent variable. Thus we obtain the function $x = x(u)$, which is defined on the interval $u \in [0, 1]$, and monotonically increases there from 0 to r. To simplify the formulas, we make a rescaling $x(u) = 12\, z(u)$, and obtain the inverted Crocco equation for the function $z(u)$

$$\frac{3}{2}\,(1-u)z(u)\frac{d^2}{du^2}z(u) - (1-u)\left(\frac{d}{du}z(u)\right)^2 + \left(\frac{d}{du}z(u)\right)^3 = 0. \qquad (14)$$

Let us recapitulate what we know so far about the function $z(u)$, apart from the fact that it satisfies Eq. (14). It is a monotonically increasing function on the interval $u \in [0, 1]$; $z(0) = 0$, $z(1) = r/12$ (which value we do not know yet);

and, by Theorem 1, $z(u)$ is a holomorphic function in the neighborhood of the origin. In addition, we have the expansion

$$z(u) = \sum_{k=1}^{\infty} c_k \, u^k, \tag{15}$$

where $c_k = b_k/12$, and b_k are defined by the formula (13).

But now we can forget about the formula (13) and redefine the coefficients c_k by the formula (15). Thus, to make heuristic manipulations in Sect. 2 legitimate, we need to prove

Theorem 2. *The solution $z(u)$ to the equation* (14) *is a holomorphic function inside the unit circle. The point $u = 1$ is a unique singularity on the boundary $|u| = 1$. The power series* (15) *converges at $u = 1$. The power series for $z'(u)$ also converges at $u = 1$, and $z'(1) = 0$.*

The rest of this section is an extended proof of Theorem 2.

We remark that the function $z(u)$ and the radius of convergence of its series (15) are defined now independently from any unknown constant. And the Blasius constant itself is expressed through the boundary value $z(1)$ of a uniquely defined holomorphic function, i.e., $s = \sqrt{1/(12\, z(1))}$.

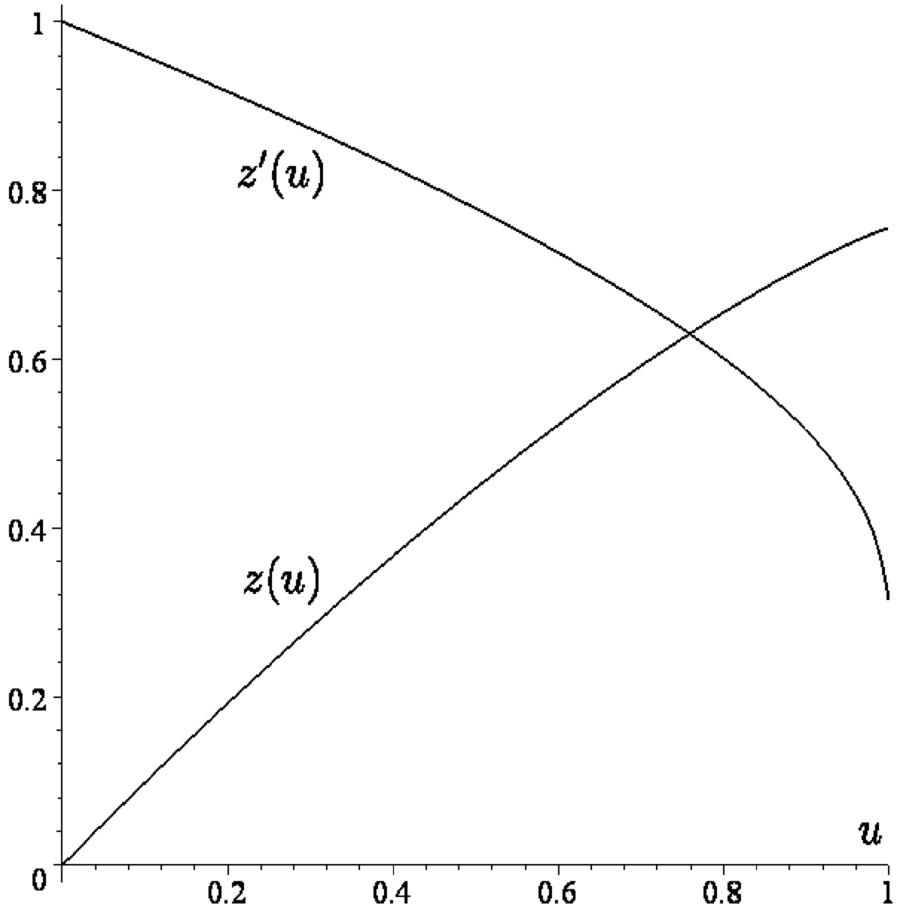

Fig 1. Functions $z(u)$ and $z'(u)$ as truncated series.

The series (15) behaves much nicer by this theorem than the series (8), since both $z(1)$ and $z'(1)$ exist. Also, it follows that $y'(r) = -\infty$ for the Crocco solution, since $z'(u) \geq 0$ for $u \in [0, 1]$.

The first priority is an effective formula for the coefficients c_n in (15). We can substitute the series (15) into Eq. (14), and, after routine but extremely tedious manipulations, obtain a recursive formula $c_n = R(n, c_1, \ldots, c_{n-1})$. This formula takes a third of a page, so we skip it. It can hardly be called effective, and it completely hides asymptotic properties of coefficients c_n. Still, it is much faster than general formulas for inversion of series, and it allowed us to compute a few hundreds of c_n.

Fig. 1 shows the plots of the functions $z(u)$ and $z'(u)$ represented by truncated series with 200 terms. The program was not able to plot $z'(u)$ in full, although we used high precision arithmetic. Assuming that Theorem 2 holds, Fig. 1 explains why it is so difficult to approximate an apparently smooth function $y(t)$ or $z(u)$ for that matter (see [3]). The problem lies below the surface, so to speak, in the first derivative $z'(u)$, which quickly degenerates as $u \to 1$.

The direct approach does not look promising, so we try a different tack. We isolate the expression $z(u)\, z''(u)/(z'(u))^2$, which is integrable, from Eq. (14). Then we integrate both sides of the transformed equation and obtain

$$\frac{z(u)}{z'(u)} = \frac{1}{3}\, u + \frac{2}{3} \int \frac{z'(u)}{1-u} du, \tag{16}$$

where the constant of integration is, obviously, zero.

The benefits of this representation become clear when we consider the Taylor coefficients of both sides of Eq. (16). We make the substitution $d_{n-1} = n\, c_n$, $n \in \mathbb{N}$, i.e., we use the function $z'(u)$ rather than $z(u)$. Then

$$z'(u) = \sum_{k=0}^{\infty} d_k\, u^k, \qquad \frac{z(u)}{z'(u)} = \sum_{n=1}^{\infty} g_n\, u^n,$$

where

$$g_1 = 1, \quad g_n = d_{n-1}/n - \sum_{j=1}^{n-1} d_{n-j}\, g_j, \; n \geq 2.$$

The last formula is easily verified with multiplication of two series. Similarly,

$$\int \frac{z'(u)}{1-u} du = \sum_{n=1}^{\infty} p_n\, u^n, \quad p_n = q_n/n, \quad q_n = \sum_{j=0}^{n-1} d_j,$$

where we introduced a new set of coefficients $\{q_n,\, n \in \mathbb{N}\}$, which will play a crucial role. Collecting terms with the same power of u, we obtain

$$\frac{(3\,n-1)\, q_n}{2\,n} - \frac{3\,(n-1)\, q_{n-1}}{2\,n} = \sum_{k=1}^{n-2} \frac{(q_k - q_{k+1})\, q_{n-k}}{n-k}, \; n > 2, \tag{17}$$

and the first 7 coefficients q_n are

$$1,\ \frac{3}{5},\ \frac{27}{50},\ \frac{279}{550},\ \frac{9351}{19250},\ \frac{7692543}{16362500},\ \frac{48494709}{105875000}.$$

Lemma 1. *The coefficients q_n monotonically decrease as $n \to \infty$, and they are positive.*

Proof. First we prove that if $q_1 > \ldots > q_n > 0$, then $q_n > q_{n+1}$.

We substitute $n \to n+1$ in the identity (17) and cut the sum by one term. We obtain the identity

$$\frac{(3\,n+2)\,q_{n+1}}{2\,(n+1)} - \frac{3\,(4\,n-1)\,q_n}{10\,(n+1)} - \frac{3\,q_{n-1}}{10} = \sum_{k=1}^{n-2} \frac{(q_k - q_{k+1})\,q_{n-k+1}}{n-k+1}. \tag{18}$$

Now we use inequalities $q_{n-k+1} < q_{n-k}$, and $1/(n-k+1) < (n-1)/n/(n-k)$ in the right hand side (rhs) of (18) and obtain an inequality out of Eq. (18). The rhs of this newly obtained inequality equals rhs(17) times $(n-1)/n$. We substitute the left hand side (lhs) of (17) instead of rhs(17) and, after some rearrangement, we obtain

$$q_{n+1} < \frac{(27\,n^3 - 8\,n^2 - 15\,n + 5)\,q_n}{5\,n^2\,(3\,n+2)} - \frac{3\,(n+1)\,(4\,n^2 - 10\,n + 5)\,q_{n-1}}{5\,n^2\,(3\,n+2)}. \tag{19}$$

Now we need to prove that rhs(19) is less than q_n. This is equivalent to

$$\frac{q_n}{q_{n-1}} < \frac{12\,n^3 - 18\,n^2 - 15\,n + 15}{12\,n^3 - 18\,n^2 - 15\,n + 5} = 1 + \frac{5}{6\,n^3} + O(\frac{1}{n^4}),$$

and thus it is true.

Now, $q_{n+1} > 0$ by the formula (18). **EoP**

Thus the sequence $\{q_n,\ n \in \mathbb{N}\}$ has a certain limit $q_\infty \geq 0$. It cannot decrease too quickly however, since lhs(17) is positive, and hence

$$\frac{3\,(n-1)}{3\,n-1} = 1 - \frac{2}{3\,n} - O(\frac{1}{n^2}) < \frac{q_n}{q_{n-1}} < 1,$$

and so the convergence rate is logarithmical.

Lemma 2. *The coefficients $q_n \to 0$ as $n \to \infty$.*

Proof. We rewrite the identity (17) in this form

$$\frac{1}{2}\left(1 - \frac{3}{n}\right)(q_{n-1} - q_n) = \sum_{k=1}^{n} \frac{q_k\,q_{n-k+1}}{n-k+1} - \sum_{k=1}^{n-1} \frac{q_k\,q_{n-k}}{n-k}. \tag{20}$$

Now we observe that if we put $n \to n-1$, $n \to n-2, \ldots n \to 1$ in Eq. (20) (with $q_0 = 0$), then rhs of these identities will telescope, i.e., cancel each other when we sum up these identities. Thus we obtain another recursive formula for q_n

$$\sum_{k=1}^{n} \frac{q_k}{k\,(k+1)} - \frac{(n-2)\,q_n}{3\,(n+1)} = \frac{2}{3} \sum_{k=1}^{n} \frac{q_k\, q_{n-k+1}}{k}. \tag{21}$$

If we suppose that $q_n >$ const > 0, then lhs(21) is bounded, but rhs(21) gives a harmonic series, which diverges. **EoP**

To give the reader a flavor of what we just proved, we computed for $m = 100000$, $q_m \approx 0.21866485524$. So q_n decreases very slowly.

We recall that $n\,c_n = d_{n-1} = q_n - q_{n-1}$, and thus $c_1 = d_0 = 1$; $c_n < 0$, $n > 1$; $d_n < 0$, $n > 0$. In addition, $d_n \to 0$ as $n \to \infty$, and the series for $z'(u)$ converges at $u = 1$ due to telescoping of its coefficients and Lemma 2. Hence the series for $z(u)$ converges as well at $u = 1$, and

$$\sum_{n=0}^{\infty} d_n = z'(1) = 0, \quad 0 < \sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{q_n - q_{n-1}}{n} = z(1) < 1. \tag{22}$$

Thus we found an estimate $0 < z(1) < 1$ without any computations. This estimate can be improved indefinitely.

Lemma 3. *For every $n \in \mathbb{N}$, we have an estimate*

$$\sum_{k=1}^{n} \frac{q_k}{k\,(k+1)} < z(1) < \sum_{k=1}^{n} \frac{q_k - q_{k-1}}{k}. \tag{23}$$

Proof. It is easily seen that lhs(23) plus $q_n/(n+1)$ equals rhs(23) symbolically. Since $q_{k-1} > q_k > 0$, lhs(23) is the lower bound that monotonically increases, and rhs(23) is the upper bound that monotonically decreases. Both sides of (23) tend to $z(1)$ as $n \to \infty$ by Lemma 2. **EoP**

Hence, both sides of Eq. (21) tend to $z(1)$ as $n \to \infty$.

Thus the truncation error for the partial sums lhs(23) is estimated by the last computed term, i.e., similar to a Leibnitz's series, although in our case both series in (23) are of constant signs.

It remains to prove a few things.

The radius of convergence of the series (15) cannot be less than 1 by Abel theorem [13, page 38], since the series converges at $u = 1$.

The point $u = 1$ cannot be regular for the function $z(u)$, which can be proved exactly as in Theorem 1 by substitution of a formal power series into Eq. (14), that leads to a contradiction. But we can obtain much more from Eq. (16).

Lemma 4. *The function $z'(u)$ decreases more slowly as $u \to 1$ than any function $(1-u)^\alpha$ for $\alpha > 0$.*

Proof. Suppose that $z'(u) = A(u)\,(1-u)^\alpha$, where $\alpha > 0$ and $A()$ is a C^∞ function such that $A(1) \neq 0$. Then rhs(16) has a finite limit as $u \to 1$, but lhs(16) tends to infinity, since $z(1) > 0$, $z'(1) = 0$. **EoP**

Thus we were justified in calling this singularity a nasty one. If we rotate Fig. 1 clockwise by 90^o, then the plot of the function $z'(u)$ will represent a flat function.

Asymptotic properties of the function $z(u)$ at $u = 1$ deserve a special study, that we plan in the future.

The series (15) converges on the boundary $|u| = 1$, since the coefficients c_n are of constant sign for $n > 1$, and the series converges at $u = 1$. Then, $z(u) \neq 0$ on the boundary $|u| = 1$, since the contrary is disproved with an estimate

$$1 = |u| \leq -\sum_{n=2}^{\infty} c_n < 1.$$

Thus, by Cauchy theorem, all points on the circle $|u| = 1$ are regular except $u = 1$. This concludes the proof of Theorem 2.

§ 4. Rational approximations

Mathematics is an experimental science.
V.I. Arnold

We found the value $z(1)$ as a convergent sum of a uniquely defined set of rational numbers. As it was stated by Knopp [14, page 25] (who, by the way, was acquainted with Prof. Blasius), a convergent series of rational numbers is, in fact, the number that the series represents or converges to. In the same way as $\sqrt{2}$ is the best that we can say about the square root of 2. Anything else would be rational approximations.

On this happy note we could have concluded the paper, since we found $z(1)$, and thus the Blasius constant $s = \sqrt{1/(12\,z(1))}$.

But rational approximations are important, so in this section we will try to extract some information from the obtained formulas numerically.

According to Fig. 1, the value $z(1)$ is the area under the plot of the function $z'(u)$. Thus, numerical integration of Eq. (14) is equivalent to a numerical quadrature. But by Lemma 4, if we integrate Eq. (14) from $u = 0$ to $u = 1 - \varepsilon$ (and we cannot do it up to $u = 1$ for obvious reasons), then we make an error of the order ε in the final result $z(1)$. This is very bad, since the error could have been of the order ε^2 with the simplest quadrature formula if the function $z(u)$ behaved better at $u = 1$. But if we look on Fig. 1 at a different angle, this drawback can be turned into advantage.

First, we differentiate Eq. (14) with respect to u and eliminate the function $z(u)$ from these equations. Thus we obtain an equation for the function $z'(u)$. Next, we take the function $v(u) = 1 - z'(u)$ as a new dependant variable. It is a monotonically increasing function, hence we can take v as a new independent variable, and thus we obtain the equation

$$2\,u''\,(1-v)\,(v-u)\,(1-u) - 2\,(1-v)^2\,\left(u'\right)^2 + (1-u)\,(5+u-6\,v)\,u = 0, \quad (24)$$

where $u = u(v)$, $u' = du/dv$, etc.

This is exactly what we did with the Crocco equation in Sect. 3. These operations are equivalent to the rotation of Fig. 1 by 90^o anticlockwise. Then the plot of the function $z'(u)$ turns into the plot of the function $u(v)$. Since the area under the plot is the same, we need to compute

$$\int_0^1 u(v)dv = z(1).$$

To avoid computation of quadratures, we introduce a new function $w(v)$ such that $u(v) = w'(v)$ (we skip the equation for $w(v)$). We need to integrate this new equation from $v = 0$ to $v = v_1$, where v_1 is such that $u(v_1)$ is close to the top, i.e., $u(v_1) \approx 1$. Then

$$z(1) = w(v_1) + (1-v_1)\,(1-\theta(v_1)\,(1-u(v_1))), \quad 0 < \theta(v_1) < 1,$$

since the remaining part of the area is close to a rectangle. Assuming that the error of integration is under control, the value $z(1)$ is computed with an error less than $1 - u(v_1)$, which is exponentially small.

Since Eq. (24) is singular at the origin, we need the expansion (15), from which we obtain the expansion for $v(u) = 1 - z'(u)$. Then we invert this expansion, etc., and use these expansions to obtain initial values at $v = v_0$, where v_0 is small.

We performed these computations using extended precision package [15] with different settings chosen such that the cumulative error of integration from $v = 0$ to $v = v_1$ was less than $1 - u(v_1)$.

For example, for $v_0 = 1/16$, $v_1 = 0.912$, we obtained $1 - u(v_1) \approx 1.14 \times 10^{-32}$, and thus found $z(1)$ with no less than 30 valid decimal places. Hence, the Blasius constant

$$s \approx 0.332057336215196298937180062010, \quad (25)$$

which agrees up to 22nd digit with the result given in [4].

Here are a few technical details of these computations.

We used an extended float arithmetic with 64 decimals mantissa. Truncated series for initial values were up to 64 terms. The values $v_0 \approx 0.1$ were taken such that the initial values at $v = v_0$ were computed with a safety margin. We cannot integrate up to $v = 1$ for obvious reasons, so the values $v_1 \approx 0.9$ were taken such that the error of numerical integration up to $v = v_1$ was less than $1 - u(v_1)$. Finally, we used highly accurate Runge-Kutta integrators of the 8th and 10th order with various integration steps that cross-validated each other.

This numerical approach can be made much more effective, if we use the original idea by Blasius, i.e., if we find an asymptotic expansion of the function $u(v)$ at $v = 1$ and match the numerically obtained part of the solution with this expansion at a midpoint. The asymptotic expansion of the function $u(v)$ at $v = 1$ would be necessarily in flat functions (see [16]). But this subject deserves a special study, that we plan in the future.

Instead, we consider a problem of extracting maximum information on asymptotics of a sequence from a finite part of this sequence. This, of course, is an ancient problem of acceleration of convergence. There are numerous acceleration techniques from Euler's method to, reputedly, very powerful Levin transformations (see [17, 18]).

We tried several of these sequence transformations for both series (23) and found them either counterproductive or ineffective. And the reason for this lack of acceleration is a very tricky asymptotics of the sequence $\{q_n,\, n \in \mathbb{N}\}$.

Before we move further, we give another recursive formula for this sequence, that is very quick and stable. We denote $\{p_n = q_n/n,\, n \in \mathbb{N}\}$. Then, based on the formula (21), we obtain

$$(3\,n - 1)\,p_n = 3 \sum_{k=1}^{n-1} \frac{p_k}{k+1} - (n+1) \sum_{k=2}^{n-1} p_k\, p_{n-k+1},\ n > 2. \tag{26}$$

There is no subtraction of small almost equal numbers here as in (17), so this formula is preferable for computations in float arithmetic. In addition, the first sum in (26) needs not be recomputed, and the second sum in (26) is a convolution, so we need to compute only half of this sum for each p_n.

We computed 1000 coefficients p_n in rational arithmetic; 100000 p_n in float arithmetic with 32 and 64 decimal places (DP); and 1000000 p_n in ordinary double float arithmetic (16 DP). Comparing this data, we found that there is no accumulation of errors, and all DP (except may be the last one in each float set) are correct.

Based on these computations, we found the following empirical asymptotics

$$q_n \asymp c/\log^h n, \quad \frac{q_n}{q_{n-1}} \asymp 1 - \frac{h}{n \log n}, \tag{27}$$

where c and h are, presumably, constants, but settle very slowly as $n \to \infty$. For $n = 100000$, we found $c \approx 0.7071891$, $h \approx 0.4803651$, where we have kept the digits that are the same for $n-1$. For $n = 1000000$, $c \approx 0.71562$, $h \approx 0.48505$. Since $z(1) \approx 0.75577512653624$ by the formula (25), we expect, hypothetically, $c = z(1)$, and $h = 1/2$, although we need not precise values of these constants.

We used two techniques for accelerating partial sums in (23). The first one is based on Padé approximations (realized in Thiele continued fractions scheme), and the second one is new as far as we know.

We denote the partial sums in lhs(23) and rhs(23) as $S_1(n)$ and $S_2(n)$ respectively. Thus $S_1(n) + q_n/(n+1) = S_2(n)$. Due to the asymptotics (27), the sum $S_2(n)$ converges much faster, but, by the last formula, we can compute the sum $S_1(n)$ instead.

The sum $S_2(n)$ is considered as a function of $1/n$ rather then n. Then we extrapolate it to zero using diagonal Padé approximants of up to 75/75th order with the data selected in various ways from the 64 DP array (with 100000 terms). In this way, we were able to compute the value $z(1)$ (or s) with 9 DP compared to (25). When we considered $S_2(n)$ as a function of $q_n - q_{n-1}$, the value s was computed with 10 DP.

Thus, Padé approximations are not very effective here, which was almost predictable given the asymptotics (27). In addition, we used previously obtained value (25) for comparison, without which the estimates should have been more conservative.

Now we present a new acceleration technique which relies precisely on the logarithmic asymptotics (27), that we believe to be true.

This method is based on the formula (23) and the observation that the slower q_n decreases the better approximation rhs(23) gives to the limit value of the sum in lhs(23). In fact, if $q_n \equiv 1$, then rhs(23) would give the correct answer immediately for any $n > 1$.

Another observation is the fact that not all q_n need be positive and monotonically decreasing in order that the sums in (23) were lower and upper estimates of the value $z(1)$. It is sufficient that q_n were such for large enough n.

Finally, the upper and lower bounds will interchange in (23), if all the signs of q_n and $z(1)$ are changed. Keeping this in mind, we denote

$$q_{n,1} = q_n; \quad q_{n,m} = 0,\ n \le 0; \quad q_{n,m} = (n+1)\,(q_{n,m-1} - q_{n-1,m-1}),\ m > 1.$$

Thus lhs(23) is $S_1(n) = S(\{q_{n,1}\})$, and rhs(23) is $S_2(n) = S(\{q_{n,2}\})$. Since $S(\{q_{n,1}\}) + q_{n,1}/(n+1) = S(\{q_{n,2}\})$, and, obviously, $q_{n,1} > 0 > q_{n,2}$, then by induction, we have $S(\{q_{n,k}\}) + q_{n,k}/(n+1) = S(\{q_{n,k+1}\})$, and

$$\ldots < S(\{q_{n,2\,k-1}\}) < S(\{q_{n,2\,k+1}\}) < z(1) < S(\{q_{n,2\,k}\}) < S(\{q_{n,2\,k-2}\}) < \ldots$$

as long as $|q_{n,k+1}| < |q_{n,k}|$.

The proof is obvious. By (27), $q_{n,1} \asymp c/\log^h n$, hence $q_{n,2} \asymp -c\,h/\log^{h+1} n$, etc.

Of course, this improvement of the estimate cannot continue indefinitely, since for every fixed part of the sequence, i.e., for a fixed n, this process is divergent. In fact, $q_{k,j}$ wildly oscillate for small k and big enough fixed j. On the other hand, we do not need $q_{k,j}$ for small k, but only for $k = n$, i.e., for the last term of each sequence.

Note that if we assume $q_{n,1} \asymp b/n^h$, then $q_{n,2} \asymp -b\,h/n^h$, and the improvement in convergence is hardly noticeable if $h < 1$, or absent if $h \geq 1$. So we have indeed used the fact (or assumption) that q_n decrease slowly.

Example. We tested this acceleration technique on a model sequence

$$q_n = \frac{(n+1)}{n\,\sqrt{\log(n+1)}}.$$

The corresponding sum (23) has no closed form expression, and Maple was not able to perform summation of the infinite series numerically. With the help of MuPAD, we obtained

$$S_\infty = \sum_{k=1}^{\infty} \frac{q_k}{k\,(k+1)} \approx 1.72521454565946845.$$

For $n = 100000$, we have $S_\infty - S_n \approx 2.8 \times 10^{-6}$. We take $n = 100$, and then

$$S_1 = \sum_{k=1}^{n} \frac{q_{k,1}}{k\,(k+1)},\ S_2 = S_1 + \frac{q_{n,1}}{n+1},\ S_3 = S_2 + \frac{q_{n,2}}{n+1}, \ldots,$$

where corrections are made while $|q_{n,j+1}| < |q_{n,j}|$, which takes 5 iterations; then $|q_{n,5}| > |q_{n,4}|$ and we stop. We take $S = (S_4 + S_5)/2$ as the result. We found $S_\infty - S \approx -2.6 \times 10^{-6}$. Thus we obtained the same accuracy with 100 terms of this sequence as one would get with $n = 100000$ by direct summation. In addition, this process has a built in error control. Using Levin transformations [19, Prog. HURRY], we were able to obtain only 4 DP for this sequence with $n \leq 100$.

Sequence transformations is not the subject of this paper, so we only remark that this technique works for slowly increasing sequences q_n as well, for instance, for $q_n = \log n$.

Now we return to the sequence q_n (17) that gives the Blasius constant. Following this example, we performed the same computations and obtained the following results.

For $n = 100$, we found the Blasius constant with 6 DP after the decimal point.

For $n = 1000$, we found the Blasius constant with 12 DP with either rational or the 64 float arithmetic. This is much better than we were able to obtain with Padé approximations using up to 100000 coefficients.

For greater n, we cannot ignore the cancellation of significant digits, since the data is in float arithmetic, but the process that we use takes care of this by itself. Corrections are stopped when the next is no better than the previous one for whatever reason.

Thus, for $n = 67000$, we found the Blasius constant with 16 DP that are guaranteed to be true and coincide with that in (25).

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