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KELDYSH INSTITUTE OF APPLIED MATHEMATICS
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Active magnetic attitude control system providing three-axis inertial attitude

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Овчинников М.Ю., Пеньков В.И., Ролдугин Д.С.
Трехосная магнитная система ориентации спутника в инерциальном пространстве

Рассматривается магнитная система ориентации спутника, обеспечивающая его произвольную наперед заданную ориентацию в инерциальном пространстве. На примере плоского движения рассматривается логика формирования алгоритма управления, схожая с конструированием ПД-регулятора. Работа алгоритма затем анализируется при помощи методов осреднения, показывается демпфирование угловой скорости аппарата и устойчивость требуемой ориентации. При помощи теории Флоке подбираются оптимальные по быстродействию параметры алгоритма

Ключевые слова: магнитная система ориентации, трехосная ориентация

Michael Ovchinnikov, Vladimir Penkov, Dmitry Roldugin
Active magnetic attitude control system providing three-axis inertial attitude

Active magnetic attitude control system providing arbitrary inertial attitude is considered. An algorithm that closely resembles the PD-controller is constructed on the basis of a planar model problem. System behavior is analyzed using averaging technique. Angular velocity damping and the desired attitude stability are proven. Optimal algorithm parameters are found.

Key words: magnetic attitude control system, three-axis stabilization

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Contents

Introduction

A three-axis active magnetic attitude control system and related algorithms are of great interest and importance if one considers small satellites. Being the low-cost, reliable and small, magnetorquers are especially attractive for these satellites. However, magnetic control is limited due to the underactuation problem. It is impossible to implement control torque along the geomagnetic field induction vector. As a result, it is impossible to implement an arbitrary in terms of the direction torque at each moment. Therefore it seems impossible to achieve necessary three-axis attitude using numerically simple locally optimal algorithms.

Nevertheless, the work is underway to overcome the underactuation issue. We can outline only one paper with a comprehensive analytical approach to the problem [1]. It is shown that the three-axis magnetic attitude is achievable, though only small vicinity of the necessary attitude is taken into account, the satellite is considered to be a spherically-symmetrical one, and the whole analysis can be hardly interpreted in order to implement on a spacecraft. Similar assumption on a spherically-symmetrical satellite was made in [2]. An asymptotical stability of the necessary attitude is shown using the Lyapunov function approach, however the analysis shows that the assumption on the satellite inertia tensor is key to its overall success. The analysis is not valid for a three-axial satellite and therefore is of limited technical importance. There is a bunch of papers on the numerical analysis of the problem, we can outline only interesting works [3] and [4]. Probably the most important paper on the three-axis magnetic control apart from [1] is [5]. Performance of the three-axis magnetic attitude control system of the Gurwin-Techsat small satellite is present. Only the necessary attitude maintenance is shown, however, the work is of great importance, showing the possibility to overcome the underactuation issue.

Present paper deals with this issue analytically. The control used is a PD-controller inspired one that is the common way to construct a three-axis magnetic control algorithm. Another construction approach is present. The control is studied analytically in order to prove the asymptotical stability of the necessary attitude. Optimal control parameters (in terms of the degree of stability) are found, numerical analysis is carried out. Control limitations are assessed and overall recommendations for the implementation are provided.

1. Problem statement

The choice of the geomagnetic field model is one of the most crucial points for the success of the work. We use the averaged geomagnetic field model (simplified direct dipole model) since it allows the most compact and simple, though rather accurate geomagnetic field model approximation. It does not allow us to take into account the non-uniformity of the geomagnetic induction vector motion (as the right dipole model does) and its diurnal change (as the inclined dipole model does) but it is considered as a good trade-off between the accuracy of modeling geomagnetic field and the possibility to get analytical result. To introduce this model we need to notify a reference system $O_aY_1Y_2Y_3$ where O_a is the Earth's center, the O_aY_3 axis is directed along with the Earth's axis, O_aY_1 lies in the Earth's equatorial plane and is directed to the ascending node of the satellite's orbit, the O_aY_2 axis is directed so the system is right-handed. If the magnetic induction vector source point is translated to the O_a then the cone is tangent to the O_aY_3 axis, its axis lies in the $O_aY_2Y_3$ plane (Fig. 1). The cone half-angle is given [6] by

$$\operatorname{tg}\Theta = \frac{3\sin 2i}{2\left(1-3\sin^2 i+\sqrt{1+3\sin^2 i}\right)} \tag{1.1}$$

where i is the orbit inclination. The geomagnetic induction vector moves uniformly on the cone side with the doubled orbital angular speed, $\chi = 2u+\chi_0$ where u is the argument of latitude, ω_0 is the orbital angular velocity. Without loss of generality we can assume $\chi_0 = 0$.

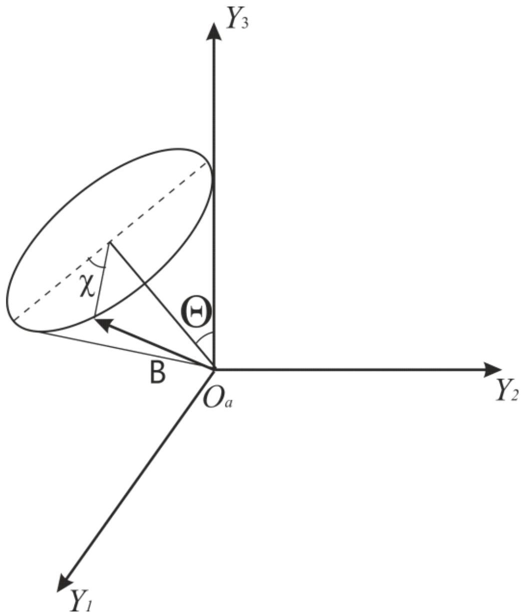

Fig. 1. Averaged geomagnetic field model

Let us introduce the reference frames.

$O_aZ_1Z_2Z_3$ is the inertial frame, got from the $O_aY_1Y_2Y_3$ turning by the angle Θ about the O_aY_1 axis.

$OL_1L_2L_3$ is the frame associated with the angular momentum of the satellite. O is the satellite's center of mass, the OL_3 axis is directed along the angular momentum, the OL_2 axis is perpendicular to OL_3 and lies in a plane parallel to the $O_aZ_1Z_2$ plane and containing O, OL_1 is directed such that the reference frame is right-handed.

$Ox_1x_2x_3$ is the bound frame, its axes are directed along the principal axes of inertia of the satellite.

Reference frames' mutual orientation is described with the direction cosine matrices $\mathbf{Q},\mathbf{A},\mathbf{D}$ expressed in the following tables

$$\begin{array}{cccc} & L_1 & L_2 & L_3 \\ Z_1 & q_{11} & q_{12} & q_{13} \\ Z_2 & q_{21} & q_{22} & q_{23} \\ Z_3 & q_{31} & q_{32} & q_{33} \end{array}, \quad \begin{array}{cccc} & x_1 & x_2 & x_3 \\ L_1 & a_{11} & a_{12} & a_{13} \\ L_2 & a_{21} & a_{22} & a_{23} \\ L_3 & a_{31} & a_{32} & a_{33} \end{array}, \quad \begin{array}{cccc} & x_1 & x_2 & x_3 \\ Z_1 & d_{11} & d_{12} & d_{13} \\ Z_2 & d_{21} & d_{22} & d_{23} \\ Z_3 & d_{31} & d_{32} & d_{33} \end{array}.$$

We introduce subscripts Z, L, x to denote the vector components in frames $O_aZ_1Z_2Z_3$, $OL_1L_2L_3$ and $Ox_1x_2x_3$ respectively. For example, for the first component of a torque in these frames we write M_{1Z}, M_{1L}, M_{1x}.

We use the Beletsky-Chernousko variables and the Euler equations to represent the motion of the satellite. The first set of variables are $L,\rho,\sigma,\varphi,\psi,\theta$ [7] where L is the angular momentum magnitude, angles ρ,σ represent its orientation with respect to the $O_aZ_1Z_2Z_3$ frame (Fig. 2). Orientation of the frame $Ox_1x_2x_3$ with respect to $OL_1L_2L_3$ is described using the Euler angles φ,ψ,θ.

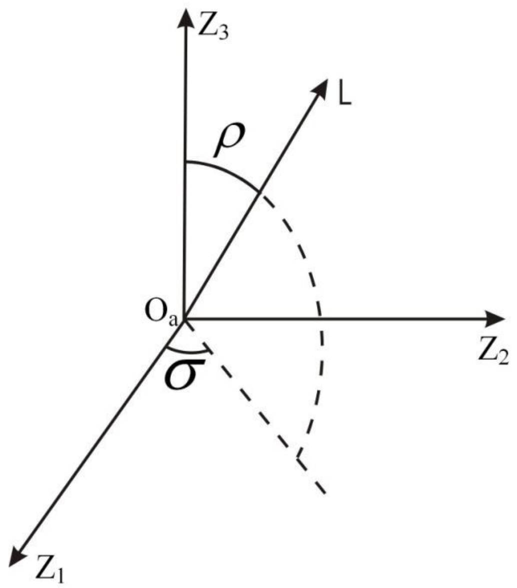

Fig. 2. Angular momentum attitude in the inertial space

Direction cosine matrices $\mathbf{Q}$ and $\mathbf{A}$ take form

$$\mathbf{Q}=\begin{pmatrix}\cos\rho\cos\sigma & -\sin\sigma & \sin\rho\cos\sigma\\ \cos\rho\sin\sigma & \cos\sigma & \sin\rho\sin\sigma\\ -\sin\rho & 0 & \cos\rho\end{pmatrix} \qquad (1.2)$$

$$\mathbf{A}=\begin{pmatrix}\cos\varphi\cos\psi-\cos\theta\sin\varphi\sin\psi & -\sin\varphi\cos\psi-\cos\theta\cos\varphi\sin\psi & \sin\theta\sin\psi\\ \cos\varphi\sin\psi+\cos\theta\sin\varphi\cos\psi & -\sin\varphi\sin\psi+\cos\theta\cos\varphi\cos\psi & -\sin\theta\cos\psi\\ \sin\theta\sin\varphi & \sin\theta\cos\varphi & \cos\theta\end{pmatrix} \qquad (1.3)$$

Inertia tensor of the satellite is $\mathbf{J}_x = diag(A,A,C)$. Angular motion of the satellite in a circular orbit is described [7] by the equations

$$\begin{gathered}\frac{dL}{dt}=M_{3L},\frac{d\rho}{dt}=\frac{1}{L}M_{1L},\frac{d\sigma}{dt}=\frac{1}{L\sin\rho}M_{2L},\\ \frac{d\theta}{dt}=\frac{1}{L}\left(M_{2L}\cos\psi-M_{1L}\sin\psi\right),\\ \frac{d\varphi}{dt}=L\cos\theta\left(\frac{1}{C}-\frac{1}{A}\right)+\frac{1}{L\sin\theta}\left(M_{1L}\cos\psi+M_{2L}\sin\psi\right),\\ \frac{d\psi}{dt}=\frac{L}{A}-\frac{1}{L}M_{1L}\cos\psi\operatorname{ctg}\theta-\frac{1}{L}M_{2L}\left(\operatorname{ctg}\rho+\sin\psi\operatorname{ctg}\theta\right).\end{gathered} \qquad (1.4)$$

where M_{1L}, M_{2L}, M_{3L} are the torque components in $OL_1L_2L_3$ frame.

In case of the Euler equations we use variables $\omega_1,\omega_2,\omega_3,\alpha,\beta,\gamma$ where ω_i are angular velocity components in the bound frame, Euler angles α,β,γ introduce the $Ox_1x_2x_3$ frame attitude with respect to the $OZ_1Z_2Z_3$ one. The direction cosine matrix $\mathbf{D}$ is

$$\mathbf{D}=\begin{pmatrix}\cos\alpha\cos\beta & \sin\beta & -\sin\alpha\cos\beta\\ -\cos\alpha\sin\beta\cos\gamma+\sin\alpha\sin\gamma & \cos\beta\cos\gamma & \sin\alpha\sin\beta\cos\gamma+\cos\alpha\sin\gamma\\ \sin\alpha\cos\gamma+\cos\alpha\sin\beta\sin\gamma & -\cos\beta\sin\gamma & -\sin\alpha\sin\beta\sin\gamma+\cos\alpha\cos\gamma\end{pmatrix} \qquad (1.5)$$

The equations of motion for the three axial satellite (inertia tensor $\mathbf{J}_x = diag(A,B,C)$) are

$$
\begin{aligned}
&A\frac{d\omega_1}{dt} = (B-C)\omega_2\omega_3 + M_{1x},\\
&B\frac{d\omega_2}{dt} = -(A-C)\omega_1\omega_3 + M_{2x},\\
&C\frac{d\omega_3}{dt} = -\left(B-A\right)\omega_1\omega_2 + M_{3x},\\
&\frac{d\alpha}{dt} = \frac{1}{\cos\beta}\left(-\omega_3\sin\gamma + \omega_2\cos\gamma\right),\\
&\frac{d\beta}{dt} = \omega_2\sin\gamma + \omega_3\cos\gamma,\\
&\frac{d\gamma}{dt} = \omega_1 - \mathrm{tg}\,\beta\left(\omega_2\cos\gamma - \omega_3\sin\gamma\right),
\end{aligned}
\tag{1.6}
$$

where M_{1x}, M_{2x}, M_{3x} are torque components in the $Ox_1x_2x_3$ frame.

2. Control construction

Consider model problem of a body rotating along the fixed in the inertial space axis. The body is subjected to the torque M. The position is characterized using only one angle φ. The equation of motion then is of the form

$$\ddot{\varphi} = M .$$

The inertia moment is omitted. The problem is to find the torque that provides the asymptotic stability of $\varphi = 0$. In order to do so we introduce the misalignment

$$\Delta = \frac{1}{2}\left(\dot{\varphi}^2 + \left(1-\cos\varphi\right)^2 + \sin^2\varphi\right).$$

This misalignment gives an insight into the positional error and the angular velocity of the model body. It can be used to find the control providing this misalignment tendency to zero. That means the torque must provide the negative derivative of the misalignment. First we divide the misalignment into two parts Δ_2 and Δ_1. These will be considered as positional and differential ones. Their variations are

$$\delta\Delta_1 = \dot{\varphi}\ddot{\varphi}\delta t,\ \delta\Delta_2 = \sin\varphi\dot{\varphi}\delta t .$$

Differential misalignment part can be minimized since its variation contains the second derivative of the angle $\ddot{\varphi}$ and therefore the torque. This part becomes controllable. This is not valid for the positional misalignment part, so we decompose the misalignment with respect to the time increment,

$$\Delta\left(t+\delta t\right) - \Delta\left(t\right) = \frac{d\Delta\left(t\right)}{dt}\delta t + \frac{d^2\Delta\left(t\right)}{dt^2}\delta t^2 + \ldots$$

Since the differential misalignment part can be minimized the angular velocity $\dot{\varphi}$ tends to zero. The first variation of the positional part then tends to zero also and we can consider its second variation after the last expression being rewritten in a form

$$\Delta(t+\delta t)-\Delta(t)=\delta\Delta_1+\frac{d^2\Delta(t)}{dt^2}\delta t^2+...$$

Positional misalignment part second variation is

$$\delta^2\Delta_2=\frac{1}{2}\left(\cos\varphi\dot{\varphi}^2+\sin\varphi\ddot{\varphi}\right)\delta t^2.$$

The first part in this expression is small again (it contains the angular velocity). The misalignment part Δ_2 change is therefore governed by the second part in the last expression if the control is considered on the enough time interval ($\dot{\varphi}$ becomes small). The misalignment increment can be written as

$$\Delta(t+\delta t)-\Delta(t)=\dot{\varphi}\ddot{\varphi}\delta t+\frac{1}{2}\sin\varphi\ddot{\varphi}\delta t^2+...$$

So in order to minimize the misalignment Δ the torque should satisfy two conditions

$$\dot{\varphi}M<0,\qquad \sin\varphi M<0. \tag{2.1}$$

We construct the torque as the sum of two components each satisfying only one of conditions (2.1),

$$M=-k_a\sin\varphi-k_\omega\dot{\varphi} \tag{2.2}$$

where k_a and k_ω are positive control gains. The equation of motion with this control torque takes the form

$$\ddot{\varphi}+k_\omega\dot{\varphi}+k_a\sin\varphi=0. \tag{2.3}$$

Equation (2.3) corresponds to the damped oscillations. Necessary position $\varphi=0$ is asymptotically stable.

This reasoning can be generalized for the satellite movement around its center of mass. In this case the misalignment components are

$$\Delta_1=\frac{1}{2}\left(\omega_1^2+\omega_2^2+\omega_3^2\right),$$

$$\Delta_2=\frac{1}{2}\left[\left(d_{11}-1\right)^2+d_{12}^2+d_{13}^2+d_{21}^2+\left(d_{22}-1\right)^2+d_{23}^2+d_{31}^2+d_{32}^2+\left(d_{33}-1\right)^2\right].$$

Clearly,

$$\Delta_2=3-d_{11}-d_{22}-d_{33}.$$

The equations of motion are written in the form

$$\mathbf{J}\dot{\boldsymbol{\omega}} = \mathbf{M}_{cntrl} - \mathbf{M}_{gir} = \mathbf{M},$$
$$\dot{\mathbf{D}} = \mathbf{DW} \qquad (2.4)$$

where

$$\mathbf{W} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

That leads to $\delta\Delta_1 = \sum \omega_i \dot{\omega}_i \delta t = \boldsymbol{\omega}\mathbf{J}^{-1}\mathbf{M}\delta t$.

Differential misalignment part becomes controllable again. For the positional part variation we have

$$\delta\Delta_2 = -\sum \dot{d}_{ii}\delta t = \left[\omega_1\left(d_{23} - d_{32}\right) + \omega_2\left(d_{31} - d_{13}\right) + \omega_3\left(d_{12} - d_{21}\right)\right]\delta t.$$

We introduce the vector $\mathbf{S} = \left(d_{23} - d_{32},\ d_{31} - d_{13},\ d_{12} - d_{21}\right)^T$. This allows the last expression to be rewritten in a form $\delta\Delta_2 = \boldsymbol{\omega}\mathbf{S}\delta t$. The positional part cannot be minimized according to the first variation. However it is possible again to state the angular velocity damping (since the differential part can be minimized). So the second variation of the positional misalignment part governs its behavior,

$$\delta^2\Delta_2 = \left(\mathbf{S}\mathbf{J}^{-1}\mathbf{M} + \dot{\mathbf{S}}\boldsymbol{\omega}\right)\delta t^2.$$

Omitting the term with the angular velocity we obtain conditions

$$\boldsymbol{\omega}\mathbf{J}^{-1}\mathbf{M} < 0, \quad \mathbf{S}\mathbf{J}^{-1}\mathbf{M} < 0 \qquad (2.5)$$

analogous to (2.1). The gyroscopic torque is omitted because of the angular velocity damping, the "weighing" matrix $\mathbf{J}^{-1}$ is omitted so the control correspond to the PD-controller inspired one. After the control torque is written in the form $\mathbf{M}_{cntrl} = \mathbf{m} \times \mathbf{B}$ and the circular permutation is performed conditions (2.5) can be written as

$$\mathbf{m}\left(\mathbf{B} \times \boldsymbol{\omega}\right) < 0, \qquad \mathbf{m}\left(\mathbf{B} \times \mathbf{S}\right) < 0.$$

These conditions lead to the control dipole moment

$$\mathbf{m} = -k_\omega \mathbf{B} \times \boldsymbol{\omega} - k_a \mathbf{B} \times \mathbf{S} \qquad (2.6)$$

analogous to the control torque (2.2).

This control is often constructed using different reasoning. It is inspired by the PD-controller construction. As a Lyapunov function candidate the expression

$$V = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{J}\boldsymbol{\omega} + k_a \sum_{i=1}^{3}\left(1 - d_{ii}\right)$$

can be used. Clearly the equilibrium position is $\boldsymbol{\omega} = 0$, $,\mathbf{D} = diag(1,1,1)$ that is the inertial and the bound reference frames coincide. The Lyapunov function candidate derivative is

$$\frac{dV}{dt} = \frac{1}{2}\left(\dot{\boldsymbol{\omega}} \cdot \mathbf{J}\boldsymbol{\omega} + \boldsymbol{\omega} \cdot \mathbf{J}\dot{\boldsymbol{\omega}}\right) - k_a \sum_{i=1}^{3} \dot{d}_{ii} .$$

This derivative equals to

$$\frac{dV}{dt} = \boldsymbol{\omega}\left(\mathbf{M}_{cntrl} + \mathbf{M}_{gir} + k_a \mathbf{S}\right) \tag{2.7}$$

after the equations of motion are taken into account. We need to find the torque leading to the Lyapunov function candidate derivative being negative everywhere except the equilibrium position. This can be achieved if $\frac{dV}{dt} = -k_\omega \boldsymbol{\omega} \cdot \boldsymbol{\omega}$. This is valid if the torque is chosen in such a way that

$$\mathbf{M}_{cntrl} + k_a \mathbf{S} = -k_\omega \boldsymbol{\omega} .$$

Note that $\boldsymbol{\omega}\mathbf{M}_{gir} = 0$. The control torque should be of the form

$$\mathbf{M}_{cntrl} = -k_\omega \boldsymbol{\omega} - k_a \mathbf{S} . \tag{2.8}$$

Expression (2.8) corresponds to the (2.2). However, the torque (2.8) cannot be achieved using magnetorquers. The common way to implement the control is to use only accessible part of (2.8). So instead of the torque (2.8) its projection on the plane perpendicular to the local geomagnetic induction vector is used,

$$\mathbf{M}'_{cntrl} = \left(\mathbf{B}_0 \times \mathbf{M}_{cntrl}\right) \times \mathbf{B}_0$$

or

$$\mathbf{M}'_{cntrl} = \frac{1}{|\mathbf{B}|} \mathbf{m} \times \mathbf{B}_0 .$$

Here the geomagnetic induction vector $\mathbf{B}_0$ is a unit one. The dipole magnetic control moment is constructed according to (2.6). In this case the Lyapunov function candidate derivative is not negative. It is governed by the relation

$$\frac{dV}{dt} = -k_\omega \left(\boldsymbol{\omega} \times \mathbf{B}_0\right)^2 + k_a |\boldsymbol{\omega}||\mathbf{S}| \cos\left(\mathbf{B}_0, \boldsymbol{\omega}\right) \cos\left(\mathbf{B}_0, \mathbf{S}\right). \tag{2.9}$$

This relation can be used to choose the control implementation moments. If the second term in the relation is negative, the control should be implemented. However the numerical analysis showed very slight control time-response gain so it is illogical to overburden the control with unnecessary condition. From (2.9) we see that the differential misalignment part is negative always so the control is perfect for the angular velocity damping. This is not valid for the positional misalignment.

In this section two ways to construct the control (2.6) are presented. However these arguments cannot be considered as the algorithm efficiency proof. They can only provide the hope that the control is close to the necessary one and can be used. So we now process to the control (2.6) analysis.

3. Transient motion analysis

An analytical study of the satellite dynamics in the arbitrary motion is a complicated problem, so we divide it into three stages. This allows us to introduce several assumptions relevant to each stage. Provided the result is obtained on each stage we can get overall comprehension on the satellite dynamics in the arbitrary state.

The first stage corresponds to the fast rotation of the satellite. We rewrite the control law (2.6) in the form

$$\mathbf{m} = -k'_\omega \mathbf{B} \times \mathbf{\Omega} - k'_a \mathbf{B} \times \mathbf{S}, \qquad (3.1)$$

where $\boldsymbol{\omega} = \omega_0 \mathbf{\Omega}$, so $\mathbf{\Omega}$ is a dimensionless angular velocity. The geomagnetic induction vector is a unit one (note that in the averaged geomagnetic field model its magnitude is constant). We assume that the control gains ratio k'_ω / k'_a is of the order of unity. Under this assumption the differential control part is prevailing for the fast tumbling satellite since $|\mathbf{\Omega}| \gg 1$. The torque may be written in an approximate form

$$\mathbf{m} = -k'_\omega \mathbf{B} \times \mathbf{\Omega}.$$

Clearly this leads to the satellite with the “-Bdot” [8] control implemented. The motion of the satellite was analyzed by the authors earlier [9], and the angular velocity is asymptotically damped.

This leads to the second stage, when the velocity becomes of the order of the orbital one, that is $|\mathbf{\Omega}| = O(1)$. Positional control part cannot be omitted. We assume the control to be rather small, so the angular momentum prevails over its change during one orbit revolution. This allows us to use the Beletsky-Chernousko variables and implement an averaging technique. In order to do so we introduce a dimensionless torque $\overline{\mathbf{M}}_L$ according to the relation

$$\mathbf{M}_L = k'_a B_0^2 \overline{\mathbf{M}}_L;$$

the argument of latitude $u=\omega_0(t-t_0)$ where t_0 is an initial moment; dimensionless angular momentum magnitude l according to $L=L_0 l$ where L_0 is an initial angular momentum. As a result the equations (1.4) can be written in a dimensionless form

$$\frac{dl}{du}=\varepsilon l\overline{M}_{3L},\frac{d\rho}{du}=\varepsilon\overline{M}_{1L},\frac{d\sigma}{du}=\frac{\varepsilon}{\sin\rho}\overline{M}_{2L},$$
$$\frac{d\theta}{du}=\varepsilon\left(\overline{M}_{2L}\cos\psi-\overline{M}_{1L}\sin\psi\right),$$
$$\frac{d\varphi}{du}=\eta_1 l\cos\theta+\frac{\varepsilon}{\sin\theta}\left(\overline{M}_{1L}\cos\psi+\overline{M}_{2L}\sin\psi\right), \tag{3.2}$$
$$\frac{d\psi}{du}=\eta_2 l-\varepsilon\overline{M}_{1L}\cos\psi\operatorname{ctg}\theta-\varepsilon\overline{M}_{2L}\left(\operatorname{ctg}\rho+\sin\psi\operatorname{ctg}\theta\right).$$

The notations $\varepsilon=\dfrac{k_a' B_0^2}{\omega_0 L_0}$, $\eta_1=\dfrac{L_0}{\omega_0}\left(\dfrac{1}{C}-\dfrac{1}{A}\right)$, $\eta_2=\dfrac{L_0}{A\omega_0}$ are introduced. Here (φ,ψ,u) are fast variables, while (l,ρ,σ,θ) are slow ones. So, we can use the averaging technique [10] to determine the slow variables evolution. In order to do it we need to average the equations in the vicinity of the undisturbed solution of equations (3.2). However, since this motion is a regular precession, we need only to average separately the equations for slow variables over the fast variables. After this we get evolutionary equations for slow variables with accuracy of the order of ε on the time interval of the order of $1/\varepsilon$. We assume that the moments of inertia A and C provide no resonance between η_1, η_2 and 1 (u rate of change) . In order to obtain averaged equations we need expressions $\langle M_{iL}\rangle_{u,\psi,\varphi}$, $\langle M_{iL}\cos\psi\rangle_{u,\psi,\varphi}$ and $\langle M_{iL}\sin\psi\rangle_{u,\psi,\varphi}$. The damping component averaging result is known[9]. We need to prove that the positional part has no influence on the damping of the angular velocity, that is

$$\langle(\mathbf{S}\times\mathbf{B})\times\mathbf{B}\rangle=0.$$

That leads to the condition

$$\langle\mathbf{S}\rangle_{\psi,\varphi}=0.$$

Clearly, $\mathbf{S}_L=\mathbf{A}\mathbf{S}_x$ and the transition matrix between the inertial and bound frames can be expressed as $\mathbf{D}=\mathbf{A}^T\mathbf{Q}^T$. Therefore we need to average the expressions $a_{ij}a_{kl}$ over ψ and φ. For the first component of the $\mathbf{S}_L$ vector

$$S_{1L}=a_{11}a_{12}q_{31}+a_{11}a_{22}q_{32}+a_{11}a_{32}q_{33}-a_{11}a_{13}q_{21}-a_{11}a_{23}q_{22}-a_{11}a_{33}q_{23}+$$

$$+a_{12}a_{13}q_{11}+a_{12}a_{23}q_{12}+a_{12}a_{33}q_{13}-a_{12}a_{11}q_{31}-a_{12}a_{21}q_{32}-a_{12}a_{31}q_{33}+$$
$$+a_{13}a_{11}q_{21}+a_{13}a_{21}q_{22}+a_{13}a_{31}q_{23}-a_{13}a_{12}q_{11}-a_{13}a_{22}q_{12}-a_{13}a_{32}q_{13}.$$

Simplifying leads to

$$S_{1L}=q_{32}\left(a_{11}a_{22}-a_{12}a_{21}\right)+q_{33}\left(a_{11}a_{32}-a_{12}a_{31}\right)+q_{22}\left(a_{13}a_{21}-a_{11}a_{23}\right)+$$
$$+q_{23}\left(a_{13}a_{31}-a_{11}a_{33}\right)+q_{12}\left(a_{12}a_{23}-a_{13}a_{22}\right)+q_{13}\left(a_{12}a_{33}-a_{13}a_{32}\right).$$

Taking into account a_{ij} (1.3) and q_{ij} (1.2) after averaging over ψ we get

$$\left\langle S_{1L}\right\rangle_{\psi}=q_{12}\sin\varphi\sin\theta$$

and in the end

$$\left\langle S_{1L}\right\rangle_{\psi,\varphi}=0.$$

The same math can be easily obtained for S_{2L} and S_{3L}.

In this section we managed to prove the asymptotical damping of the angular velocity of the satellite regardless of the positional torque part. This allows passing to the last stage in the satellite dynamics.

4. Stability analysis

Previous section proved that the control law (2.6) leads to the angular velocity damping and the angular momentum magnitude tends to zero exponentially [11], [12]. So we now assume that the control gains k'_{ω} and k'_{a} are of different magnitude, but $|\mathbf{\Omega}|=o(1)$. The equations of motion (1.6) may be written in a dimensionless form

$$\begin{aligned}
&\frac{d\Omega_1}{du}=\frac{(B-C)}{A}\Omega_2\Omega_3+\varepsilon_1\overline{M}_{1x},\\
&\frac{d\Omega_2}{du}=-\frac{(A-C)}{B}\Omega_1\Omega_3+\varepsilon_1\frac{A}{B}\overline{M}_{2x},\\
&\frac{d\Omega_3}{du}=-\frac{(B-A)}{C}\Omega_1\Omega_2+\varepsilon_1\frac{A}{C}\overline{M}_{3x},\\
&\frac{d\alpha}{du}=\frac{1}{\cos\beta}\left(-\Omega_3\sin\gamma+\Omega_2\cos\gamma\right),\\
&\frac{d\beta}{du}=\Omega_2\sin\gamma+\Omega_3\cos\gamma,\\
&\frac{d\gamma}{du}=\Omega_1-\operatorname{tg}\beta\left(\Omega_2\cos\gamma-\Omega_3\sin\gamma\right),
\end{aligned}\qquad(4.1)$$

where $\varepsilon_1 = \dfrac{k_a' B_0^2}{A\omega_0^2}$ is a new small parameter. In order to make these equations convenient for the averaging we introduce another small parameter characterizing the angular velocity magnitude, $\mathbf{\Omega} = \varepsilon_2 \mathbf{w}$ where $\varepsilon_2 = \left|\mathbf{\Omega}(0)\right|$. We introduce vector $\mathbf{x} = (\alpha, \beta, \gamma, w_1, w_2, w_3)$ and the equations (4.1) are of the form

$$\dot{\mathbf{x}} = \varepsilon_2 \mathbf{X}(\mathbf{x}, u, \varepsilon, \mu). \tag{4.2}$$

Here

$$\mathbf{X} = \begin{pmatrix} \dfrac{1}{\cos\beta}(-w_3 \sin\gamma + w_2 \cos\gamma) \\ w_2 \sin\gamma + w_3 \cos\gamma \\ w_1 - \operatorname{tg}\beta(w_2 \cos\gamma - w_3 \sin\gamma) \\ \dfrac{(B-C)}{A} w_2 w_3 + \dfrac{\varepsilon_1}{\varepsilon_2^2} \overline{M}_{1x} \\ -\dfrac{(A-C)}{B} w_1 w_3 + \dfrac{\varepsilon_1}{\varepsilon_2^2} \dfrac{A}{B} \overline{M}_{2x} \\ -\dfrac{(B-A)}{C} w_1 w_2 + \dfrac{\varepsilon_1}{\varepsilon_2^2} \dfrac{A}{C} \overline{M}_{3x} \end{pmatrix}.$$

We assume $\dfrac{\varepsilon_1}{\varepsilon_2^2} = \kappa = O(1)$ for the analysis simplification. The equation in the form (4.2) can be used for the formal averaging over the time (argument of latitude in our case). Since the stability of the interest in this section, we first linearize the equation of motion (4.1),

$$\begin{aligned}
\frac{dw_1}{du} &= \varepsilon_2 \kappa \Big(-(B_2^2 + B_3^2)\chi w_1 + B_1 B_2 \chi w_2 + B_1 B_3 \chi w_3 + \\
&+ 2B_1 B_2 \alpha + 2B_1 B_3 \beta - 2(B_2^2 + B_3^2)\gamma \Big), \\
\frac{dw_2}{du} &= \varepsilon_2 \kappa \frac{A}{B} \Big(B_1 B_2 \chi w_1 - (B_1^2 + B_3^2)\chi w_2 + B_2 B_3 \chi w_3 - \\
&- 2(B_1^2 + B_3^2)\alpha + 2B_2 B_3 \beta + 2B_1 B_2 \gamma \Big), \\
\frac{dw_3}{du} &= \varepsilon_2 \kappa \frac{A}{C} \Big(B_1 B_3 \chi w_1 + B_2 B_3 \chi w_2 - (B_1^2 + B_2^2)\chi w_3 + \\
&+ 2B_2 B_3 \alpha - 2(B_1^2 + B_2^2)\beta + 2B_1 B_3 \gamma \Big),
\end{aligned} \tag{4.3}$$

$$\frac{d\alpha}{du} = \varepsilon_2 w_2,\ \frac{d\beta}{du} = \varepsilon_2 w_3,\ \frac{d\gamma}{du} = \varepsilon_2 w_1,$$

where $\chi = k'_\omega / k'_a$. The averaged linearized equations are

$$\ddot{\gamma} + \varepsilon_1 \chi (p+q)\dot{\gamma} + 2\varepsilon_1 (p+q)\gamma = 0,$$

$$\ddot{\alpha} + \varepsilon_1 \chi (p+q)\frac{A}{B}\dot{\alpha} + 2\varepsilon_1 (p+q)\frac{A}{B}\alpha = 0, \quad (4.4)$$

$$\ddot{\beta} + 2\varepsilon_1 \chi p \frac{A}{C}\dot{\beta} + 4\varepsilon_1 p \frac{A}{C}\beta = 0$$

where $p = \frac{1}{2}\sin^2\Theta,\ q = \cos^2\Theta$. The equations (4.4) introduce damping oscillations for each angle, so the necessary attitude is stable. Equations (4.4) allow us to assess the control gains values that provide the optimal algorithm time-response. The characteristic polynomials roots are

$$\lambda_{1,2} = \frac{1}{2}\left(-\varepsilon_1 \chi (p+q) \pm \sqrt{\varepsilon_1^2 \chi^2 (p+q)^2 - 8\varepsilon_1 (p+q)}\right),$$

$$\lambda_{3,4} = \frac{1}{2}\left(-\varepsilon_1 \chi (p+q)\frac{A}{B} \pm \sqrt{\varepsilon_1^2 \chi^2 (p+q)^2 \left(\frac{A}{B}\right)^2 - 8\varepsilon_1 (p+q)\frac{A}{B}}\right),$$

$$\lambda_{5,6} = -\varepsilon_1 \chi p \frac{A}{C} \pm \sqrt{\varepsilon_1^2 \chi^2 p^2 \left(\frac{A}{C}\right)^2 - 4\varepsilon_1 p \frac{A}{C}}.$$

We introduce parameters

$$\theta_1 = B/A,\ \theta_2 = C(p+q)/2pA$$

and new control gains

$$K_\omega = \frac{B_0^2}{A\omega_0^2}(p+q)k'_\omega,\ K_a = \frac{B_0^2}{A\omega_0^2}(p+q)k'_a.$$

The roots are then rewritten in the form

$$\lambda_{1,2} = \frac{1}{2}\left(-K_\omega \pm \sqrt{K_\omega^2 - 8K_a}\right),$$

$$\lambda_{3,4} = \frac{1}{2}\left(-\frac{K_\omega}{\theta_1} \pm \frac{1}{\theta_1^2}\sqrt{K_\omega^2 - 8\theta_1 K_a}\right), \quad (4.5)$$

$$\lambda_{5,6} = \frac{1}{2}\left(-\frac{K_\omega}{\theta_2} \pm \frac{1}{\theta_2^2}\sqrt{K_\omega^2 - 8\theta_2 K_a}\right).$$

The bound frame is chosen in such a way that $C > B > A$. In this case $\theta_1 > 1$ and it is necessary to examine three cases.

I. $\theta_2 > \theta_1 > 1$. This situation takes place for the low inclination orbit where q prevails over p. Control gains are from one of the three following areas.

1. $8K_a > K_\omega^2$. All radicands in (4.5) are negative and the degree of stability is

$$\xi = \min\left(\frac{1}{2}K_\omega, \frac{1}{2\theta_1}K_\omega, \frac{1}{2\theta_2}K_\omega\right).$$

From the restriction I we have

$$\xi_1 = \frac{1}{2\theta_2}K_\omega .$$

2. $8\theta_2 K_a < K_\omega^2$. All radicands are positive and

$$\xi = \frac{1}{2}\min\left(K_\omega - \sqrt{K_\omega^2 - 8K_a}, \frac{K_\omega}{\theta_1} - \frac{1}{\theta_1^2}\sqrt{K_\omega^2 - 8\theta_1 K_a}, \frac{K_\omega}{\theta_2} - \frac{1}{\theta_2^2}\sqrt{K_\omega^2 - 8\theta_2 K_a}\right).$$

That leads to

$$\xi_2 = K_\omega - \sqrt{K_\omega^2 - 8K_a} .$$

3. $8K_a < K_\omega^2 < 8\theta_2 K_a$. In this case either one or two radicands in (4.5) are positive and the degree of stability equals ξ_1 or ξ_2.

The case when $\xi_1 = \xi_2$ is of the utmost interest since it provides the maximum degree of stability with the minimum control gains (for each degree)

$$K_\omega^2 = \frac{8\theta_2^2}{2\theta_2 - 1}K_a ,$$

or in the initial expressions

$$k_a' = \frac{2\theta_2 - 1}{8\theta_2^2}\frac{B_0^2}{A\omega_0^2}(p+q)k_\omega'^2 . \qquad (4.6)$$

Fig. 3 shows the degree of stability isolines with respect to both control gains for the satellite with inertia tensor $\mathbf{J} = diag(1, 1.5, 2)$ kg·m^2 on the orbit with 30° inclination and 350 km attitude. The parabola (4.6) is highlighted by the thick line.

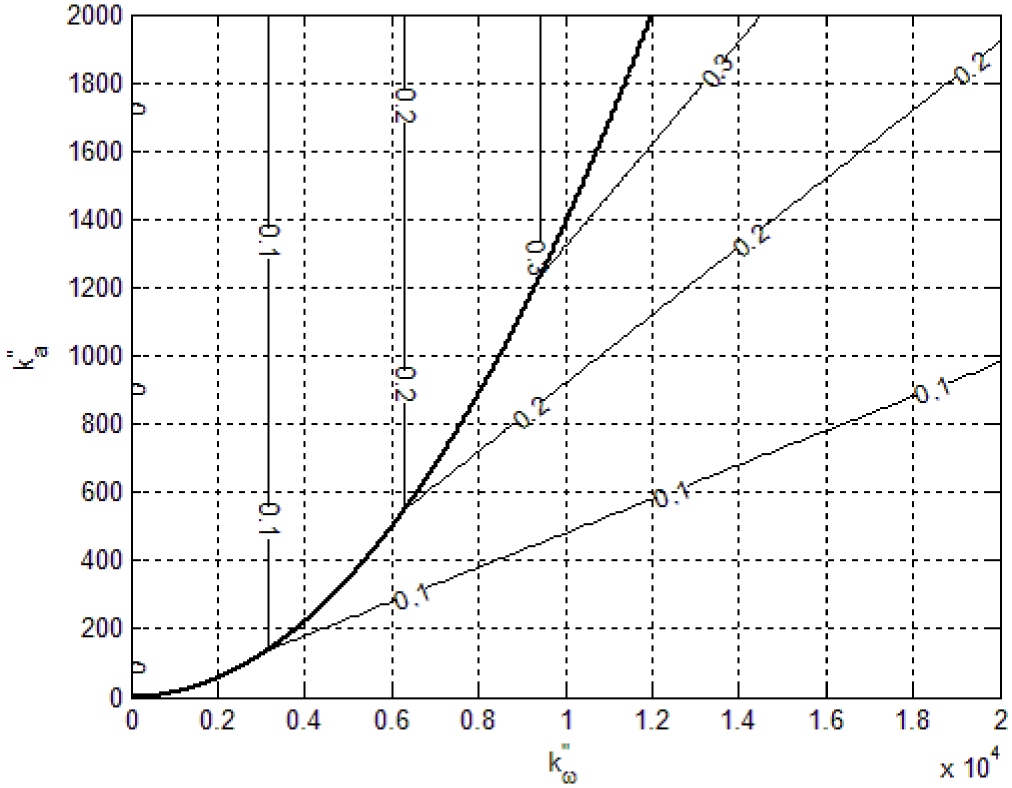

Fig. 3. Optimal control gains parabola

Fig. 3 reveals the parabola (4.6) point. If one chooses the degree of stability, this parabola offers the minimum possible control gains. This parabola also corresponds to the equality between three roots real part for each of the equations in (4.5). We now pass to other cases.

II. $\theta_1 > \theta_2 > 1$. The reasoning is the same, the optimal parabola is defined by

$$k'_a = \frac{2\theta_1 - 1}{8\theta_1^2}\frac{B_0^2}{A\omega_0^2}(p+q)k'^2_\omega \tag{4.7}$$

III. $\theta_1 > 1 > \theta_2$. This relation is valid for the high inclined orbits.

1. $8\theta_2 K_a > K_\omega^2$. All radicands are negative, the degree of stability is

$$\xi_1 = \frac{1}{2\theta_1}K_\omega.$$

2. $8\theta_1 K_a < K_\omega^2$. All radicands are positive,

$$\xi_2 = \frac{K_\omega}{\theta_2} - \frac{1}{\theta_2^2}\sqrt{K_\omega^2 - 8\theta_2 K_a}\ .$$

3. $8\theta_1 K_a < K_\omega^2 < 8\theta_2 K_a$. There are both positive and negative radicands, and the optimal parabola is defined by the expression

$$k_a' = \frac{1}{8\theta_1}\left(2 - \frac{\theta_2}{\theta_1}\right)\frac{B_0^2}{A\omega_0^2}(p+q)k_\omega'^2. \tag{4.8}$$

Fig. 4 brings the degree of stability isolines for the same satellite but for the orbit inclination of 70 degrees.

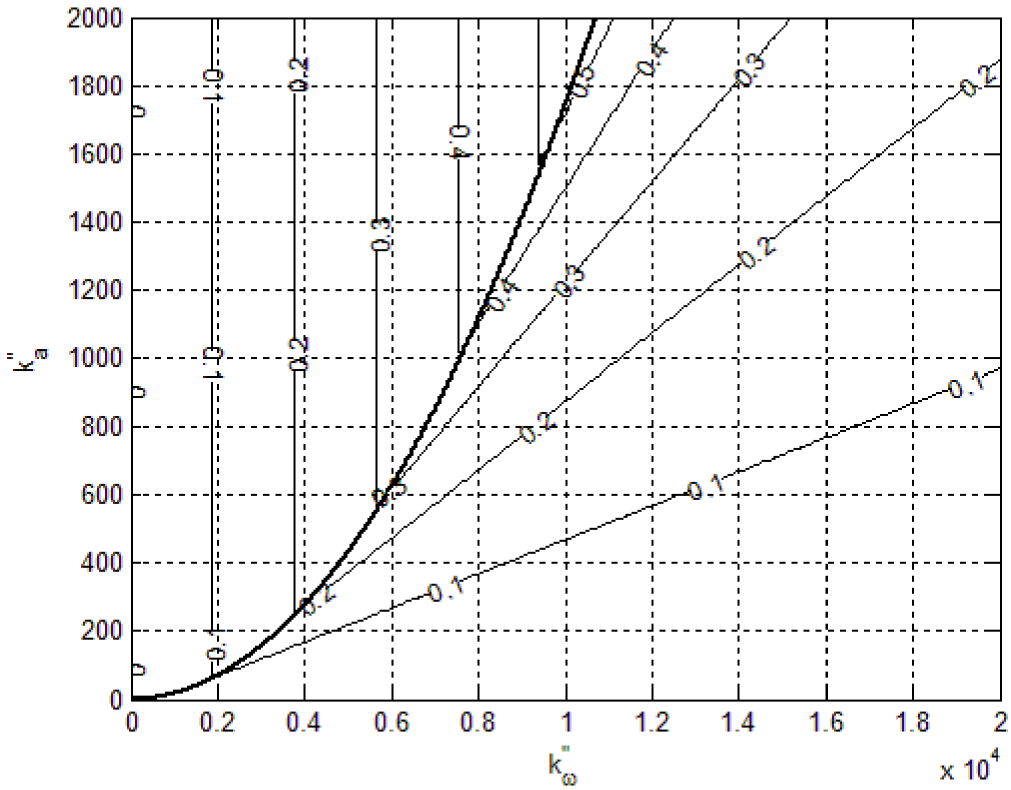

Fig. 4. Optimal control gains parabola for the highly inclined orbit

All three parabolas are close for low and high inclined orbits, but in the latter case the damping control component should be slightly greater.

Consider equations (4.2) again. The reasoning above is valid only if $\frac{\varepsilon_1}{\varepsilon_2^2} = \kappa = O(1)$. Assume that this relation is not valid and $\varepsilon_1 = \varepsilon_2^n$ instead with arbitrary n. Notation $\mathbf{y} = (\alpha, \beta, \gamma)$ allows the equations (4.1) to be written in the form

$$\dot{\boldsymbol{\omega}} = \varepsilon_2 \mathbf{f}_1(\boldsymbol{\omega}) + \varepsilon_2^{n-1}\mathbf{f}_2(\boldsymbol{\omega}, \mathbf{y}),$$
$$\dot{\mathbf{y}} = \varepsilon_2 \mathbf{f}_3(\boldsymbol{\omega}, \mathbf{y}).$$

The averaged equations are valid on the time interval $u \sim 1/\varepsilon_2$ with the accuracy ε_2 if $n > 2$. On this time interval only angles change significantly while the angular velocity components stay almost constant. If $n < 2$ the angular velocity components change while angles not, and the averaging is valid on the interval $u \sim 1/\varepsilon_2^{n-1}$. However both cases allow the averaging and the resulting equations are asymptotically stable. That leads to the asymptotically stable limit cycle of the initial equations. The solution of initial equations is in the vicinity of the averaged solution with the accuracy of ε_2^k where $k = \min(1, n-1)$. This allows the averaging result to be prolonged on the infinite time interval, so the result obtained for $n = 2$ is valid for arbitrary n value.

5. Numerical analysis

According to previous sections, the necessary attitude is stable if the control gains are small enough. This result conflicts with the obvious underactuation problem: there should be unstable areas. To dismiss this problem we use the numerical analysis with the Floquet theory (the isolines are characteristic multipliers in this case). Fig. 5 shows the discrepancy between the analytical and numerical results for the control gains valid for the analytical results (same satellite, 70 degree inclination orbit).

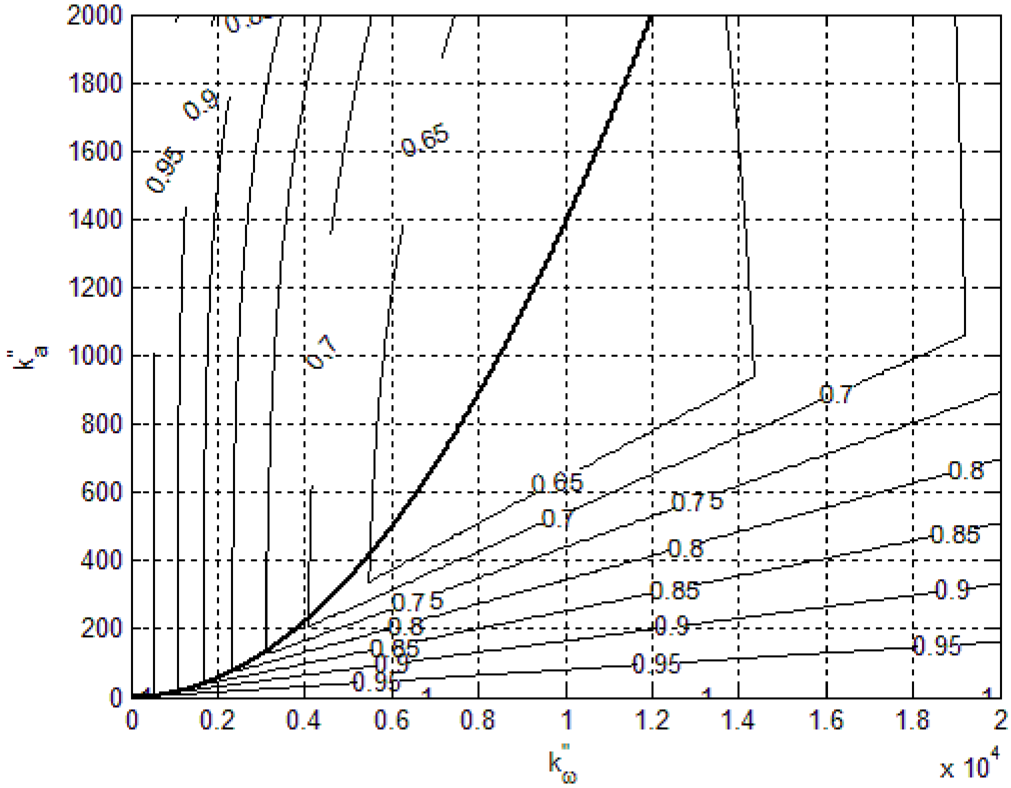

Fig. 5. Analytical and numerical results comparison

As seen from Fig. 5 the discrepancy raises as the control gains rise. The accuracy of the analytical result is of the order of the small parameter depending on the gains. Fig. 6 shows the unstable control gains area.

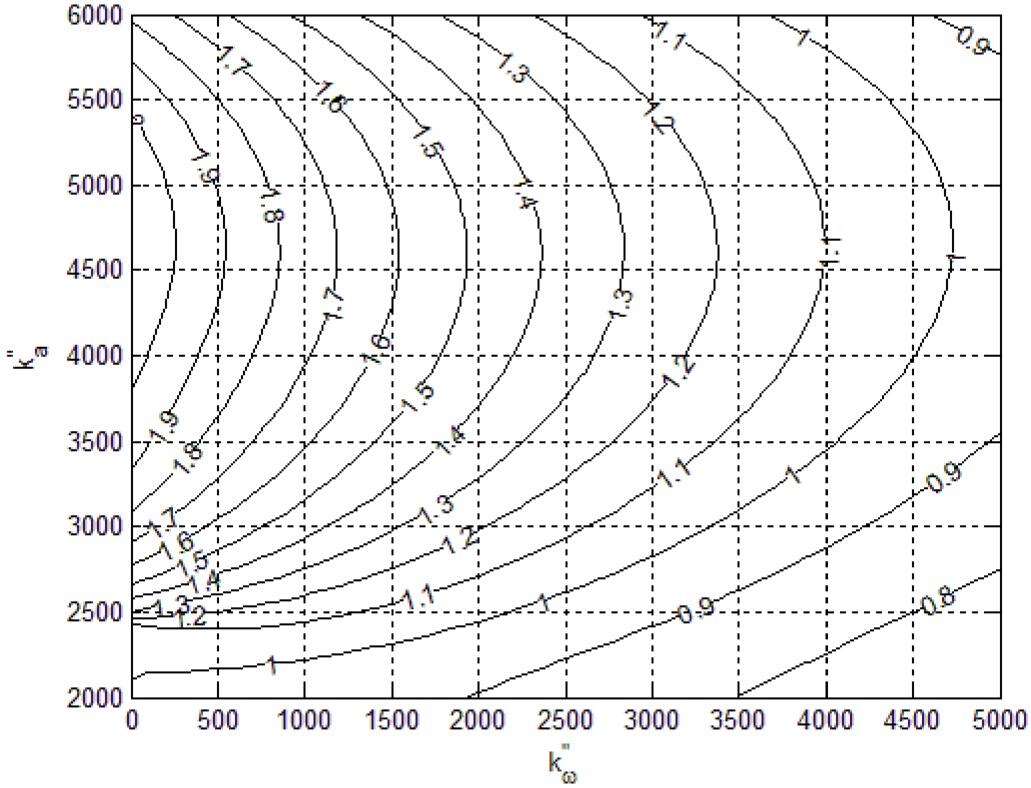

Fig. 6. Unstable area for the strong positional torque part

Fig. 6 corresponds to the prevailing positional torque component. This strong positional torque rotates the satellite in a wrong direction, “missing” the point where it should be stopped, and small damping torque cannot neglect this rotation. Fig. 7 presents the numerical modeling example for the same satellite with control gains $k'_{\omega} = 9580$, $k'_{a} = 1510$, orbit inclination 70 degrees.

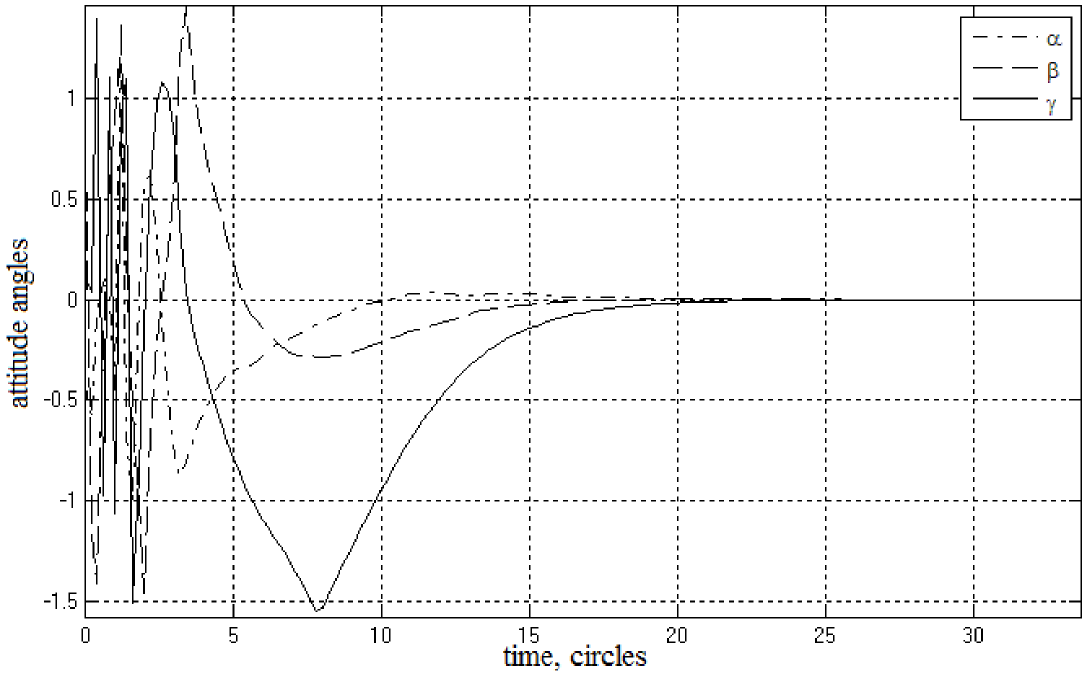

Fig. 7. Attitude angles

Note that the small control torque restriction is not unnatural. It is induced in order to perform the asymptotical analysis with the small parameter. However, as seen from Fig. 6, it is essential to have small control torque. If the torque is too strong, the satellite will acquire the error greater or comparable to the one before each control iteration. The torque should ob the order of about $5 \cdot 10^{-6}$ N·m for the satellite considered. However it is only 2-4 times greater than the gravitational one which results in poor attitude accuracy, slightly better than 10 degrees. With different satellite masses and inertia tensors the accuracy of about few degrees is achievable. Nevertheless the control (2.6) should be modified. The weighing matrix ($\mathbf{J}^{-1}$ for example) may be introduced or varying control gains used [13] in order to maintain stable control torque magnitude when the angular velocity and the positional error diminish.

Conclusion

The satellite equipped with the active magnetic attitude control system providing three-axis stabilization is considered. The algorithm is constructed and analytically studied. Angular velocity damping and necessary attitude stability is proven using the asymptotical methods. Optimal (in terms of the degree of stability) control parameters are provided. Algorithm limitations are present.

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