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We consider an ordinary differential equation (ODE) which can be written as a polynomial in variables and derivatives. Several types of asymptotic expansions of its solutions can be found by algorithms of 2D Power Geometry. They are power, power-logarithmic, exotic and complicated expansions. Here we develop 3D Power Geometry and apply it for calculation power-elliptic expansions of solutions to an ODE. Among them we select regular power-elliptic expansions and give a survey of all such expansions in solutions of the Painlevé equations $P_1, \ldots, P_6$.

**Key words:** Power Geometry, asymptotic expansion, Painlevé equations

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1. Universal Nonlinear Analysis

We develop a new Calculus based on Power Geometry \[1\textendash}4\]. Now it allows to compute local and asymptotic expansions of solutions to nonlinear equations of three classes: (A) algebraic, (B) ordinary differential, (C) partial differential, as well as to systems of such equations.

Principal ideas and algorithms are common for all classes of equations. Computation of asymptotic expansions of solutions consists of 3 following steps (we describe them for one equation \(f = 0\)).

1. Isolation of truncated equations \(\hat{f}_j^{(d)} = 0\) by means of faces of the convex polyhedron \(\Gamma(f)\) which is a generalization of the Newton polyhedron. The first term of the expansion of a solution to the initial equation \(f = 0\) is a solution to the corresponding truncated equation \(\hat{f}_j^{(d)} = 0\).

2. Finding solutions to a truncated equation \(\hat{f}_j^{(d)} = 0\) which is quasihomogenous. Using power and logarithmic transformations of coordinates we can reduce the equation \(\hat{f}_j^{(d)} = 0\) to such simple form that can be solved. Among the solutions found we must select appropriate ones which give the first terms of asymptotic expansions.

3. Computation of the tail of the asymptotic expansion. Each term in the expansion is a solution to a linear equation which can be written down and solved.

Applications

Class A. 1. Sets of stability of multiparameter problems \[5,6\].
Class B. 2. Asymptotic forms and expansions of solutions to the Painlevé equations \[4,7,8\].
3. Periodic motions of a satellite around its mass center moving along an elliptic orbit \[9\].
4. New properties of motion of a top \[10\].
5. Families of periodic solutions of the restricted three-body problem and distribution of asteroids \[11,12\].
6. Integrability of ODE systems \[13\].
Class C. 7. Boundary layer on a needle \[14\].
8. Evolution of the turbulent flow \[15\].

A survey of these applications see in \[24\].

2. Introduction

Let \(w(u)\) be a formal elliptic asymptotic form of a solution to an ODE. The form \(w(u)\) is suitable if it can be extended into power asymptotic expansion
\[ v = w(u) + \sum_{j=1}^{\infty} b_j u^{-j}, \text{ where } b_j = b_j(u) \text{ are some functions.} \]

The expansion is regular, if all \( b_j \) are not branching functions of \( w(u) \) and its derivatives. If all functions \( b_j(w, \dot{w}) \) have no branching, then they are elliptic functions with the same periods as \( w(u) \). Selection of such cases is our aim. For given \( w(u) \) and fixed point \( w^0 \) (including infinity), we can compute power-logarithmic expansions of functions \( b_j(w, \dot{w}) \) near \( w = w^0 \). In these expansions logarithmic branching can appear, only if \( w^0 \) is a singular point, and algebraic branching (of finite order) can be for subsingular points \( w^0 \). To each singular point \( w^0 \) and suitable asymptotic form \( w(u) \), we assign unique regular expansion \( v = \tilde{v}(w^0, w(u)) \), so called basic, and we are looking for such basic expansions near singular point \( w^0 \), which have not branching.

We propose algorithms for: (1) finding all formal elliptic asymptotic forms, (2) finding all suitable elliptic asymptotic forms, (3) calculation of power-logarithmic expansions of functions \( b_j(w, \dot{w}) \) near a singular point \( w^0 \) and selection of basic expansions without branching. All algorithms are based on 3D Power Geometry.

Application of these algorithms to the Painlevé equations \( P_1, \ldots, P_6 \) gives following.

1. \( P_1, P_2, P_4 \) have continuum of 2-parameter families of elliptic asymptotic forms each, \( P_3 \) has three and \( P_5 \) has two of them. \( P_6 \) does not have.

2. \( P_1, P_2, P_4 \) have countable sets of families of suitable asymptotic forms each, and all 5 forms of \( P_3 \) and \( P_5 \) are suitable.

3. Basic expansions for all suitable forms have not branching for \( P_1 \), for \( P_2 \) if the independent variable tends to infinity, for \( P_3 \) if condition C is fulfilled and for \( P_5 \) if condition D is fulfilled and \( a = b = 0, d \neq 0 \).

Expansions are formal, their convergence is not considered.

Here we consider application of Power Geometry to calculation of elliptic expansions of solutions to the Painlevé equations.

A hundred years ago, Boutroux \[16\] found 2 families of elliptic asymptotic forms of solutions to the Painlevé equations \( P_1 \) and \( P_2 \). During last 5 years we found 6 additional families of elliptic asymptotic forms of solutions to \( P_3 \) (three) \[17, 18\], \( P_4 \) (one) \[19\], \( P_5 \) (two) \[20\]. Moreover the Painlevé equations \( P_1, P_2, P_4 \) have continuum of families of elliptic asymptotic forms each, and I proposed a criterion for selection suitable asymptotic forms, witch can be extended as asymptotic expansions. All 8 known elliptic asymptotic forms are suitable. Solutions to the equation \( P_6 \) have not elliptic asymptotic forms at all.

Near infinity of the independent variable, the Painlevé equations \( P_1-P_5 \) have 12 families of suitable asymptotic forms and near zero of the independent variable equations \( P_1, P_2, P_4 \) have countable sets of such families each. Next I extend
these suitable elliptic asymptotic forms \( w(u) \) into power-elliptic expansions
\[
v = w(u) + \sum_{j=1}^{\infty} b_j u^{-j},
\]
where coefficients \( b_j \) are functions of the corresponding elliptic asymptotic forms and their derivatives. To each family of suitable elliptic asymptotic forms, I put in correspondence unique basic formal power-elliptic expansion near \( w^0 = \infty \) for \( P_1 - P_5 \), near \( w^0 = 0 \) for \( P_2 - P_5 \) and near \( w^0 = 1 \) for \( P_5 \). Obstacles (logarithmic branching) in calculations of these basic expansions appeared only for \( P_2 \) if the independent variable tends to zero, for \( P_4 \) and for \( P_5 \) if \(|a| + |b| \neq 0 \) or \( d = 0 \).

Thus, near infinity of the independent variable there are 10 families of regular (i.e. without branching) elliptic expansions of solutions to equations
\( P_1 - P_6 \):
4 for \( P_1 \),
2 for \( P_2 \),
3 for \( P_3 \),
and 1 for \( P_5 \).

Existence of these expansions for two Boutroux families of asymptotic forms was proven in [21]. Near zero of the independent variable there is countable set of families of such expansions for \( P_1 \). The results were obtained by means of algorithms of 3D Power Geometry [17–23], realized in very cumbersome calculations.

Here I introduce the third variant of 3D Power Geometry. The first was in [17, 19, 23], the second was in [18, 20–22].

Main applications of the Painlevé equations: many soliton equations of Mathematical Physics can be reduced to Painlevé equations. For example:

- the Korteweg-de Vries equation is reduced to \( P_1 \) and \( P_2 \);
- the nonlinear Schrödinger equation is reduced to \( P_2 \) and \( P_4 \);
- the Sine-Gordon equation is reduced to \( P_3 \) and \( P_4 \);
- the Boussinesq and Kadomtsev-Petviashvili equations are reduced to \( P_1 \), \( P_2 \) and \( P_4 \);
- the Ernst equations are reduced to \( P_3 \), \( P_5 \) and \( P_6 \).

3. 3D Power Geometry

Let \( x \) be independent and \( y \) be dependent variables, \( x, y \in \mathbb{C} \). A differential monomial \( a(x, y) \) is a product of an ordinary monomial \( c x^{r_1} y^{r_2} \), where \( c = \text{const} \in \mathbb{C}, (r_1, r_2) \in \mathbb{R}^2 \), and a finite number of derivatives of the form \( d^l y/dx^l \), \( l \in \mathbb{N} \). The sum of differential monomials
\[
f(x, y) = \sum a_i(x, y)
\]
is called the differential sum. Let \( n \) be the maximal value of \( l \) in \( f(x, y) \).

In [24] it was shown that as \( x \to 0 \) (\( \omega = -1 \)) or as \( x \to \infty \) (\( \omega = 1 \)) solutions \( y = \varphi(x) \) to the ODE \( f(x, y) = 0 \), where \( f(x, y) \) is a differential sum, can be found by means of algorithms of Plane (2D) Power Geometry, if
\[
p_\omega \left( d^l \varphi/dx^l \right) = p_\omega(\varphi(x)) - l, \quad l = 1, \ldots, n,
\]
where the order
\[ p_\omega(\varphi(x)) = \omega \limsup_{x_\omega \to \infty} \frac{\log |\varphi(x)|}{\omega \log |x|} \]
on a ray \( \arg x = \text{const} \) and \( n \) is the maximal order of derivatives in \( f(x, y) \).

Order of the power function \( \varphi(x) = x^\alpha \) with \( \alpha \in \mathbb{C} \) is \( p_\omega(x^\alpha) = \Re \alpha \).

Here we introduce algorithms, which allow to calculate solutions \( y = \varphi(x) \) with the property
\[ p_\omega \left( \frac{d^l \varphi}{dx^l} \right) = p_\omega(\varphi(x)) - l\gamma_\omega, \quad l = 1, \ldots, n, \quad (2) \]
where \( \gamma_\omega \in \mathbb{R} \).

**Theorem 1.** \( \omega - \omega \gamma_\omega \geq 0 \).

For example, \( \gamma_1 = 0 \) for \( \varphi = \sin x \) and \( \gamma_{-1} = 2 \) for \( \varphi = \sin(1/x) \). Note, that in Plane Power Geometry we had \( \gamma_\omega = 1 \), i.e. \( \omega - \omega \gamma_\omega = 0 \). So, new interesting possibilities correspond to \( \omega - \omega \gamma_\omega > 0 \).

**Problem**
Select leading terms in the sum (1) after substitution \( y = \varphi(x) \) with property (2).

Below we describe algorithms for solution of the problem. To each differential monomial \( a_i(x, y) \), we assign its \((3D) \) power exponent \( Q(a_i) = (q_1, q_2, q_3) \in \mathbb{R}^3 \) by the following rules:

- \( q_3 = \) sum of orders of all derivatives;
- \( q_2 = \) order of \( y \);
- \( q_1 = \) difference of order of \( x \) and \( q_3 \).

Then the 2D vector \( Q = (q_1, q_2) \) is the same as in 2D Power Geometry [2–4] and \( q_3 \) corresponds to the total order of derivatives. The power exponent of the product of differential monomials is the sum of power exponents of factors: \( Q(a_1 a_2) = Q(a_1) + Q(a_2) \).

The set \( \hat{S}(f) \) of power exponents \( Q(a_i) \) of all differential monomials \( a_i(x, y) \) presented in the differential sum \( f(x, y) \) is called the \( 3D \) support of the sum \( f(x, y) \). Obviously, \( \hat{S}(f) \subset \mathbb{R}^3 \). The convex hull \( \Gamma(f) \) of the support \( \hat{S}(f) \) is called the \textit{polyhedron} of the sum \( f(x, y) \). The boundary \( \partial \Gamma(f) \) of the polyhedron \( \Gamma(f) \) consists of the vertices \( \Gamma^{(0)}_j \), the edges \( \Gamma^{(1)}_j \) and the faces \( \Gamma^{(2)}_j \). They are called \textit{(generalized) faces} \( \Gamma^{(d)}_j \), where the upper index indicates the dimension of the face, and the lower one is its number. Each face \( \Gamma^{(d)}_j \) corresponds to the 3D \textit{truncated sum}
\[ \hat{f}^{(d)}_j(x, y) = \sum a_i(x, y) \text{ over } Q(a_i) \in \Gamma^{(d)}_j \cap \hat{S}(f). \]
All these definitions are applied to differential equation

\[ f(x, y) = 0. \]  \hspace{2cm} (3)

Thus, each generalized face \( \Gamma_j^{(d)} \) corresponds to the truncated equation

\[ \tilde{f}_j^{(d)}(x, y) = 0. \]

Let \( N_j = (n_1, n_2, n_3) \) be the external normal to two-dimensional face \( \Gamma_j^{(2)} \). We will consider only normals with \( n_1 \neq 0 \).

**Example 1.** Consider the second Painlevé equation \( P_2 \)

\[ f(x, y) \overset{\text{def}}{=} -y'' + 2y^3 + xy + a = 0, \]  \hspace{2cm} (4)

where \( a \) is the complex parameter.

If \( a \neq 0 \), the 3D support \( \tilde{S}(f) \) consists of 4 points

\[ Q_1 = (-2,1,2), Q_2 = (0,3,0), Q_3 = (1,1,0), Q_4 = 0. \]

They are shown in Figure 1. Their convex hull \( \Gamma(f) \) is a tetrahedron. It has 4 vertices \( Q_1-Q_4 \), 6 edges \( \Gamma_j^{(1)} \) and 4 faces \( \Gamma_j^{(2)} \). Face \( \Gamma_1^{(2)} = [Q_1, Q_2, Q_3] \) is distinguished in Fig. 1, its external normal \( N_1 = (2,1,3) \) and its truncated equation

\[ \tilde{f}_1^{(2)}(x, y) \overset{\text{def}}{=} -y'' + 2y^3 + xy = 0. \]

Edge \( \Gamma_1^{(1)} = [Q_1, Q_2] \) is also distinguished in Fig. 1, its truncated equation

\[ \tilde{f}_1^{(1)}(x, y) \overset{\text{def}}{=} -y'' + 2y^3 = 0. \]

Let \( y = \varphi(x) \) be a solution to equation (3) with property (2) and \( p = p_\omega(\varphi), \gamma = \gamma_\omega(\varphi) \), then the order of a monomial \( a(x,y) \) with \( Q(a) = (q_1, q_2, q_3) \) is

\[ q_1 + q_2 + q_3(1 - \gamma) = \langle P, Q \rangle, \]

where \( P = (1, p, 1 - \gamma) \) and \( \langle \cdot, \cdot \rangle \) is the scalar product. Leading terms of the sum (1) after substitution \( y = \varphi(x) \) are monomials \( a(x,y) \), for which \( \omega \langle P, Q \rangle = \langle \omega P, Q \rangle \) reaches the maximal value on the support \( \tilde{S}(f) \). Here \( \omega P = (\omega, \omega p, \omega(1 - \gamma_\omega)) \) and \( \omega(1 - \gamma_\omega) \geq 0 \) according to Theorem 1.
Figure 1. 3D support $\tilde{S}(f)$ and polyhedron $\Gamma(f)$ of equation $P_2$ \( \square \). The grey face is $\Gamma^{(2)}_1$, the grey edge is $\Gamma^{(1)}_1$. Projection on the plane $(q_1, q_2)$ is shown by dotted lines.

The support $\tilde{S}(f) = \{Q_i\}$ maximum of the scalar product $\langle \omega P, Q_i \rangle$ is achieved on a generalized face $\Gamma^{(d)}_j$ of the polyhedron $\Gamma(f)$.

By $\mathbb{R}^3$ we denote the 3D real space, where by power exponents $Q$, and by $\mathbb{R}^*_3$ we denote the space dual (conjugate) to $\mathbb{R}^3$. We will denote points in $\mathbb{R}^*_3$ as $R = (r_1, r_2, r_3)$. Then we have the scalar product

$$\langle Q, R \rangle = q_1 r_1 + q_1 r_2 + q_3 r_3.$$

Each face $\Gamma^{(d)}_j$ corresponds to its normal cone $\mathbb{U}^{(d)}_j$

$$\mathbb{U}^{(d)}_j = \left\{ R : \begin{array}{l} \langle Q', R \rangle = \langle Q'', R \rangle, Q', Q'' \in \Gamma^{(d)}_j, \\ \langle Q', R \rangle > \langle Q'', R \rangle, Q'' \in \Gamma \setminus \Gamma^{(d)}_j \end{array} \right\}.$$

Thus, normal cone $\mathbb{U}^{(2)}_j$ of the face $\Gamma^{(2)}_j$ is a ray spanned on the exterior normal $N_j$ of the face $\Gamma^{(2)}_j$, normal cone $\mathbb{U}^{(1)}_j$ of the edge $\Gamma^{(1)}_j$ is 2D angle spanned on rays $U^{(2)}_k$ and $U^{(2)}_l$, where $\Gamma^{(1)}_j = \Gamma^{(2)}_k \cap \Gamma^{(2)}_l$; normal cone $\mathbb{U}^{(0)}_j$ of the vertex
Γ_{j}(0) is a 3D angle spanned on exterior normals N_k of all 2D faces Γ_{k}(2) containing the vertex Γ_{j}(0) (see [2]).

Thus, selection of the truncated sums \( \hat{f}_{j}(d)(x, y) \) can be made by following method. First we compute the support \( \tilde{S}(f) \) of the initial sum \( f(x, y) \). Using support \( \tilde{S}(f) \), we compute the polyhedron \( \Gamma(f) \) of sum \( f(x, y) \), i.e. all its vertices \( \Gamma_{j}(0) \), edges \( \Gamma_{j}(1) \) and faces \( \Gamma_{j}(2) \). Next we compute their normal cones \( U_{j}(d) \) and select only such truncated equations \( \hat{f}_{j}(d)(x, y) = 0 \) for which the intersection \( U_{j}(d) \cap \{p_{3} \geq 0\} \neq \emptyset \). But truncated equations \( \hat{f}_{j}(d)(x, y) = 0 \) with \( p_{3} = 0 \) can be studied by algorithms of 2D Power Geometry. So 3D Power Geometry studies truncated equations \( \hat{f}_{j}(d)(x, y) = 0 \) with nonempty intersection \( U_{j}(d) \cap \{p_{3} > 0\} \).

Example 2 (continuation of Example 1). Polyhedron \( \Gamma(f) \) for equation \( P_{2}(4) \) has 4 following faces with exterior normals

\[
\begin{align*}
\Gamma_{1}(2) &= [Q_{1}, Q_{2}, Q_{3}], & N_{1} &= (2,1,3), \\
\Gamma_{2}(2) &= [Q_{1}, Q_{3}, Q_{4}], & N_{2} &= (2, -2,3), \\
\Gamma_{3}(2) &= [Q_{1}, Q_{2}, Q_{4}], & N_{3} &= (-1,0,-1), \\
\Gamma_{4}(2) &= [Q_{2}, Q_{3}, Q_{4}], & N_{4} &= (0,0,-1).
\end{align*}
\]

Only two of them, \( N_{1} \) and \( N_{2} \), have \( r_{3} > 0 \). Hence, all edges exept \( \Gamma_{6}(1) = [Q_{2}, Q_{4}] \) and all vertices \( \Gamma_{j}(0) \) have vectors \( R = (r_{1}, r_{2}, r_{3}) \) with \( r_{3} > 0 \) in their normal cones \( U_{j}(1) \) and \( U_{j}(0) \).

4. Power transformations

If the face \( \Gamma_{j}(d) \) has the normal \( N_{j} = (1,0,1) \) then the corresponding truncation \( \hat{f}_{j}(d)(x, y) = x^{q}g(y) \), where the differential sum \( g(y) \) contains \( y \) and its derivatives but does not contain \( x \). In that case the full sum \( f(x, y) \) can be written as \( f(x, y) = x^{q}g(y) + x^{q-r}h(x, y) \), where \( r > 0 \) and \( h(x, y) \) is a differential sum.

Remark 1

If \( y(x) \) is a solution to the equation \( g(y) = 0 \) with the property

\[ 0 < \varepsilon < |y(x)|, |y'(x)|, \ldots, |y^{(n)}(x)| < \varepsilon^{-1}, \]

then \( y(x) \) can be the asymptotic form of the solutions to the full equation \( (3) \). Here \( \varepsilon \) is a small real number. We call \( y(x) \) as formal asymptotic form.
Let the power transformation of variables $x, y \rightarrow u, v$:

$$y = x^\alpha v, \quad u = \frac{1}{\beta} x^\beta,$$  \hspace{1cm} (6)

transform $f(x, y)$ into $f^*(u, v)$: $f^*(u, v) = f(x, y)$.

**Theorem 2.** Let the face $\Gamma_i^{(d)}$ of $\Gamma(f)$ have the exterior normal $N_i = (n_1, n_2, n_3)$ with

$$n_1 \neq 0, \quad n_3 > 0,$$  \hspace{1cm} (7)

then the power transformation (6) with $\alpha = n_2/n_1, \beta = n_3/n_1$ transforms the truncation $\tilde{f}_i^{(d)}(x, y)$ of $f(x, y)$ into the truncation

$$\tilde{f}_i^{*(d)}(u, v) = u^q g(v)$$

of $f^*(u, v)$, corresponding to the face $\Gamma_i^{*(d)}$ of $\Gamma(f^*)$ with the exterior normal $N_i^* = (1,0,1)$. Here $\tilde{f}_i^{*(d)}(u, v)$ equals $\tilde{f}_i^{(d)}(x, y)$ after substitution

$$w^{[\alpha + l(\beta - 1)]/\beta} \frac{dv}{dx^l}$$

instead of $y^{(l)} = d^l y^1 dx^l$.

So, if $v = \varphi(u)$ is a solution to the equation $g(v) = 0$ and $|\varphi(u)|$ is bounded from zero and infinity as $|y|$ in (5), then the initial equation $f(x, y) = 0$ can have a solution with the asymptotic form

$$y \sim x^\alpha \varphi(x^\beta / \beta), \quad x^\omega \rightarrow \infty.$$

Herewith the power transformation (6) induces the following formulas for derivatives:

$$y' = x^{\alpha + \beta - 1} \dot{v} + \alpha x^{\alpha - 1} v,$$
$$y'' = x^{2\beta + \alpha - 2} \ddot{v} + (2\alpha + \beta - 1)x^{\beta + \alpha - 2} \dot{v} + \alpha(\alpha - 1)x^\alpha v,$$  \hspace{1cm} (8)

where $\dot{v} = dv/du$.

**Theorem 3.** Let an equation of order $n$

$$g(v) + \sum_{j=1}^{m} h_j(v) u^{-j} = 0$$  \hspace{1cm} (9)

have a solution of the form

$$v = w + \sum_{j=1}^{\infty} b_j(w) u^{-j},$$  \hspace{1cm} (10)
where \( w = w(u) \) is the solution to the truncated equation

\[
\frac{\partial g}{\partial v} \bigg|_{v=w} = 0
\]

(11)

with the property

\[
0 < \varepsilon < |w|, \left| \frac{dw}{du} \right|, \ldots, \left| \frac{d^n w}{du^n} \right| < \frac{\varepsilon}{\varepsilon} < \infty.
\]

Then \( b_j(w) \) satisfies the linear equation

\[
\mathcal{L}(u)b_j(w) + \theta_j(w) = 0,
\]

(12)

where \( \mathcal{L}(u) = \frac{\partial g}{\partial v} \bigg|_{v=w} \), \( \theta_j(w) \) is a polynomial on \( w^{(l)} \) depending on \( g(w), h_i(w) \) and \( b_i^{(l)}(w) \) for \( i < j \) and \( l = 0,1,2,\ldots,n \). \( \delta g/\delta v \) is the first variation.

Solution \( v = \psi(u) \) to the transformed equation \( f^*(u,v) = 0 \) is expanded into series (10) with integer \( j \) only if the transformed equation \( f^*(u,v) = 0 \) divided by \( u^q \) has form (9) with integer \( j \). In that case, solutions \( v = w(u) \) to the truncated equation \( g(v) = 0 \) are suitable asymptotic forms for continuation by power expansion (10) and corresponding normal \( N_i \) is also suitable.

External normal \( N_i = (n_1,n_2,n_3) \) to 2D face \( \Gamma_i^{(2)} \) is unique up to positive scalar factor. Hence, power transformation (6) of Theorem 2 is unique and we must only check that the transformed equation has form (9) with integer \( j \). The external normal \( N = (n_1,n_2,n_3) \) to 1D edge \( \Gamma_i^{(1)} \) belongs to the normal cone \( U_i^{(1)} \). Hence, in the cone \( U_i^{(1)} \) we must select suitable vectors \( N \) with mentioned property of integer \( j \). Things for a vertex \( \Gamma_j^{(0)} \) are the same, but usually solutions \( v = w(u) \) to corresponding equation \( g(v) = 0 \) are so simple, that do not give interesting expansion.

Let \( S(f) = \{Q_1,\ldots,Q_M\} \), \( \tilde{S}(f_j^{(d)}) = \{Q_1,\ldots,Q_L\} \), \( 0 < L < M \), \( N = (n_1,n_2,n_3) \subset U_i^{(d)} \) and \( n_1 \neq 0, n_3 > 0 \). Denote

\[
\hat{Q}_l = Q_{L+l} - Q_1, \quad l = 1,\ldots,M-L
\]

and \( \hat{N} = (n_1/n_3, n_2/n_3, 1) \).

**Theorem 4.** The transformed equation (9) has the property of integer \( j \) iff all numbers

\[
-\left\langle \hat{N}, \hat{Q}_l \right\rangle, \quad l = 1,\ldots,M-L
\]

are natural.
There are 8 essentially different polyhedrons for Painlevé equations $P_1$–$P_5$. Each of them has exactly one 2D face which truncated equation has elliptic solutions. It was shown that all those elliptic asymptotic forms are suitable. Among 8 polyhedrons only 3 have an edge which truncated equation has elliptic solutions. These are $P_1$, $P_2$ and $P_4$. No truncated equations corresponding to vertices of these 8 polyhedrons have elliptic solutions.

Example 3 (continuation of examples 1, 2). Polyhedron $\Gamma(f)$ of equation $P_2$ has edge $\Gamma^{(1)}_1 = [Q_1, Q_2]$ with truncated equation $f^{(1)}_1(x, y) \overset{\text{def}}{=} y'' + 2y^3 = 0$. Its first integral is
\[ y'^2 = y^4 + C_0 \overset{\text{def}}{=} P(y), \quad (13) \]
where $C_0$ is arbitrary constant. If $C_0 \neq 0$, solutions to equation (13) are elliptic functions. The same will be true after any power transformation (6). Let us apply Theorem 4 to the edge $\Gamma^{(1)}_1$. The edge $\Gamma^{(1)}_1 = \Gamma^{(2)}_1 \cap \Gamma^{(2)}_3$. So normal cone $U^{(1)}_1$ is the conic hull of two normals $N_1 = (2, 1, 3)$ and $N_3 = (-1, 0, -1)$, i.e. up to positive scalar factor, vectors $N \in U^{(1)}_1$ have the form
\[ N = \kappa N_1 + (1 - \kappa)N_3 = (3\kappa - 1, \kappa, 4\kappa - 1), 0 < \kappa < 1. \]

Here $M = 4$, $L = 2$, $Q_1 = (3, 0, -2)$, $Q_2 = -Q_1 = (2, -1, -2)$,
\[ N = \left(\frac{3\kappa - 1}{4\kappa - 1}, \frac{\kappa}{4\kappa - 1}, 1\right). \]

Conditions of Theorem 4 are
\[ \langle N, Q_1 \rangle = \frac{3(3\kappa - 1)}{4\kappa - 1} - 2 = \frac{\kappa - 1}{4\kappa - 1} = -k, \]
\[ \langle N, Q_2 \rangle = \frac{2(3\kappa - 1)}{4\kappa - 1} - \frac{\kappa}{4\kappa - 1} - 2 = -\frac{3\kappa}{4\kappa - 1} = -l, \]
where $k$ and $l$ are natural numbers. Hence $\kappa = \frac{k + 1}{4k + 1} = \frac{l}{4l - 3}$, i.e. $l = k + 1$, $k = 1, 2, \ldots$

We can write $N' = (2 - k, k + 1, 3)$. Condition (7) of Theorem 2 means that $k \neq 2$. If $k = 1$, then $n_1 > 0$, i.e. $x \to \infty$; if $k > 2$, then $n_1 < 0$, i.e. $x \to 0$. So there is a countable set of suitable normals $N'$ to edge $\Gamma^{(1)}_1$. According to Theorem 2, here
\[ \alpha = \frac{k + 1}{2 - k}, \quad \beta = \frac{3}{2 - k} = \alpha + 1. \quad (14) \]
5. Computation of expansions

Below we consider the case when the truncated equation \( g(w) = 0 \) has the first integral of the form

\[
\dot{w}^2 = P(w) \overset{\text{def}}{=} \sum_{k=0}^{\lambda} p_k w^k, \quad p_k = \text{const} \in \mathbb{C}.
\]

(15)

Differentiating with respect to \( u \) and dividing by \( 2 \dot{w} \), we obtain

\[
\ddot{w} = \frac{1}{2} P'(w).
\]

(16)

Here and below the prime denotes the derivative with respect to \( w \).

Using the equations (15) and (16), any power series \( R \) of \( w \) and its derivatives \( d^l w/du^l \) can be written as the sum \( R = R^*(w) + \dot{w} R^{**}(w) \), where \( R^*(w) \) and \( R^{**}(w) \) are power series only of \( w \). Let \( b_j(w) = F_j(w) + \dot{w} G_j(w) \), where \( F_j \) and \( G_j \) are functions only of \( w \). Then, omitting the index \( j \), by (15) and (16), we obtain

\[
\dot{b} = F' \dot{w} + P G' + \frac{1}{2} P' G,
\]

(17)

\[
\ddot{b} = P F'' + \frac{1}{2} P' F' + \dot{w} \left( P G'' + \frac{3}{2} P' G' + \frac{1}{2} P'' G \right).
\]

Further derivatives of \( b \) does not need us here, because we consider only equations (9) of the second order. In our case

\[
\mathcal{L} b = \mathcal{F}(w) F(w) + \dot{w} \mathcal{G}(w) G(w).
\]

Thus, the equation (12) splits in two

\[
\mathcal{F}(w) F_j(w) + \theta_j^*(w) = 0,
\]

\[
\mathcal{G}(w) G_j(w) + \theta_j^{**}(w) = 0,
\]

(18)

where \( \theta_j(w) = \theta_j^*(w) + \dot{w} \theta_j^{**}(w) \). Note that in equations (18) differential operators \( \mathcal{F}(w) \) and \( \mathcal{G}(w) \) are operators on \( w \) and do not depend on \( u \). If polynomial \( P(w) \) in (15) does not have multiple roots and its degree \( \lambda \) is greater than one, i.e.

\[
\lambda > 1 \quad \text{and} \quad \Delta(P) \neq 0,
\]

where \( \Delta(P) \) is discriminant of the polynomial \( P(w) \), then solution \( w(u) \) to the truncated equation (11) is periodic (if \( \lambda = 2 \)), or elliptic (if \( \lambda = 3 \) or \( 4 \)) or hyperelliptic (if \( \lambda \geq 5 \) function).
Near some point $w = w^0$ we will compute asymptotic expansions of functions $F_j(w)$ and $G_j(w)$

$$
F_j = \sum_{i=-a_j}^{\infty} \varphi_{ji} \xi^i, \quad G_j = \sum_{i=-b_j}^{\infty} \gamma_{ji} \xi^i,
$$

where $\xi = w - w^0$ if $w^0 \neq \infty$ and $\xi = w^{-1}$ if $w^0 = \infty$. If initial equation (9) is a differential sum then according to Theorem 3.1 [3] coefficients $\varphi_{ji}$ and $\gamma_{ji}$ are either constants or polynomial of log $\xi$, i.e. expansions (19) are either power or power-logarithmic [3]. Moreover, according to Theorem 3.4 [3] (see proof in Theorem 1.7.2. [4]) power expansions (19) converge for small $|\xi|$.

If the solutions $F_j(w)$ and $G_j(w)$ to the system (18) have no branching, then they are also periodic or (hyper)elliptic functions. Finally, if for the sequence of equations (18) with $j = 1, 2, \ldots$, there exist solutions $F_j(w)$ and $G_j(w)$ without branching, the solutions to the equation (9) have a regular asymptotic expansion (10).

Let operators $\mathcal{F}^{-1}(w)$ and $\mathcal{G}^{-1}(w)$ be inverse to operators $\mathcal{F}(w)$ and $\mathcal{G}(w)$ respectively. Then the solutions of the equations (12) are of forms

$$
F_j(w) = -\mathcal{F}^{-1}(w)\theta_j^*(w), \quad G_j(w) = -\mathcal{G}^{-1}(w)\theta_j^{**}(w).
$$

In our case the initial ODE (9) has order two. Hence operators $\mathcal{F}(w)$ and $\mathcal{G}(w)$ are of the second order. Moreover, in our case factors of $F''$ in $\mathcal{F}$ and of $G''$ in $\mathcal{G}$ are the same. Denote it as $R(w)$. Singular points $w^0$ of operators $\mathcal{F}$ and $\mathcal{G}$ are roots of $R(w)$. Indeed $R(w) = r(w)P(w)$, where $r(w)$ is a simple polynomial. So roots $w^0$ of $r(w)$ and $w^0 = \infty$ will be singular points of operators $\mathcal{F}$ and $\mathcal{G}$, but roots $w^0$ of polynomial $P(w)$ different of singular points will be their subsingular points.

**Theorem 5.** If functions $\theta_j^*(w)$ and $\theta_j^{**}(w)$ are regular then the solutions to the equations (20) can have logarithmic branching only at infinity $w = \infty$ and at singular points of the operators $\mathcal{F}(w)$ and $\mathcal{G}(w)$ but they can have algebraic branching can be in singular and subsingular points only.

For the existence of a regular expansion (10) we need to prove the existence of a sequence of functions $F_j(w)$ and $G_j(w)$ that do not have branching. From other side, if it is shown that $F_j(w)$ or $G_j(w)$ have branching, then it proves the absence of regular expansion.

In [18, 22], for each polyhedron of the Painlevé equations, we selected suitable 2D faces, for each of them we wrote the equation (9), operators $\mathcal{F}(w)$ and $\mathcal{G}(w)$ and inverse ones $\mathcal{F}^{-1}(w)$ and $\mathcal{G}^{-1}(w)$. We found their singular points and the conditions on the parameters of the equation and on the solution $w(u)$.
under which the functions $F_1(w)$ and $G_1(w)$ do not have logarithmic branching, as well as the conditions under which at least one of these functions has such branching. It is wonder that for each Painlevé equation $P_l$ the operators $\mathcal{F}$ and $\mathcal{G}$ are expressed in the same way in terms of polynomial $P(w)$ and different cases distinguish only by this polynomial. At the same time, for all cases of faces $\Gamma_i^{(d)}$ of five Painlevé equations $P_1 – P_5$, there are only four different pairs of operators $\mathcal{F}$ and $\mathcal{G}$.

Singular point of operators $\mathcal{F}$ and $\mathcal{G}$ are $w_0 = \infty$ for $P_1 – P_5$ and $w_0 = 1$ for $P_3 – P_5$, and $w_0 = 1$ for $P_5$.

Our aim: to show existence or nonexistence of regular basic expansions by means of calculation of expansions (19) near the singular points.

6. Expansions for $P_2$

Details of calculation of expansions (10) will be explained for equation $P_2$

$$f(x, y) \overset{\text{def}}{=} -y'' + 2y^3 + xy + a = 0$$

and its truncated equation

$$f_1^{(1)}(x, y) \overset{\text{def}}{=} -y'' + 2y^3 = 0.$$ 

First, according to (14) and Theorem 2 we make power transformation $y = x^\alpha v$, $u = x^\beta / \beta$ (6) using formulas (8), and obtain equation $P_2$ (21) in the form (9)

$$g(v) + h_1(v)u^{-1} + h_2(v)u^{-2} + h_k(v)u^{-k} + h_{k+1}(v)u^{-k-1} = 0,$$

where

$$g(v) = -\dot{v} + 2v^3, \quad h_1(v) = -\frac{3\alpha}{\beta} \dot{v}, \quad h_2 = -\frac{\alpha(\alpha - 1)}{\beta^2}v,$$

$$h_k(v) = \beta^{-k}v, \quad h_{k+1}(v) = a\beta^{-k-1},$$

$$P(w) = w^4 + C_0, \quad C_0 \neq 0.$$ 

Here $\dot{v} = dv/du$, and $C_0$ is arbitrary complex constant.

Operators $-\mathcal{F}^{-1}$ and $-\mathcal{G}^{-1}$ (20) are

$$F_j = P^{1/2} \int \frac{1}{P^{3/2}} \int \theta_j^* \text{d}wdw, \quad G_j = \int \frac{1}{P^{3/2}} \int P^{1/2} \theta_j^* \text{d}wdw.$$

Here $r(w) \equiv 1$ (22) and singular points of operators (24) are only infinity. Let us introduce a function

$$H(w) = \int P^{-3/2}dw = \text{const} \cdot w^{-5} + \text{const} \cdot w^{-6} + \ldots$$
Here the integral is determined by mentioned asymptotic expansion near \( w = \infty \). Solutions of system (18) or (24) have 4 arbitrary constants \( C_1 \text{--} C_4 \):

\[
F = C_1 P^{1/2} + C_2 P^{1/2} H + F^0, \quad G = C_3 + C_4 H + G^0, \tag{26}
\]

where \( F^0 \) and \( G^0 \) are fixed solutions. Here expansions near \( w = \infty \) are

\[
P^{1/2} = \text{const} \cdot w^2 + \ldots, \quad P^{1/2} H = \text{const} \cdot w^{-3} + \ldots
\]

So we will assume that power expansion for \( F^0 \) does not contain terms \( \text{const} \cdot w^2 \) and \( \text{const} \cdot w^{-3} \) but expansion for \( G^0 \) does not contain terms \( \text{const} \) and \( \text{const} \cdot w^{-5} \). If it is necessary we can change constants \( C_1 \text{--} C_4 \). Now the functions \( F_j^0 \) and \( G_j^0 \) are unique and expansion (10) is called basic if there all \( b_j = F_j^0 + w G_j^0 \). Below we compute these basic expansion only.

**Lemma 1.** If \( C_1 = C_4 = 0 \), then solutions (26) to equations (24) for \( P_2 \) are regular in subsingular points (if \( \theta_j^* \) and \( \theta_j^{**} \) are also regular in them).

Let \( \theta_j^*(w) \) and \( \theta_j^{**}(w) \) be power series on decreasing power exponents of \( w \) and \( A_j w^\sigma_j \) and \( B_j w^\tau_j \) be their terms with maximal power exponents \( \sigma_j \) and \( \tau_j \) correspondingly, \( 0 \neq A_j, B_j \in \mathbb{C}, \sigma_j, \tau_j \in \mathbb{R} \). \( F_j \) and \( G_j \) contain \( \log w \), if

\[
\sigma_j = -1 \text{ or } 4 \text{ and } \tau_j = -3 \text{ or } 2. \tag{27}
\]

So these numbers are critical for operators \( \mathcal{F}^{-1} \) and \( \mathcal{G}^{-1} \).

We will compute \( \theta_j(w), \theta_j^*, \theta_j^{**} \) as functions of \( b_i = F_i + w' G_i, h_i \) for \( i < j \) and also will compute leading terms of \( F_j \) and \( G_j \), i.e. power exponents \( \sigma_j \) and \( \tau_j \) and constants \( A_j \) and \( B_j \).

For that we will use following expansions

\[
v = w + \frac{b_1}{u} + \frac{b_2}{u^2} + \frac{b_3}{u^3} + \frac{b_4}{u^4} + \ldots, \]

\[
\dot{v} = \dot{w} + \frac{\dot{b}_1}{u} + \frac{\dot{b}_2 - b_1}{u^2} + \frac{\dot{b}_3 - 2b_2}{u^3} + \frac{\dot{b}_4 - 3b_3}{u^4} + \ldots, \]

\[
\ddot{v} = \ddot{w} + \frac{\ddot{b}_1}{u} + \frac{\ddot{b}_2 - 2b_1}{u^2} + \frac{\ddot{b}_3 - 4b_2}{u^3} + \frac{\ddot{b}_4 - 6b_3 + 6b_2}{u^4} + \ldots, \]

\[
v^3 = w^3 + \frac{3w^2 b_1}{u} + \frac{3w^2 b_2}{u^2} + \frac{3w^2 b_3}{u^3} + \frac{3w^2 b_4}{u^4} + \frac{6w b_1 b_2}{u^2} + \frac{b_1^2}{u^3} + \frac{3w^2 b_4}{u^4} + 6wb_1 b_3 + \frac{3wb_2^2}{u^3} + \frac{3b_1^2 b_2}{u^4} + \ldots
\]
Case $k > 4$. According to (22), $h_1(v) = -\frac{3\alpha}{\beta} \dot{v}$, hence, $\theta_1^* = 0$, $\theta_1^{**} = -3\alpha/\beta$.

According to (23) and (24) we obtain $F_1 = 0$, $G_1 = \alpha / (2\beta) w^2 + \ldots$. Next,

$$
\theta_2 = 2b_1 + 6wb_1^2 - \frac{3\alpha}{\beta} b_1 - \frac{\alpha(\alpha - 1)}{\beta^2} w.
$$

Hence, according to (17)

$$
\theta_2^* = \left( 2 - \frac{3\alpha}{\beta} \right) \left( \frac{1}{2} P'G_1 + PG'_1 \right) + 6wG_1^2 P - \frac{\alpha(\alpha - 1)}{\beta^2} w
= -\frac{\alpha(\alpha + 2)}{2\beta^2} w + \ldots, \quad \theta_2^{**} = 0.
$$

According to (24), $F_2 = -\frac{\alpha(\alpha + 2)}{12\beta^2} w^{-1} + \ldots$, $G_2 = 0$. Next,

$$
\theta_3 = 4b_2 - 2b_1 + 2 \left( 6wb_1b_2 + b_1^3 \right) - \frac{3\alpha}{\beta} \left( b_2 - b_1 \right) - \frac{\alpha(\alpha - 1)}{\beta^2} b_1.
$$

Hence, $\theta_3^* = 0$, according to (17),

$$
\theta_3^{**} = \frac{\alpha + 4}{\beta} F'_2 - \frac{2(\alpha + 1)^2 - 3\alpha(\alpha + 1) + \alpha(\alpha - 1)}{\beta^2} G_1 + 12wG_1F_2 + 2PG_1^3 = -\frac{\alpha(\alpha + 2)}{6\beta^2} w^{-2} + \ldots.
$$

According to (24), $F_3 = 0$, $G_3 = \frac{\alpha(\alpha + 2)}{24\beta^2} w^{-4} + \ldots$.

Next,

$$
\theta_4 = 6b_3 - 6b_2 + 2 \left( 3wb_3^2 + 6wb_1b_3 + 3b_1^2b_2 \right) - \frac{3\alpha}{\beta} \left( b_3 - 2b_2 \right) - \frac{\alpha(\alpha - 1)}{\beta^2} b_2.
$$

Hence, according to (17)

$$
\theta_4^* = \frac{3(\alpha + 2)}{\beta} \left( \frac{1}{2} P'G_3 + PG'_3 \right) + 12wPG_1G_3 - \frac{(\alpha + 2)(\alpha + 3)}{\beta^2} F_2 + 6wF_2^2 + 6PF_2G_1^2 = 0w^{-1} + \ldots \overset{\text{def}}{=} A_4 w^{-1} + \ldots
$$

Here power exponent $-1$ of leading term in $\theta_4^*$ is critical for operator $\mathcal{F}^{-1}$ but $A_4 = 0$. Hence $F_4$ has not logarithmic branching.
Now we take in account terms $h_k(v)$ and $h_{k+1}(v)$ from (23). For $j = 4, \ldots, k-1$ power exponents $\sigma_j$ and $\tau_j$ for $F_j$ and $G_j$ are small enough to neglect them. So

$$v = w + \frac{b_1}{u} + \frac{b_2}{u^2} + \frac{b_k}{u^k} + \frac{b_{k+1}}{u^{k+1}} + \frac{b_{k+2}}{u^{k+2}} + \ldots$$

(28)

We can write corresponding expansions for $\dot{v}$, $\ddot{v}$, $v^3$. Then

$$\theta_k^* = \beta^{-k}w + \ldots, \quad \theta_k^{**} = 0,$$ hence $F_k = -\frac{1}{6\beta^k}w^{-1} + \ldots, G_k = 0.$

$$\theta_{k+1} = (k-1)b_k + 12wb_1b_k + \frac{a}{\beta^{k+1}} + \frac{b_1}{\beta^k}, \text{ hence }$$

$$\theta_{k+1}^* = \frac{a}{\beta^{k+1}} + \ldots \text{ and } F_{k+1} = -\frac{a}{4\beta^{k+1}}w^{-2} + \ldots,$$

$$\theta_{k+1}^{**} = (k-1)F'_k + 12wG_1F_k + \frac{1}{\beta^k}G_1 = -\frac{1}{3\beta^k}w^{-2} + \ldots \text{ and } G_{k+1} = \frac{1}{12\beta^k}w^{-4} + \ldots;$$

$$\theta_{k+2} = 2(k+1)b_{k+1}-k(k+1)b_k + 12wb_1b_{k+1} + 12wb_2b_k + 6b^2_1b_k - (k+1)\left(b_{k+1} - kb_k\right) - \frac{\alpha(\alpha + 1)}{\beta^2}b_k + \frac{1}{\beta^k}b_2.$$

Hence $\theta_{k+2} = (k+1)\left(\frac{1}{2}P'G_{k+1} + PG'_1\right) - \frac{\alpha(\alpha - 1)}{\beta^2}F_k + \frac{1}{\beta^k}F_2 + 12wPG_1G_{k+1} + 12wF_2F_k + 6PG_1^2F_k = 0 \cdot w^{-1} + \ldots,$

$$\theta_{k+2}^{**} = (k + 1)F'_{k+1} - \frac{\alpha(\alpha - 1)}{\beta^2}G_k + 12wG_1F_{k+1} + 12wF_2G_k + 6PG_1^2G_k = 0 \cdot w^{-3} + \ldots.$$

It means that $F_{k+2}$ and $G_{k+2}$ have not branching at $w = \infty$ and $\sigma_j < -1$ and $\tau_j < -3$ for $k + 2 < j < 2k$.

So we neglect $b_j$ for $j = k + 2, \ldots, 2k - 1$ and consider

$$v = w + \frac{b_1}{u} + \frac{b_2}{u^2} + \frac{b_k}{u^k} + \frac{b_{k+1}}{u^{k+1}} + \frac{b_{k+2}}{u^{k+2}} + \frac{b_{2k}}{u^{2k}} + \ldots$$

We have

$$\theta_{2k} = 6wb^2_k + \ldots$$

Hence, according to results after (28),

$$\theta_{2k}^* = 6wF^2_k + \ldots = \frac{6}{36\beta^{2k}}w^{-1} + \ldots = A_{2k}w^{-1} + \ldots,$$

where $A_{2k} = \frac{1}{6\beta^{2k}} \neq 0$, and $F_{2k}$ has the logarithmic branching, i.e. the regular expansion does not exist.

For $k = 4$, we must add $\beta^{-4}w$ to the computed value of $\theta_4^*$, but it does not change result on existence of logarithmic branching in $F_8$. 
Case $k = 3$ is close to the case $k \geq 4$ and it has branching in $F_6$.
Case $k = 1$ was calculated separately. It has not branching.
Case $k = 0$ corresponds to 2D face $\Gamma^{(2)}_1$. It has not branching.

Thus, for equation $P_2$ (21), basic formal expansions are regular for two suitable asymptotic forms with $k = 0$ and $k = 1$ when $x \to \infty$.

**Theorem 6.** For $P_2$, the regular basic families of formal power-elliptic expansions exist only for two suitable elliptic asymptotic forms with $k = 0$ and $k = 1$, i.e. when $x \to \infty$.

It is possible to prescribe power exponents $\sigma_j$ and $\tau_j$ of leading terms in $\theta_j^*$ and $\theta_j^{**}$. So we can compute such numbers $j^*$ and $j^{**}$, that $\sigma_j < -1$ for $j > j^*$ and $\tau_j < -3$ for $j > j^{**}$. Here $-1$ and $-3$ are smaller critical values (27) of operators $\mathcal{F}^{-1}$ and $\mathcal{G}^{-1}$. And it is enough to calculate $F_j$ and $G_j$ up to $j = \max(j^*, j^{**})$.

7. Nonbasic expansions for $P_2$

Basic expansions (10) were defined by formulas (24), (26) with $C_1 = C_2 = C_3 = C_4 = 0$. According to Lemma 1, condition $C_1 = C_4 = 0$ guarantees regularity of $F_j$ and $G_j$ in subsingular points. Now we want to study cases with nonzero $C_3$.

**Example 4.** Let us show that $C_3 \neq 0$ in $G_j$ gives the logarithmic branching in $w = \infty$ for $G_{j+2}$. For $j = 1$, we put $C_3 = A \neq 0$. According to formulas for case $k \geq 4$, we obtain

$F_1 = 0$, $G_1 = A + \frac{\alpha}{2\beta}w^{-2} + \ldots$,

$\theta_2^* = \frac{2 - \alpha}{2\beta} \left( \frac{1}{2} P'G_1 + PG_1' \right) + 6wG_1^2P - \frac{\alpha(\alpha - 1)}{\beta^2}w + \ldots$

$= \frac{2 - \alpha}{2\beta} 2w^3A + 6w^5 \left( A + \frac{\alpha}{2\beta}w^{-2} \right)^2 - \frac{\alpha(\alpha - 1)}{\beta^2}w + \ldots$

$= 6A^2w^5 + \frac{5\alpha + 2}{\beta}Aw^3 + \frac{\alpha^2 + 2\alpha}{2\beta^2}w + \ldots$

Hence, $F_2 = A^2w^3 - \frac{5\alpha + 2}{4\beta}Aw + \ldots$, $G_2 = F_3 = 0.$
Next,

$$\theta_3^{**} = \frac{\alpha + 4}{\beta} \left( 3A^2w^2 - \frac{5\alpha + 2}{4\beta} \right) - \frac{2}{\beta^2} \left( A + \frac{\alpha}{2\beta}w^{-2} \right)$$

$$+ 12wG_1F_2 + 2PG_1^3 + \ldots = \frac{\alpha + 4}{\beta} 3A^2w^2 - \frac{(\alpha + 4)(\alpha + 2)}{4\beta} A - \frac{2}{\beta^2} A$$

$$+ 12w \left( A + \frac{\alpha}{2\beta}w^{-2} \right) \left( A^2w^3 - \frac{5\alpha + 2}{4\beta} Aw \right) + 2w^4 \left( A + \frac{\alpha}{2\beta}w^{-2} \right)^3 + \ldots$$

Power exponent 2 is critical for \( G^{-1} \) (see (27)). Coefficient for \( w^2 \) in \( \theta_3^{**} \) is

$$-3(\alpha - 2)A^2.$$

It is equal to zero only for \( \alpha = 2 \), but \( \alpha = \frac{k + 1}{2 - k} \), i.e. \( k = 1 \). But \( k \geq 4 \), then \( G_3 \) has logarithmic branching.

8. Equation \( P_1 \)

$$f(x, y) \overset{\text{def}}{=} -y'' + 3y^2 + x = 0.$$  

Support \( \tilde{S}(f) \) consists of 3 points \( Q_1 = (-2,1,2), Q_2 = (0,2,0), Q_3 = (1,0,0). \) Its polyhedron \( \Gamma(f) \) is a triangle with normal \( N = (4,2,5) \). So the equation is its own truncation. The edge \( \Gamma^{(1)} = [Q_1, Q_2] \) of the triangle \( \Gamma \) corresponds to the truncated equation

$$\tilde{f}^{(1)}_1(x, y) \overset{\text{def}}{=} -y'' + 3y^2 = 0,$$

which has the first integral

$$y'^2 = 2(y^3 + C_0)$$

with elliptic solutions.

Suitable normals \( N \) to the edge \( \Gamma^{(1)} \) are \( N_k = (4-k,2(k+1),5), k = 1,2,\ldots \) and \( n_1 \neq 0 \) if \( k \neq 4 \). Here \( \alpha = \frac{2(k + 1)}{4 - k}, \beta = \frac{5}{4 - k} \) and \( \alpha = 2(\beta - 1), \gamma = 2\beta = \alpha + 2 \), the transformed equation is

$$-\dot{v} + 3v^2 - \frac{5\alpha}{\gamma} \dot{u}u^{-1} - \frac{4\alpha(\alpha - 1)}{\gamma^2} vu^{-2} + 2^k \gamma^{-k} u^{-k} = 0,$$

\( P = 2(w^3 + C_0) \), operators \( F^{-1} \) and \( G^{-1} \) are again (24) and \( r(w) = 1 \) (22). Hence there is only one singular point \( w^0 = \infty \) and Lemma 1 is true for \( P_1 \). Here \( H(w) = \text{const} \cdot w^{-7/2} + \ldots \) and integral critical numbers are \( \sigma_j = -1 \) and
\( \tau_j = 1 \). Formulas (24)–(26) again define basic expansions. If \( k > 6 \) then \( F_1 = 0, \ G_1 = \alpha \gamma w - 1 + \ldots, F_2 = \frac{\alpha(\alpha - 8)}{6\gamma^2} + \ldots, G_2 = F_3 = 0, \ G_3 = \frac{\alpha(\alpha + 4)}{3\gamma^3} w^{-2} + \ldots, F_4 = -\frac{\alpha(\alpha + 4)(\alpha^2 + 24\alpha + 48)}{60\gamma^4} w^{-1} + \ldots, G_4 = F_5 = 0, \ G_5 = \frac{\alpha(\alpha + 4)(3\alpha^3 + 56\alpha^2 + 200\alpha + 192)}{180\gamma^5} w^{-3} + \ldots, \theta_6^* = 0 \cdot w^{-1} + \ldots \) defined \( \theta_j^* = \frac{2^k}{\gamma^k} + \ldots, F_k = -\frac{2^k}{5\gamma^k} w^{-1} + \ldots, G_k = F_{k+1} = 0, G_{k+1} = \frac{(k + 11)2^k}{75\gamma^k} w^{-3} + \ldots, \theta_{k+2}^* = 0 \cdot w^{-1} + \ldots, \sigma_j < -1, \tau_j < 1 \) for \( j > k + 2 \) and the regular expansion exists. If \( 4 < k < 7 \), then the regular expansion exists, the same is true for \( k = 1,2,3 \). Case \( k = 0 \) corresponds to 2D face and to other \( P = 2 (w^3 + w + C_0) \), but \( A_6 = 0 \). Thus, equation \( P_1 \) has regular basic families of elliptic expansions corresponding to all suitable asymptotic forms. Thus we have

**Theorem 7.** To each suitable elliptic asymptotic form of \( P_1 \) there corresponds the basic family of formal power-elliptic expansions, which is regular.

### 9. Equation \( P_3 \)

\[
f(x, y) \overset{\text{def}}{=} -x y y'' + x y'^2 - y y' + a y^3 + b y + c x y^4 + d x = 0 \tag{29}
\]

has 3 different polyhedrons depending on values of coefficients \( a, b, c, d \) \[18,22\].

**Case \( cd \neq 0 \), Figure 2.**

Here only one truncated equation

\[-x y y'' + x y'^2 + c x y^4 + d x = 0\]

corresponding to the distinguished 2D face in Figure 2 has elliptic solutions. Here the power transformation (6) is identical.

The equation (29) with \( cd \neq 0 \) is of the form (9) with \( m = 1 \), where

\[
g(v) \overset{\text{def}}{=} -v v' + v^2 + c v^4 + d = 0, \quad h_1 = -v v' + a v^3 + b v,
\]

\[
P(w) = c w^4 + C_0 w^2 - d, \quad \Delta(P) = -c d (C_0^2 + 4 c d)^2 / 16 \neq 0.
\]

Solutions to the equations (18) are of the form

\[
F_j = P^{1/2} \int \frac{w^2}{P^{3/2}} \int \frac{\theta_1^*}{w^3} d w d w, \quad G_j = \int \frac{w^2}{P^{3/2}} \int \frac{P^{1/2} \theta_1^{**}}{w^3} d w d w. \tag{30}
\]
Here $r(w) = w^2$, so there are 2 singular points $w^0 = \infty$ and $w^0 = 0$. This is true for all cases of $P_3$. Near the singular point $w^0 = \infty$, $H(w) = \int \frac{w^2}{P^{3/2}} dw = \text{const} \cdot w^3 + \ldots$. So $P^{1/2} = \text{const} \cdot w^2 + \ldots$, $P^{1/2} H = \text{const} \cdot w^{-1} + \ldots$ and expansions for $F_j^0$ and $G_j^0$ do not contain terms $\text{const} \cdot w^2$, $\text{const} \cdot w^{-1}$ and $\text{const} \cdot w^0$, $\text{const} \cdot w^{-3}$ correspondingly. Critical numbers for $\theta_j^*$ and $\theta_j^{**}$ are 2, 5 and 0, 3 correspondingly. Moreover, $\theta_2^* = 0 \cdot w^2 + \ldots$, $\theta_2^{**} = 0 \cdot w + \ldots$ and $\sigma_j < 2$, $\tau_j < 0$ for $j > 2$. So expansion has not logarithmic branching at $w = \infty$.

Near the singular point $w^0 = 0$ we have $H^0(w) = \int \frac{w^2}{P^{3/2}} dw = \text{const} \cdot w^3 + O(w^4)$. Here we have 4 constants $C_1^0, \ldots, C_4^0$ and basic expansion if all $C_i^0 = 0$. Here Lemma [1] is correct for $P_3$. 

Figure 2. 3D support $\tilde{S}(f)$ and polyhedron $\Gamma(f)$ of equation $P_3$ with all $a, b, c, d \neq 0$. The grey face is $\Gamma_1^{(2)}$. All dotted lines are in the plane $q_1$, $q_2$, they show projections of $\Gamma(f)$ on the plane $(q_1, q_2)$. 

Here Lemma 1 is correct for $P_3$. 

2

\[ q_3 \]

\[ Q_1 \]

\[ Q_2 \]

\[ 0 \]

\[ 1 \]

\[ q_1 \]

\[ q_2 \]

\[ Q_6 \]

\[ Q_5 \]

\[ Q_4 \]
Condition C. \( \int_0^\infty \frac{w^2}{P^{3/2}} dw = 0. \)

**Theorem 8.** If the Condition C is satisfied then basic expansions for \( P_3 \) are regular.

**Case c = 0, ad ≠ 0.** After the power transformation \( y = x^{1/3} v, u = \frac{3}{2} x^{2/3} \), the equation (29) with \( c = 0 \) takes the form (9) with \( m = 1 \), where \( g(v) = -v^2 + 3v^3 \) \( h_1 = \frac{3}{2} bv - v^2 \), \( P(w) = 2aw^3 + C_0w^2 \) \( \Delta(P) = 4d (C_0^3 - 27a^2 d) \neq 0. \)

Formula (30) is valid here. At \( w = \infty \), \( \theta_j^* \) and \( \theta_j^{**} \) have critical number 2. \( \theta_2 = 0 \cdot w^2 + \ldots \) and orders of \( \theta_j^*, \theta_j^{**} \) are less than 2 for \( j > 2 \).

The same is at \( w^0 = 0 \). Thus, here formal basic expansion is regular. Lemma 1 and Theorem 8 are true.

**Case c = d = 0, ab ≠ 0.** After the power transformation \( y = v, u = 2x^{1/2} \), equation (29) with \( c = d = 0 \) takes the form (9) with \( m = 1 \), where \( g(v) = -v^2 + 3v^3 + bv \) \( h_1 = -v^2/2 \), \( P(w) = 2 \left(Aw^3 + C_0w^2 - bw \right) \), \( \Delta(P) = 4b^2 (C_0^3 + 16ab) \neq 0. \)

At \( w^0 = \infty \) critical values for \( \theta_j^* \) and \( \theta_j^{**} \) are 2, \( \theta_j^* = 0 \cdot w^2 + \ldots \), \( \sigma_j, \tau_j < 2 \) for \( j > 2 \). So here basic expansion has not branching.

The same is at \( w^0 = 0 \). Lemma 1 and theorem 8 are true.

Each of 3 polyhedrons has exactly one 2D face corresponding to a truncated equation with elliptic solutions \([17, 18, 22]\). They have different first integrals \( (\ddot{w})^2 = P(w) \), but common operators \( \mathscr{F}^{-1} \) and \( \mathscr{G}^{-1} \) with singularities in two points \( w = 0 \) and \( w = \infty \).

**10. Equation \( P_4 \)**

\[ f(x, y) \overset{\text{def}}{=} -2yy'' + y'^2 + 3y^4 + 8xy^3 + 4 \left(x^2 - a\right) y^2 + 2b = 0. \]

If complex parameters \( a, b \neq 0 \), its support \( \mathcal{S}(f) \) consists of 6 points, polyhedron \( \Gamma(f) \) is a tetrahedron and has one 2D face \( \Gamma^{(2)}_1 \) and one edge \( \Gamma^{(1)}_1 \) with truncated equations

\[ f_1^{(2)} \overset{\text{def}}{=} -2yy'' + (y')^2 + 3y^4 + 8xy^3 + 4x^2y^2 = 0, \]

\[ f_1^{(1)} \overset{\text{def}}{=} -2yy'' + (y')^2 + 3y^4 = 0, \]
having elliptic solutions [18,19,22]. Normal to \( \Gamma^{(2)}_1 \) are \( N_k = (1-k,k+1,2), k = 2,3, \ldots \) After power transformation (6) with \( \alpha = \frac{k+1}{1-k}, \beta = \frac{2}{1-k} = \alpha + 1 \) we obtain the equation (9) with \( m = 6 \)

\[-2v\dot{v} + v^2 + 3v^4 - \frac{4\alpha}{\beta}v^2u^{-1} + \frac{\alpha(2 - \alpha)}{\beta^2}v^2u^{-2} + \frac{8}{\beta^3}v^3u^{-k} - \frac{4a}{\beta^{k+1}}v^2u^{-(k+1)} + \]

\[+ \frac{4}{\beta^{2k}}v^2u^{-2k} + \frac{2b}{\beta^{2(k+1)}}u^{-2(k+1)} = 0, \quad P(w) = w^4+C_0w, \quad C_0 \neq 0, \quad k = 2,3, \ldots \]

Here solutions to equations (18) are

\[F_j = \frac{1}{2}P^{1/2} \int \frac{w}{P^{3/2}} \int \frac{\theta^*_j}{w^2}dw dw, \quad G_j = \frac{1}{2} \int \frac{w}{P^{3/2}} \int \frac{P^{1/2}\theta^*_{j}}{w^2}dw dw, \]

\[r(w) = w \text{ [22]}, \text{ so there are two singular points } w^0 = \infty \text{ and } w^0 = 0. \text{ Near } w^0 = \infty H = \int \frac{w}{P^{3/2}} \int dw = \text{const} \cdot w^{-4} + \ldots \text{ Critical numbers for } \theta^* \text{ and } \theta^{**} \text{ are } 1,5 \text{ and } -1,3 \text{ correspondingly. If } k > 3, \quad F_1 = 0, \quad G_1 = \frac{\alpha}{2\beta}w^{-2} + \ldots, \quad F_2 = \]

\[\frac{-\alpha(\alpha + 2)}{12\beta^2}w^{-1} + \ldots, \quad G_2 = 0, \quad F_3 = 0, \quad \theta_3^{**} = 0 \cdot w^{-1} + \ldots \]

Now we compute expansion of the form (28). Then \( F_k = -\frac{1}{\beta^k} + \ldots, G_k = 0, \)

\[F_{k+1} = \frac{2a}{3\beta^{k+1}}w^{-1} + \ldots \quad G_{k+1} = \frac{1}{3\beta^{k+1}}w^{-3} + \ldots, \quad \theta^*_{k+2} = \frac{4\alpha(2\alpha - 1)}{\beta^{k+2}}w + \ldots, \]

\[\theta^{**}_{k+2} = 0 \cdot w^{-1} + \ldots \text{ Thus, } A_{k+2} = \frac{4\alpha(2\alpha - 1)}{\beta^{k+2}} = 0 \text{ only if } 2\alpha - 1 = 0, \text{ i.e. } k = -1/3, \text{ that is impossible. Thus, } F_{k+2} \text{ has logarithmic branching and the regular basic expansion is absent. The same is true for } k = 3,2 \text{ and for } k = 0, \text{ when } P = w^4 + 4w^3 + 4w^2 + C_0w. \]

11. Equation \( P_5 \)

\[f(x, y) \overset{\text{def}}{=} x^2y(y-1)y'' + x^2 \frac{3y-1}{2}y' - xy(y-1)y' +
\]

\[+ (y-1)^3(ay^2 + b) + cy^2(y-1) + dx^2y^2(y+1) = 0, \]

where \( a, b, c, d \) are complex parameters, has two different polyhedrons depending on values of parameter \( d \) [20,22]. Each of the polyhedrons has only one 2D face with elliptic solutions.
Case d ≠ 0. Here transformation (6) is identical \( y = v, \ x = u \). So, in equation (9) \( m = 2 \),
\[
    g(v) = -v(v - 1)\ddot{v} + (3v - 1)\dot{v}^2/2 + dv^2(v + 1),
    
    h_1 = -v(v - 1)\dot{v} + cv^2(v - 1), \quad h_2 = (v - 1)^3(av^2 + b),
    
    P = -2dw \left[ C_0(w - 1)^2 + w \right], \quad \Delta(P) = (2d)^4C_0^2(1 - 4C_0) \neq 0.
\]

Solutions to equations (18) are
\[
    F_j = P^{1/2} \int \frac{w(w - 1)^2}{P^{3/2}} \int \frac{\theta_j^*}{w^2(w - 1)^3} dw dw,
    
    G_j = \int \frac{w(w - 1)^2}{P^{3/2}} \int \frac{P^{1/2}\theta_j^{**}}{w^2(w - 1)^3} dw dw.
\]
(31)

Here \( r(w) = w(w - 1)^2 \) [22], so singular points are \( w^0 = \infty, 0, 1 \) Near the singular point \( w^0 = \infty \)
\[
    H = \int \frac{w(w - 1)^2}{P^{3/2}} dw = \text{const} \cdot w^{-1/2} + \ldots
\]
critical numbers for \( \theta_j^* \) and \( \theta_j^{**} \) are 4 and 3 correspondingly. If \( a \neq 0 \), then \( \theta_2^* \) contains the term \(-3aw^4\) and \( F_2 \) has logarithmic branching. If \( a = 0 \), then \( \sigma_j < 4 \) and \( \tau_j < 3 \) for all \( j > 0 \). Thus, the basic expansion is regular. Similarly basic expansions are regular near \( w^0 = 0 \) iff \( b = 0 \) and near \( w^0 = 1 \) without restrictions.

Condition D.
\[
    \int_0^1 \frac{w(w - 1)^2}{P^{3/2}} dw = \int_1^\infty \frac{w(w - 1)^2}{P^{3/2}} dw = 0.
\]

Theorem 9. If in equation \( P_5 \) with \( d \neq 0 \) and with \( a = b = 0 \) Condition D is fulfilled then basic expansions are regular. If one of these conditions is violated then all basic expansions are nonregular.

Case d = 0, c ≠ 0. After the change \( y = v, \ u = 2x^{1/2} \), equation \( P_5 \) takes the form (9) with \( m = 2 \), where
\[
    g(v) = -v(v - 1)\ddot{v} + \frac{3v - 1}{2}\dot{v}^2 + cv^2(v - 1),
    
    h_1 = -v(v - 1)\dot{v}, \quad h_2 = (v - 1)^3(av^2 + b),
    
    P = -2cw(w - 1) \left[ C_0(w - 1) + 1 \right], \quad \Delta(P) = (C_0 - 1)^2 \neq 0, C_0 \neq 0.
\]

Formulas (31) are again valid. Here basic expansions near \( w^0 = \infty \) are regular iff \( a = 0 \), near \( w^0 = 0 \) iff \( b = 0 \) and near \( w = 1 \) are always non regular.
12. Equation $P_6$

in generic case has polyhedron $\Gamma$ with ten 2D faces $\Gamma_i^{(2)}$, but all external normals to them $N = (n_1, n_2, n_3)$ do not satisfy conditions (7) $n_1 \neq 0$, $n_3 > 0$. Moreover, all edges $\Gamma_i^{(1)}$ have no suitable normals. The same is true for degenerate cases.

13. Summary

Thus, all basic expansions are regular for $P_1$ without additional restrictions (Theorem 7), for $P_2$ if $x \to \infty$ (Theorem 6), for $P_3$ under Condition C (Theorem 8) and for $P_5$ with $a = b = 0$ and $d \neq 0$ under Condition D (Theorem 9).

As next step it is necessary to study convergence of found regular formal power-elliptic expansions.

References


Contents

1 Universal Nonlinear Analysis ........................................... 3
2 Introduction ...................................................................... 3
3 3D Power Geometry .......................................................... 5
4 Power transformations ....................................................... 9
5 Computation of expansions ............................................... 13
6 Expansions for $P_2$ .......................................................... 15
7 Nonbasic expansions for $P_2$ ............................................ 19
8 Equation $P_1$ .................................................................. 20
9 Equation $P_3$ .................................................................. 21
10 Equation $P_4$ ................................................................. 23
11 Equation $P_5$ ................................................................. 24
12 Equation $P_6$ ................................................................. 26
13 Summary ....................................................................... 26
References ..................................................................... 26