ISSN 2071-2898 (Print)
ISSN 2071-2901 (Online)

Dudnikova T.V.

Infinite non homogeneous chain of harmonic oscillators: Large-time behavior of solutions

Recommended form of bibliographic references: Dudnikova T.V. Infinite non homogeneous chain of harmonic oscillators: Large-time behavior of solutions // Keldysh Institute Preprints. 2017. No. 109. 35 p. doi:10.20948/prepr-2017-109-e
URL: http://library.keldysh.ru/preprint.asp?id=2017-109&lg=e

О р д е н а   Л е н и н а
ИНСТИТУТ ПРИКЛАДНОЙ МАТЕМАТИКИ
имени М. В. КЕЛДЫША
Р о с с и й с к о й   а к а д е м и и   н а у к

T. V. Dudnikova

Infinite non homogeneous chain of harmonic oscillators: Large-time behavior of solutions

Москва — 2017

Дудникова Т.В.

Бесконечная неоднородная цепочка гармонических осцилляторов: Поведение решений при больших временах

Рассматривается задача Коши для бесконечной одномерной цепочки гармонических осцилляторов с различными массами. Изучается поведение решений при больших временах, и выводятся дисперсионные оценки для них.

Ключевые слова: бесконечная одномерная неоднородная цепочка гармонических осцилляторов, задача Коши, преобразование Фурье–Лапласа, ряды Пьюзо, дисперсионные оценки

Работа выполнена при поддержке гранта РНФ (грант N. 14-21-00025)

Tatiana Vladimirovna Dudnikova

Infinite non homogeneous chain of harmonic oscillators: Large-time behavior of solutions

We consider the Cauchy problem for infinite one-dimensional chain of harmonic oscillators with different masses. We study the large time behavior of solutions and derive the dispersive bounds for them.

Key words: infinite one-dimensional non homogeneous chain of harmonic oscillators, Cauchy problem, Fourier–Laplace transform, Puiseux expansion, dispersive estimates

This work was done with the financial support from the Russian Science Foundation (grant N. 14-21-00025).

Contents

1. Introduction

We consider the infinite chain of oscillators having harmonic nearest-neighbor interactions. We assume that the oscillators have identical masses $m_x = m_-$ in positions $x < 0$ and $m_x = m_+$ in positions $x > 0$ with some positive numbers $m_\pm$. The displacement of the x-th oscillator from its equilibrium position obeys the following equations:

$$m_+\ddot{u}(x,t) = (\nu_+^2\Delta_L - \kappa_+^2)u(x,t), \quad x \geq 1, \quad t > 0 \tag{1.1}$$
$$m_0\ddot{u}(0,t) = \nu_+^2(u(1,t)-u(0,t))+\nu_-^2(u(-1,t)-u(0,t))-\kappa_0^2u(0,t)-\gamma\dot{u}(0,t) \tag{1.2}$$
$$m_-\ddot{u}(x,t) = (\nu_-^2\Delta_L - \kappa_-^2)u(x,t), \quad x \leq -1, \quad t > 0 \tag{1.3}$$

with the initial data

$$u(x,0) = u_0(x), \quad \dot{u}(x,0) = v_0(x), \quad x \in \mathbb{Z}. \tag{1.4}$$

Here $u(x,t) \in \mathbb{R}$, $m_\pm, m_0, \nu_\pm > 0$, $\kappa_\pm, \kappa_0, \gamma \geq 0$, Δ_L denotes the second derivative on $\mathbb{Z}$:

$$\Delta_L u(x) = u(x+1) - 2u(x) + u(x-1), \quad x \in \mathbb{Z}.$$

If $\gamma = 0$, then formally the system (1.1)–(1.3) is Hamiltonian with the Hamiltonian functional

$$\begin{aligned} \mathrm{H}(u,v) &= \frac{1}{2}\sum_{x\in\mathbb{Z}}\Big(\frac{|v(x)|^2}{m_x} + \nu_x^2|u(x+1)-u(x)|^2 + \kappa_x^2|u(x)|^2\Big) \\ &= \frac{1}{2}\sum_{x\geq 1}\Big(\frac{|v(x)|^2}{m_+} + \nu_+^2|u(x-1)-u(x)|^2 + \kappa_+^2|u(x)|^2\Big) \\ &\quad +\frac{1}{2m_0}|v(0)|^2 + \frac{1}{2}\kappa_0^2|u(0)|^2 \\ &\quad +\frac{1}{2}\sum_{x\leq -1}\Big(\frac{|v(x)|^2}{m_-} + \nu_-^2|u(x+1)-u(x)|^2 + \kappa_-^2|u(x)|^2\Big), \end{aligned} \tag{1.5}$$

where $v(x,t) = m_x\dot{u}(x,t)$ is momentum of x-th oscillator, $\dot{u}(x,t)$ is its velocity, m_x its mass; $\kappa_x = \kappa_+$ for $x \geq 1$, $\kappa_x = \kappa_-$ for $x \leq -1$, $\nu_x = \nu_+$ for $x \geq 0$, $\nu_x = \nu_-$ for $x \leq -1$. We divide (1.1) by m_+, (1.2) by m_0 and (1.3) by m_- and rename the rest constants. Then, without loss of generality, we can put $m_+ = m_0 = m_- = 1$. Also, we assume that $\kappa_- \leq \kappa_+$.

On the constants $\gamma, \kappa_\pm, \kappa_0, \nu_\pm$ of the system we impose condition **C** or $\mathbf{C}_0$. In formulating these conditions we consider various cases of mutual disposition of points $\kappa_\pm$ and $a_\pm = \sqrt{4\nu_\pm^2 + \kappa_\pm^2}$ (see also page 34) and in each case we impose restrictions on γ and κ_0. At first, we introduce the following notations.

$$\bar\kappa := \left((\kappa_-^2+\kappa_+^2)/2\right)^{1/2} \tag{1.6}$$

$$K_\pm(\omega) := \bar\kappa^2 + \frac{1}{2}\sqrt{\omega^2-\kappa_\pm^2}\sqrt{\omega^2-a_\pm^2}, \quad |\omega|\ge a_\pm > \kappa_\pm \tag{1.7}$$

$$K_0(\omega) := \bar\kappa^2 - \frac{1}{2}\sqrt{\kappa_+^2-\omega^2}\sqrt{a_+^2-\omega^2}, \quad |\omega|\le \kappa_+ \quad (\text{if} \ \ \kappa_+>0) \tag{1.8}$$

$$F_\pm(\omega) := \frac{1}{2\omega}\sqrt{\omega^2-\kappa_\pm^2}\sqrt{a_\pm^2-\omega^2}, \quad \kappa_\pm\le\omega\le a_\pm \tag{1.9}$$

$$F(\omega) := F_-(\omega)+F_+(\omega), \quad \kappa_+\le\omega\le a_m, \ \ a_m:=\min(a_-,a_+) \tag{1.10}$$

$$\gamma_1 := F_-(\kappa_+), \ \ \gamma_2 := F_-(a_+), \ \ \gamma_3 := F_+(a_-) \tag{1.11}$$

$$\gamma_\pm^{cr} := \max F_\pm(\omega) = F_\pm(\sqrt{\kappa_- a_-}) = \frac{a_\pm-\kappa_\pm}{2} \tag{1.12}$$

$$F_{\max} := \max F(\omega), \ \ F_{\min} := \min F(\omega) \tag{1.13}$$

For $\gamma\in(0,\gamma_-^{cr})$ ($\gamma\in(0,\gamma_+^{cr})$), introduce numbers $P_\pm\in(\kappa_-,a_-)$ (numbers $Q_\pm\in(\kappa_+,a_+)$, resp.) as follows

$$P_\pm^2 := \kappa_-^2+p_\pm, \quad p_\pm := 2(\nu_-^2-\gamma^2)\pm 2\sqrt{(\nu_-^2-\gamma^2)^2-\kappa_-^2\gamma^2}\in(0{,}4\nu_-^2), \tag{1.14}$$

$$Q_\pm^2 := \kappa_+^2+q_\pm, \quad q_\pm := 2(\nu_+^2-\gamma^2)\pm 2\sqrt{(\nu_+^2-\gamma^2)^2-\kappa_+^2\gamma^2}\in(0{,}4\nu_+^2). \tag{1.15}$$

Note that $\gamma_1\le\gamma_-^{cr}$, $\gamma_2\le\gamma_-^{cr}$, $\gamma_3\le\gamma_+^{cr}$, $F_{\max}\le\gamma_-^{cr}+\gamma_+^{cr}$. $\gamma_1=\gamma_-^{cr}$ iff $\kappa_+=\sqrt{\kappa_-a_-}$, $\gamma_2=\gamma_-^{cr}$ iff $a_+=\sqrt{\kappa_-a_-}$, $\gamma_3=\gamma_+^{cr}$ iff $a_-=\sqrt{\kappa_+a_+}$. If $\kappa_-=0$, then $a_-=2\nu_-$, $\gamma_1=\sqrt{a_-^2-\kappa_+^2}/2$, $\gamma_2=\sqrt{a_-^2-a_+^2}/2$, $\gamma_-^{cr}=\nu_-$. If $\kappa_+=0$, then $a_+=2\nu_+$, and $\gamma_3=\sqrt{a_+^2-a_-^2}/2$.

Condition C is the following.

(i) Let $\kappa_-=\kappa_+=0$ and $\nu_-\ne\nu_+$. Then $\kappa_0\ne0$, and, in addition,
if $\gamma=0$, then $\kappa_0^2<2\max(\nu_-,\nu_+)\sqrt{|\nu_-^2-\nu_+^2|}$;
if $\gamma\in(0,\sqrt{|\nu_-^2-\nu_+^2|})$, then $\kappa_0^2\ne2\sqrt{\max(\nu_-^2,\nu_+^2)-\gamma^2}\sqrt{|\nu_-^2-\nu_+^2|-\gamma^2}$.

(ii) Let $\kappa_-=\kappa_+=0$ and $\nu_-=\nu_+$. Then $\kappa_0\ne0$ and $\gamma\ne0$.

(iii) Let $\kappa_-=\kappa_+\ne0$ and $\nu_-=\nu_+$. Then $\gamma\ne0$.
In addition, if $\gamma\in(0,a_--\kappa_-]$, then $\kappa_0\ne\kappa_-$.

(iv) Let $\kappa_-=\kappa_+\ne0$ and $\nu_-\ne\nu_+$. If $\gamma=0$, then
$\kappa_0^2\in(\kappa_-^2,\kappa_-^2+2\max(\nu_-,\nu_+)\sqrt{|\nu_-^2-\nu_+^2|})$.
In addition, if $\gamma\in(0,F_{\max}]$, then $\kappa_0\ne\bar\kappa$.
Moreover, in the case when $\nu_+<\nu_-$, it is assumed that if $\gamma\in(0,\gamma_2]$, then $\kappa_0^2\ne K_+(P_+)$, if $\gamma\in(\gamma_2,\gamma_-^{cr}]$ and $a_+<\sqrt{\kappa_-a_-}$, then $\kappa_0^2\ne K_+(P_\pm)$.
In the case when $\nu_+>\nu_-$, we assume that if $\gamma\in(0,\gamma_3]$, then $\kappa_0^2\ne K_-(Q_+)$, if $\gamma\in(\gamma_3,\gamma_+^{cr}]$ and $a_-<\sqrt{\kappa_+a_+}$, then $\kappa_0^2\ne K_-(Q_\pm)$.

(v) Let $\kappa_- = 0$, $\kappa_+ > 0$, and $a_+ < a_-$. If $\gamma = 0$, then $\kappa_0^2 < K_+(a_-)$.
In addition, we assume that if $\gamma \in [\gamma_2, F_{\max}]$, then $\kappa_0 \neq \bar{\kappa}$.
Moreover, if $\gamma \in (0, \gamma_2)$, then $\kappa_0^2 \neq K_+(2\sqrt{\nu_-^2 - \gamma^2})$,
if $\gamma \in (\gamma_1, \nu_-)$, then $\kappa_0^2 \neq K_0(2\sqrt{\nu_-^2 - \gamma^2})$.

(vi) Let $0 < \kappa_- < \kappa_+$, and $a_+ < a_-$. If $\gamma = 0$, then $\kappa_0^2 \in (K_0(\kappa_-), K_+(a_-))$.
In addition, if $\gamma \in [F_{\min}, F_{\max}]$, then $\kappa_0 \neq \bar{\kappa}$ (where $F_{\min} = \min(\gamma_1, \gamma_2)$).
Moreover, if $\gamma \in (0, \gamma_1]$, then $\kappa_0^2 \neq K_0(P_-)$;
if $\gamma \in (\gamma_1, \gamma_-^{cr}]$ and $\kappa_+ > \sqrt{\kappa_- a_-}$, then $\kappa_0^2 \neq K_0(P_\pm)$.
If $\gamma \in (0, \gamma_2]$, then $\kappa_0^2 \neq K_+(P_+)$;
if $\gamma \in (\gamma_2, \gamma_-^{cr}]$ and $a_+ < \sqrt{\kappa_- a_-}$, then $\kappa_0^2 \neq K_+(P_\pm)$.

(vii) Let $\kappa_- = 0$, $\kappa_+ > 0$, $a_- = a_+$. Then if $\gamma = 0$, then $\kappa_0^2 < K_+(a_-) = \bar{\kappa}^2$.
In addition, we assume that if $\gamma \in (0, F_{\max}]$, then $\kappa_0 \neq \bar{\kappa}$.
Moreover, if $\gamma \in (\gamma_1, \nu_-)$, then $\kappa_0^2 \neq K_0(2\sqrt{\nu_-^2 - \gamma^2})$.

(viii) Let $\kappa_- \neq 0$, $\kappa_+ > \kappa_-$, and $a_- = a_+$. If $\gamma = 0$, then $\kappa_0^2 \in (K_0(\kappa_-), \bar{\kappa}^2)$.
In addition, we assume that if $\gamma \in (0, F_{\max}]$, then $\kappa_0 \neq \bar{\kappa}$.
Moreover, if $\gamma \in (0, \gamma_1]$, then $\kappa_0^2 \neq K_0(P_-)$;
if $\gamma \in (\gamma_1, \gamma_-^{cr}]$ and $\kappa_+ > \sqrt{\kappa_- a_-}$, then $\kappa_0^2 \neq K_0(P_\pm)$.

(ix) Let $\kappa_- = 0$ and $0 < \kappa_+ < a_- < a_+$. If $\gamma = 0$, then $\kappa_0^2 < K_-(a_+)$.
In addition, if $\gamma \in [F_{\min}, F_{\max}]$, then $\kappa_0 \neq \bar{\kappa}$ (where $F_{\min} = \min(\gamma_1, \gamma_3)$).
Moreover, if $\gamma \in (\gamma_1, \nu_-)$, then $\kappa_0^2 \neq K_0(2\sqrt{\nu_-^2 - \gamma^2})$.
if $\gamma \in (0, \gamma_3]$, then $\kappa_0^2 \neq K_-(Q_+)$;
if $\gamma \in (\gamma_3, \gamma_+^{cr}]$ and $a_- < \sqrt{\kappa_+ a_+}$, then $\kappa_0^2 \neq K_-(Q_\pm)$.

(x) Let $0 < \kappa_- < \kappa_+ < a_- < a_+$. If $\gamma = 0$, then $\kappa_0^2 \in (K_0(\kappa_-), K_-(a_+))$.
In addition, if $\gamma \in [F_{\min}, F_{\max}]$, then $\kappa_0 \neq \bar{\kappa}$ (where $F_{\min} = \min(\gamma_1, \gamma_3)$).
Moreover, if $\gamma \in (0, \gamma_1]$, then $\kappa_0^2 \neq K_0(P_-)$;
if $\gamma \in (\gamma_1, \gamma_-^{cr}]$ and $\kappa_+ > \sqrt{\kappa_- a_-}$, then $\kappa_0^2 \neq K_0(P_\pm)$.
If $\gamma \in (0, \gamma_3]$, then $\kappa_0^2 \neq K_-(Q_+)$;
if $\gamma \in (\gamma_3, \gamma_+^{cr}]$ and $a_- < \sqrt{\kappa_+ a_+}$, then $\kappa_0^2 \neq K_-(Q_\pm)$.

(xi) Let $\kappa_- = 0$, $\kappa_+ = a_-$. If $\gamma = 0$, then $\kappa_0^2 < K_-(a_+)$ and $\kappa_0 \neq \bar{\kappa}$.
If $\gamma \in (0, \nu_-)$, then $\kappa_0^2 \neq K_0(2\sqrt{\nu_-^2 - \gamma^2})$.
Also, if $\gamma \in (0, \gamma_+^{cr}]$, then $\kappa_0^2 \neq K_-(Q_\pm)$.

(xii) Let $\kappa_- \neq 0$, $\kappa_+ = a_-$. If $\gamma = 0$, then $\kappa_0^2 \in (K_0(\kappa_-), K_-(a_+))$ and $\kappa_0 \neq \bar{\kappa}$.
Also, if $\gamma \in (0, \gamma_-^{cr}]$, then $\kappa_0^2 \neq K_0(P_\pm)$. If $\gamma \in (0, \gamma_+^{cr}]$, then $\kappa_0^2 \neq K_-(Q_\pm)$.

(xiii) Let $\kappa_- = 0$, $a_- < \kappa_+$. If $\gamma = 0$, then either $\kappa_0^2 \in (K_-(\kappa_+), K_-(a_+))$ or $\kappa_0^2 < K_0(a_-)$ (if $K_0(a_-) > 0$).
If $\gamma \in (0, \nu_-)$, then $\kappa_0^2 \neq K_0(2\sqrt{\nu_-^2 - \gamma^2})$.
Also, if $\gamma \in (0, \gamma_+^{cr}]$, then $\kappa_0^2 \neq K_-(Q_\pm)$.

(xiv) Let $\kappa_- \neq 0$, $a_- < \kappa_+$. Then if $\gamma = 0$, then either $\kappa_0^2 \in (K_-(\kappa_+), K_-(a_+))$ or $\kappa_0^2 \in (K_0(\kappa_-), K_0(a_-))$ (if $K_0(a_-) > 0$)
If $\gamma \in (0, \gamma_-^{cr}]$, then $\kappa_0^2 \neq K_0(P_\pm)$. Also, if $\gamma \in (0, \gamma_+^{cr}]$, then $\kappa_0^2 \neq K_-(Q_\pm)$.

Remark 1.1. *Condition* **C** *looks complicate especially in the cases* **(iv)**–**(xiv)** *and for* $\gamma \neq 0$. *Note that in these cases instead of restrictions of condition* **C** *it suffices to impose a stronger condition. Namely, in the cases* **(iv)**–**(xiv)**, *it is enough to assume that either* $\gamma > \gamma_-^{cr} + \gamma_+^{cr}$ *or* $\gamma = 0$ *and* κ_0 *satisfies the following restrictions.*

$$\begin{array}{l} \kappa_0^2 < K_+(a_-), \ \textit{if} \ \ a_- \geq a_; \quad \kappa_0^2 < K_-(a_+), \ \textit{if} \ \ a_+ \geq a_-; \\ \kappa_0^2 > K_0(\kappa_-), \ \textit{if} \ \ \kappa_- \neq 0; \\ \kappa_0^2 > K_-(\kappa_+) \ \textit{or} \ \kappa_0^2 < K_0(a_-), \ \ \textit{if} \ a_- \leq \kappa_+. \end{array}$$

We also study the behavior of the system under the following condition.
Condition $\mathbf{C}_0$: One of the following restrictions is fulfilled.
(i) $a_- > a_+$, $\gamma = 0$, $\kappa_0^2 = K_+(a_-)$.
(ii) $a_- < a_+$, $\gamma = 0$, $\kappa_0^2 = K_-(a_+)$.
The particular case of the condition **(i)** or **(ii)** is
$\kappa_- = \kappa_+$, $\nu_- \neq \nu_+$, $\gamma = 0$, $\kappa_0^2 = \kappa_-^2 + 2\max(\nu_-, \nu_+)\sqrt{|\nu_-^2 - \nu_+^2|}$.
(iii) $a_+ = a_-$, $(\kappa_-, \kappa_+) \neq (0{,}0)$, $\gamma = 0$, $\kappa_0 = \bar{\kappa}$.
(iv) $\kappa_- \neq 0$, $\gamma = 0$, $\kappa_0^2 = K_0(\kappa_-)$ (if $K_0(\kappa_-) \geq 0$).
The particular case of the condition **(iv)** is
$\kappa_- = \kappa_+ \neq 0$, $\gamma = 0$, $\kappa_0^2 = K_0(\kappa_-) = \bar{\kappa}^2$.
The particular case of the conditions **(iii)** or **(iv)** is
$\kappa_- = \kappa_+ \neq 0$, $\nu_- = \nu_+$, $\gamma = 0$, $\kappa_0 = \kappa_+$.
(v) $a_- \leq \kappa_+$, $\gamma = 0$, $\kappa_0^2 = K_-(\kappa_+)$.
In particular, if $a_- = \kappa_+$ and $\gamma = 0$, then $\kappa_0^2 = K_-(\kappa_+) = \bar{\kappa}^2$.
(vi) $a_- < \kappa_+$, $\gamma = 0$, $\kappa_0^2 = K_0(a_-)$ (if $K_0(a_-) \geq 0$).
(vii) $\kappa_- < \kappa_+ < a_+ < a_-$, $\gamma = \min(\gamma_1, \gamma_2)$, $\kappa_0 = \bar{\kappa}$.
(viii) $\kappa_- < \kappa_+ < a_- < a_+$, $\gamma = \min(\gamma_1, \gamma_3)$, $\kappa_0 = \bar{\kappa}$.

Our objective is to study the long-time behavior of solutions. Write $Y(t) = (u(\cdot, t), \dot{u}(\cdot, t))$, $Y_0(x) \equiv (Y_0^0(x), Y_0^1(x)) = (u_0(x), v_0(x))$. We assume that the initial state $Y_0(x)$ belongs to the Hilbert space $\mathcal{H}_\alpha$, $\alpha \in \mathbb{R}$, defined below.

Definition 1.2. $\ell_\alpha^2 \equiv \ell_\alpha^2(\mathbb{Z})$, $\alpha \in \mathbb{R}$, *is the Hilbert space of sequences* $u(x)$, $x \in \mathbb{Z}$, *with norm* $\|u\|_\alpha^2 = \sum\limits_{x \in \mathbb{Z}} \langle x \rangle^{2\alpha} |u(x)|^2 < \infty$, $\langle x \rangle := (1 + x^2)^{1/2}$.
$\mathcal{H}_\alpha = \ell_\alpha^2 \otimes \ell_\alpha^2$ *is the Hilbert space of pairs* $Y = (u, v)$ *of sequences equipped with norm* $\|Y\|_\alpha^2 = \|u\|_\alpha^2 + \|v\|_\alpha^2 < \infty$.

We prove that for any initial state $Y_0 \in \mathcal{H}_\alpha$ with $\alpha > 3/2$, the solution $Y(t)$ of the system (1.1)–(1.4) obeys the following bound

$$\|Y(t)\|_{-\alpha} \leq C(1 + |t|)^{-\beta/2} \|Y_0\|_\alpha, \quad t \in \mathbb{R}, \tag{1.16}$$

where $\beta = 3$ if condition $\mathbf{C}$ holds and $\beta = 1$ if condition $\mathbf{C}_0$ holds. We specify the long-time behavior of the solutions in Theorem 2.4. If conditions $\mathbf{C}$ and $\mathbf{C}_0$ are not fulfilled, then the bound (1.16) is not true, in general, see Remark 4.5.

For the solutions of the linear discrete Schrodinger and Klein–Gordon equations, the dispersive estimates of the type (1.16) were obtained by Shaban and Vainberg [10], Komech, Kopylova and Kunze [8] and Pelinosky and Stefanov [9]. The wave operators for the discrete Schrodinger operators were studied by Cuccagna [1]. In [5], we studied the long-time behavior of the chain of oscillators on the half-line with *random* initial data $Y_0 \in \mathcal{H}_\alpha$ and $\alpha < -3/2$. In [3], we considered the linear Hamiltonian system consisting of the discrete Klein–Gordon field coupled to a particle and obtained the similar results on the long–time behavior of the solutions.

2. Main Results

The following theorem can be proved by a similar way as [5, Theorem 2.2].

Theorem 2.1. *(i) Let* $\gamma, \kappa_\pm, \kappa_0 \geq 0$, $\nu_\pm > 0$*, and let* $Y_0 \in \mathcal{H}_\alpha$, $\alpha \in \mathbb{R}$*. Then the problem (1.1)–(1.4) has a unique solution* $Y(t) \in C(\mathbb{R}, \mathcal{H}_\alpha)$*.*
(ii) The operator $U(t) : Y_0 \to Y(t)$ *is continuous on* $\mathcal{H}_\alpha$*. Moreover, there exist constants* $C, B < \infty$ *such that* $\|U(t)Y_0\|_\alpha \leq Ce^{B|t|}\|Y_0\|_\alpha$*.*
(iii) Let $Y_0 \in \mathcal{H}_0$*. Then*

$$\mathrm{H}(Y(t)) + \gamma \int_0^t |\dot{u}(0,s)|^2\, ds = \mathrm{H}(Y_0), \quad t \geq 0, \tag{2.1}$$

where $\mathrm{H}(Y(t))$ *is defined in (1.5).*

The proof is based on the following representation for the solutions of the problem (1.1)–(1.3):

$$u(x,t) = \begin{cases} z_+(x,t) + r(x,t), & x \geq 0, \quad t > 0, \\ z_-(x,t) + r(x,t), & x \leq 0, \quad t > 0, \end{cases} \tag{2.2}$$

where $z_\pm(x,t)$ are solutions of the mixing problems with zero boundary condition

$$\ddot{z}_\pm(x,t) = (\nu_\pm^2 \Delta_L - \kappa_\pm^2) z_\pm(x,t), \quad \pm x \geq 1, \quad t > 0, \tag{2.3}$$

$$z_\pm(0,t) = 0, \quad t > 0, \tag{2.4}$$

$$z_\pm(x,0) = u_0(x), \quad \dot{z}_\pm(x,0) = v_0(x), \quad \pm x \geq 1. \tag{2.5}$$

Therefore, $r(x,t)$ is a solution of the following problem

$$\ddot{r}(x,t) = (\nu_\pm^2 \Delta_L - \kappa_\pm^2) r(x,t), \quad \pm x \geq 1, \quad t > 0, \tag{2.6}$$

$$\begin{aligned}\ddot{r}(0,t) &= \nu_+^2(r(1,t) - r(0,t)) + \nu_-^2(r(-1,t) - r(0,t)) - \kappa_0^2 r(0,t) \\ &\quad -\gamma \dot{r}(0,t) + \nu_+^2 z_+(1,t) + \nu_-^2 z_-(-1,t), \quad t > 0,\end{aligned} \tag{2.7}$$

$$r(x,0) = 0, \quad \dot{r}(x,0) = 0, \quad x \neq 0, \tag{2.8}$$

$$r(0,0) = u_0(0), \quad \dot{r}(0,0) = v_0(0). \tag{2.9}$$

At first we state results concerning the solutions of the problem (2.3)–(2.5). Write $Z_\pm(t) \equiv (z_\pm(\cdot,t), \dot{z}_\pm(\cdot,t))$. Introduce a Hilbert space $\ell^2_{\alpha,\pm} \equiv \ell^2_{\alpha,\pm}(\mathbb{Z}_\pm)$, $\alpha \in \mathbb{R}$, with norm $\|u\|^2_{\alpha,\pm} = \sum\limits_{\pm x \geq 0} \langle x \rangle^{2\alpha} |u(x)|^2 < \infty$, $\mathbb{Z}_\pm = \{x \in \mathbb{Z} : \pm x \geq 0\}$.

Let $\mathcal{H}_{\alpha,\pm} = \ell^2_{\alpha,\pm} \otimes \ell^2_{\alpha,\pm}$ be a Hilbert space of pairs $Y = (u,v)$ equipped with norm $\|Y\|^2_{\alpha,\pm} = \|u\|^2_{\alpha,\pm} + \|v\|^2_{\alpha,\pm} < \infty$.

Lemma 2.2. *(see Lemma 2.7 in [2]) Assume that* $\alpha \in \mathbb{R}$. *Then*
(i) for any initial data $Y_0 \in \mathcal{H}_{\alpha,\pm}$*, there exists a unique solution* $Z_\pm(t) \in C(\mathbb{R}, \mathcal{H}_{\alpha,\pm})$ *to the problem (2.3)–(2.5);*
(ii) the operator $W_\pm(t) : Y_0 \mapsto Z_\pm(t)$ *is continuous on* $\mathcal{H}_{\alpha,\pm}$. *Furthermore, the following bound holds,*

$$\|W_\pm(t)Y_0\|_{\alpha,\pm} \le C\langle t\rangle^\sigma \|Y_0\|_{\alpha,\pm} \tag{2.10}$$

with some constants $C = C(\alpha), \sigma = \sigma(\alpha) < \infty$.

The proof of Lemma 2.2 is based on the following formula for the solutions $z_\pm(x,t)$ of the problem (2.3)–(2.5):

$$z_\pm^{(i)}(x,t) = \sum_{y\ge 0} G_{t,\pm}^{ij}(x,y)Y_0^j(y), \quad \pm x \ge 1, \quad i = 0{,}1, \tag{2.11}$$

where $z_\pm^{(0)}(x,t) \equiv z_\pm(x,t)$, $z_\pm^{(1)}(x,t) \equiv \dot z_\pm(x,t)$, $Y_0^0(x) \equiv u_0(x)$, $Y_0^1(x) \equiv v_0(x)$, the Green function $G_{t,\pm}(x,y) = (G_{t,\pm}^{ij}(x,y))_{i,j=0}^1$ is a matrix-valued function of a form

$$G_{t,\pm}(x,y) := \mathcal{G}_{t,\pm}(x-y) - \mathcal{G}_{t,\pm}(x+y),\ \mathcal{G}_{t,\pm}(x) \equiv \frac{1}{2\pi}\int_{\mathbb{T}} e^{-ix\theta}\hat{\mathcal{G}}_{t,\pm}(\theta)d\theta, \tag{2.12}$$

$\mathbb{T} \equiv \mathbb{R}/(2\pi\mathbb{Z})$ denotes torus,

$$\begin{aligned} \hat{\mathcal{G}}_{t,\pm}(\theta) &= \left(\hat{\mathcal{G}}_{t,\pm}^{ij}(\theta)\right)_{i,j=0,1} = \begin{pmatrix} \cos(\phi_\pm(\theta)t) & \sin(\phi_\pm(\theta)t)/\phi_\pm(\theta) \\ -\phi_\pm(\theta)\sin(\phi_\pm(\theta)t) & \cos(\phi_\pm(\theta)t) \end{pmatrix} \\ \phi_\pm(\theta) &= \sqrt{\nu_\pm^2(2-2\cos\theta)+\kappa_\pm^2}. \end{aligned} \tag{2.13}$$

In particular, $\phi_\pm(\theta) = 2\nu_\pm|\sin(\theta/2)|$ if $\kappa_\pm = 0$. We see that $z_\pm(0,t) \equiv 0$ for any t, since $\mathcal{G}_{t,\pm}^{ij}(-x) = \mathcal{G}_{t,\pm}^{ij}(x)$.

For the solutions of the problem (2.3)–(2.5), the following bound is true.

Theorem 2.3. *Let* $Y_0 \in \mathcal{H}_{\alpha,\pm}$ *and* $\alpha > 3/2$. *Then*

$$\|W_\pm(t)Y_0\|_{-\alpha,\pm} \le C\langle t\rangle^{-3/2}\|Y_0\|_{\alpha,\pm}, \quad t \in \mathbb{R}. \tag{2.14}$$

This theorem can be proved by a similar way as Remark 3.3 in [4].

To formulate the main result, introduce the following notations.

(i) Write

$$\begin{aligned} \mathbf{g}_\pm^i(y,t) &:= \left(G_{t,\pm}^{i0}(\pm1,y), G_{t,\pm}^{i1}(\pm1,y)\right) \\ &= \left(\mathcal{G}_{t,\pm}^{i0}(\pm1-y) - \mathcal{G}_{t,\pm}^{i0}(\pm1+y), \mathcal{G}_{t,\pm}^{i1}(\pm1-y) - \mathcal{G}_{t,\pm}^{i1}(\pm1+y)\right), \end{aligned} \tag{2.15}$$

$y \in \mathbb{Z}$, $i = 0{,}1$, $t \in \mathbb{R}$, $\mathcal{G}^{ij}_{t,\pm}$ is defined in (2.12) and (2.13).
(ii) $\mathbf{G}^j_\pm(y)$, $j = 0{,}1$, denotes the vector valued function

$$\mathbf{G}^j_\pm(y) = \int_0^{+\infty} N^{(j)}(s)\mathbf{g}^0_\pm(y,-s)\,ds, \quad y \in \mathbb{Z}, \tag{2.16}$$

where $N^{(0)}(s) \equiv N(s)$, $N^{(1)}(s) \equiv \dot N(s)$, $N(s)$ is introduced in (3.12), $\mathbf{g}^0_\pm(y,s)$ is defined in (2.15). Note that $\mathbf{g}^0_\pm(0,t) = 0$ and $\mathbf{G}^j_\pm(0) = 0$. Introduce

$$\overline{\mathbf{G}}^j(y) = \begin{cases} \nu_+^2 \mathbf{G}^j_+(y), & y \ge 0, \\ \nu_-^2 \mathbf{G}^j_-(y), & y \le 0, \end{cases} \qquad j = 0{,}1. \tag{2.17}$$

(iii) Denote by $W'_\pm(t)$ the operator adjoint to $W_\pm(t)$, $t \in \mathbb{R}$,

$$\langle Y, W'_\pm(t)\Psi\rangle_\pm = \langle W_\pm(t)Y, \Psi\rangle_\pm, \ \ Y \in \mathcal{H}_{\alpha,\pm}, \ \ \Psi = (\Psi^0, \Psi^1) \in [S(\mathbb{Z}_\pm)]^2.$$

Here $S(\mathbb{Z}_\pm)$ denotes the class of rapidly decreasing sequences in $\mathbb{Z}_\pm$, $\langle\cdot,\cdot\rangle_\pm$ stands for the inner product in $\mathcal{H}_{0,\pm}$ or for its different extensions. Below we also use the notation $\langle\cdot,\cdot\rangle$ for the inner product in $\mathcal{H}_0$ or for its different extensions. Applying the Green function $G_{t,\pm}$, we rewrite $W'_\pm(t)\Psi$ in the form

$$(W'_\pm(t)\Psi)^j(y) = \sum_{i=0,1}\sum_{\pm x \ge 0} G^{ij}_{t,\pm}(x,y)\Psi^i(x), \quad t \in \mathbb{R}, \ \ \pm y \ge 0, \ \ j = 0{,}1.$$

In particular (see (2.15)),

$$\mathbf{g}^0_\pm(y,t) = (W'_\pm(t)Y_0)(y) \quad \text{with} \quad Y_0(x) = (\delta_{\pm 1 x}, 0), \tag{2.18}$$

where δ_{ij} denotes the Kronekker symbol.
(iv) Introduce a vector-valued function $\overline{\mathbf{K}}^j(x,y)$ $j = 0{,}1$, $x,y \in \mathbb{Z}$, by the rule

$$\overline{\mathbf{K}}^j(x,y) = \begin{cases} \nu_\pm^2 \displaystyle\int_0^{+\infty} K^+_x(s)\Big(W'_\pm(-s)\mathbf{G}^j_\pm\Big)(y)\,ds, & \text{if } x \ge 1, \quad \pm y \ge 0, \\ \nu_\pm^2 \displaystyle\int_0^{+\infty} K^-_x(s)\Big(W'_\pm(-s)\mathbf{G}^j_\pm\Big)(y)\,ds, & \text{if } x \le -1, \ \pm y \ge 0, \\ \overline{\mathbf{G}}^j(y), & \text{if } x = 0, \quad y \in \mathbb{Z}, \end{cases} \tag{2.19}$$

where $K^\pm_x(s)$ is defined in (3.4), $\mathbf{G}^j_\pm$ is introduced in (2.16). In particular, $\overline{\mathbf{K}}^j(x,0) = 0$ for any $x \in \mathbb{Z}$.
(v) Define an operator $\overline{W}(t)$, $t \in \mathbb{R}$, by the rule

$$\left(\overline{W}(t)Y_0\right)(x) = \begin{cases} \left(W_+(t)Y_0\right)(x), & \text{if } x \ge 0, \\ \left(W_-(t)Y_0\right)(x), & \text{if } x \le 0. \end{cases} \tag{2.20}$$

In particular, $\big(\overline{W}(t)Y_0\big)(0) = 0$ for any t.
(vi) Introduce an operator $\Omega : \mathcal{H}_\alpha \to \mathcal{H}_{-\alpha}$, $\alpha > 3/2$, by the rule

$$\Omega : Y \to Y(x) + \Big(\langle Y(\cdot), \overline{\mathbf{K}}^0(x,\cdot)\rangle, \langle Y(\cdot), \overline{\mathbf{K}}^1(x,\cdot)\rangle\Big). \tag{2.21}$$

The properties of the functions $\overline{\mathbf{K}}^j$ and $\mathbf{G}^j_\pm$ and the operator Ω are given in Remarks 4.2 and 4.4.

Theorem 2.4. *Let $Y_0 \in \mathcal{H}_\alpha$, $\alpha > 3/2$, and conditions* $\mathbf{C}$ *or* $\mathbf{C}_0$ *hold. Then the following assertions are fulfilled.*
(i) $U(t)Y_0 = \Omega(\overline{W}(t)Y_0) + \delta(t)$, where $\|\delta(t)\|_{-\alpha} \le C\langle t\rangle^{-\beta/2}\|Y_0\|_\alpha$, $\beta = 3$ if condition $\mathbf{C}$ *holds and $\beta = 1$ if condition* $\mathbf{C}_0$ *holds.*
(ii) The solution $Y(t) = U(t)Y_0$ obeys the bound (1.16).

This theorem is proved in Section 4 using the technique of Jensen and Kato [7] which was developed in the works of Komech *at al.* [8]. If conditions $\mathbf{C}$ and $\mathbf{C}_0$ are not fulfilled, then the bound (1.16) is not true for *any* initial data $Y_0 \in \mathcal{H}_\alpha$ (see Remark 4.5 below).

3. Fourier–Laplace transform

In this section, we construct the solution $r(x,t)$ of the problem (2.6)–(2.9) using the Fourier–Laplace transform.

Definition 3.1. *Let $|r(t)| \le Ce^{Bt}$. The Fourier–Laplace transform of $r(t)$ is given by the formula*

$$\tilde{r}(\omega) = \int_0^{+\infty} e^{i\omega t} r(t)\,dt, \quad \Im\omega > B.$$

The Gronwall inequality implies standard a priori estimates for the solutions $r(x,t)$, $x \in \mathbb{Z}$. In particular, there exist constants $A, B < \infty$ such that

$$\sum_{x\in\mathbb{Z}} (|r(x,t)|^2 + |\dot{r}(x,t)|^2) \le Ce^{Bt} \quad \text{as} \quad t \to +\infty.$$

Hence, the Fourier–Laplace transform of $r(x,t)$ with respect to t-variable, $r(x,t) \to \tilde{r}(x,\omega)$, exists at least for $\Im\omega > B$ and satisfies the following equation

$$(-\nu_\pm^2\Delta_L + \kappa_\pm^2 - \omega^2)\tilde{r}(x,\omega) = 0 \quad \text{for} \quad \pm x \ge 1, \quad \Im\omega > B. \tag{3.1}$$

Now we construct the solution of (3.1). We first note that the Fourier transform of the lattice operator $-\nu_\pm^2\Delta_L + \kappa_\pm^2$ is the operator of multiplication by the function

$\phi^2_\pm(\theta) = \nu^2_\pm(2 - 2\cos\theta) + \kappa^2_\pm$. Thus, $-\nu^2_\pm \Delta_L + \kappa^2_\pm$ is a self-adjoint operator and its spectrum is absolutely continuous and coincides with the range of $\phi^2_\pm(\theta)$, i.e., with the segment $[\kappa^2_\pm, a^2_\pm]$, where $a^2_\pm := \kappa^2_\pm + 4\nu^2_\pm$. Denote by $\Lambda_\pm$ the *critical set*, $\Lambda_\pm := [-a_\pm, -\kappa_\pm] \cup [\kappa_\pm, a_\pm]$, and by $\Lambda^0_\pm = \{-a_\pm, -\kappa_\pm, \kappa_\pm, a_\pm\}$ the set of "spectral edges".

Lemma 3.2. *(see Lemma 2.1 in [8]) For given $\omega \in \mathbb{C} \setminus \Lambda_\pm$, the equation*

$$\nu^2_\pm(2 - 2\cos\theta) = \omega^2 - \kappa^2_\pm \tag{3.2}$$

has the unique solution $\theta_\pm(\omega)$ in the domain $\{\theta \in \mathbb{C} : \Im\theta > 0,\ \Re\theta \in (-\pi, \pi]\}$. Moreover, $\theta_+(\omega)$ ($\theta_-(\omega)$) is an analytic function in $\mathbb{C}\setminus\Lambda_+$ (in $\mathbb{C}\setminus\Lambda_-$, respectively).

Since we seek the solution $r(\cdot, t) \in \ell^2_\alpha$ with some α, $\tilde{r}(x, \omega)$ has a form

$$\tilde{r}(x,\omega) = \begin{cases} \tilde{r}(0,\omega)e^{i\theta_+(\omega)x}, & x \ge 0, \\ \tilde{r}(0,\omega)e^{-i\theta_-(\omega)x}, & x \le 0. \end{cases}$$

We put $\tilde{K}^\pm_x(\omega) = e^{\pm i\theta_\pm(\omega)x}$, $\pm x \ge 1$. Applying the inverse Fourier–Laplace transform with respect to ω-variable, we write the solution of (2.6) in the form

$$(r(x,t), \dot{r}(x,t)) = \int_0^t K^\pm_x(t-s)\big(r(0,s), \dot{r}(0,s)\big)\, ds \quad \text{for } \pm x \ge 1,\ \ t > 0, \tag{3.3}$$

where

$$K^\pm_x(t) := \frac{1}{2\pi} \int\limits_{-\infty+i\mu}^{+\infty+i\mu} e^{-i\omega t}\tilde{K}^\pm_x(\omega)\, d\omega,\ \ \pm x \ge 1,\ \ t > 0,\ \ \text{with some } \mu > 0. \tag{3.4}$$

Put

$$\|f(x)\|'_{\alpha,\pm} := \Big(\sum_{\pm x \ge 1} \langle x\rangle^{2\alpha}|f(x)|^2\Big)^{1/2}. \tag{3.5}$$

Theorem 3.3. *(see [5, Theorem 3.3]) For any $\alpha > 3/2$, the following bound holds,*

$$\|K^+_x(t)\|'_{-\alpha,+} \le C\langle t\rangle^{-3/2}, \quad \|K^-_x(t)\|'_{-\alpha,-} \le C\langle t\rangle^{-3/2}, \qquad t > 0. \tag{3.6}$$

In particular,

$$|K^+_1(t)| \le C(1+t)^{-3/2}, \quad |K^-_{-1}(t)| \le C(1+t)^{-3/2}, \quad t > 0. \tag{3.7}$$

Using (3.3), we rewrite Eqn (2.7) in the form

$$\ddot r(0,t) = -(\nu_+^2+\nu_-^2+\kappa_0^2)r(0,t) - \gamma\dot r(0,t) + \nu_+^2 z_+(1,t) + \nu_-^2 z_-(-1,t) + \int_0^t \Big(\nu_+^2 K_1^+(t-s) + \nu_-^2 K_{-1}^-(t-s)\Big) r(0,s)\,ds, \quad t>0. \tag{3.8}$$

At first, we study the solutions of the corresponding homogeneous equation

$$\ddot r(0,t) = -(\nu_+^2+\nu_-^2+\kappa_0^2)r(0,t) - \gamma\dot r(0,t) + \int_0^t \big(\nu_+^2 K_1^+(t-s)+\nu_-^2 K_{-1}^-(t-s)\big)\, r(0,s)\,ds, \tag{3.9}$$

with the initial data

$$r(0,t)|_{t=0} = u_0(0), \quad \dot r(0,t)|_{t=0} = v_0(0). \tag{3.10}$$

Applying the Fourier–Laplace transform to the solutions of (3.9), we obtain

$$\tilde r(0,\omega) = \tilde N(\omega)\left[-i\omega u_0(0) + u_0(0)\gamma + v_0(0)\right] \quad \text{for} \;\; \Im\omega > B,$$

where, by definition, $\tilde N(\omega) := [\tilde D(\omega)]^{-1}$ and

$$\tilde D(\omega) := \kappa_0^2 - \omega^2 + \nu_+^2(1-e^{i\theta_+(\omega)}) + \nu_-^2(1-e^{i\theta_-(\omega)}) - i\omega\gamma, \quad \omega\in\mathbb{C}. \tag{3.11}$$

The analytic properties of $\tilde D(\omega)$ and $\tilde N(\omega)$ are studied in Appendix. In particular, we prove that $\tilde N(\omega)$ is analytic in the upper half-space. Denote

$$N(t) = \frac{1}{2\pi}\int_{-\infty+i\mu}^{+\infty+i\mu} e^{-i\omega t}\tilde N(\omega)\,d\omega, \quad t\ge 0, \quad \text{with some } \mu>0. \tag{3.12}$$

Theorem 3.4. *Let condition* $\mathbf{C}$ *or* $\mathbf{C}_0$ *hold. Then*

$$|N^{(k)}(t)| \le C(1+t)^{-\beta/2}, \quad t\ge 0, \quad k=0,1,2, \tag{3.13}$$

where $\beta=3$ *if condition* $\mathbf{C}$ *holds and* $\beta=1$ *if condition* $\mathbf{C}_0$ *holds.*

We prove Theorem 3.4 in Appendix.

Corollary 3.5. *Denote by* $S(t)$ *a solving operator of the Cauchy problem (3.9), (3.10). Then the variation constants formula gives the following representation for the solution of the problem (3.8), (3.10):*

$$\begin{pmatrix} r(0,t)\\ \dot r(0,t)\end{pmatrix} = S(t)\begin{pmatrix} u_0(0)\\ v_0(0)\end{pmatrix} + \int_0^t S(\tau)\begin{pmatrix} 0\\ \nu_+^2 z_+(1,t-\tau)+\nu_-^2 z_-(-1,t-\tau)\end{pmatrix} d\tau.$$

Evidently, $S(0)=I$*. The matrix* $S(t)$ *has a form* $\begin{pmatrix} \dot N(t)+\gamma N(t) & N(t)\\ \ddot N(t)+\gamma\dot N(t) & \dot N(t)\end{pmatrix}$*.* $|S(t)|\le C(1+t)^{-\beta/2}$*, by Theorem 3.4.*

4. Asymptotic behavior of solutions

Set $r^{(0)}(x,t)=r(x,t)$, $r^{(1)}(x,t)=\dot r(x,t)$.

Proposition 4.1. *Let $Y_0\in\mathcal{H}_\alpha$, $\alpha>3/2$, condition* **C** *or* $\mathbf{C}_0$ *hold, and $r(0,t)$ be a solution of the problem (3.8), (3.10). Then for $t>0$, $j=0,1$,*

$$r^{(j)}(0,t) = \sum_{\pm}\nu_\pm^2\langle W_\pm(t)Y_0,\mathbf{G}^j_\pm\rangle_\pm+\delta_j(t)$$
$$= \langle\overline{W}(t)Y_0,\overline{\mathbf{G}}^j\rangle+\delta_j(t),\quad |\delta_j(t)|\le C\langle t\rangle^{-\beta/2}\|Y_0\|_\alpha, \qquad (4.1)$$

where the vector valued functions $\mathbf{G}^j_\pm$, $\overline{\mathbf{G}}^j$ and the operator $\overline{W}(t)$ are defined in (2.16), (2.17) and (2.20), respectively, β is introduced in Theorem 3.4.

Proof Applying Corollary 3.5 and the bound (3.13), we obtain

$$r^{(j)}(0,t)=\sum_{\pm}\nu_\pm^2\int_0^t N^{(j)}(\tau)z_\pm(\pm1,t-\tau)\,d\tau+O((1+t)^{-\beta/2}),\quad t>0.$$

Furthermore, the bounds (2.14) and (3.13) give

$$\Big|\int_t^{+\infty}N^{(j)}(\tau)z_\pm(\pm1,t-\tau)d\tau\Big|\le C\int_t^{+\infty}\langle\tau\rangle^{-\beta/2}\langle t-\tau\rangle^{-3/2}d\tau\le C\langle t\rangle^{-\beta/2}\|Y_0\|_\alpha.$$

Using (2.18) and the equality $W_\pm(t-s)=W_\pm(t)W_\pm(-s)$, we have

$$z_\pm(\pm1,t-\tau)=\Big\langle W_\pm(t-\tau)Y_0,\begin{pmatrix}\delta_{\pm1x}\\0\end{pmatrix}\Big\rangle_\pm=\langle W_\pm(t)Y_0,\mathbf{g}^0_\pm(\cdot,-\tau)\rangle_\pm.$$

This implies the representation (4.1). ■

Remark 4.2. Now we list the properties of the functions $\mathbf{G}^j_\pm$ and $\overline{\mathbf{G}}^j$.
(i) By (2.12) and (2.13), $\mathbf{g}^0_\pm(x,t)$ is odd w.r.t. $x\in\mathbb{Z}$. Then, $\mathbf{G}^j_\pm$ is odd. Formulas (2.13) and the Parseval identity give

$$\|\mathbf{g}^0_\pm(\cdot,t)\|_0^2=C\int_{-\pi}^{\pi}\Big(\cos^2(\phi_\pm(\theta)t)+\frac{\sin^2(\phi_\pm(\theta)t)}{\phi^2_\pm(\theta)}\Big)\sin^2(\theta)\,d\theta\le C<\infty.\quad (4.2)$$

(ii) Let condition **C** or $\mathbf{C}_0$ hold. The action of group $W'_\pm(t)$ coincides with action of group $W_\pm(t)$, up to order of the components. Hence, using (2.18) and the bound (2.14), we have

$$\|W'_\pm(t)\mathbf{G}^j_\pm\|_{-\alpha,\pm}\le\int_0^{+\infty}|N^{(j)}(s)|\|W'_\pm(t-s)(\delta_{\pm1x},0)\|_{-\alpha,\pm}\,ds\le C\langle t\rangle^{-\beta/2}\quad (4.3)$$

for any $\alpha > 3/2$. Therefore,

$$|\langle W_\pm(t)Y_0, \mathbf{G}^j_\pm\rangle_\pm| \le \|Y_0\|_\alpha \|W'_\pm(t)\mathbf{G}^j_\pm\|_{-\alpha,\pm} \le C\langle t\rangle^{-\beta/2}\|Y_0\|_\alpha.$$

Since

$$\langle \overline{W}(t)Y_0, \overline{\mathbf{K}}^j(0,\cdot)\rangle = \langle \overline{W}(t)Y_0, \overline{\mathbf{G}}^j\rangle = \sum_\pm \nu^2_\pm \langle W_\pm(t)Y_0, \mathbf{G}^j_\pm\rangle_\pm, \tag{4.4}$$

by (2.17) and (2.19), we obtain the following estimate

$$|\langle \overline{W}(t)Y_0, \overline{\mathbf{G}}^j\rangle| \le C\langle t\rangle^{-\beta/2}\|Y_0\|_\alpha, \quad \alpha > 3/2. \tag{4.5}$$

(iii) Let condition **C** hold. Then, $\mathbf{G}^j_\pm \in \mathcal{H}_0$ by (4.2) and (3.13). Moreover, since $W'_\pm(t)\mathbf{g}^0_\pm(x,-s) = \mathbf{g}^0_\pm(x,t-s)$, we have

$$\sup_{t\in\mathbb{R}} \|W'_\pm(t)\mathbf{G}^j_\pm\|_{0,\pm} \le \sup_{t\in\mathbb{R}} \int_0^{+\infty} |N^{(j)}(s)|\|\mathbf{g}^0_\pm(\cdot,t-s)\|_0\, ds \le C < \infty, \tag{4.6}$$

by the bounds (3.13) and (4.2).

Now we study the large time behavior of $r(x,t)$ for $x \neq 0$.

Lemma 4.3. *Assume that $Y_0 \in \mathcal{H}_\alpha$, $\alpha > 3/2$, condition* **C** *or* $\mathbf{C}_0$ *hold. Then the solution $r(x,t)$, with $x \neq 0$, of the problem (2.6), (2.8) admits the following representation*

$$r^{(j)}(x,t) = \langle \overline{W}(t)Y_0, \overline{\mathbf{K}}^j(x,\cdot)\rangle + \delta_j(x,t), \quad j = 0{,}1, \quad t > 0, \tag{4.7}$$

where $\overline{\mathbf{K}}^j$ is introduced in (2.19), $\|\delta_j(\cdot,t)\|'_{-\alpha,\pm} \le C\langle t\rangle^{-\beta/2}\|Y_0\|_\alpha$. The norm $\|\cdot\|'_{-\alpha,\pm}$ is defined in (3.5).

Proof At first, by (3.3) and (4.1),

$$r^{(j)}(x,t) = \int_0^t K^\pm_x(t-s)\langle \overline{W}(s)Y_0, \overline{\mathbf{G}}^j\rangle\, ds + \delta'_j(x,t) \quad \text{for} \quad \pm x \ge 1, \tag{4.8}$$

where $\|\delta'_j(\cdot,t)\|'_{-\alpha,\pm} \le C\langle t\rangle^{-\beta/2}$. Indeed, by (4.1) and (3.6),

$$\begin{aligned}\|\delta'_j(\cdot,t)\|'_{-\alpha,\pm} \;&=\; \Big\| \int_0^t K^\pm_x(t-s)\delta_j(s)\, ds \Big\|'_{-\alpha,\pm} \le \int_0^t \big\|K^\pm_x(t-s)\big\|'_{-\alpha,\pm} |\delta_j(s)| ds \\ &\le\; C\int_0^t (1+t-s)^{-3/2}(1+s)^{-\beta/2}\, ds \le C_1\langle t\rangle^{-\beta/2}.\end{aligned}$$

Second, using (2.19), we have

$$\langle\overline{W}(t)Y_0,\overline{\mathbf{K}}^j(x,\cdot)\rangle=\begin{cases}\displaystyle\sum_{\pm}\nu_\pm^2\int_0^{+\infty}K_x^+(s)\langle W_\pm(t-s)Y_0,\mathbf{G}_\pm^j\rangle_\pm ds,\ x\ge 1,\\ \displaystyle\sum_{\pm}\nu_\pm^2\int_0^{+\infty}K_x^-(s)\langle W_\pm(t-s)Y_0,\mathbf{G}_\pm^j\rangle_\pm ds,\ x\le -1,\end{cases}$$

$$=\int_0^{+\infty}K_x^\pm(s)\langle\overline{W}(t-s)Y_0,\overline{\mathbf{G}}^j\rangle\,ds,\quad \pm x\ge 1. \tag{4.9}$$

Therefore, the first term in the r.h.s. of (4.8) has a form

$$\int_0^t K_x^\pm(s)\langle\overline{W}(t-s)Y_0,\overline{\mathbf{G}}^j\rangle\,ds=\langle\overline{W}(t)Y_0,\overline{\mathbf{K}}^j(x,\cdot)\rangle+\delta_j''(x,t),\ \ \pm x\ge 1, \tag{4.10}$$

where, by definition, $\delta_j''(x,t)=-\displaystyle\int_t^{+\infty}K_x^\pm(s)\langle\overline{W}(t-s)Y_0,\overline{\mathbf{G}}^j\rangle ds$ for $\pm x\ge 1$. The bounds (3.6) and (4.5) yield

$$\begin{aligned}\|\delta_j''(\cdot,t)\|'_{-\alpha,\pm}\ &\le\ \int_t^{+\infty}\|K_x^\pm(\cdot,s)\|'_{-\alpha,\pm}\left|\langle\overline{W}(t-s)Y_0,\overline{\mathbf{G}}^j\rangle\right|ds\\ &\le\ C\langle t\rangle^{-\beta/2}\|Y_0\|_\alpha.\end{aligned} \tag{4.11}$$

Hence, the bounds (4.8), (4.10) and (4.11) imply (4.7) with $\delta_j(x,t)=\delta_j'(x,t)+\delta_j''(x,t)$. ■

Remark 4.4. (i) Assume that condition **C** or $\mathbf{C}_0$ hold. Then $\|\overline{\mathbf{K}}^j(x,\cdot)\|_{-\alpha}\in\mathcal{H}_{-\alpha}$ with $\alpha>3/2$, since

$$\begin{aligned}\Big\|\|\overline{\mathbf{K}}^j(x,\cdot)\|_{-\alpha}\Big\|_{-\alpha}\le\int_0^{+\infty}\Big(\sum_{\pm}\|K_x^\pm(s)\|'_{-\alpha,\pm}\Big)\Big(\sum_{\pm}\nu_\pm^2\left\|W'_\pm(-s)\mathbf{G}_\pm^j\right\|_{-\alpha,\pm}\Big)\,ds\\ +\|\overline{\mathbf{G}}^j\|_{-\alpha}<\infty,\end{aligned}$$

due to (2.19), (3.6) and (4.3). Hence,

$$\|\langle Y,\overline{\mathbf{K}}^j(x,\cdot)\rangle\|_{-\alpha}\le C\|Y\|_\alpha\quad\text{for }\ \alpha>3/2. \tag{4.12}$$

Then, using (2.21), we obtain

$$\|\Omega Y\|_{-\alpha}\le\|Y\|_{-\alpha}+\sum_{j=0,1}\|\langle Y,\overline{\mathbf{K}}^j(x,\cdot)\rangle\|_{-\alpha}\le C\|Y\|_\alpha.$$

(ii) Assume that condition **C** holds. Then, $\|\overline{\mathbf{K}}^j(x,\cdot)\|_0 \in \mathcal{H}_{-\alpha}$ with $\alpha > 3/2$. Indeed,

$$\Big\| \|\overline{\mathbf{K}}^j(x,\cdot)\|_0 \Big\|_{-\alpha} \le \int_0^{+\infty} \Big(\sum_{\pm} \|K_x^{\pm}(s)\|'_{-\alpha,\pm} \Big) \Big(\sum_{\pm} \nu_{\pm}^2 \Big\| W'_{\pm}(-s)\mathbf{G}_{\pm}^j \Big\|_{0,\pm} \Big) ds$$
$$+ \|\overline{\mathbf{G}}^j\|_0 < \infty,$$

by (3.6) and (4.6). Hence, $\|\langle Y, \overline{\mathbf{K}}^j(x,\cdot)\rangle\|_{-\alpha} \le C\|Y\|_0$ for $\alpha > 3/2$. Therefore, $\|\Omega Y\|_{-\alpha} \le C\|Y\|_0$ for any $Y_0 \in \mathcal{H}_\alpha$, $\alpha > 3/2$.

Proof of Theorem 2.4. The item (i) follows from the representation (2.2) and the bounds (2.14), (4.1) and (4.7). Further, definition (2.21) implies

$$\|\Omega(\overline{W}(t)Y_0)\|_{-\alpha} \le \|\overline{W}(t)Y_0\|_{-\alpha} + \sum_{j=0}^{1} \|\langle \overline{W}(t)Y_0, \overline{\mathbf{K}}^j(x,\cdot)\rangle\|_{-\alpha}, \tag{4.13}$$

where

$$\|\overline{W}(t)Y_0\|_{-\alpha} \le C\langle t\rangle^{-3/2}\|Y_0\|_\alpha \quad \text{for } \alpha > 3/2, \tag{4.14}$$

by (2.20) and (2.14). Using (4.4), (4.9), (3.6) and (4.5), we have

$$\|\langle \overline{W}(t)Y_0, \overline{\mathbf{K}}^j(x,\cdot)\rangle\|_{-\alpha} \le \int_0^{+\infty} \Big(\sum_{\pm} \|K_x^{\pm}(s)\|'_{-\alpha,\pm} \Big) \Big| \langle \overline{W}(t-s)Y_0, \overline{\mathbf{G}}^j\rangle \Big| ds$$
$$+ |\langle \overline{W}(t)Y_0, \overline{\mathbf{G}}^j\rangle| \le C\langle t\rangle^{-\beta/2}\|Y_0\|_\alpha \tag{4.15}$$

for $\alpha > 3/2$. Thus, (4.13)–(4.15) yield

$$\|\Omega(\overline{W}(t)Y_0)\|_{-\alpha} \le C\langle t\rangle^{-\beta/2}\|Y_0\|_\alpha, \quad \alpha > 3/2. \tag{4.16}$$

Finally, the bound (1.16) follows from the part (i) of Theorem 2.4 and the bound (4.16). ■

Remark 4.5. In Appendix, we prove that condition **C** is a necessary and sufficient condition under which $\tilde{D}(\omega) \ne 0$ for any $\omega \in \mathbb{R}$. If $\gamma \ne 0$, condition **C** includes the restrictions on γ and κ_0 (for different cases of mutual disposition of $\nu_\pm$ and $\kappa_\pm$) under which $\tilde{D}(\omega - i0) \ne 0$ for values of ω from the singular set $\cup_\pm \Lambda_\pm \setminus (\cup_\pm \Lambda_\pm^0)$. If $\gamma = 0$, we impose such restrictions on γ and κ_0 that $\tilde{D}(\omega) \ne 0$ for "regular" real points $\omega \in \mathbb{R} \setminus (\cup_\pm \Lambda_\pm)$. Moreover, condition **C** implies that the asymptotic behavior of $\tilde{D}(\omega)$ in the neighborhood of the points $\omega_0 \in \Lambda_\pm^0$ is of a form $\tilde{D}(\omega) \sim C_1 + C_2(\omega^2 - \omega_0^2)^{1/2} + \ldots$ as $\omega \to \omega_0$ with some constants $C_1, C_2 \ne 0$. If condition $\mathbf{C}_0$ holds, then $\tilde{D}(\omega) \ne 0$ for any $\omega \in \mathbb{R} \setminus (\cup_\pm \Lambda_\pm^0)$. However, the asymptotics of $\tilde{D}(\omega)$ in the neighborhood of the points $\omega_0 \in \Lambda^0_\pm$ is of a form $\tilde{D}(\omega) \sim C_1(\omega^2-\omega_0^2)^{1/2}$ with $C_1 \neq 0$.

Now we consider the different cases when conditions $\mathbf{C}$ and $\mathbf{C}_0$ do not hold and show that the bound (1.16) is not true for *any* initial data $Y_0 \in \mathcal{H}_\alpha$.

(i) Let $\kappa_- = \kappa_+ = \kappa_0 = 0$ and $\gamma \geq 0$. Then $\tilde{N}(\omega)$ has a simple pole at zero (see (5.35)), and any constant is a solution of the system (1.1)–(1.3).

(ii) Let $\gamma = 0$, and $\kappa_0^2 > K_+(a_-)$ if $a_- \geq a_+$, or $\kappa_0^2 > K_-(a_+)$ if $a_- \leq a_+$, where $K_\pm$ are defined in (1.7). In particular, this is true in the following cases:

$$
\begin{array}{l}
1)\ \gamma = 0,\ \kappa_- = \kappa_+ = 0,\ \nu_- \neq \nu_+,\ \kappa_0^2 > 2\max(\nu_-,\nu_+)\sqrt{|\nu_+^2-\nu_-^2|}\\
2)\ \gamma = 0,\ \kappa_- = \kappa_+ = 0,\ \nu_- = \nu_+,\ \kappa_0 \neq 0\\
3)\ \gamma = 0,\ a_- = a_+ \neq 0,\ \kappa_0^2 > \bar{\kappa}^2 \equiv (\kappa_-^2+\kappa_+^2)/2
\end{array}
$$

Then, by Remark 5.4 (i), there exists a point $\omega_0 > \max(a_-, a_+)$ such that $\tilde{N}(\omega)$ has simple poles at the points $\omega = \pm\omega_0$. Note that $\Im\theta_\pm(\omega_0) > 0$, $\Re\theta_\pm(\omega_0) = \pi$. Hence, the function of the form

$$
u(x,t) = \left\{ \begin{array}{ll} e^{i\theta_+(\omega_0)x}\sin(\omega_0 t), & x \geq 0 \\ e^{-i\theta_-(\omega_0)x}\sin(\omega_0 t), & x \leq 0 \end{array} \right| \quad t \geq 0, \tag{4.17}
$$

is a solution of the system (1.1)–(1.3), and $|u(x,t)| \leq e^{-\Im\theta_+(\omega_0)|x|}$. Therefore, the bound (1.16) is not true.

(iii) Let $\gamma = 0$, $\kappa_- \neq 0$ and $\kappa_0^2 < K_0(\kappa_-)$, where K_0 is defined in (1.8). Then, by Remark 5.4 (ii), there exists a point $\omega_0 \in (0, \kappa_-)$ such that $\tilde{N}(\omega)$ has simple poles at the points $\omega = \pm\omega_0$. Note that $\Im\theta_\pm(\omega_0) > 0$, $\Re\theta_\pm(\omega_0) = -\pi$. Hence, the function (4.17) is a solution of the system (1.1)–(1.3).

The particular case of the cases (i) and (ii) is $\gamma = 0$, $\kappa_- = \kappa_+ \neq 0$, $\nu_- = \nu_+$, $\kappa_0 \neq \kappa_+$. Then $K_-(a_+) = K_+(a_-) = K_0(\kappa_+) = \bar{\kappa}^2 = \kappa_-^2$. Hence, if $\kappa_0 > \kappa_-$, then see item (ii) above, if $\kappa_0 < \kappa_-$, see item (iii).

(iv) Let $\gamma = 0$, $a_- < \kappa_+$, and $\kappa_0^2 \in (K_0(a_-), K_-(\kappa_+))$. Then, by Remark 5.4 (iii), there exists a point $\omega_0 \in (a_-, \kappa_+)$ such that $\tilde{N}(\omega)$ has simple poles at the points $\omega = \pm\omega_0$. Note that $\Im\theta_\pm(\omega_0) > 0$, $\Re\theta_-(\omega_0) = \pi$, $\Re\theta_+(\omega_0) = -\pi$. Hence, the function (4.17) is a solution of the system (1.1)–(1.3).

(v) Let $\gamma \neq 0$, and condition $\mathbf{C}$ or $\mathbf{C}_0$ be not fulfilled. Then, there exists a point $\omega_0 \in \cup_\pm\Lambda_\pm \setminus (\cup_\pm\Lambda^0_\pm)$ such that $\tilde{D}(\omega_0 - i0) = 0$. Write $\theta^0_\pm := \lim_{\varepsilon\to+0}\theta_\pm(\omega_0 + i\varepsilon)$, $\theta^0_\pm \in \mathbb{R}$. Hence, the function of the form

$$
u(x,t) = \left\{ \begin{array}{ll} \sin(\theta^0_+ x + \omega_0 t), & x \geq 0 \\ \sin(\theta^0_- x + \omega_0 t), & x \leq 0 \end{array} \right| \quad t \geq 0,
$$

is a solution of the system (1.1)–(1.3), and the bound (1.16) does not hold.

5. Appendix: Proof of Theorem 3.4

For the chain of the harmonic oscillators in the half-line, Theorem 3.4 was proved in [5]. For the model studied in this paper, the proof is more complex since the singular set $\Lambda_- \cup \Lambda_+$ contains two segments and we have to study carefully the mutual positioning of these segments.

5.1. Properties of $e^{i\theta_\pm(\omega)}$ for $\omega \in \mathbb{C}$. In this subsection, for simplicity, we omit indices + and − in the notations $a_\pm$, $\nu_\pm$, $\kappa_\pm$, $\Lambda_\pm$, $\Lambda^0_\pm$, $\theta_\pm(\omega)$ ($\theta_\pm(\omega)$ is introduced in Lemma 3.2). We indicate the properties of the function $e^{i\theta(\omega)}$ for $\omega \in \mathbb{C} \setminus \Lambda$, $\omega \in \Lambda \setminus \Lambda^0$ and $\omega \in \Lambda^0 = \{\pm\kappa, \pm a\}$ applying the technique of [8, 3].

For $\omega \in \mathbb{C} \setminus \Lambda$, $\Im\theta(\omega) > 0$, and $e^{i\theta(\omega)}$ is an analytic function. Moreover, by (3.2) and the condition $\Im\theta(\omega) > 0$, we have

$$\left|e^{i\theta(\omega)}\right| \le C|\omega|^{-2} \quad \text{as} \quad |\omega| \to \infty. \tag{5.1}$$

Furthermore, if $\omega \in \mathbb{R}$ and $|\omega| > a$, then $e^{i\theta(\omega)} = -e^{-\Im\theta(\omega)}$. If $\omega \in \mathbb{R}$ and $|\omega| < \kappa$ (with $\kappa \ne 0$), then $\Re\theta(\omega) = 0$ and $e^{i\theta(\omega)} = e^{-\Im\theta(\omega)}$. In both cases, $e^{i\theta(\omega_1)} < e^{i\theta(\omega_2)}$ if $|\omega_1| < |\omega_2|$.

For $\omega \in \Lambda \setminus \Lambda^0$, put $\theta(\omega \pm i0) = \lim\limits_{\varepsilon \to +0} \theta(\omega \pm i\varepsilon)$. Since $\overline{\theta(\omega)} = -\theta(\bar\omega)$ for $\omega \in \mathbb{C} \setminus \Lambda$, then $e^{i\theta(\omega - i0)} = \overline{e^{i\theta(\omega+i0)}}$ for $\omega \in \Lambda \setminus \Lambda^0$. Further, if $\omega \in \Lambda \setminus \Lambda^0$, then

$$e^{i\theta(\omega+i0)} = \begin{cases} 1 - \frac{\omega^2-\kappa^2}{2\nu^2} + \frac{i}{2\nu^2}\,\mathrm{sign}\,\omega\,\sqrt{\omega^2-\kappa^2}\sqrt{a^2-\omega^2}, & \text{if} \quad \kappa \ne 0, \\ 1 - \frac{\omega^2}{2\nu^2} + \frac{i\omega}{2\nu^2}\sqrt{4\nu^2-\omega^2}, & \text{if} \quad \kappa = 0. \end{cases}$$

Now we study the behavior of $e^{i\theta(\omega)}$ near the points $\omega \in \Lambda^0$. Eqn (3.2) implies

$$e^{i\theta(\omega)} = \cos\theta(\omega) + i\sin\theta(\omega) = 1 - \frac{\omega^2-\kappa^2}{2\nu^2} + \frac{i}{2\nu^2}\sqrt{(\omega^2-\kappa^2)(a^2-\omega^2)} \tag{5.2}$$

for $\omega \in \mathbb{C} \setminus \Lambda$. The Taylor expansion implies

$$e^{i\theta(\omega)} = 1 + \frac{i}{\nu}\sqrt{\omega^2-\kappa^2} - \frac{1}{2\nu^2}(\omega^2-\kappa^2) - \frac{i}{8\nu^3}(\omega^2-\kappa^2)^{3/2} + \ldots \text{ as } \omega \to \pm\kappa, \tag{5.3}$$

$\omega \in \mathbb{C}_+ := \{\omega \in \mathbb{C} : \Im\omega > 0\}$, $\Im\sqrt{\omega^2-\kappa^2} > 0$. Moreover, $\mathrm{sgn}(\Re\sqrt{\omega^2-\kappa^2}) = \mathrm{sgn}(\Re\omega)$ for $\Im\omega > 0$. This choice of the branch of the complex root $\sqrt{\omega^2-\kappa^2}$ follows from the condition $\Im\theta(\omega) > 0$. Similarly,

$$e^{i\theta(\omega)} = -1 + \frac{i}{\nu}\sqrt{a^2-\omega^2} + \frac{1}{2\nu^2}(a^2-\omega^2) - \frac{i}{8\nu^3}(a^2-\omega^2)^{3/2} + \ldots \text{ as } \omega \to \pm a, \tag{5.4}$$

$\omega \in \mathbb{C}_+$, $\Im\sqrt{a^2-\omega^2} < 0$. Here $\sqrt{a^2-\omega^2}$ is the complex root and we choose the branch of this root such that $\mathrm{sgn}(\Re\sqrt{a^2-\omega^2}) = \mathrm{sgn}(\Re\omega)$ by the condition $\Im\theta(\omega) > 0$.

If $\kappa = 0$, then (5.2) and the Taylor expansion give

$$e^{i\theta(\omega)} = \begin{cases} 1 + \frac{i\omega}{\nu} - \frac{\omega^2}{2\nu^2} - \frac{i\omega^3}{8\nu^3} + \dots & \text{as} \quad \omega \to 0, \\ -1 + i\sqrt{4\nu^2-\omega^2}/\nu + \dots & \text{as} \quad \omega \to \pm 2\nu, \end{cases} \qquad \omega \in \mathbb{C}_+. \tag{5.5}$$

5.2. Properties of $\tilde{D}(\omega)$ and $\tilde{N}(\omega)$ for $\omega \in \mathbb{C}$.

Lemma 5.1. *(i) $\tilde{N}(\omega)$ is meromorphic for $\omega \in \mathbb{C} \setminus (\Lambda_- \cup \Lambda_+)$.*
(ii) $|\tilde{N}(\omega)| = O(|\omega|^{-2})$ as $|\omega| \to \infty$.
(iii) $\tilde{D}(\omega) \neq 0$ for all $\omega \in \mathbb{C}_+$.
(iv) If $\gamma = 0$, then $\tilde{D}(\omega) \neq 0$ for any $\omega \in \mathbb{C}_- = \{\omega \in \mathbb{C} : \Im\omega < 0\}$.

Proof The first assertion of the lemma follows from the formula (3.11) and the analyticity of $\tilde{D}(\omega)$ for $\omega \in \mathbb{C} \setminus (\Lambda_- \cup \Lambda_+)$. The assertion (ii) follows from (3.11) and (5.1). To prove the third assertion, we assume that $\tilde{D}(\omega_0) = 0$ for some $\omega_0 \in \mathbb{C}_+$. Introduce a function $u_*(x,t)$, $x \in \mathbb{Z}$, $t \geq 0$, as

$$u_*(x,t) = \begin{cases} e^{i\theta_+(\omega_0)x} e^{-i\omega_0 t}, & x \geq 0, \quad t \geq 0, \\ e^{-i\theta_-(\omega_0)x} e^{-i\omega_0 t}, & x \leq 0, \quad t \geq 0. \end{cases}$$

It is easy to check that $u_*(x,t)$ is a solution of the problem (1.1)–(1.3) with the initial data $Y_*(x) = e^{\pm i\theta_\pm(\omega_0)x}(1, -i\omega_0)$ for $\pm x \geq 0$. Therefore, the Hamiltonian (see (1.5)) is

$$\mathrm{H}(u_*(\cdot,t), \dot{u}_*(\cdot,t)) = e^{2t\,\Im\omega_0}\mathrm{H}(Y_*) \quad \text{for any } t > 0, \quad \text{where } \ \mathrm{H}(Y_*) > 0.$$

Since $\Im\omega_0 > 0$ and $Y_* \in \mathcal{H}_{0,+}$, this exponential growth contradicts the energy estimate (2.1). Hence, $\tilde{D}(\omega) \neq 0$ for any $\omega \in \mathbb{C}_+$.

If $\gamma = 0$, then $\overline{\tilde{D}(\omega)} = \tilde{D}(\bar{\omega})$, because $\overline{\theta(\omega)} = -\theta(\bar{\omega})$ for $\omega \in \mathbb{C} \setminus \Lambda$. Therefore, item (iv) follows from item (iii) of Lemma 5.1. ∎

Theorem 5.2. *Let condition* **C** *or* $\mathbf{C}_0$ *hold. Then $\tilde{D}(\omega) \neq 0$ for $\omega \in \mathbb{R} \setminus (\cup_\pm \Lambda_\pm)$, $\tilde{D}(\omega \pm i0) \neq 0$ for $\omega \in (\Lambda_- \cup \Lambda_+) \setminus \cup_\pm \Lambda^0_\pm$.*

We study $\tilde{D}(\omega)$ at first for $\omega \notin (\cup_\pm \Lambda_\pm)$ in Lemma 5.3 and then for $\omega \in (\Lambda_- \cup \Lambda_+) \setminus (\cup_\pm \Lambda^0_\pm)$ in Lemmas 5.5 and 5.6.

Let $\omega \in \mathbb{R} \setminus (\cup_\pm \Lambda_\pm)$. We consider three regions of such values of ω:
(I.1) $|\omega| > \max(a_-, a_+)$, **(I.2)** $|\omega| < \min(\kappa_-, \kappa_+) = \kappa_-$ (if $\kappa_- \neq 0$),
(I.3) $a_- < |\omega| < \kappa_+$ (if $a_- < \kappa_+$).

Lemma 5.3. **(I.1)** *For* $|\omega| > \max(a_-, a_+)$, $\tilde{D}(\omega) \neq 0$ *iff either* $\gamma \neq 0$ *or* $\gamma = 0$ *and one of two following inequalities hold (see (1.7))*

$$\kappa_0^2 \le K_+(a_-), \quad \text{if} \quad a_- \ge a_+. \tag{5.6}$$
$$\kappa_0^2 \le K_-(a_+), \quad \text{if} \quad a_+ \ge a_-. \tag{5.7}$$

(I.2) *If* $\kappa_- \neq 0$, *then* $\tilde{D}(\omega) \neq 0$ *for* $|\omega| < \kappa_-$ *iff either* $\gamma \neq 0$ *or* $\gamma = 0$ *and*

$$\kappa_0^2 \ge K_0(\kappa_-). \tag{5.8}$$

(I.3) *If* $a_- < \kappa_+$, *then* $\tilde{D}(\omega) \neq 0$ *for* $|\omega| \in (a_-, \kappa_+)$ *iff either* $\gamma \neq 0$ *or* $\gamma = 0$ *and one of two inequalities holds*

$$\kappa_0^2 \ \ge \ K_-(\kappa_+), \tag{5.9}$$
$$\kappa_0^2 \ \le \ K_0(a_-), \quad \text{if} \ \ K_0(a_-) \ge 0. \tag{5.10}$$

Proof. Case (I.1): For $|\omega| > \max(a_-, a_+)$,

$$\tilde{D}(\omega) = \kappa_0^2 - \omega^2 + \sum_{\pm} \nu_\pm^2 \left(1 - e^{i\theta_\pm(\omega)}\right) - i\omega\gamma, \quad \text{where} \ \ e^{i\theta_\pm(\omega)} = -e^{-\Im\theta_\pm(\omega)}.$$

Hence, $\Im\tilde{D}(\omega) \neq 0$ iff $\gamma \neq 0$. If $\gamma = 0$, then $\tilde{D}(\omega) \neq 0$ iff

$$\tilde{D}(\omega) = \Re\tilde{D}(\omega) = \kappa_0^2 - \omega^2 + \sum_{\pm} \nu_\pm^2 \left(1 + e^{-\Im\theta_\pm(\omega)}\right) \neq 0.$$

Note that $\Re\tilde{D}(\omega_1) < \Re\tilde{D}(\omega_2)$ for $|\omega_1| > |\omega_2| \ge \max(a_-, a_+)$, and $\Re\tilde{D}(\omega) \to -\infty$ as $|\omega| \to \infty$. Therefore, $\Re\tilde{D}(\omega) \neq 0$ for such values of ω iff

$$\Re\tilde{D}(\max a_\pm) \le 0. \tag{5.11}$$

Now we find conditions on $\kappa_\pm, \kappa_0, \nu_\pm$ such that (5.11) holds.

Let $\max(a_-, a_+) = a_-$. Then

$$\Re\tilde{D}(a_-) = \kappa_0^2 - a_-^2 + 2\nu_-^2 + \nu_+^2 \left(1 + e^{-\Im\theta_+(a_-)}\right). \tag{5.12}$$

We calculate $r := -e^{i\theta_+(a_-)} = e^{-\Im\theta_+(a_-)}$. By (3.2), we have

$$r + \frac{1}{r} = -2\cos\theta_+(a_-) = \frac{a_-^2 - \kappa_+^2}{\nu_+^2} - 2.$$

Since $r < 1$, then

$$r = \frac{b_+ - 2 - \sqrt{b_+^2 - 4b_+}}{2}, \quad \text{where} \ \ b_+ := \frac{a_-^2 - \kappa_+^2}{\nu_+^2}. \tag{5.13}$$

Note that $b_+ > 0$ and $b_+ - 4 = (a_-^2 - a_+^2)/\nu_+^2 \geq 0$. Therefore,

$$\nu_+^2\left(1 + e^{-\Im\theta_+(a_-)}\right) = \nu_+^2(1+r) = \frac{1}{2}\left(a_-^2 - \kappa_+^2 - \sqrt{a_-^2 - \kappa_+^2}\sqrt{a_-^2 - a_+^2}\right). \quad (5.14)$$

We substitute (5.14) in (5.12) and obtain

$$\Re\tilde{D}(a_-) = \kappa_0^2 - K_+(a_-). \quad (5.15)$$

Thus, $\Re\tilde{D}(a_-) \leq 0$ iff (5.6) holds.

In the case when $\max(a_-, a_+) = a_+$, we have

$$\Re\tilde{D}(a_+) = \kappa_0^2 - a_+^2 + 2\nu_+^2 + \nu_-^2(1 + e^{-\Im\theta_-(a_+)}),$$

$$e^{-\Im\theta_-(a_+)} = \frac{b_- - 2 - \sqrt{b_-^2 - 4b_-}}{2}, \quad \text{where} \quad b_- := \frac{a_+^2 - \kappa_-^2}{\nu_-^2}. \quad (5.16)$$

Thus, $\Re\tilde{D}(a_+) \leq 0$ iff (5.7) holds.

Case (I.2): Since we assume that $\kappa_- \leq \kappa_+$, then $\min(\kappa_-, \kappa_+) = \kappa_-$. Moreover, for $|\omega| < \kappa_-$ ($\kappa_- \neq 0$),

$$\tilde{D}(\omega) = \kappa_0^2 - \omega^2 + \sum_\pm \nu_\pm^2\left(1 - e^{i\theta_\pm(\omega)}\right) - i\omega\gamma, \quad \text{where} \quad e^{i\theta_\pm(\omega)} = e^{-\Im\theta_\pm(\omega)}.$$

Note that $\tilde{D}(0) = \kappa_0^2 + \sum_\pm \nu_\pm^2\left(1 - e^{-\Im\theta_\pm(0)}\right) > \kappa_0^2 \geq 0$, since

$$e^{-\Im\theta_\pm(0)} = 4\left(\frac{\kappa_\pm}{\nu_\pm} + \sqrt{4 + \frac{\kappa_\pm^2}{\nu_\pm^2}}\right)^{-2} < 1, \quad \text{if} \quad \kappa_\pm > 0. \quad (5.17)$$

Hence, $\tilde{D}(0) \neq 0$. Therefore, if $\omega \neq 0$, then $\Im\tilde{D}(\omega) \neq 0$ iff $\gamma \neq 0$. Assume that $\omega \neq 0$ and $\gamma = 0$ and find restrictions on $\kappa_\pm, \kappa_0, \nu_\pm$ such that $\tilde{D}(\omega) \neq 0$. Note that $\Re\tilde{D}(\omega_1) > \Re\tilde{D}(\omega_2)$ for $|\omega_1| < |\omega_2| < \min(\kappa_-, \kappa_+)$, and $\Re\tilde{D}(0) > 0$. Therefore, $\Re\tilde{D}(\omega) \neq 0$ for such values of ω iff $\Re\tilde{D}(\kappa_-) \geq 0$. Since $\gamma = 0$,

$$\Re\tilde{D}(\kappa_-) = \tilde{D}(\kappa_-) = \kappa_0^2 - \kappa_-^2 + \nu_+^2(1 - e^{i\theta_+(\kappa_-)}). \quad (5.18)$$

We calculate $r := e^{i\theta_+(\kappa_-)} = e^{-\Im\theta_+(\kappa_-)}$. By (3.2), we have

$$r + \frac{1}{r} = 2\cos\theta_+(\kappa_-) = 2 - \frac{\kappa_-^2 - \kappa_+^2}{\nu_+^2}.$$

Since $r < 1$, then $r = (d_+ + 2 - \sqrt{d_+^2 + 4d_+})/2$, where $d_+ := \frac{\kappa_+^2 - \kappa_-^2}{\nu_+^2}$. Therefore,

$$\nu_+^2\left(1 - e^{i\theta_+(\kappa_-)}\right) = \nu_+^2(1-r) = \frac{1}{2}\left(\kappa_-^2 - \kappa_+^2 + \sqrt{\kappa_+^2 - \kappa_-^2}\sqrt{a_+^2 - \kappa_-^2}\right). \quad (5.19)$$

We substitute (5.19) in (5.18) and obtain

$$\Re\tilde{D}(\kappa_-) = \kappa_0^2 - K_0(\kappa_-). \tag{5.20}$$

Thus, $\Re\tilde{D}(\kappa_-) \geq 0$ iff (5.8) holds.

Case (I.3): For $\omega \in \mathbb{R}$: $a_- < |\omega| < \kappa_+$, $e^{i\theta_+(\omega)} = e^{-\Im\theta_+(\omega)}$, $e^{i\theta_-(\omega)} = -e^{-\Im\theta_-(\omega)}$. If $\gamma \neq 0$, then $\Im\tilde{D}(\omega) = -\omega\gamma \neq 0$. Assume that $\gamma = 0$ and find the restrictions on $\kappa_\pm, \kappa_0, \nu_\pm$ such that $\tilde{D}(\omega) \equiv \Re\tilde{D}(\omega) \neq 0$. Note that $\Re\tilde{D}(\omega_1) < \Re\tilde{D}(\omega_2)$ for $|\omega_1| > |\omega_2|$. Hence, $\Re\tilde{D}(\omega) \neq 0$ iff $\Re\tilde{D}(\kappa_+) \geq 0$ or $\Re\tilde{D}(a_-) \leq 0$. Calculate $\Re\tilde{D}(\kappa_+)$ and $\Re\tilde{D}(a_-)$.

$$\Re\tilde{D}(\kappa_+) = \kappa_0^2 - \kappa_+^2 + \nu_-^2(1 - e^{i\theta_-(\kappa_+)}),$$

where $r := -e^{i\theta_-(\kappa_+)} = e^{-\Im\theta_-(\kappa_+)}$ is a solution of the following equation

$$r + \frac{1}{r} = -2\cos\theta_-(\kappa_+) = \frac{\kappa_+^2 - \kappa_-^2}{\nu_-^2} - 2.$$

Since $r < 1$, then $r = (d_- - 2 - \sqrt{d_-^2 - 4d_-})/2$, with $d_- := (\kappa_+^2 - \kappa_-^2)/\nu_-^2$. Therefore,

$$\nu_-^2\left(1 - e^{i\theta_-(\kappa_+)}\right) = \nu_-^2(1 + r) = \frac{1}{2}\left(\kappa_+^2 - \kappa_-^2 - \sqrt{\kappa_+^2 - \kappa_-^2}\sqrt{\kappa_+^2 - a_-^2}\right).$$

Hence,

$$\Re\tilde{D}(\kappa_+) = \kappa_0^2 - K_-(\kappa_+). \tag{5.21}$$

Thus, $\Re\tilde{D}(\kappa_+) \geq 0$ iff (5.9) holds.

On the other hand, $\Re\tilde{D}(a_-) = \kappa_0^2 - a_-^2 + 2\nu_-^2 + \nu_+^2\left(1 - e^{-\Im\theta_+(a_-)}\right)$. Let us calculate $r := e^{i\theta_+(a_-)} = e^{-\Im\theta_+(a_-)}$ for $a_+ > a_-$. By (3.2), we have (cf (5.13))

$$e^{-\Im\theta_+(a_-)} = \frac{c_+ + 2 - \sqrt{c_+^2 + 4c_+}}{2}, \quad \text{where} \quad c_+ := \frac{\kappa_+^2 - a_-^2}{\nu_+^2}.$$

Therefore,

$$\Re\tilde{D}(a_-) = \kappa_0^2 - K_0(a_-). \tag{5.22}$$

Hence, $\Re\tilde{D}(a_-) \leq 0$ iff (5.10) holds. Note that

$$K_0(a_-) \geq 0 \Longleftrightarrow \nu_+^2 \leq (\kappa_-^2 + 2\nu_-^2)(\kappa_+^2 - 2\nu_-^2)(\kappa_+^2 - a_-^2)^{-1} \ (\text{where} \ \ \kappa_+ > a_+).$$

Remark 5.4. (i) Let $\gamma = 0$ and condition (5.6) (if $a_- \geq a_+$) or condition (5.7) (if $a_- \leq a_+$) be not true. Then, there exists a point $\omega_0 > \max(a_-, a_+)$ such that

$\tilde{D}(\pm\omega_0) = 0$. Note that $\tilde{D}'(\omega_0) < 0$. Therefore, $\tilde{N}(\omega)$ has simple poles at the points $\omega = \pm\omega_0$.

(ii) Let $\gamma = 0$, $\kappa_- \neq 0$ and condition (5.8) be not true. Then, there exists a point $\omega_0 \in (0, k_-)$ such that $\tilde{D}(\pm\omega_0) = 0$. Moreover, $\tilde{D}'(\omega_0) < 0$. Therefore, $\tilde{N}(\omega)$ has simple poles at the points $\omega = \pm\omega_0$.

(iii) Let $\gamma = 0$, $a_- < \kappa_+$, and conditions (5.9) and (5.10) be not true. Then, there exists a point $\omega_0 \in (a_-, \kappa_+)$ such that $\tilde{D}(\pm\omega_0) = 0$ and $\tilde{D}'(\pm\omega_0) \neq 0$. Hence, $\tilde{N}(\omega)$ has simple poles at the points $\omega = \pm\omega_0$.

Lemma 5.5. *For any* $\omega \in \cup_\pm\Lambda_\pm \setminus \overline{(\cup_\pm\Lambda^0_\pm)}$, $\tilde{D}(\omega + i0) \neq 0$.

If $\gamma = 0$, *then* $\tilde{D}(\omega - i0) = \overline{\tilde{D}(\omega + i0)} \neq 0$.

Proof For $\omega \in \Lambda_- \setminus \Lambda^0_-$ and $\omega \notin \Lambda_+$,

$$\begin{aligned}\Im\tilde{D}(\omega + i0) &= -\omega\gamma - \nu_-^2 \sin\theta_-(\omega + i0)\\ &= \begin{cases} -\mathrm{sign}(\omega)\left(|\omega|\gamma + \frac{1}{2}\sqrt{\omega^2 - \kappa_-^2}\sqrt{a_-^2 - \omega^2}\right), & \text{if } \kappa_- \neq 0\\ -\omega\left(\gamma + \frac{1}{2}\sqrt{4\nu_-^2 - \omega^2}\right), & \text{if } \kappa_- = 0\end{cases}\end{aligned}$$

Hence, $\Im\tilde{D}(\omega + i0) \neq 0$ for such values of ω. Similarly, we can check that for $\omega \in \Lambda_+ \setminus \Lambda^0_+$ and $\omega \notin \Lambda_-$, $\Im\tilde{D}(\omega + i0) \neq 0$. For $\omega \in (\Lambda_- \cap \Lambda_+) \setminus \cup_\pm\Lambda^0_\pm$,

$$\begin{aligned}\Im\tilde{D}(\omega + i0) &= -\omega\gamma - \nu_-^2 \sin\theta_-(\omega + i0) - \nu_+^2 \sin\theta_+(\omega + i0)\\ &= \begin{cases} -\mathrm{sign}(\omega)\Big(|\omega|\gamma + \frac{1}{2}\sum\limits_{\pm}\sqrt{\omega^2 - \kappa_\pm^2}\sqrt{a_\pm^2 - \omega^2}\Big), & \text{if } \kappa_\pm \neq 0\\ -\mathrm{sign}(\omega)\Big(|\omega|\gamma + |\omega|\sqrt{\nu_-^2 - \omega^2/4} + \frac{1}{2}\sqrt{\omega^2 - \kappa_+^2}\sqrt{a_+^2 - \omega^2}\Big), & \\ & \text{if } \kappa_- = 0, \kappa_+ \neq 0\\ -\omega\Big(\gamma + \sum\limits_{\pm}\sqrt{\nu_\pm^2 - \omega^2/4}\Big), & \text{if } \kappa_- = \kappa_+ = 0\end{cases}\end{aligned}$$

Therefore, $\Im\tilde{D}(\omega + i0) \neq 0$ for such values of ω. ■

Now we consider the case when $\gamma \neq 0$ and find necessary and sufficient conditions on the constants of the system under which $\tilde{D}(\omega - i0) \neq 0$ for $\omega \in (\cup_\pm\Lambda_\pm) \setminus (\cup_\pm\Lambda^0_\pm)$. Below we consider four regions of such values of ω:
(II.1) $\omega \in \Lambda_- \setminus \Lambda^0_-$, $\omega \notin \Lambda_+$, $|\omega| < \kappa_+$, i.e. $\kappa_- < |\omega| < \min(\kappa_+, a_-)$ $(\kappa_+ > \kappa_-)$,
(II.2) $\omega \in \Lambda_- \setminus \Lambda^0_-$, $\omega \notin \Lambda_+$, $|\omega| > a_+$, i.e., $a_+ < |\omega| < a_-$ (if $a_+ < a_-$),
(II.3) $\omega \in \Lambda_+ \setminus \Lambda^0_+$, $\omega \notin \Lambda_-$, i.e., $\max(\kappa_+, a_-) < |\omega| < a_+$ (if $a_- < a_+$),
(II.4) $\omega \in (\Lambda_- \cap \Lambda_+) \setminus (\cup_\pm\Lambda^0_\pm)$, i.e., $\kappa_+ < |\omega| < \min(a_-, a_+)$ (if $\kappa_+ < a_-$).

Lemma 5.6. *Let* $\gamma > 0$. *In the case* **(II.1)**, $\tilde{D}(\omega - i0) \neq 0$ *iff the following conditions hold.*

(i) If $\kappa_- = 0$ *and* $\kappa_+ \geq a_- \equiv 2\nu_-$*, then either (1)* $\gamma \geq \nu_-$ *or (2)* $\gamma \in (0, \nu_-)$ *and*

$$\kappa_0^2 \neq K_0(\omega_*) \tag{5.23}$$

with $\omega_* = 2\sqrt{\nu_-^2 - \gamma^2}$*.*

(ii) If $\kappa_- = 0$ *and* $\kappa_+ < a_-$*, then either (1)* $\gamma \geq \nu_-$ *or (2)* $\gamma \in (\gamma_1, \nu_-)$ *and* (5.23) *holds with* $\omega_* = 2\sqrt{\nu_-^2 - \gamma^2}$ *or (3)* $\gamma \in (0, \gamma_1]$*.*

(iii) If $\kappa_- \neq 0$ *and* $\kappa_+ \geq a_-$*, then either (1)* $\gamma > \gamma_-^{cr}$ *or (2)* $\gamma \in (0, \gamma_-^{cr}]$ *and* (5.23) *holds with* $\omega_* = P_\pm$*, where* $P_\pm$ *from* (1.14)*. In particular, if* $\gamma = \gamma_-^{cr}$*, then* $\omega_* = \sqrt{\kappa_- a_-}$*.*

(iv) If $\kappa_- \neq 0$ *and* $\kappa_+ \in \left(\sqrt{\kappa_- a_-}, a_-\right)$*, then either (1)* $\gamma > \gamma_-^{cr}$ *or (2)* $\gamma \in (\gamma_1, \gamma_-^{cr}]$ *and* (5.23) *holds with* $\omega_* = P_\pm$ *(if* $\gamma = \gamma_-^{cr}$*, then* $P_\pm = \sqrt{\kappa_- a_-}$*) or (3)* $\gamma \in (0, \gamma_1]$ *and* (5.23) *holds with* $\omega_* = P_-$*.*

(v) If $\kappa_- \neq 0$ *and* $\kappa_+ \in (\kappa_-, \sqrt{\kappa_- a_-}]$*, then either (1)* $\gamma \geq \gamma_1$ *or (2)* $\gamma \in (0, \gamma_1)$ *and* (5.23) *holds with* $\omega_* = P_-$*. Note that if* $\kappa_+ = \sqrt{\kappa_- a_-}$*, then* $\gamma_-^{cr} = \gamma_1$*. In the all cases (i)–(v),* $\omega_* \in (\kappa_-, \min(\kappa_+, a_-))$*.*

In the case **(II.2)***,* $\tilde{D}(\omega - i0) \neq 0$ *iff the following conditions hold.*

(i) If $\kappa_- = 0$*, then either (1)* $\gamma \geq \gamma_2$ *or (2)* $\gamma \in (0, \gamma_2)$ *and*

$$\kappa_0^2 \neq K_+(\omega_*) \tag{5.24}$$

with $\omega_* = 2\sqrt{\nu_-^2 - \gamma^2}$*.*

(ii) If $\kappa_- \neq 0$ *and* $a_+ \geq \sqrt{\kappa_- a_-}$*, then either (1)* $\gamma \geq \gamma_2$ *or (2)* $\gamma \in (0, \gamma_2)$ *and* (5.24) *holds with* $\omega_* = P_+$*.*

(iii) If $\kappa_- \neq 0$ *and* $a_+ < \sqrt{\kappa_- a_-}$*, then either (1)* $\gamma > \gamma_-^{cr}$ *or (2)* $\gamma \in (\gamma_2, \gamma_-^{cr}]$ *and* (5.24) *holds with* $\omega_* = P_\pm$ *(if* $\gamma = \gamma_-^{cr}$*, then* $P_\pm = \sqrt{\kappa_- a_-}$*) or (3)* $\gamma \in (0, \gamma_2]$ *and* (5.24) *holds with* $\omega_* = P_+$*.*

In the all cases (i)–(iii), $\omega_* \in (a_+, a_-)$*.*

In the case **(II.3)***,* $\tilde{D}(\omega - i0) \neq 0$ *iff the following conditions hold.*

(i) If $\kappa_- = \kappa_+ = 0$*, then either (1)* $\gamma \geq \gamma_3$ *or (2)* $\gamma \in (0, \gamma_3)$ *and*

$$\kappa_0^2 \neq K_-(\omega_*) \tag{5.25}$$

with $\omega_* = 2\sqrt{\nu_+^2 - \gamma^2}$*.*

(ii) If $\kappa_+ \neq 0$ *and* $a_- \in [\sqrt{\kappa_+ a_+}, a_+)$*, then either (1)* $\gamma \geq \gamma_3$ *or (2)* $\gamma \in (0, \gamma_3)$ *and* (5.25) *holds with* $\omega_* = Q_+$*, where* $Q_\pm$ *from* (1.15)*. In particular, if* $a_- = \sqrt{\kappa_+ a_+}$*, then* $\gamma_3 = \gamma_+^{cr}$ *and* $\omega_* = \sqrt{\kappa_+ a_+}$*.*

(iii) If $\kappa_+ \neq 0$ *and* $a_- \in (\kappa_+, \sqrt{\kappa_+ a_+})$*, then either (1)* $\gamma > \gamma_+^{cr}$ *or (2)* $\gamma \in (\gamma_3, \gamma_+^{cr}]$ *and* (5.25) *holds with* $\omega_* = Q_\pm$ *(if* $\gamma = \gamma_+^{cr}$*, then* $\omega_* = \sqrt{\kappa_+ a_+}$*) or (3)* $\gamma \in (0, \gamma_3]$ *and* (5.25) *holds with* $\omega_* = Q_+$*.*

(iv) If $a_- \leq \kappa_+$*, then either (1)* $\gamma > \gamma_+^{cr}$ *or (2)* $\gamma \in (0, \gamma_+^{cr}]$ *and* (5.25) *holds with* $\omega_* = Q_\pm$ *(if* $\gamma = \gamma_+^{cr}$*, then* $Q_\pm = \sqrt{\kappa_+ a_+}$*).*

In the all cases (i)–(iv), $\omega_* \in (\max(\kappa_+, a_-); a_+)$.

In the case **(II.4)**, $\tilde{D}(\omega - i0) \neq 0$ *iff the following conditions hold.*

(i) If $\kappa_+ = \kappa_- = 0$, *then either (1)* $\gamma \geq \nu_+ + \nu_-$ *or (2)* $\gamma \in (\sqrt{|\nu_-^2 - \nu_+^2|}, \nu_+ + \nu_-)$ *and* $\kappa_0 \neq 0$ *or (3)* $\gamma \in (0, \sqrt{|\nu_-^2 - \nu_+^2|}\,]$. *If, in addition,* $\nu_+ = \nu_-$, *then the case (3) is excluded.*

(ii) If $\kappa_+ = \kappa_- \neq 0$ *or* $a_+ = a_-$, *then either (1)* $\gamma > F_{\max}$ *(*$F_{\max}$ *is defined in (1.13)) or (2)* $\gamma \in (0, F_{\max}]$ *and* $\kappa_0 \neq \bar{\kappa}$. *In particular, if* $\kappa_+ = \kappa_- \neq 0$ *and* $a_+ = a_-$, *then* $F_{\max} = a_- - \kappa_-$.

(iii) If $\kappa_+ \neq \kappa_-$ *and* $a_+ \neq a_-$, *then either (1)* $\gamma > F_{\max}$ *or (2)* $\gamma \in (F_{\min}, F_{\max}]$ *and* $\kappa_0 \neq \bar{\kappa}$ *or (3)* $\gamma \in (0, F_{\min}]$. *Here* $F_{\min} = \min(\gamma_1, \gamma_2)$ *if* $a_+ < a_-$, *and* $F_{\min} = \min(\gamma_1, \gamma_3)$ *if* $a_+ > a_-$.

Proof Since $\tilde{D}(\omega - i0) = \overline{\tilde{D}(\omega + i0)} - 2i\omega\gamma$, then for $\omega \in \Lambda_-$,

$$\begin{aligned} \Re\tilde{D}(\omega - i0) &= \kappa_0^2 - \omega^2 + \nu_+^2(1 - e^{i\theta_+(\omega)}) + \nu_-^2(1 - \cos\theta_-(\omega + i0)) \\ \Im\tilde{D}(\omega - i0) &= \nu_-^2 \sin\theta_-(\omega + i0) - \omega\gamma. \end{aligned}$$

(II.1): In this case, we have

$$e^{i\theta_+(\omega)} = e^{-\Im\theta_+(\omega)} = 1 + \frac{\kappa_+^2 - \omega^2}{2\nu_+^2} - \frac{1}{2\nu_+^2}\sqrt{\kappa_+^2 - \omega^2}\sqrt{a_+^2 - \omega^2}.$$

Hence, $\tilde{D}(\omega - i0) = 0$ for such values of ω iff

$$\kappa_0^2 = K_0(\omega), \quad \omega^2 \in (\kappa_-^2, \min(\kappa_+^2, a_-^2)) \tag{5.26}$$

$$\sqrt{\omega^2 - \kappa_-^2}\sqrt{a_-^2 - \omega^2} = 2|\omega|\gamma. \tag{5.27}$$

If $\kappa_- = 0$, (5.27) becomes $\sqrt{4\nu_-^2 - \omega^2} = 2\gamma$ and the assertion (i) is obvious, since $\omega = \pm\omega_* = \pm 2\sqrt{\nu_-^2 - \gamma^2}$ are the solutions of (5.27) if $\gamma \in (0, \nu_-)$. The assertion (ii) can be checked by a similar way.

Let $\kappa_- \neq 0$ and $p := \omega^2 - \kappa_-^2$. Then, p is a solution of the equation

$$p^2 + 4p(\gamma^2 - \nu_-^2) + 4\kappa_-^2\gamma^2 = 0 \tag{5.28}$$

in the region $p \in (\min(\kappa_+^2 - \kappa_-^2), 4\nu_-^2)$. Eqn (5.28) has solutions iff $(\gamma^2 - \nu_-^2)^2 - \kappa_-^2\gamma^2 \geq 0$ and $0 < \gamma < \nu_-$. This is equivalent to the condition $\gamma \in (0, \gamma_-^{cr}]$. Therefore, if $\kappa_- \neq 0$ and $\gamma \in (0, \gamma_-^{cr}]$, then Eqn (5.28) has solutions $p_\pm \in (0, 4\nu_-^2)$ with $p_\pm$ from (1.14). Hence, Eqn (5.27) has the solutions of the form

$$\omega = \pm\omega_*, \quad \text{where } \omega_*^2 = P_\pm^2 = \kappa_-^2 + p_\pm \in (\kappa_-^2, a_-^2). \tag{5.29}$$

Therefore, if $\kappa_+ \geq a_-$, then the system (5.26)–(5.27) has no solutions in the interval **(II.1)** iff the conditions (iii) are fulfilled.

Let $\kappa_- \neq 0$ and $\kappa_+ < a_-$. Then, $\kappa_- < |\omega| < \kappa_+$. For the function $F_-(\omega)$ (see (1.9)), $\max_{(\kappa_-, a_-)} F_-(\omega) = F_-(\omega_0) = \gamma_-^{cr}$, where $\omega_0 = \sqrt{a_- \kappa_-}$. We calculate the minimum and maximum of the function $F_-(\omega)$ in the interval $\omega \in (\kappa_-, \kappa_+)$ and deduce the rest conditions (iii)–(v).

(II.2): Let $\omega \in \Lambda_- \setminus \Lambda_-^0$, $\omega \notin \Lambda_+$, and $a_+ < |\omega| < a_-$. Then,

$$e^{i\theta_+(\omega)} = -e^{-\Im\theta_+(\omega)} = 1 - \frac{\omega^2 - \kappa_+^2}{2\nu_+^2} + \frac{1}{2\nu_+^2}\sqrt{\omega^2 - \kappa_+^2}\sqrt{\omega^2 - a_+^2}.$$

Hence, $\tilde{D}(\omega - i0) = 0$ iff Eqn (5.27) holds and $\kappa_0^2 = K_+(\omega)$. $F_-(a_-) = 0$, $F_-(a_+) = \gamma_2$. By the similar reasoning as before, we can derive conditions under which $\tilde{D}(\omega - i0) \neq 0$ in the region $|\omega| \in (a_+, a_-)$.

(II.3): In this case, $e^{i\theta_-(\omega)} = -e^{-\Im\theta_-(\omega)}$. This case is similar to the case **(II.2)** and all arguments remain valid up to the replacement of the plus and minus indices.

(II.4): Let $\omega \in (\Lambda_- \cap \Lambda_+) \setminus \cup_\pm \Lambda_\pm^0$. $\tilde{D}(\omega - i0) \neq 0$ in this interval iff one of the following two equalities is not fulfilled

$$\kappa_0 = \bar{\kappa} \tag{5.30}$$
$$F(\omega) = \gamma \quad \text{for} \quad \omega \in (\kappa_+, a_m), \ a_m := \min(a_-, a_+), \ \gamma > 0, \tag{5.31}$$

where $F(\omega)$ is defined in (1.10).

(i) Assume first that $\kappa_+ = 0$. Then, $\kappa_+ = \kappa_- = 0$, $a_\pm = 2\nu_\pm$, and Eqn (5.31) becomes $\sum_\pm \sqrt{4\nu_\pm^2 - \omega^2}/2 = \gamma$. In particular, if $\kappa_+ = \kappa_- = 0$ and $\nu_- = \nu_+$, then Eqn (5.31) becomes $\sqrt{4\nu_-^2 - \omega^2} = \gamma$. Moreover, $F_{\max} = F(0) = \nu_- + \nu_+$, $F_{\min} = F(a_m) = \sqrt{|\nu_-^2 - \nu_+^2|}$. This implies assertion (i).

(ii) Let $\kappa_+ \neq 0$. Then, $F'(\omega) \to +\infty$ as $\omega \to \kappa_+ + 0$, $F'(\omega) \to -\infty$ as $\omega \to a_m - 0$, $F''(\omega) \leq 0$ for $\omega \in (\kappa_+, a_m)$. There exists a unique point $\omega_0 \in (\kappa_+, a_m)$ such that $F'(\omega_0) = 0$, and $F_{\max} = F(\omega_0)$.

If $\kappa_+ = \kappa_- \neq 0$ or $a_+ = a_-$, then $F_{\min} = 0$. This implies assertion (ii).

(iii) If $\kappa_+ \neq \kappa_-$ and $a_+ \neq a_-$, then $F_{\min} = \min(\gamma_1, \gamma_2)$ if $a_+ < a_-$, and $F_{\min} = \min(\gamma_1, \gamma_3)$ if $a_+ > a_-$, where $\gamma_1, \gamma_2, \gamma_3$ are defined in (1.11). This implies assertion (iii). ■

Proof of Theorem 5.2 For each mutual disposition of the segments Λ_+ and Λ_- (or of the points $\kappa_\pm$, $a_\pm$), we first divide $\mathbb{R} \setminus (\cup_\pm \Lambda_\pm^0)$ into the intervals **(I.1)**–**(I.3)** and **(II.1)**–**(II.4)** (if they exist). Then, using Lemmas 5.3 and 5.6, we deduce the corresponding restrictions on γ and κ_0 (see conditions **C** and $\mathbf{C}_0$) such that $\tilde{D}(\omega) \neq 0$ for $\omega \in \mathbb{R} \setminus (\cup_\pm \Lambda_\pm)$ and $\tilde{D}(\omega \pm i0) \neq 0$ for $\omega \in \cup_\pm \Lambda_\pm \setminus (\cup_\pm \Lambda_\pm^0)$.

We split these restrictions into conditions **C** and $\mathbf{C}_0$ because of the difference in the asymptotic behaviour of $\tilde{D}(\omega)$ at the points $\Lambda^0_\pm$ (see Lemma 5.7 below). ■

5.3. Behavior of $\tilde{D}(\omega)$ and $\tilde{N}(\omega)$ near singular points in $\Lambda^0_-\cup\Lambda^0_+$. Now we study the behavior of $\tilde{N}(\omega)$ in the neighborhood of the singular points in $\Lambda^0_-\cup\Lambda^0_+$. This behavior will be used in the proof of Theorem 3.4 below.

Lemma 5.7. *Let $\omega_0\in\Lambda^0_-\cup\Lambda^0_+$ and $\omega_0\neq 0$. If condition **C** holds, then*

$$\tilde{N}(\omega)=C_1+iC_2(\omega^2-\omega_0^2)^{1/2}+\dots\quad as\quad \omega\to\pm\omega_0,\quad \omega\in\mathbb{C}_\pm, \tag{5.32}$$

with some $C_1,C_2\neq 0$. If condition $\mathbf{C}_0$ holds, then either (5.32) is true or

$$\tilde{N}(\omega)=iC_1(\omega^2-\omega_0^2)^{-1/2}+C_2+\dots\quad as\quad \omega\to\pm\omega_0,\quad \omega\in\mathbb{C}_\pm, \tag{5.33}$$

with some $C_1\neq 0$.

*Let $\omega_0=0\in\Lambda^0_-\cup\Lambda^0_+$ and conditions **C** or $\mathbf{C}_0$ hold. Then*

$$\tilde{N}(\omega)=C_1+iC_2\omega+\dots\quad as\quad \omega\to 0,\quad \omega\in\mathbb{C}_\pm, \tag{5.34}$$

with some $C_1,C_2\neq 0$.

To prove Lemma 5.7, we consider various cases of mutual disposition of the points $\kappa_\pm$ and $a_\pm$.

(1) Let $\omega_0=\pm\kappa_-$. We consider the following cases: **(1.1)** $\kappa_-=\kappa_+=0$; **(1.2)** $\kappa_-=\kappa_+\neq 0$; **(1.3)** $\kappa_-=0$, $\kappa_+>0$; **(1.4)** $0<\kappa_-<\kappa_+$.

(1.1) If $\kappa_-=\kappa_+=0$, then we apply the representation (5.5) to $e^{i\theta_\pm(\omega)}$ and obtain

$$\begin{aligned}\tilde{D}(\omega)&=\kappa_0^2-\omega^2+\sum_\pm\nu_\pm^2\Big(-\frac{i\omega}{\nu_\pm}+\frac{\omega^2}{2\nu_\pm^2}+\frac{i\omega^3}{8\nu_\pm^3}-\dots\Big)-i\omega\gamma\\&=\kappa_0^2-i\omega(\nu_-+\nu_++\gamma)+i\omega^3\Big(\frac{1}{8\nu_-}+\frac{1}{8\nu_+}\Big)-\dots\quad\text{as}\quad\omega\to 0.\end{aligned}$$

In particular, $\tilde{D}(0)=\kappa_0^2\neq 0$ by condition **C** or $\mathbf{C}_0$. Hence, (5.34) is true with $C_1=\kappa_0^{-2}$ and $C_2=(\nu_-+\nu_++\gamma)\kappa_0^{-4}$. Note that if $\kappa_-=\kappa_+=\kappa_0=0$ (this case is excluded by conditions **C** and $\mathbf{C}_0$), then $\tilde{N}(\omega)$ has a simple pole at zero,

$$\tilde{N}(\omega)=\frac{i}{\omega(\nu_-+\nu_++\gamma)}+\frac{i\omega}{8(\nu_-+\nu_++\gamma)^2}\Big(\frac{1}{\nu_-}+\frac{1}{\nu_+}\Big)+\dots,\quad\omega\to 0. \tag{5.35}$$

(1.2) If $\kappa_-=\kappa_+\neq 0$, then we apply the representation (5.3) to $e^{i\theta_\pm(\omega)}$ and obtain

$$\begin{aligned}\tilde{D}(\omega)&=\kappa_0^2-\omega^2+\sum_\pm\nu_\pm^2\Big(-\frac{i}{\nu_\pm}\sqrt{\omega^2-\kappa_-^2}+\frac{1}{2\nu_\pm^2}(\omega^2-\kappa_-^2)+\dots\Big)-i\omega\gamma\\&=[\kappa_0^2-\kappa_-^2\mp i\kappa_-\gamma]-i(\nu_-+\nu_+)(\omega^2-\kappa_-^2)^{1/2}-i(\omega\mp\kappa_-)\gamma+\dots\end{aligned}$$

as $\omega \to \pm\kappa_-$, $\omega \in \mathbb{C}_+$. In particular, $\tilde{D}(\pm\kappa_-) = \kappa_0^2 - \kappa_-^2 \mp i\kappa_-\gamma \neq 0$ iff either $\gamma \neq 0$ or $\gamma = 0$ and $\kappa_0 \neq \kappa_-$. Hence, if condition **C** holds, then (5.32) is true with $\omega_0 = \pm\kappa_-$, $C_1 = (\kappa_0^2 - \kappa_-^2 \mp i\kappa_-\gamma)^{-1}$, $C_2 = C_1^2(\nu_- + \nu_+)$. If condition $\mathbf{C}_0$ (iv) holds, then (5.33) is true with $\omega_0 = \pm\kappa_-$, $C_1 = (\nu_- + \nu_+)^{-1}$.

(1.3) If $\kappa_- = 0$ and $\kappa_+ > 0$, then the function $e^{i\theta_+(\omega)}$ is an analytic function in a small neighborhood of origin. Applying the representation (5.5) to $e^{i\theta_-(\omega)}$, we obtain

$$\begin{aligned}\tilde{D}(\omega) &= \kappa_0^2 - \omega^2 + \nu_-^2\Big(-\frac{i\omega}{\nu_-} + \frac{\omega^2}{2\nu_-^2} + \ldots\Big) + \nu_+^2\Big(1 - e^{-\Im\theta_+(\omega)}\Big) - i\omega\gamma \\ &= \Big[\kappa_0^2 + \nu_+^2\Big(1 - e^{-\Im\theta_+(0)}\Big)\Big] + c_1\omega + c_2\omega^2 + \ldots, \quad \omega \to 0,\end{aligned}$$

with some complex constants c_1, c_2. In particular, by (5.17),

$$\tilde{D}(0) = \kappa_0^2 + \nu_+^2\Big(1 - e^{-\Im\theta_+(0)}\Big) > 0.$$

Hence, (5.34) holds with $C_1 = (\tilde{D}(0))^{-1} > 0$.

(1.4) If $0 < \kappa_- < \kappa_+$, then $e^{i\theta_+(\omega)}$ is an analytic function in a small neighborhood of the points $\omega = \pm\kappa_-$. Applying the representation (5.3) to $e^{i\theta_-(\omega)}$, we obtain

$$\begin{aligned}\tilde{D}(\omega) &= \kappa_0^2 - \omega^2 + \nu_-^2\Big(-\frac{i}{\nu_-}\sqrt{\omega^2 - \kappa_-^2} + \frac{1}{2\nu_-^2}(\omega^2 - \kappa_-^2) + \ldots\Big) \\ &\quad + \nu_+^2\big(1 - e^{-\Im\theta_+(\omega)}\big) - i\omega\gamma \\ &= \tilde{D}(\pm\kappa_-) - i\nu_-(\omega^2 - \kappa_-^2)^{1/2} + O(|\omega \mp \kappa_-|), \ \ \omega \to \pm\kappa_-, \ \ \omega \in \mathbb{C}_+,\end{aligned}$$

where (5.20) gives

$$\tilde{D}(\pm\kappa_-) = \kappa_0^2 - \kappa_-^2 + \nu_+^2(1 - e^{-\Im\theta_+(\kappa_-)}) \mp i\kappa_-\gamma = \kappa_0^2 - K_0(\kappa_-) \mp i\kappa_-\gamma.$$

Hence, $\tilde{D}(\pm\kappa_-) \neq 0$ iff either $\gamma \neq 0$ or $\gamma = 0$ and $\kappa_0^2 \neq K_0(\kappa_-)$. Therefore, if condition **C** holds, then (5.32) is true with $\omega_0 = \pm\kappa_-$. If condition $\mathbf{C}_0$ (iv) holds, then (5.33) is true with $\omega_0 = \pm\kappa_-$ and $C_1 = \nu_-^{-1}$.

(2) Let $\omega_0 = \pm\kappa_+$. There are five cases: **(2.1)** $\kappa_+ > a_-$; **(2.2)** $\kappa_+ = a_-$; **(2.3)** $\kappa_- < \kappa_+ < a_-$; **(2.4)** $\kappa_- = \kappa_+ = 0$ (see case **(1.1)**); **(2.5)** $\kappa_- = \kappa_+ \neq 0$ (see case **(1.2)**).

(2.1) Let $\kappa_+ > a_-$. Then the function $e^{i\theta_-(\omega)} = -e^{-\Im\theta_-(\omega)}$ is analytic in a small neighborhood of the points $\omega_0 = \pm\kappa_+$. Applying the representation (5.3) to $e^{i\theta_+(\omega)}$, we obtain

$$\begin{aligned}\tilde{D}(\omega) &= \kappa_0^2 - \omega^2 + \nu_+^2\Big(-\frac{i}{\nu_+}\sqrt{\omega^2 - \kappa_+^2} - \frac{1}{2\nu_+^2}(\omega^2 - \kappa_+^2) + \ldots\Big) \\ &\quad + \nu_-^2\big(1 + e^{-\Im\theta_-(\omega)}\big) - i\omega\gamma \\ &= \tilde{D}(\pm\kappa_+) - i\nu_+(\omega^2 - \kappa_+^2)^{1/2} + O(|\omega \mp \kappa_+|), \ \ \omega \to \pm\kappa_+.\end{aligned}$$

Using (5.21), we obtain

$$\tilde{D}(\pm\kappa_+) = \kappa_0^2 - \kappa_+^2 + \nu_-^2\big(1 + e^{-\Im\theta_-(\kappa_+)}\big) \mp i\kappa_+\gamma = \kappa_0^2 - K_-(\kappa_+) \mp i\kappa_+\gamma.$$

Hence, $\tilde{D}(\pm\kappa_+) \neq 0$ iff either $\gamma \neq 0$ or $\gamma = 0$ and $\kappa_0^2 \neq K_-(\kappa_+)$. Therefore, condition **C** implies (5.32) with $\omega_0 = \pm\kappa_+$, $C_1 = (\kappa_0^2 - K_-(\kappa_+) \mp i\kappa_-\gamma)^{-1}$, $C_2 = C_1^2\nu_+$. If condition $\mathbf{C}_0$ (v) holds, then (5.33) is true with $\omega_0 = \pm\kappa_+$.

(2.2) If $\kappa_+ = a_-$, then we apply (5.3) and (5.4) to $e^{i\theta_+(\omega)}$ and $e^{i\theta_-(\omega)}$, respectively, and obtain

$$\begin{aligned}\tilde{D}(\omega) &= \kappa_0^2 - \omega^2 + \nu_+^2\Big(-\frac{i}{\nu_+}\sqrt{\omega^2 - \kappa_+^2} - \frac{1}{2\nu_+^2}(\omega^2 - \kappa_+^2) + \dots\Big) \\ &\quad + \nu_-^2\Big(2 - \frac{i}{\nu_-}\sqrt{a_-^2 - \omega^2} - \frac{1}{2\nu_-^2}(a_-^2 - \omega^2) + \dots\Big) - i\omega\gamma \\ &= \tilde{D}(\pm\kappa_+) - (\kappa_+^2 - \omega^2)^{1/2}(\nu_+ + i\nu_-) + O(|\omega \mp \kappa_+|), \;\; \omega \to \pm\kappa_+,\end{aligned}$$

$\omega \in \mathbb{C}_+$. Here $\tilde{D}(\pm\kappa_+) = \kappa_0^2 - \bar{\kappa}^2 \mp i\kappa_+\gamma$. Therefore, condition **C** implies (5.32) with $\omega_0 = \pm\kappa_+$. If condition $\mathbf{C}_0$ (v) holds, then (5.33) is true with $\omega_0 = \pm\kappa_+$.

(2.3) If $\kappa_+ \in (\kappa_-, a_-)$, then we apply (5.3) to $e^{i\theta_+(\omega)}$ and obtain

$$\begin{aligned}\tilde{D}(\omega) &= \kappa_0^2 - \omega^2 + \nu_+^2\Big(-\frac{i}{\nu_+}\sqrt{\omega^2 - \kappa_+^2} - \frac{1}{2\nu_+^2}(\omega^2 - \kappa_+^2) + \dots\Big) \\ &\quad + \nu_-^2\big(1 - e^{i\theta_-(\omega)}\big) - i\omega\gamma \\ &= \tilde{D}(\kappa_+ \pm i0) - i\nu_+(\omega^2 - \kappa_+^2)^{1/2} + O(|\omega - \kappa_+|), \;\; \omega \to \kappa_+ \pm i0,\end{aligned}$$

where $\tilde{D}(\kappa_+ \pm i0) = \kappa_0^2 - \kappa_+^2 + \nu_-^2\left(1 - e^{\pm i\theta_-(\kappa_+ + i0)}\right) - i\kappa_+\gamma$. Therefore, $\Re\tilde{D}(\kappa_+ \pm i0) = \kappa_0^2 - \bar{\kappa}^2$ and

$$\begin{aligned}\Im\tilde{D}(\kappa_+ \pm i0) &= -\kappa_+\gamma \mp \nu_-^2 \sin\theta_-(\kappa_+ + i0) \\ &= \begin{cases} -\Big(\kappa_+\gamma \pm \frac{1}{2}\sqrt{\kappa_+^2 - \kappa_-^2}\sqrt{a_-^2 - \kappa_+^2}\Big), & \text{if } \kappa_- > 0 \\ -\kappa_+\Big(\gamma \pm \frac{1}{2}\sqrt{a_-^2 - \kappa_+^2}\Big), & \text{if } \kappa_- = 0 \end{cases}\end{aligned}$$

Then, $\Im\tilde{D}(\kappa_+ \pm i0) = -\kappa_+(\gamma \pm \gamma_1)$ (see (1.11)). Hence, $\tilde{D}(\kappa_+ + i0) \neq 0$ since $\Im\tilde{D}(\kappa_+ + i0) \neq 0$. $\tilde{D}(\kappa_+ - i0) \neq 0$ iff either $\gamma \neq \gamma_1$ or $\kappa_0 \neq \bar{\kappa}$. Hence, condition **C** implies (5.32). In the cases when either $\gamma_1 \leq \gamma_2$ and $a_+ < a_-$ or $\gamma_1 \leq \gamma_3$ and $a_- < a_+$, condition $\mathbf{C}_0$ **(vii)**, **(viii)** implies (5.33).

(3) Let $\omega_0 = \pm a_-$. There are the following cases: **(3.1)** $a_- > a_+$; **(3.2)** $a_- = a_+$; **(3.3)** $a_- \in (\kappa_+, a_+)$; **(3.4)** $a_- = \kappa_+$ (see the case **(2.2)**); **(3.5)** $a_- < \kappa_+$.

(3.1) If $a_- > a_+$, then the function $e^{i\theta_+(\omega)} = -e^{-\Im\theta_+(\omega)}$ is analytic in a small neighborhood of the points $\omega_0 = \pm a_-$. We apply (5.4) to $e^{i\theta_-(\omega)}$ and obtain

$$\begin{aligned}\tilde{D}(\omega) &= \kappa_0^2 - \omega^2 + \nu_-^2\Big(2 - \frac{i}{\nu_-}\sqrt{a_-^2 - \omega^2} - \frac{1}{2\nu_-^2}(a_-^2 - \omega^2) + \dots\Big) \\ &\quad +\nu_+^2\big(1 + e^{-\Im\theta_+(\omega)}\big) - i\omega\gamma \\ &= \tilde{D}(\pm a_-) - i\nu_-(a_-^2 - \omega^2)^{1/2} + O(|\omega \mp a_-|),\ \ \omega \to \pm a_-,\ \omega \in \mathbb{C}_+,\end{aligned}$$

where (see (5.15)) $\tilde{D}(\pm a_-) = \kappa_0^2 - K_+(a_-) \mp i a_-\gamma$. Hence, $\tilde{D}(\pm a_-) \neq 0$ iff either $\gamma \neq 0$ or $\gamma = 0$ and $\kappa_0^2 \neq K_+(a_-)$. If condition **C** holds, then (5.32) is true with $\omega_0 = \pm a_-$, $C_1 = (\tilde{D}(\pm a_-))^{-1}$, $C_2 = C_1^2\nu_-$. If condition $\mathbf{C}_0$ (i) holds, then (5.33) is true with $\omega_0 = \pm a_-$.

(3.2) In the case when $a_- = a_+$, we apply (5.4) to $e^{i\theta_\pm(\omega)}$ and obtain

$$\begin{aligned}\tilde{D}(\omega) &= \kappa_0^2 - \omega^2 + \nu_-^2\Big(2 - \frac{i}{\nu_-}\sqrt{a_-^2 - \omega^2} - \frac{1}{2\nu_-^2}(a_-^2 - \omega^2) + \dots\Big) \\ &\quad +\nu_+^2\Big(2 - \frac{i}{\nu_+}\sqrt{a_-^2 - \omega^2} - \frac{1}{2\nu_+^2}(a_-^2 - \omega^2) + \dots\Big) - i\omega\gamma \\ &= \tilde{D}(\pm a_-) - i(\nu_- + \nu_-)(a_-^2 - \omega^2)^{1/2} + O(|\omega \mp a_-|),\ \ \omega \to \pm a_-,\end{aligned}$$

where $\tilde{D}(\pm a_-) = \kappa_0^2 - \bar{\kappa}^2 \mp i a_-\gamma$. Hence, $\tilde{D}(\pm a_-) \neq 0$ iff either $\gamma \neq 0$ or $\gamma = 0$ and $\kappa_0 \neq \bar{\kappa}$. Condition **C** implies (5.32) with $\omega_0 = \pm a_-$. If condition $\mathbf{C}_0$ (iii) holds, then (5.33) is true with $\omega_0 = \pm a_-$.

(3.3) If $a_- \in (\kappa_+, a_+)$, then we apply (5.4) to $e^{i\theta_-(\omega)}$ and obtain

$$\begin{aligned}\tilde{D}(\omega) &= \kappa_0^2 - \omega^2 + \nu_-^2\Big(2 - \frac{i}{\nu_-}\sqrt{a_-^2 - \omega^2} - \frac{1}{2\nu_-^2}(a_-^2 - \omega^2) + \dots\Big) \\ &\quad +\nu_+^2\big(1 - e^{i\theta_+(a_- \pm i0)}\big) - i\omega\gamma \\ &= \tilde{D}(a_- \pm i0) - i\nu_-(a_-^2 - \omega^2)^{1/2} + O(|\omega - a_-|),\ \ \omega \to a_- \pm i0,\end{aligned}$$

where $\omega \in \mathbb{C}_\pm$, $\tilde{D}(a_- \pm i0) = \kappa_0^2 - a_-^2 + 2\nu_-^2 + \nu_+^2\left(1 - e^{\pm i\theta_+(a_- + i0)}\right) - i a_-\gamma$ with $a_- \in \Lambda_+ \setminus \Lambda_+^0$ and $\Im\theta_+(a_- + i0) = 0$. Hence, $\Re\tilde{D}(a_- \pm i0) = \kappa_0^2 - \bar{\kappa}^2$,

$$\begin{aligned}\Im\tilde{D}(a_- \pm i0) &= -a_-\gamma \mp \nu_+^2 \sin\theta_+(a_- + i0) \\ &= \begin{cases} -(a_-\gamma \pm \frac{1}{2}\sqrt{a_-^2 - \kappa_+^2}\sqrt{a_+^2 - a_-^2}), & \text{if } \kappa_+ > 0 \\ -a_-(\gamma \pm \sqrt{\nu_+^2 - \nu_-^2}), & \text{if } \kappa_+ = \kappa_- = 0 \end{cases}\end{aligned}$$

Then, $\Im\tilde{D}(a_- \pm i0) = -a_-(\gamma \pm \gamma_3)$. Hence, $\tilde{D}(a_- + i0) \neq 0$. $\tilde{D}(a_- - i0) \neq 0$ iff $\gamma \neq \gamma_3$ or $\kappa_0 \neq \bar{\kappa}$. Hence, condition **C** implies (5.32). If $\gamma_3 \leq \gamma_1$ and condition $\mathbf{C}_0$ **(viii)** holds, then (5.33) is true. Similarly, for $\omega \to -a_- \pm i0$.

(3.5) If $a_- < \kappa_+$, then $e^{i\theta_+(\omega)}$ is an analytic function in a small neighborhood of the points $\omega_0 = \pm a_-$. Applying the representation (5.4) to $e^{i\theta_-(\omega)}$, we obtain

$$\begin{aligned}\tilde{D}(\omega) &= \kappa_0^2 - \omega^2 + \nu_-^2\Big(2 - \frac{i}{\nu_-}\sqrt{a_-^2-\omega^2} - \frac{1}{2\nu_-^2}(a_-^2-\omega^2) + \dots\Big)\\ &\quad + \nu_+^2\big(1 - e^{-\Im\theta_+(\pm a_-)}\big) - i\omega\gamma\\ &= \tilde{D}(\pm a_-) - i\nu_-(a_-^2-\omega^2)^{1/2} + O(|\omega \mp a_-|),\ \ \omega \to \pm a_-,\end{aligned}$$

where $\tilde{D}(\pm a_-) = \kappa_0^2 - a_-^2 + 2\nu_-^2 + \nu_+^2\left(1 - e^{-\Im\theta_+(\pm a_-)}\right) \mp i a_-\gamma$. Therefore, by (5.22), $\tilde{D}(\pm a_-) \neq 0$ iff either $\gamma \neq 0$ or $\gamma = 0$ and $\kappa_0^2 \neq K_0(a_-)$. Condition **C** implies (5.32) with $\omega_0 = \pm a_-$. If condition $\mathbf{C}_0$ (vi) holds, then (5.33) is true with $\omega_0 = \pm a_-$.

(4) Let $\omega_0 = \pm a_+$. There are the following cases:

(4.1) $a_+ < a_-$; **(4.2)** $a_+ = a_-$ (see the case **(3.2)**); **(4.3)** $a_+ > a_-$.

(4.1) If $a_+ < a_-$, then $a_+ \in \Lambda_- \setminus \Lambda_-^0$. Applying the representation (5.4) to $e^{i\theta_+(\omega)}$, we obtain

$$\begin{aligned}\tilde{D}(\omega) &= \kappa_0^2 - \omega^2 + \nu_+^2\Big(2 - \frac{i}{\nu_+}\sqrt{a_+^2-\omega^2} - \frac{1}{2\nu_+^2}(a_+^2-\omega^2) + \dots\Big)\\ &\quad + \nu_-^2\big(1 - e^{\pm i\theta_-(a_+ + i0)}\big) - i\omega\gamma\\ &= \tilde{D}(a_+ \pm i0) - i\nu_-(a_+^2-\omega^2)^{1/2} + O(|\omega - a_+|),\ \ \omega \to a_+ \pm i0.\end{aligned}$$

Here $\tilde{D}(a_+ \pm i0) = \kappa_0^2 - a_+^2 + 2\nu_+^2 + \nu_-^2\left(1 - e^{\pm i\theta_-(a_+ + i0)}\right) - i a_+\gamma$, where $\Im\theta_-(a_+ \pm i0) = 0$. Hence, $\Re\tilde{D}(a_+ \pm i0) = \kappa_0^2 - \bar{\kappa}^2$ and

$$\begin{aligned}\Im\tilde{D}(a_+ \pm i0) &= -a_+\gamma \mp \nu_-^2 \sin\theta_-(a_+ + i0)\\ &= \begin{cases} -\left(a_+\gamma \pm \frac{1}{2}\sqrt{a_+^2-\kappa_-^2}\sqrt{a_-^2-a_+^2}\right), & \text{if } \kappa_- > 0\\ -a_+\left(\gamma \pm \frac{1}{2}\sqrt{a_-^2-a_+^2}\right), & \text{if } \kappa_- = 0\end{cases}\end{aligned}$$

Then, $\Im\tilde{D}(a_+ \pm i0) = -a_+(\gamma \pm \gamma_2)$. Hence, $\tilde{D}(a_+ + i0) \neq 0$. $\tilde{D}(a_+ - i0) \neq 0$ iff either $\gamma \neq \gamma_2$ or $\kappa_0 \neq \bar{\kappa}$. Hence, condition **C** implies (5.32). If $\gamma_2 \leq \gamma_1$ and condition $\mathbf{C}_0$ **(vii)** holds, then (5.33) is true. Similarly, for $\omega \to -a_+ \pm i0$.

(4.3) If $a_+ > a_-$, then $e^{i\theta_-(\omega)}$ is an analytic function in a small neighborhood of the points $\omega_0 = \pm a_+$. Using (5.4), we have

$$\begin{aligned}\tilde{D}(\omega) &= \kappa_0^2 - \omega^2 + \nu_+^2\Big(2 - \frac{i}{\nu_+}\sqrt{a_+^2-\omega^2} - \frac{1}{2\nu_+^2}(a_+^2-\omega^2) + \dots\Big)\\ &\quad + \nu_-^2\big(1 + e^{-\Im\theta_-(\pm a_+)}\big) - i\omega\gamma\\ &= \tilde{D}(\pm a_+) - i\nu_-(a_+^2-\omega^2)^{1/2} + O(|\omega \mp a_+|),\ \ \omega \to \pm a_+.\end{aligned}$$

where, by (5.16), $\tilde{D}(\pm a_+) = \kappa_0^2 - a_+^2 + 2\nu_+^2 + \nu_-^2\left(1 + e^{-\Im\theta_-(\pm a_+)}\right) \mp ia_+\gamma = \kappa_0^2 - K_-(a_+) \mp ia_+\gamma$. Therefore, $\tilde{D}(\pm a_+) \neq 0$ iff either $\gamma \neq 0$ or $\gamma = 0$ and $\kappa_0^2 \neq K_-(a_+)$. Condition **C** implies (5.32) with $\omega_0 = \pm a_+$. If condition $\mathbf{C}_0$ (ii) holds, then (5.33) is true with $\omega_0 = \pm a_+$. Lemma 5.7 is proved. ■

Proof of Theorem 3.4 Using Lemma 5.1, we vary the integration contour in (3.12):

$$N(t) = -\frac{1}{2\pi}\int_{|\omega|=R} e^{-i\omega t}\tilde{N}(\omega)\,d\omega, \quad t > 0, \tag{5.36}$$

where R is chosen enough large such that $\tilde{N}(\omega)$ has no poles in the region $\mathbb{C}_- \cap \{|\omega| \geq R\}$. Note that if $\gamma = 0$, then $\tilde{N}(\omega)$ has no poles in $\mathbb{C}_-$ by Lemma 5.1 (iv). Denote by σ_j the poles of $\tilde{N}(\omega)$ in $\mathbb{C}_-$ (if they exist). By Lemmas 5.1 and 5.2, there exists a $\delta > 0$ such that $\tilde{N}(\omega)$ has no poles in the region $\Im\omega \in (-\delta,0)$. Hence, we can rewrite $N(t)$ as

$$N(t) = -i\sum_{j=1}^{K}\mathrm{Res}_{\omega=\sigma_j}\left[e^{-i\omega t}\tilde{N}(\omega)\right] - \frac{1}{2\pi}\int_{\Lambda_\varepsilon} e^{-i\omega t}\tilde{N}(\omega)\,d\omega, \quad t > 0,$$

where $\varepsilon \in (0,\delta)$, the contour Λ_ε surrounds segments of $\Lambda_\pm$ and belongs to an ε-neighborhood of $\Lambda_- \cup \Lambda_+$ (Λ_ε is oriented anticlockwise). Passing to a limit as $\varepsilon \to 0$, we obtain

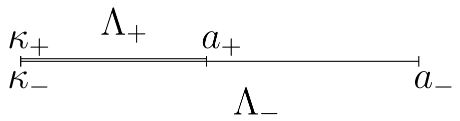
$$\begin{aligned} N(t) &= \frac{1}{2\pi}\int_{\Lambda_-\cup\Lambda_+} e^{-i\omega t}\left(\tilde{N}(\omega + i0) - \tilde{N}(\omega - i0)\right)d\omega + o(t^{-N}) \\ &= \sum_{\pm}\sum_{j=1}^{2}\frac{1}{2\pi}\int_{\Lambda_-\cup\Lambda_+} e^{-i\omega t}P_j^\pm(\omega)\,d\omega + o(t^{-N}), \quad t \to +\infty, \end{aligned}$$

with any $N > 0$. Here $P_j^\pm(\omega) := \zeta_j^\pm(\omega)(\tilde{N}(\omega + i0) - \tilde{N}(\omega - i0))$, $j = 1{,}2$, where $\zeta_j^\pm(\omega)$ are smooth functions such that $\sum\limits_{\pm,j}\zeta_j^\pm(\omega) = 1$, $\omega \in \mathbb{R}$, $\operatorname{supp}\zeta_1^\pm \subset \mathcal{O}(\pm\kappa_\pm)$, $\operatorname{supp}\zeta_2^\pm \subset \mathcal{O}(\pm a_\pm)$ ($\mathcal{O}(b)$ denotes a neighborhood of the point $\omega = b$). In the case $\kappa_\pm = 0$, instead of $\zeta_1^\pm$ ($P_1^\pm$) we introduce the function ζ_1 (respectively, P_1) with $\operatorname{supp}\zeta_1 \subset \mathcal{O}(0)$. Then Lemma 5.7 implies the bound (3.13) with $k = 0$. Here we use the following estimate (with $j = \pm 1$)

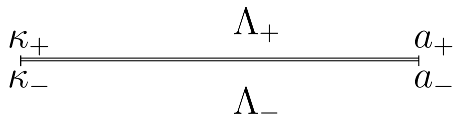
$$\left|\int_{\mathbb{R}} \zeta(\omega)e^{-i\omega t}(\omega^2 - \omega_0^2)^{j/2}\,d\omega\right| \leq C(1+t)^{-1-j/2} \quad \text{as } t \to +\infty, \ j \text{ is odd},$$

$\zeta(\omega)$ is a smooth function, and $\zeta(\omega) = 1$ for $|\omega - \omega_0| \leq \delta$ with some $\delta > 0$ (see, for example, [11, Lemma 2]). The bound (3.13) with $k = 1{,}2$ can be proved by a similar way. ■

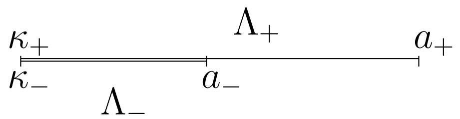
Below, we show various cases of mutual disposition of segments $\Lambda_\pm$. For simplicity, we draw only the positive part of the segments $\Lambda_\pm$.

$(P1)$ 

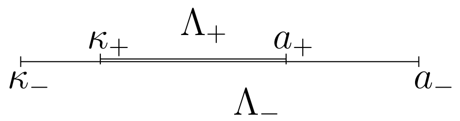
$\kappa_- = \kappa_+,\ a_+ < a_-$

$(P2)$ 

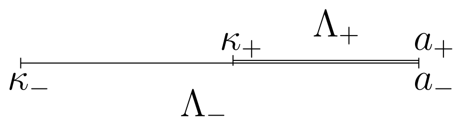
$\kappa_- = \kappa_+,\ a_+ = a_-$

$(P3)$ 

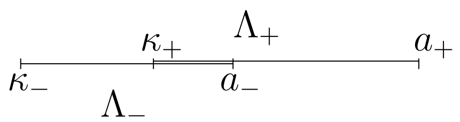
$\kappa_- = \kappa_+,\ a_+ > a_-$

$(P4)$ 

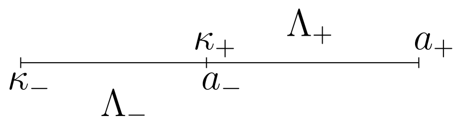
$\kappa_- < \kappa_+,\ a_- > a_+$

$(P5)$ 

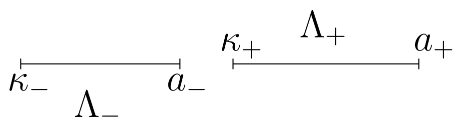
$\kappa_- < \kappa_+,\ a_- = a_+$

$(P6)$ 

$\kappa_- < \kappa_+,\ \kappa_+ < a_- < a_+$

$(P7)$ 

$\kappa_- < \kappa_+,\ a_- = \kappa_+$

$(P8)$ 

$\kappa_- < \kappa_+,\ a_- < \kappa_+$

References

[1] Cuccagna S.: L^p continuity of wave operators in $\mathbb{Z}$, *J. Math. Anal. Appl.* **354** (2009), 594–605.

[2] Dudnikova T.V.: On the asymptotical normality of statistical solutions for harmonic crystals in half-space, *Russian J. Math. Phys.* **15** (2008), no.4, 460–472.

[3] Dudnikova T.V.: Long-time asymptotics of solutions to a Hamiltonian system on a lattice, *Problems in Mathematical Analysis* **219** (2016), 69–85. [English transl. from *Problemy Matematicheskogo Analiza* **85** (2016), 69–82.]

[4] Dudnikova T.V.: Scattering theory for a discrete Hamiltonian system, *Keldysh Institute Preprints* N. 97 (2016), 26 pp.
URL: http://library.keldysh.ru/preprint.asp?id=2016-97

[5] Dudnikova T.V.: On convergence to equilibrium for one-dimensional chain of harmonic oscillators on the half-line, *J. Math. Phys.* **58** (2017), no.4, 043301.

[6] Islami H., Vainberg B.: Large time behavoir of the solutions to difference wave operators, *Commun. in Partial Dif. Eq.* **31** (2006), no.3, 397–416.

[7] Jensen A., Kato T.: Spectral properties of Schrodinger operators and time-decay of the wave functions, *Duke Math. J.* **46** (1979), 583–611.

[8] Komech A.I., Kopylova E.A., Kunze M.: Dispersive estimates for 1D discrete Schrodinger and Klein–Gordon equations, *Applicable Anal.* **85** (2006), no.12, 1487–1508.

[9] Pelinosky D.E., Stefanov A.: On the spectral theory and dispersive estimates for a discrete Schrödinger equation in one dimension, *J. Math. Phys.* **49** (2008), no.11, 113501.

[10] Shaban W., Vainberg B.R.: Radiation conditions for the difference Schrödinger operators, *Applicable Anal.* **80** (2001), 525–556.

[11] Vainberg B.R.: Behavior for large time of solutions of the Klein–Gordon equation, *Trudi Mosk. Mat. Obs.* **30** (1974), 139–158. [English transl. from *Trudy Moskovskogo Matematicheskogo Obshchestva*]