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ИНСТИТУТ ПРИКЛАДНОЙ МАТЕМАТИКИ
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А. И. Аптекарев, А. Дро, Д. Н. Туляков

Асимптотики точных констант в неравенствах Маркова - Бернштейна для соболевских пространств с когерентными весами

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Асимптотики точных констант в неравенствах Маркова - Бернштейна для соболевских пространств с когерентными весами. Препринт Института прикладной математики им. М.В. Келдыша РАН, Москва, 2017

Рассматриваются соболевские пространства с непрерывными и дискретными весами, составляющими когерентные пары. Условие позитивности скалярного произведения эквивалентно неравенству Маркова - Бернштейна в интегральных нормах с весами. Исследованы асимптотики точных констант в этих неравенствах при стремлении степени многочленов к бесконечности.

Ключевые слова: Неравенства Маркова-Бернштейна; соболевские ортогональные многочлены; непрерывные и дискретные веса; когерентные пары; асимптотики решений разностных уравнений.

Aptekarev A. I., Draux A., Tulyakov D. N.

On asymptotics of the sharp constants of the Markov-Bernshtein inequalities for the Sobolev spaces with coherent weights. Keldysh Institute of Applied Mathematics RAS, Moscow, 2017

The Sobolev spaces with continuous and discrete coherent pairs of weights are considered. The positivity of the inner product is equivalent to the Markov - Bernstein inequality for the weighted integral norm. Asymptotics of the sharp constants for these inequalities when degree of polynomials goes to infinity are obtained.

Key words: Markov - Bernstein inequalities; Sobolev orthogonal polynomials; continuous and discrete weights; coherent pairs; asymptotics of solutions of difference equations.

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Contents

1. Introduction

In 1889 Andrei Andreevich Markov proved an inequality [1], relating norms of polynomials and its derivative. Namely, for any polynomial Q, one has:

$$\|Q'\|_{C[-1,1]} \le n^2 \|Q\|_{C[-1,1]}, \qquad \mathsf{deg} Q \le n, \tag{1.1}$$

where $\|Q\|_{C[-1,1]} = \max\limits_{-1\le x\le 1} |Q(x)|$. The constant n^2 is sharp, because for the Chebychëv polynomials the inequality becomes the equality. In 1912 S.N. Bernshtein proved an analogues inequality for trigonometric polynomials [2].

Nowadays inequality (1.1) is considered in general setting. Let $\mathcal{P}_n$ be the set of polynomials of degree at most n, let X, $\tilde{X}$ be metric spaces and for any n, $\mathcal{P}_n \subset X \cap \tilde{X}$. The inequality

$$\|Q'\|_X \le M_n \|Q\|_{\tilde{X}}, \quad \mathsf{deg} Q \le n \tag{1.2}$$

is called *Markov-Bernstein inequality* **(M-BI)** and the problem of interest is to find the *sharp constant* M_n in the inequality for a given n.

In this paper we continue the study of the asymptotical behavior of the sharp constants in the Markov-Bernshtein inequalities with classical weighted integral metrics (see [3, 4, 5, 6, 7, 8]).

Suppose X is an inner product functional space, with inner product (f,g). Then one can construct the sequence of orthonormal polynomials π_n such that $(\pi_k, \pi_l) = \delta_{k,l}, \quad k,l = 0,1,2,\ldots$. In this case one has for any $Q \in \mathcal{P}_n$

$$\begin{array}{ll} Q = u_0\pi_0 + u_1\pi_1 + \ldots + u_n\pi_n, & \|Q\|^2 = |u_0|^2 + |u_1|^2 + \ldots + |u_n|^2 \\ Q' = v_0\pi_0 + v_1\pi_1 + \ldots + v_{n-1}\pi_{n-1}, & \|Q'\|^2 = |v_0|^2 + |v_1|^2 + \ldots + |v_{n-1}|^2 \end{array}$$

The sharp constant in Markov-Bernstein inequality is

$$M_n^2 = \sup_{\mathsf{deg} Q \le n} \frac{\|Q'\|^2}{\|Q\|^2} \tag{1.3}$$

It is sufficient to consider the subspace with $u_0 = 0$ ($|u_0|^2 + |u_1|^2 + \ldots + |u_n|^2 \ge |u_1|^2 + \ldots + |u_n|^2$). In this case the linear transformation $(u_1, u_2, \ldots, u_n) \to (v_0, v_1, \ldots, v_{n-1})$ is bijective on R^n. Denote by $\mathcal{A}$ the matrix of transformation $v = \mathcal{A}u$, $u = (u_1, u_2, \ldots, u_n)$, $v = (v_0, v_1, \ldots, v_{n-1})$ and $\mathcal{B} = \mathcal{A}^{-1}$. Then one has

$$M_n = \sup_{u\ne 0} \frac{\|\mathcal{A}u\|}{\|u\|} = \|\mathcal{A}\| = \sqrt{\lambda_{\max}(\mathcal{A}\mathcal{A}^T)} = \frac{1}{\sqrt{\lambda_{\min}(\mathcal{B}^T\mathcal{B})}}.$$

If $X = \tilde{X} = L^2_w$ for the classical weights w (Hermite, Laguerre, Jacobi, Meixner, Charlier), then * the matrix $(\mathcal{B}^T\mathcal{B})$ is a banded symmetric matrix.

*because there are recurrence relations connecting classical orthogonal polynomials and their derivatives.

Thus $\lambda_{\min}(\mathcal{B}^T\mathcal{B})$ is the smallest zero of the characteristic polynomial

$$A_n(\lambda) := \mathsf{det}(\mathcal{B}^T\mathcal{B} - \lambda I), \tag{1.4}$$

which is defined by means of a finite terms recurrence relations and for asymptotics of $\lambda_{\min}(\mathcal{B}^T\mathcal{B})$ the theory of local asymptotics of the polynomial solutions of finite difference equations (see [9, 10, 11]) can be employed. We have to mention that the asymptotical analysis of the higher order difference equations is rather complicated (see [8]). However, as it was proven in [12, 13, 14] for the case when $X = L^2_{w_1}$, $\tilde{X} = L^2_{w_2}$ and the classical weights $\{w_j\}$ compound a *coherent pair* then these difference equation have order two (i.e. three terms recurrence relations) and asymptotical analysis of the sharp constant for (1.2) becomes much more transparent. In this paper we study this particular case.

2. Coherent pairs

2.1. Continuous weights. The notion of coherent pairs of measures was introduced by Iserles et al [15] in 1991. These measures are defined as follows

Definition 1. *Let c_0 and c_1 be two quasi definite linear functionals. Let $\{P_n\}_{n\geq 0}$ (resp. $\{T_n\}_{n\geq 0}$) be the sequence of monic orthogonal polynomials with respect to c_0 (resp. c_1). (c_0, c_1) is called coherent pair if and only if there exists a sequence $\{\sigma_n\}_{n\geq 1}$, $\sigma_n \in \mathbb{R}$, $\sigma_n \neq 0$, such that*

$$T_n(x) = \frac{P'_{n+1}(x)}{n+1} - \sigma_n \frac{P'_n(x)}{n}, \quad \forall n \geq 1. \tag{2.1}$$

In the case $X = L^2_{\mu_1}$, $\tilde{X} = L^2_{\mu_2}$ the linear functionals mean integration with respect to measures $\{\mu_j\}$ and the Markov-Bernstein inequality (1.2) in L^2 norm that imply two measures have the following form

$$c_1((p')^2) \leq M_n^2 c_0(p^2), \quad \forall p \in \mathcal{P}_n. \tag{2.2}$$

Let us denote the square norm of P_n (resp. T_n) by $k_n^{(0)}$ (resp. $k_n^{(1)}$)

$$k_n^{(0)} = c_0(P_n^2) \text{ and } k_n^{(1)} = c_1(T_n^2), \quad \forall n \geq 0.$$

In [12, 13] the general expression of a three term-recurrence relation for polynomials (1.4) was obtained

$$A_n(\lambda) = (\lambda - \frac{k_n^{(0)}}{n^2 k_{n-1}^{(1)}} - \frac{\sigma_{n-1}^2}{(n-1)^2}\frac{k_{n-1}^{(0)}}{k_{n-1}^{(1)}})A_{n-1}(\lambda) - \frac{\sigma_{n-1}^2 (k_{n-1}^{(0)})^2}{(n-1)^4 k_{n-1}^{(1)} k_{n-2}^{(1)}} A_{n-2}(\lambda) \tag{2.3}$$

with $A_0(\lambda) = 1$ and $A_1(\lambda) = \lambda - \frac{k_1^{(0)}}{k_0^{(1)}}$.

All the kinds of coherent pairs (c_0, c_1) were described by Meijer [16] in 1997. There are in total seven cases contained in [16]. For five of these cases explicit expressions for the coefficients of the three term-recurrence relation (2.3) were obtained in [13]. These particular cases are given below.

1. **Laguerre case:** $\Omega =]0, +\infty[$
   (a) c_0 corresponds to the measure $(x+\xi)x^{\alpha-1}e^{-x}dx$ with $\alpha > 0$ and $\xi > 0$.
   c_1 corresponds to the Laguerre-Sonin measure $x^\alpha e^{-x}dx$.
   (b) c_0 corresponds to the measure $e^{-x}dx + M\delta(0)$ with $M \geq 0$. δ is the Dirac measure.
   c_1 corresponds to the Laguerre measure $e^{-x}dx$.
2. **Jacobi case:** $\Omega =]-1, +1[$
   (a) c_0 corresponds to the measure $\mid x - \xi \mid (1-x)^{\alpha-1}(1+x)^{\beta-1}dx$ with $\alpha > 0$, $\beta > 0$ and $\mid \xi \mid > 1$.
   c_1 corresponds to the Jacobi measure $(1-x)^\alpha(1+x)^\beta dx$.
   (b) c_0 corresponds to the measure $(1+x)^{\beta-1}dx + M\delta(1)$ with $\beta > 0$ and $M \geq 0$.
   c_1 corresponds to the Jacobi measure $(1+x)^\beta dx$.
   (c) c_0 corresponds to the measure $(1-x)^{\alpha-1}dx + M\delta(-1)$ with $\alpha > 0$.
   c_1 corresponds to the Jacobi measure $(1-x)^\alpha dx$.

We note, that the cases 2(b) and 2(c) are the same (up to the symmetry). For all nontrivial cases 1(a), 1(b), 2(a) and 2(b) the asymptotics of the sharp constants M_n of the Markov-Bernstein inequality (2.2) will be obtained in Section 3.

2.2. Discrete weights. The extension of the notion of coherent pairs of measures to the case of a discrete variable was realized by Area et al in 2000 and 2003 ([17], [18]). These authors gave in particular all the kinds of Δ-coherent pairs.
In the discrete case the forward difference operator Δ is used on $\mathcal{P}$ which is the vector space of real polynomials in one variable.

$$\Delta p(x) = p(x+1) - p(x).$$

Among the different kinds of Δ-coherent pairs we are only interested by those for which the support Ω is $]0, +\infty[$. In this case we have the following definition of Δ-coherent pairs.

Definition 2. *Let c_0 and c_1 be two quasi definite linear functionals. Let $\{P_n\}_{n\geq 0}$ (resp. $\{T_n\}_{n\geq 0}$) be the sequence of monic orthogonal polynomials with respect to c_0 (resp. c_1). (c_0, c_1) is called Δ-coherent pair if and only if there exists a sequence $\{\sigma_n\}_{n\geq 1}$, $\sigma_n \in \mathbb{R}$, $\sigma_n \neq 0$, such that*

$$T_n(x) = \frac{\Delta P_{n+1}(x)}{n+1} - \sigma_n \frac{\Delta P_n(x)}{n}, \quad \forall n \geq 1. \tag{2.4}$$

In [14] it was proved that the general expression of a three term-recurrence relation for polynomials (1.4) in the discrete case is the same as for the continuous case, namely (2.3) holds for the discrete case too.

All the kinds of Δ-coherent pairs (c_0, c_1) are described in [17]. We only give those for which the support Ω is $]0, +\infty[$ and for which we have explicit expressions for the coefficients of the three term-recurrence relation (2.3) obtained in [13]. One of both functionals is classical (Charlier or Meixner).

1. **Charlier case:** $\Omega =]0, +\infty[$
   (a) c_0 corresponds to the following linear functional $c_0 = (x - \xi)c_1$ with $\xi \leq 0$.
   c_1 corresponds to the Charlier linear functional.
2. **Meixner case:** $\Omega =]0, +\infty[$
   (a) c_0 corresponds to the following linear functional $c_0 = c_1 + M\delta(0)$ with $M \geq 0$.
   c_1 corresponds to the Meixner linear functional $c^{(1,c)}$ with $0 < c < 1$.
   (b) c_0 corresponds to the following linear functional $c_0 = (x - \xi)c^{(\beta-1,c)}$ with $\beta > 1$, $0 < c < 1$ and $\xi \leq 0$.
   c_1 corresponds to the Meixner linear functional $c^{(\beta,c)}$.

For these cases the asymptotics of the sharp constants M_n of the Markov-Bernstein inequality (2.2) will be obtained in Section 4.

3. Asymptotics of the sharp constants for the continuous coherent pairs

3.1. Laguerre case (a). The measure associated to c_0 is $(x + \xi)x^{\alpha-1}e^{-x}dx$ with $\alpha > 0$ and $\xi > 0$, and the one associated to c_1 is the Laguerre-Sonin measure $x^\alpha e^{-x}dx$. Therefore $T_n(x) = L_n^\alpha(x)$. From [13] we know that the following Markov-Bernstein inequality holds:

$$c_1((p')^2) \leq \frac{1}{\mu_{1,n}} c_0(p^2), \quad \forall p \in \mathcal{P}_n \tag{3.1}$$

where $\mu_{1,n}$ is the smallest zero of the polynomials $A_n(\lambda)$ satisfying the following three term recurrence relation

$$A_n(\lambda) = (\lambda - 2 - \frac{\xi + \alpha}{n})A_{n-1}(\lambda) - (1 + \frac{\alpha}{n-1})A_{n-2}(\lambda), \quad \forall n \geq 2. \tag{3.2}$$

$A_0(\lambda) = 1$ and $A_1(\lambda) = \lambda - \alpha - 1 - \xi - \frac{\xi}{\xi+\alpha}$.

A similar to (3.2) recurrence relation was studied in [19]. If we put in (3.2) $\xi = 0$, then corresponding polynomials A_n define the sharp constant for the

Markov-Bernstein inequality for L^2 norms with Laguerre case, i.e. $c_0 = c_1$ associated with the Laguerre measure $x^\alpha e^{-x}dx$. This case was studied in [4].

Proposition 3.1. *For the* ***Laguerre case (a)*** *with* $\alpha > 0$ *and* $\xi > 0$ *the limiting behavior of the sharp constant* (1.2)-(1.3) *of* ***M-BI*** *in* (3.1) *is*

$$M_n := \frac{1}{\mu_{1,n}} = \frac{n}{\xi}(1 + o(1)). \tag{3.3}$$

Numerical illustration of (3.3) see in Table 1. We see: convergence is slow.

Table 1. Laguerre case (a): $\mu_{1,n}$ and ξ/n

ξ	α		$n=20$	$n=50$	$n=100$	$n=500$
1		ξ/n	0.050000000	0.020000000	0.010000000	0.002000000
	1	$\mu_{1,n}$	0.112897230	0.037171728	0.016457958	0.002685591
	5	$\mu_{1,n}$	0.138937249	0.040820898	0.017260304	0.002710230
	10	$\mu_{1,n}$	0.208305786	0.052573298	0.020096435	0.002805712
2		ξ/n	0.100000000	0.040000000	0.020000000	0.004000000
	1	$\mu_{1,n}$	0.189023461	0.065130795	0.029626630	0.005050771
	5	$\mu_{1,n}$	0.212570115	0.068443246	0.030364003	0.005074147
	10	$\mu_{1,n}$	0.277666398	0.079264585	0.032963605	0.005163413
3		ξ/n	0.150000000	0.060000000	0.030000000	0.006000000
	1	$\mu_{1,n}$	0.259898520	0.091596782	0.042221527	0.007353154
	5	$\mu_{1,n}$	0.282581911	0.094799329	0.042938319	0.007376144
	10	$\mu_{1,n}$	0.345170893	0.105142156	0.045426263	0.007462710
4		ξ/n	0.200000000	0.080000000	0.040000000	0.008000000
	1	$\mu_{1,n}$	0.327946861	0.117252935	0.054503741	0.009621010
	5	$\mu_{1,n}$	0.350310352	0.120416126	0.055213470	0.009643881
	10	$\mu_{1,n}$	0.411279789	0.130469717	0.057636357	0.009728921
5		ξ/n	0.250000000	0.100000000	0.050000000	0.010000000
	1	$\mu_{1,n}$	0.394135819	0.142374212	0.066578849	0.011865872
	5	$\mu_{1,n}$	0.416426272	0.145529229	0.067287455	0.011888744
	10	$\mu_{1,n}$	0.476290222	0.155392565	0.069668381	0.011972814
10		ξ/n	0.500000000	0.200000000	0.100000000	0.020000000
	1	$\mu_{1,n}$	0.709637196	0.263516998	0.125220106	0.022896386
	5	$\mu_{1,n}$	0.732753391	0.266778854	0.125951166	0.022919824
	10	$\mu_{1,n}$	0.790389028	0.276267964	0.128251090	0.023002066

Proof. Here we give just a sketch of proof. The transform connecting of the two consecutive ratios $W_n(\lambda) = A_n/A_{n-1}$ of the solutions $A_n = \tilde{B}_n(\lambda)A_{n-1} - C_nA_{n-2}$ is

$$W_n = \tilde{B}_n(\lambda) - C_n/W_{n-1}.$$

If the discriminant of the fixed points of the transform: $D_n := (\tilde{B}_n(\lambda)/2)^2 + C_n$ for the large n, $D_n > 0$, then W_n does not vanish, otherwise it oscillates when $n \to \infty$. For the critical case

$$\mu_{1,n} := \lambda : \quad D_n(\lambda) = 0 \tag{3.4}$$

we get an approximate position for the extreme zero $\mu_{1,n}$ of A_n.

For the case (3.2) we have the relation

$$\mu_{1,n} - 2 + \frac{\xi - \alpha}{n} = -2\left(1 + \frac{\alpha}{n-1}\right)^{1/2} + \mathcal{O}\left(\frac{1}{n^2}\right),$$

which leads to (3.3). ∎

3.2. Laguerre case (b). The measure associated to c_0 is $e^{-x}dx + M\delta(0)$ with $M \geq 0$, and the one associated to c_1 is the Laguerre measure $e^{-x}dx$. Therefore $T_n(x)$ is the monic classical Laguerre polynomial $L_n(x)$. From [13] we know that the following Markov-Bernstein inequality holds:

$$c_1((p')^2) \leq \frac{1}{\mu_{1,n}} c_0(p^2), \quad \forall p \in \mathcal{P}_n \tag{3.5}$$

where $\mu_{1,n}$ is the smallest zero of the polynomials $A_n(\lambda)$ satisfying the following three term recurrence relation

$$A_n(\lambda) = (\lambda - 2)A_{n-1}(\lambda) - A_{n-2}(\lambda), \quad \forall n \geq 2. \tag{3.6}$$

$A_0(\lambda) = 1$ and $A_1(\lambda) = \lambda - \frac{1+2M}{1+M}$.

Proposition 3.2. *For the* ***Laguerre case (b)*** *with* $M > 0$ *the limiting behavior of the sharp constant* (1.2)-(1.3) *of* ***M-BI*** *in* (3.5) *is*

$$M_n := \frac{1}{\mu_{1,n}} = \frac{n^2}{\pi^2}(1 + o(1)). \tag{3.7}$$

Proof. It is clear that the solution of the recurrence relations (3.6) is a linear combination of the Chebyshev polynomials of the first and second kind. The fact that the constant $\frac{1+2M}{1+M}$ is positive implies that all zeros of A_n are positive. Therefore the smallest zero of A_n asymptotically (in the sense of the main term) behaves like the smallest zeros of the Chebyshev polynomials of the first and second, which indeed behaves like π^2/n^2. ∎

Numerical illustration of (3.7) see in Table 2.

Table 2. Laguerre case (b): $\mu_{1,n}$ and π^2/n^2

M		n			
		20	50	100	500
	π^2/n^2	0.024674011	0.003947841	0.000986960	0.000039478
1	$\mu_{1,n}$	0.020393972	0.003649401	0.000948578	0.000039164
5		0.021922153	0.003763805	0.000963616	0.000039289
10		0.022128520	0.003778527	0.000965523	0.000039305
50		0.022296105	0.003790372	0.000967052	0.000039317

3.3. Jacobi case (a). The measure associated to c_0 is $\mid x-\xi \mid (1-x)^{\alpha-1}(1+x)^{\beta-1}dx$ with $\alpha > 0$, $\beta > 0$ and $\mid \xi \mid > 1$, and the one associated to c_1 is the Jacobi measure $(1-x)^{\alpha}(1+x)^{\beta}dx$. Therefore $T_n(x)$ is the monic Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$. For convenience the measure associated to c_0 will be written as $\varepsilon(x-\xi)(1-x)^{\alpha-1}(1+x)^{\beta-1}dx$ with $\varepsilon = 1$ if $\xi < -1$ and $\varepsilon = -1$ if $\xi > 1$.

From [13] we know that the following Markov-Bernstein inequality holds:

$$c_1((p')^2) \leq \frac{1}{\mu_{1,n}} c_0(p^2), \quad \forall p \in \mathcal{P}_n \tag{3.8}$$

where $\mu_{1,n}$ is the smallest zero of the polynomials $A_n(\lambda)$ satisfying the following three term recurrence relation

$$A_n(\lambda) = (\lambda + B_n)A_{n-1}(\lambda) - C_n A_{n-2}(\lambda), \quad \forall n \geq 2. \tag{3.9}$$

where

$$\begin{aligned} B_n &= \frac{\varepsilon}{n(n+\alpha+\beta-1)}\left(\xi - \frac{(\beta-\alpha)(\alpha+\beta-2)}{(2n+\alpha+\beta-2)(2n+\alpha+\beta)}\right), \\ C_n &= \frac{4(n+\alpha-1)(n+\beta-1)}{(n-1)(n+\alpha+\beta-1)(2n+\alpha+\beta-1)(2n+\alpha+\beta-2)^2(2n+\alpha+\beta-3)}. \end{aligned}$$

with $A_0(\lambda) = 1$ and

$$\begin{aligned} A_1(\lambda) &= \lambda + \frac{\varepsilon}{\alpha+\beta}\frac{P_2^{(\alpha-1,\beta-1)}(\xi)}{P_1^{(\alpha-1,\beta-1)}(\xi)} \\ &= \lambda + \frac{\varepsilon}{\alpha+\beta}\left(\xi - \frac{(\beta-\alpha)(\alpha+\beta-2)}{(2+\alpha+\beta)(\alpha+\beta)} - \frac{4\alpha\beta}{(\alpha+\beta)_2(\xi(\alpha+\beta)-\beta+\alpha)}\right) \\ &= \lambda + \varepsilon\frac{(1+\alpha+\beta)_2\xi^2 + 2(\alpha-\beta)(1+\alpha+\beta)\xi + (\alpha-\beta)^2 - (2+\alpha+\beta)}{(1+\alpha+\beta)_2(\xi(\alpha+\beta)-\beta+\alpha)} \end{aligned}$$

Proposition 3.3. *For the **Jacobi case (a)** with $\alpha > 0$, $\beta > 0$ and $|\ \xi\ | > 1$ the limiting behavior of the sharp constant* (1.2)-(1.3) *of **M-BI** in* (3.8) *is*

$$M_n := \frac{1}{\mu_{1,n}} = \frac{n^2}{(|\ \xi\ | - 1)}(1 + o(1)). \tag{3.10}$$

Proof. For this case we follow the same line as in the proof of the Proposition 3.3. In accordance with (3.4) we have that an approximate position for the extreme zero $\mu_{1,n}$ of A_n from (3.9) is defined by

$$\mu_{1,n} := \lambda : \qquad D_n(\lambda) = 0, \quad \text{where} \quad D_n := \left(\frac{\lambda + B_n}{2}\right)^2 + C_n.$$

The main terms of asymptotics of the coefficients of the recurrence relations (3.9) are (see the next page):

Table 3. Jacobi case (a): $\mu_{1,n}$ and $(|\ \xi\ | - 1)/n^2$ for $\alpha = 1$, $\beta = 2$, $\xi > 0$

ξ		n			
		20	50	100	500
1.1	$(\|\ \xi\ \| - 1)/n^2$	0.000250000	0.000040000	0.000010000	0.000000400
	$\mu_{1,n}$	0.000467169	0.000058563	0.000012880	0.000000438
1.5	$(\|\ \xi\ \| - 1)/n^2$	0.001250000	0.000200000	0.000050000	0.000002000
	$\mu_{1,n}$	0.001662021	0.000241596	0.000056988	0.000002104
2	$(\|\ \xi\ \| - 1)/n^2$	0.002500000	0.000400000	0.000100000	0.000004000
	$\mu_{1,n}$	0.003007773	0.000456913	0.000110002	0.000004157
5	$(\|\ \xi\ \| - 1)/n^2$	0.010000000	0.001600000	0.000400000	0.000016000
	$\mu_{1,n}$	0.010427303	0.001687402	0.000418295	0.000016344
10	$(\|\ \xi\ \| - 1)/n^2$	0.022500000	0.003600000	0.000900000	0.000036000
	$\mu_{1,n}$	0.022171156	0.003675681	0.000922050	0.000036518
20	$(\|\ \xi\ \| - 1)/n^2$	0.047500000	0.007600000	0.001900000	0.000076000
	$\mu_{1,n}$	0.045156622	0.007585103	0.001918247	0.000076711
50	$(\|\ \xi\ \| - 1)/n^2$	0.122500000	0.019600000	0.004900000	0.000196000
	$\mu_{1,n}$	0.113513739	0.019182629	0.004879491	0.000196885

$$B_n = \frac{\varepsilon}{n^2 + \mathcal{O}(n)} \left(\xi + \frac{\mathcal{O}(1)}{4n^2 + \mathcal{O}(n)} \right),$$

and

$$C_n = \frac{1 + \mathcal{O}\left(\frac{1}{n}\right)}{4\,n^4 + \mathcal{O}(n^3)} .$$

Thus for the case (3.9) we have the relation

$$\mu_{1,n} + \frac{\varepsilon\,\xi}{n^2} = -\frac{1}{n^2} + \mathcal{O}\left(\frac{1}{n^3}\right),$$

which leads to (3.10). ■

Numerical illustration of (3.10) see in Tables 3 - 6.

Table 4. Jacobi case (a): $\mu_{1,n}$, and $(\mid \xi \mid -1)/n^2$ for $\alpha = 1$, $\beta = 2$, $\xi < 0$

ξ		n			
		20	50	100	500
-1.1	$(\mid \xi \mid -1)/n^2$	0.000250000	0.000040000	0.000010000	0.000000400
	$\mu_{1,n}$	0.000473466	0.000058689	0.000012887	0.000000438
-1.5	$(\mid \xi \mid -1)/n^2$	0.001250000	0.000200000	0.000050000	0.000002000
	$\mu_{1,n}$	0.001665991	0.000241693	0.000056994	0.000002104
-2	$(\mid \xi \mid -1)/n^2$	0.002500000	0.000400000	0.000100000	0.000004000
	$\mu_{1,n}$	0.003011250	0.000457003	0.000110008	0.000004157
-5	$(\mid \xi \mid -1)/n^2$	0.010000000	0.001600000	0.000400000	0.000016000
	$\mu_{1,n}$	0.010430180	0.001687483	0.000418300	0.000016344
-10	$(\mid \xi \mid -1)/n^2$	0.022500000	0.003600000	0.000900000	0.000036000
	$\mu_{1,n}$	0.022173847	0.003675758	0.000922055	0.000036518
-20	$(\mid \xi \mid -1)/n^2$	0.047500000	0.007600000	0.001900000	0.000076000
	$\mu_{1,n}$	0.045159234	0.007585178	0.001918252	0.000076711
-50	$(\mid \xi \mid -1)/n^2$	0.122500000	0.019600000	0.004900000	0.000196000
	$\mu_{1,n}$	0.113516323	0.019182703	0.004879496	0.000196885

Table 5. Jacobi case (a): $\mu_{1,n}$ and $(|\xi|-1)/n^2$ for $\alpha = 1$, $\beta = 10$

ξ		n			
		20	50	100	500
1.1	$(\mid \xi \mid -1)/n^2$	0.000250000	0.000040000	0.000010000	0.000000400
	$\mu_{1,n}$	0.000321785	0.000049832	0.000011866	0.000000431
1.5	$(\mid \xi \mid -1)/n^2$	0.001250000	0.000200000	0.000050000	0.000002000
	$\mu_{1,n}$	0.001180356	0.000207384	0.000052661	0.000002070
2	$(\mid \xi \mid -1)/n^2$	0.002500000	0.000400000	0.000100000	0.000004000
	$\mu_{1,n}$	0.002156030	0.000393217	0.000101738	0.000004091
5	$(\mid \xi \mid -1)/n^2$	0.010000000	0.001600000	0.000400000	0.000016000
	$\mu_{1,n}$	0.007574853	0.001457510	0.000387354	0.000016086
10	$(\mid \xi \mid -1)/n^2$	0.022500000	0.003600000	0.000900000	0.000036000
	$\mu_{1,n}$	0.016185415	0.003179622	0.000854283	0.000035942
20	$(\mid \xi \mid -1)/n^2$	0.047500000	0.007600000	0.001900000	0.000076000
	$\mu_{1,n}$	0.033048201	0.0065678783	0.0017779070	0.000075503
50	$(\mid \xi \mid -1)/n^2$	0.122500000	0.019600000	0.004900000	0.000196000
	$\mu_{1,n}$	0.083184416	0.016620609	0.004523892	0.000193790

Table 6. Jacobi case (a): $\mu_{1,n}$ and $(|\xi|-1)/n^2$ for $\alpha = 10$, $\beta = 10$

ξ		n			
		20	50	100	500
1.1	$(\mid \xi \mid -1)/n^2$	0.000250000	0.000040000	0.000010000	0.000000400
	$\mu_{1,n}$	0.000349237	0.000047925	0.000011293	0.000000424
1.5	$(\mid \xi \mid -1)/n^2$	0.001250000	0.000200000	0.000050000	0.000002000
	$\mu_{1,n}$	0.000976140	0.000183137	0.000048853	0.000002034
2	$(\mid \xi \mid -1)/n^2$	0.002500000	0.000400000	0.000100000	0.000004000
	$\mu_{1,n}$	0.001713256	0.000343878	0.000094131	0.000004020
5	$(\mid \xi \mid -1)/n^2$	0.010000000	0.001600000	0.000400000	0.000016000
	$\mu_{1,n}$	0.005859797	0.001267367	0.000357894	0.000015805
10	$(\mid \xi \mid -1)/n^2$	0.022500000	0.003600000	0.000900000	0.000036000
	$\mu_{1,n}$	0.012479282	0.002763878	0.000789328	0.000035316
20	$(\mid \xi \mid -1)/n^2$	0.047500000	0.007600000	0.001900000	0.000076000
	$\mu_{1,n}$	0.025453805	0.005710133	0.001642977	0.000074189
50	$(\mid \xi \mid -1)/n^2$	0.122500000	0.019600000	0.004900000	0.000196000
	$\mu_{1,n}$	0.064025164	0.014452828	0.004181368	0.000190423

3.4. Jacobi case (b). In the case (b) the measure associated to c_0 is $(1+x)^{\beta-1}dx + M\delta(1)$ with $\beta > 0$ and $M \geq 0$, and the one associated to c_1 is the Jacobi measure $(1+x)^{\beta}dx$. For more convenience and to give unified proofs for both cases (b) and (c) we will adopt a specific notation. The measure associated to c_0 will be $(1+\varepsilon x)^{\gamma-1}dx+M\delta(\varepsilon)$ with $\gamma > 0$ and $M \geq 0$, and the one associated to c_1 will be $(1+\varepsilon x)^{\gamma}dx$. If $\varepsilon = 1$ then $\gamma = \beta$, and if $\varepsilon = -1$ then $\gamma = \alpha$.

From [13] we know that the following Markov-Bernstein inequality holds:

$$c_1((p')^2) \leq \frac{1}{\mu_{1,n}} c_0(p^2), \quad \forall p \in \mathcal{P}_n \tag{3.11}$$

where $\mu_{1,n}$ is the smallest zero of the polynomials $A_n(\lambda)$ satisfying the following three term recurrence relation

$$A_n(\lambda) = (\lambda + B_n)A_{n-1}(\lambda) - C_n A_{n-2}(\lambda), \quad \forall n \geq 2. \tag{3.12}$$

where

$$\begin{aligned}
B_n &= -\frac{2^\gamma + (n+1)M(n+\gamma)}{2^\gamma + nM(n-1+\gamma)} \frac{2}{(2n+\gamma)(2n+\gamma-1)} \\
&\quad - \frac{(\gamma 2^\gamma + M(n-1+\gamma)(4(n-1)-\gamma(n-2)))^2}{2(2^\gamma + nM(n-1+\gamma))(2^\gamma + (n-1)M(n-2+\gamma))} \times \\
&\quad \frac{1}{(2n+\gamma-1)(2n+\gamma-2)(n-1+\gamma)^2}, \\
C_n &= \frac{(\gamma 2^\gamma + M(n-1+\gamma)(4(n-1)-\gamma(n-2)))^2}{(2^\gamma + (n-1)M(n-2+\gamma))^2(n+\gamma-1)^2(2n+\gamma-1)(2n+\gamma-2)^2(2n+\gamma-3)}
\end{aligned}$$

with $A_0(\lambda) = 1$ and $A_1(\lambda) = \lambda - \frac{2(\gamma+1)^2}{\gamma+2} \frac{2^\gamma + 2M(1+\gamma)}{2^\gamma + M\gamma}$.

Proposition 3.4. *For the **Jacobi case (b)-(c)** with $\gamma > 0$ and $M \geq 0$ the limiting behavior of the sharp constant* (1.2)-(1.3) *of **M-BI** in* (3.11) *is*

$$M_n := \frac{1}{\mu_{1,n}} = 2\,n^2\,(1 + o(1)). \tag{3.13}$$

Proof. For this case we follow the same line as in the proof of the Propositions 3.1 and 3.3. The main terms of asymptotics of the coefficients of the recurrence relations (3.12) are:

$$B_n = \frac{1}{2\,n^2} + \mathcal{O}\left(\frac{1}{n^5}\right), \qquad C_n = \mathcal{O}\left(\frac{1}{n^7}\right).$$

Thus for the case (3.12) we have the relation

$$\mu_{1,n} = \frac{1}{2n^2} + \mathcal{O}\left(\frac{1}{n^{3.5}}\right),$$

which leads to (3.13). ∎

Numerical illustration of (3.13) see in Tables 7.

Table 7. Jacobi case (b) and (c): $\mu_{1,n}$ and $1/(2n^2)$

γ	M		n = 20	50	100
1		$1/(2n^2)$	0.001250000	0.000200000	0.000050000
	1	$\mu_{1,n}$	0.001287477	0.000202344	0.000050290
	5		0.001319626	0.000203104	0.000050336
	10		0.001359396	0.000204050	0.000050394
	50		0.001671577	0.000211610	0.000050855
2		$1/(2n^2)$	0.001250000	0.000200000	0.000050000
	1	$\mu_{1,n}$	0.001249135	0.000200110	0.000050019
	5		0.001256341	0.000200281	0.000050029
	10		0.001264358	0.000200483	0.000050042
	50		0.001327490	0.000202092	0.000050142
3		$1/(2n^2)$	0.001250000	0.000200000	0.000050000
	1	$\mu_{1,n}$	0.001205353	0.000197387	0.000049681
	5		0.001208370	0.000197437	0.000049684
	10		0.001209960	0.000197474	0.000049686
	50		0.001220902	0.000197747	0.000049703
4		$1/(2n^2)$	0.001250000	0.000200000	0.000050000
	1	$\mu_{1,n}$	0.001154153	0.000193985	0.000049250
	5		0.001157218	0.000194021	0.000049251
	10		0.001157630	0.000194026	0.000049251
	50		0.001158165	0.000194031	0.000049251
5		$1/(2n^2)$	0.001250000	0.000200000	0.000050000
	1	$\mu_{1,n}$	0.001098243	0.000189912	0.000048715
	5		0.001101619	0.000189951	0.000048717
	10		0.001102222	0.000189967	0.000048718
	50		0.001104455	0.000190073	0.000048726
10		$1/(2n^2)$	0.001250000	0.000200000	0.000050000
	1	$\mu_{1,n}$	0.000787635	0.000170562	0.000045615
	5		0.000863678	0.000167229	0.000045381
	10		0.000853601	0.000166642	0.000045363
	50		0.000854359	0.000167582	0.000045471

4. Asymptotics of the sharp constants for the discrete coherent pairs

4.1. Charlier case (a). The linear functional c_0 is $(x-\xi)c_1$ with $\xi \le 0$ where c_1 is the Charlier linear functional, related to the discrete measure:

$$\sum_{n=0}^{\infty} \delta(x-n)\frac{\boldsymbol{e}^{-\alpha}\alpha^x}{x!}, \qquad \alpha > 0.$$

Therefore $T_n(x)$ in (2.4) is the monic Charlier polynomial $C_n^\alpha(x)$:

$$C_{n+1}^\alpha(x) \;=\; (x-n-\alpha)\,C_n^\alpha(x) \;-\; n\,\alpha\, C_{n-1}^\alpha(x), \qquad C_0^\alpha = 1, \quad C_{-1}^\alpha = 0.$$

From [14] we know that the following Markov-Bernstein inequality holds:

$$c_1((\Delta p)^2) \le \frac{1}{\mu_{1,n}} c_0(p^2), \quad \forall p \in \mathcal{P}_n \tag{4.1}$$

where $\mu_{1,n}$ is the smallest zero of the polynomials $A_n(\lambda)$ satisfying the following three term recurrence relation

$$A_n(\lambda) = (\lambda - \alpha + \frac{\alpha}{n}(\xi-\alpha))A_{n-1}(\lambda) - \frac{\alpha^3}{n-1}A_{n-2}(\lambda), \quad \forall n \ge 2, \tag{4.2}$$

where $A_0(\lambda) = 1$ and $A_1(\lambda) = \lambda + \alpha(\xi-\alpha) - \frac{\alpha\xi}{\xi-\alpha}$.

Proposition 4.1. *For the **Charlier case (a)** with $\alpha > 0$ and $\xi < 0$ the limiting behavior of the sharp constant* (1.2)-(1.3) *of **M-BI** in* (4.1) *is*

$$M_n := \frac{1}{\mu_{1,n}} = \frac{\alpha-\xi}{\alpha\xi}\,(1+o(1)). \tag{4.3}$$

Proof. It is well known (and easy to see) that solution of the recurrence relations

$$q_n \;=\; -\,(a_{n-1}+b_n)\,q_{n-1} - \; a_{n-1}b_{n-1}\,q_{n-2}, \qquad q_0 = 1, \quad q_1 = -b_1, \tag{4.4}$$

is

$$q_n \;=\; (-1)^n \prod_{k=1}^{n} b_k.$$

We see, that at the point $\lambda := \frac{\alpha\xi}{\xi-\alpha}$ the recurrences (4.2) takes the form of (4.4). Therefore for the polynomial A_n at this point we have:

$$A_n\left(\frac{\alpha\xi}{\xi-\alpha}\right) = (-1)^n \frac{\alpha^n(\alpha-\xi)^n}{n!}. \tag{4.5}$$

From the interlacing properties of orthogonal polynomials it follows that the polynomial A_n has no zeros on the interval $(-\infty, \frac{\alpha\xi}{\xi-\alpha})$. Finally, taking into account that A_n is the monic polynomial from (4.5) we conclude that the point $\lambda := \frac{\alpha\xi}{\xi-\alpha}$ attracts the extreme zero of the polynomial A_n. ■

Numerical illustration of (4.3) see in Table 8.

Table 8. Charlier case (a): $\mu_{1,n}$

α	ξ	n				$\frac{\alpha\xi}{\xi-\alpha}$
		20	50	100	250	
0.5	-0.5	0.250000000	0.250000000	0.250000000	0.250000000	0.250000000
	-1	0.333333340	0.333333333	0.333333333	0.333333333	0.333333333
	-10	0.640547514	0.513326784	0.484031182	0.476200701	0.476190476
1	-0.5	0.333333333	0.333333333	0.333333333	0.333333333	0.333333333
	-1	0.500000003	0.500000000	0.500000000	0.500000000	0.500000000
	-10	1.181961621	0.952732052	0.912520466	0.909090909	0.909090909
10	-0.5	0.495836962	0.476190476	0.476190476	0.476190476	0.476190476
	-1	0.946997021	0.909090909	0.909090909	0.909090909	0.909090909
	-10	7.103502828	5.020883655	5.000000000	5.000000000	5.000000000
100	-0.5	227.7989201	31.35564047	1.658097013	0.497512437	0.497512437
	-1	230.6621066	32.48673152	2.239107967	0.990099009	0.990099009
	-10	281.9616214	52.73205293	12.52046699	9.090909090	9.090909090

If $\xi = 0$, then $\frac{\alpha\xi}{\xi-\alpha} = 0$ and $\mu_{1,n}$ decreases very rapidly to 0 when n is increasing.

For example if $\alpha = 1$, $\xi = 0$ we have:

for $n = 5$, $\mu_{1,n} = 0.002466147487$
for $n = 10$, $\mu_{1,n} = 9.110651525546 \times 10^{-8}$;
for $n = 20$, $\mu_{1,n} = 1.436284278434 \times 10^{-19}$;
for $n = 100$, $\mu_{1,n} = 3.902443375880 \times 10^{-159}$;
for $n = 250$, $\mu_{1,n} = 1.133387537522 \times 10^{-493}$.

4.2. Meixner case (a). The linear functional c_0 is $c_1 + M\delta(0)$ where c_1 is the Meixner linear functional $c^{(1,c)}$ with $0 < c < 1$ and $M \geq 0$. The monic polynomials $T_n(x)$, orthogonal with respect to $c_1 = c^{(1,c)}$, are the monic Meixner polynomials $m_n^{(1,c)}(x)$. From [14] we know that the following Markov-Bernstein inequality holds:

$$c_1((\Delta p)^2) \leq \frac{1}{\mu_{1,n}} c_0(p^2), \quad \forall p \in \mathcal{P}_n$$

where $\mu_{1,n}$ is the smallest zero of the polynomials $A_n(\lambda)$ satisfying the following three term recurrence relation with $A_0(\lambda) = 1$ and $A_1(\lambda) = \lambda - \frac{c}{(c-1)^2}\frac{1-M(c^2-1)}{1-M(c-1)}$.

$$A_n(\lambda) = (\lambda - \frac{c(c+1)}{(c-1)^2})A_{n-1}(\lambda) - \frac{c^3}{(c-1)^4}A_{n-2}(\lambda), \quad \forall n \geq 2. \tag{4.6}$$

Proposition 4.2. *For the* ***Meixner case (a)*** *with* $0 < c < 1$ *and* $M \geq 0$ *the limiting behavior of the sharp constant* (1.2)-(1.3) *of* ***M-BI*** *in* (3.5) *is* $M_n := \frac{1}{\mu_{1,n}}$*:*

$$\mu_{1,n} = \sigma + \frac{\pi^2}{n^2}(\tau + o(1)), \qquad \sigma := \frac{c}{(1+\sqrt{c})^2}, \quad \tau := \frac{c^{3/2}}{(1-c)^2}. \tag{4.7}$$

Table 9. Meixner case (a): $\mu_{1,n}$ and their asymptotics (4.7)

M	c	n			
		20	50	100	500
0.5	0.1	0.058571127	0.057868028	0.057759100	0.057723072
	0.3	0.132271214	0.126476450	0.125557766	0.125250834
	0.6	0.245831995	0.200802693	0.193236681	0.190638371
	0.9	1.314548751	0.474200953	0.306498709	0.240241716
1	0.1	0.058577416	0.057868481	0.057759159	0.057723073
	0.3	0.132404496	0.126486409	0.125559074	0.125250845
	0.6	0.248559243	0.201029820	0.193267539	0.190638634
	0.9	1.464013772	0.496483692	0.310352793	0.240281200
10	0.1	0.058590948	0.057869443	0.057759283	0.057723074
	0.3	0.132676805	0.126506090	0.125561633	0.125250866
	0.6	0.254302376	0.201464961	0.193324956	0.190639113
	0.9	1.993218347	0.549762189	0.318152325	0.240351562
$\sigma + \frac{\pi^2}{n^2}\tau$	0.1	0.058684824	0.057875664	0.057760070	0.057723080
	0.3	0.133511875	0.126561555	0.125568652	0.125250923
	0.6	0.262196506	0.201992424	0.193391841	0.190639655
	0.9	2.343710344	0.574078989	0.321274510	0.240377077

Proof. Follows the same line as in the proof of the Proposition 3.2. ■

4.3. Meixner case (b). The linear functional c_0 is $(x-\xi)c^{(\beta-1,c)}$ where $c^{(\beta-1,c)}$ is the Meixner linear functional of parameters $\beta-1>0$ and c, $0<c<1$, with $\xi\le 0$. The linear functional c_1 is the Meixner linear functional $c^{(\beta,c)}$ of parameters β and c. Therefore the monic polynomials $T_n(x)$, orthogonal with respect to c_1, are the monic Meixner polynomials $m_n^{(\beta,c)}(x)$ orthogonal with respect to the discrete measure

$$\sum_{n=0}^{\infty}\delta(x-n)\frac{\Gamma(\beta+x)\,c^x}{x!},\qquad 0<c<1\,.$$

From [14] we know that the following Markov-Bernstein inequality holds:

$$c_1((\Delta p)^2)\le\frac{1}{\mu_{1,n}}c_0(p^2),\quad \forall p\in\mathcal{P}_n$$

where $\mu_{1,n}$ is the smallest zero of the polynomials $A_n(\lambda)$ satisfying the following three term recurrence relation $\forall n\ge 2$

$$A_n(\lambda)=(\lambda+B_n)A_{n-1}(\lambda)-C_nA_{n-2}(\lambda),\quad \forall n\ge 2, \tag{4.8}$$

$$B_n:=\frac{\beta-1}{n}\frac{c}{1-c}(\xi-\frac{n(c+1)+(\beta-1)c}{1-c}),\qquad C_n:=\frac{c^3(\beta-1)^2(\beta+n-2)}{(n-1)(1-c)^4},$$

with $A_0(\lambda)=1$ and $A_1(\lambda)=\lambda-\frac{c(\beta-1)(\beta^2c^2+(c-1)^2\xi(\xi-1)-\beta c(c+2\xi-2c\xi))}{(c-1)^2(\xi(c-1)+c(\beta-1))}$.

Proposition 4.3. *For the **Meixner case (b)** with $\beta-1>0$, $0<c<1$ and $\xi\le 0$ the limiting behavior of the sharp constant* (1.2)-(1.3) *of **M-BI** in* (3.5) *is*

$$M_n:=\frac{1}{\mu_{1,n}},\qquad \mu_{1,n}=\begin{cases}\sigma\;+\;o(1) & \textit{for}\quad \dfrac{-\xi}{\beta-1}\ge\dfrac{\sqrt{c}}{1+\sqrt{c}}\\[2ex] \tau\;+\;\mathcal{O}(q^n),\ \exists q<1 & \textit{for}\quad \dfrac{-\xi}{\beta-1}<\dfrac{\sqrt{c}}{1+\sqrt{c}}\end{cases}, \tag{4.9}$$

where

$$\sigma:=\frac{c(\beta-1)}{(1+\sqrt{c})^2},\qquad \tau:=\frac{\xi c(\beta-1+\xi)}{\xi\;-\;c(\beta-1+\xi)}.$$

Proof. At first we notice (as in the proof of the Propositions 4.1) that at the point $\lambda:=\tau$ recurrences (4.8) have explicit solution

$$A_n\left(\frac{\xi c(\beta-1+\xi)}{\xi\;-\;c(\beta-1+\xi)}\right)\;=\;\frac{(\beta)_n\,c^n\,(\xi\;-\;c(\beta-1+\xi))^n}{n!\,(c-1)^{2n}}.$$

Since $\xi-c(\beta-1+\xi)=\xi\,(1-c)-c\,(\beta-1)\le 0$ (for parameters β, c and ξ under considerations), then $A_n(\tau)$ changes its sign when $n=0,1,2,3,\cdots$. Therefore, from the interlacing property we have $A_n(\lambda)\ne 0$ for $\lambda\in(-\infty,\tau)$.

However, unlike (4.5), the point τ does not attract the extreme zero of polynomial A_n for all values of the parameters. To clarify a possible set of parameters we turn to the orthonormal polynomials $\widehat{A}_n := A_n / \prod_{k=1}^{n-1} C_k$, see (4.8)

$$\widehat{A}_n^2(\tau) = \frac{[(\beta)_n]^2 c^{2n}(\xi - c(\beta-1+\xi))^{2n} n!(1-c)^{4n}}{(n!)^2\,(c-1)^{4n}\,c^{3n}\,(\beta-1)^{2n}\,(\beta)_n} = \frac{(\xi - c(\beta-1+\xi))^{2n}}{c^n\,(\beta-1)^{2n}}(1+o(1))$$

We see that to have convergence of $\sum \widehat{A}_n^2(\tau)$ the parameters have to satisfy

$$\frac{(\xi\,(1-c) - c\,(\beta-1))^2}{c\,((\beta-1)^2} < 1 \quad \Rightarrow \quad \frac{-\xi}{\beta-1} < \frac{\sqrt{c}}{1+\sqrt{c}}$$

In this case τ contains a mass which attracts $\mu_{1,n}$ (see Table 11). In opposite case $\mu_{1,n}$ tends to the end point of the continuous spectra σ (see Table 10). ■

Table 10. Meixner case (b): $\mu_{1,n}$ for $\beta = 2$

c	ξ	n				
		20	50	100	500	σ
0.1	-0.5	0.061287875	0.058869491	0.058222890	0.057801537	
	-1	0.065560279	0.060434849	0.058959088	0.057933433	0.057721539
0.3	-0.5	0.138315815	0.128964921	0.126743622	0.125448037	
	-1	0.158309497	0.135973717	0.129950727	0.125996107	0.125237684
0.6	-0.5	0.245605736	0.202998709	0.194920666	0.191005312	
	-1	0.336400026	0.233897017	0.208540917	0.193167700	0.190524980
0.9	-0.5	1.418119006	0.437924131	0.289883611	0.239767194	
	-1	2.128977665	0.700198326	0.408146412	0.257241318	0.237006350

Table 11. Meixner case (b): $\mu_{1,n}$ for $\beta = 20$

c	ξ	n				
		20	50	100	500	τ
0.1	-0.5	0.393617183	0.393617021	0.393617021	0.393617021	0.393617021
	-1	0.642882463	0.642857142	0.642857142	0.642857142	0.642857142
0.3	-0.5	0.484604767	0.458677686	0.458677685	0.458677685	0.458677686
	-1	0.903593234	0.843750013	0.843750000	0.843750000	0.843750000
0.6	-0.5	5.815331207	0.565973780	0.478448966	0.478448275	0.478448275
	-1	6.859745365	1.055915624	0.915259980	0.915254237	0.915254237
0.9	-0.5	421.7989420	64.01564842	11.31039595	0.485424648	0.4854227405
	-1	427.5607865	66.30888262	12.47404432	0.941870495	0.9418604651

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