ISSN 2071-2898 (Print)
ISSN 2071-2901 (Online)

Bobylev A.V., Potapenko I.F.

Long wave asymptotics for the Vlasov-Poisson-Landau equation

Recommended form of bibliographic references: Bobylev A.V., Potapenko I.F. Long wave asymptotics for the Vlasov-Poisson-Landau equation // Keldysh Institute Preprints. 2018. No. 121. 16 p. doi:10.20948/prepr-2018-121-e
URL: http://library.keldysh.ru/preprint.asp?id=2018-121&lg=e

Ордена Ленина
ИНСТИТУТ ПРИКЛАДНОЙ МАТЕМАТИКИ
имени М.В. Келдыша
Российской академии наук

A.V. Bobylev, I.F. Potapenko

Long wave asymptotics for the Vlasov–Poisson–Landau equation

Москва
2018

А.В. Бобылев, И.Ф. Потапенко

Длинноволновые асимптотики для уравнения Власова–Пуассона–Ландау

Работа посвящена некоторым математическим проблемам динамики столкновительной плазмы. Сложность заключается в том, что в случае плазмы мы имеем по крайней мере три различных масштаба: радиус Дебая r_D, длину свободного пробега l и макроскопическую длину L. Это справедливо даже для простейшей модели (электронная плазма с нейтрализующим фоном бесконечно тяжелых ионов), рассматриваемой в данной работе. Мы изучаем на формальном уровне математической строгости решения уравнения Власова–Пуассона–Ландау, имеющие типичную длину порядка $l >> r_D$, и выясняем некоторые математические вопросы, относящиеся к соответствующему пределу. В частности, мы изучаем существование предела для электрического поля и показываем, что, вообще говоря, он не существует из-за быстро осциллирующих членов. Всё же предельные уравнения, которые используются во многих публикациях физиков, могут приводить в некоторых случаях к правильным результатам для функции распределения. Мы также исследуем корректность этих уравнений и формулируем соответствующий критерий для различных классов слабо неоднородных начальных данных. Показано, что ситуация с корректностью в нашем случае качественно сходна с подобной проблемой для уравнения Власова–Дирака–Бенни, которое было изучено детально в недавних публикациях Бардоса и др.

Ключевые слова: уравнение Власова–Пуассона–Ландау, кулоновские столкновения, квазинейтральный предел

A.V. Bobylev, I.F. Potapenko

Long wave asymptotics for the Vlasov–Poisson–Landau equation

The work is devoted to some mathematical problems of dynamics of collisional plasma. The difficulty is that in plasma case we have at least three different length scales: Debye radius r_D, mean free pass l and macroscopic length L. This is true even for the simplest model (plasma of electrons with a neutralizing background of infinitely heavy ions), considered in the paper. We study at the formal level of mathematical rigour solutions of the VLPE, having the typical length of the order $l >> r_D$, and try to clarify some mathematical questions related to corresponding limit. In particular, we study the existence of the limit for electric field and show that, generally speaking, it does not exist because of rapidly oscillating terms. Still the limiting equations, which are used in many publications by physicists, can lead in some cases to correct results for the distribution function. We also study the well-posedness of these equations and formulate the corresponding criterion for different classes of weakly inhomogeneous initial data. It is shown that the situation with well-posedness in our case is qualitively similar to the same problem for Vlasov–Dirac–Benney equation, which was studied in detail in recent publications of Bardos et al.

Key words: Vlasov–Poisson–Landau kinetic equation, Coulomb collisions, quasi neutral limit

Работа выполнена при поддержке Российского фонда фундаментальных исследований, Грант N 17-51-52007 МНТ_а.

1 Introduction

The work is devoted to some mathematical problems of dynamics of collisional plasma. These problems are related to the general question of different length and time scales in plasma physics. In particular, the smallest important length scale for plasma is defined by the Debye radius r_D. What happens if the typical length for the one-particle distribution function is much bigger than r_D? For example, it can be of order of the mean free path $l >> r_D$. This is what we call below the long wave asymptotics for the Vlasov–Poisson–Landau kinetic equation (VPLE). Ideologically it is very close to the quasi neutral limit for Vlasov–Poisson equation [1],[2] when the Debye radius vanishes.

Our work is partly motivated by series of papers by physicists published in last two decades. In particular, see [3] – [8] where the authors consider a simplified model of electrons with a neutralizing background of infinitely heavy ions. Then they implicitly make a formal transition to certain limit and solve numerically the limiting equations. These equations have some advantages, since the limiting electrical field is given by explicit formula, not via solution of Poisson equation.

Can this limit be rigorously justified? This is an important question, which, to our knowledge, was not studied before. The present work is just the first step in clarification of this and similar questions. By using rather elementary mathematical methods we show (at the formal level) in Section 5 and 6 that the situation with limiting equation is not very simple. In particular, (1) the electric field rapidly oscillates near the limit and (2) the limiting equations are not always well-posed. The "collisionless" part of the limiting equations is, to some extent, similar to Vlasov–Dirac–Benney equations (VDBE) studied recently by Bardos and Besse [2],[9] (see also [10]). The well-posedness problem can be investigated by methods of these papers. It is done in Section 6.

Another group of recent mathematical publications which should be mentioned is related to the quasi neutral limit for the Vlasov–Poisson equation. We mean, in particular, the paper by Han-Kwan and Rousset [11] and references therein. The physical model from that paper is quite different from ours: collisionless plasma of ions in the presence of massless electrons. In that case the limiting equations coincide with Vlasov–Poisson–Benney equations (VPBE) from [2, 9]. The paper [11] contains very deep mathematical results based on some new ideas. Similar methods can be probably used for our problem despite the fact that physical models are quite different in these two cases.

The paper is organized as follows. The physical model and the statement of the problem in dimensionless variables are explained in Section 2. The Vlasov–Poisson–Landau system is presented in Section 3. The long wave asymptotics and corresponding limiting equations are discussed in Section 4. The asymptotic behavior of electric field is studied in Section 5. The well-posedness of limiting equations is discussed in Section 6. Main results are formulated in **Proposition** 2 and **Proposition** 3 (at the formal level) and briefly discussed in Conclusions.

2 Statement of the problem

We denote by $f(x, v, t)$ a one-particle distribution function, where $x \in \mathbb{R}^3$, $v \in \mathbb{R}^3$, $t \in \mathbb{R}_+$ stand for position, velocity and time variables respectively. We will also use below the notation $m > 0$ for the particle mass. A general kinetic equation (Boltzmann-type, Vlasov-type, etc.) can be written as

$$\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} + \frac{1}{m} F(f) \cdot \frac{\partial f}{\partial v} = C(f), \tag{1}$$

where $F(f)$ and $C(f)$ denote the self-consistent force field and the collision term respectively. Here and below dots stand for scalar product in $\mathbb{R}^3$. We consider the Cauchy problem for $t > 0$ in the whole phase space $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$ with initial conditions

$$f|_{t=0} = \rho_0\, \theta_0^{-3/2}\, \tilde{f}_0 \left(\frac{x}{L_0}, \frac{v}{\theta_0^{1/2}} \right), \quad \theta_0 = \frac{T_0}{m}, \tag{2}$$

where ρ_0 and T_0 denote respectively some typical values of initial density and absolute temperature (expressed in units of energy). Note that the length L_0 is considered here as a free parameter. It is not assumed to be large from the very beginning. Usually we also assume that

$$\tilde{f}_0(\tilde{x}, \tilde{v}) \xrightarrow[|\tilde{x}| \to \infty]{} M(|\tilde{v}|) = (2\pi)^{-3/2} \exp(-|\tilde{v}|^2/2). \tag{3}$$

Moreover the same constant Maxwellian at infinity for dimensionless problem is assumed for all $t > 0$. This assumption will be considered below as the boundary conditions at infinity. To specify the dimensionless problem we denote

$$f(x, v, t) = \rho_0 \theta_0^{-3/2}\, \tilde{f}(\tilde{x}, \tilde{v}, \tilde{t}),\ F(f) = m F_0\, \tilde{F}(\tilde{f}),\ C(f) = C_0\, \tilde{C}(\tilde{f}), \tag{4}$$

with appropriate constants F_0 (force) and C_0,

$$\tilde{x} = x/L_0,\ \tilde{v} = v/\theta_0^{1/2},\ \tilde{t} = t/t_0,\ t_0 = L_0/\theta_0^{1/2}. \tag{5}$$

Omitting tildes, we obtain

$$f_t + v \cdot f_x + A\,F(f) \cdot f_v = B\,C(f); \quad x \in \mathbb{R}^3,\ v \in \mathbb{R}^3;$$
$$f|_{t=0} = f(x,v), \quad f(x,v,t) \underset{|x|\to\infty}{\longrightarrow} (2\pi)^{-3/2} \exp(-|v|^2/2),\ t \ge 0. \tag{6}$$

Hence, we finally have only two dimensionless parameters

$$A = A(L_0;\, \rho_0,\, T_0, ...), \quad B = B(L_0;\, \rho_0,\, T_0, ...),$$

where dots stand for other (microscopic, like particle mass m, charge e, diameter d, etc) parameters. We can always choose such notations that $A \ge 0$ and $B \ge 0$.

In order to illustrate the existence of different length scales we first consider the well-known example of the Boltzmann equation. Then

$$A = 0, \quad B = \frac{L_0}{l} = \frac{1}{Kn}\,,$$

where $l(\rho_0, T_0, ...)$ and Kn, denote respectively the mean free path and the Knudsen number. For example, $l = (\pi \rho_0\, d^2)^{-1}$ for hard spheres with diameter d. The long wave asymptotics for the Boltzmann equation means simply the hydrodynamic limit $Kn \to 0$. This obviously means that the typical length L_0 of our solution is much greater than the mean free path.The behavior of sotutions to the Boltzmann equation near that limit is well studied in literature. The plasma case described by Vlasov–Poisson–Landau equation is more complicated because $A > 0$. Moreover $A >> B$ in that case, as we shall see in the next section.

3 Vlasov–Poisson–Landau Equation (VPLE)

We consider a simplified physical model of plasma: gas of electrons with neutralizing background of infinitely heavy positive ions ($Z = 1$ for simplicity) distributed in space with constant density ρ_0. Then the dimensionless VPLE reads

$$f_t + v \cdot f_x - A\varphi_x \cdot f_v = B\,C(f), \quad C(f) = Q(f,f) + K(f), \tag{7}$$

$$\triangle\varphi = 1 - \rho, \quad \rho = \int\limits_{\mathbb{R}^3} dv f(x,v,t). \tag{8}$$

where $Q(f,f)$ the nonlinear Landau collision integral [12] (for e-e collisions) can be written as

$$
\begin{aligned}
Q(f,f) &= \partial_{v_\alpha} \int\limits_{\mathbb{R}^3} dw T_{\alpha\beta}(v-w)(\partial_{v_\beta} - \partial_{w_\beta})\, f(v) f(w), \\
T_{\alpha\beta}(u) &= \frac{|u|^2\delta_{\alpha\beta} - u_\alpha u_\beta}{|u|^3}; \quad \alpha,\beta = 1,2,3.
\end{aligned} \tag{9}
$$

The usual summation rule over repeating Greek indices is assumed here and below.

The linear collision term $K(f)$ for e-i collisions reads

$$
K(f) = \partial_{v_\alpha} T_{\alpha\beta}(v) \partial_{v_\beta} f(v) \tag{10}
$$

All quantities in above equation are assumed to be of order one, except for two positive dimensionless parameters A and B. These parameters have the following form:

$$
A = \left(\frac{L_0}{r_D}\right)^2, \quad B = \frac{L_0}{l} = \frac{1}{Kn}, \tag{11}
$$

where

$$
\begin{aligned}
r_D &= \left(\frac{4\pi\rho_0 e^2}{T_0}\right)^{-1/2} && \text{– Debye radius} \\
l &= \frac{T_0^2}{2\pi e^4 \rho_0^{1/3} \Lambda} && \text{– mean free path} \\
\Lambda &= \log \frac{r_D T_0}{e^2} && \text{– Coulomb logarithm}
\end{aligned} \tag{12}
$$

denote respectively the Debye radius, the mean free path and the Coulomb logarithm.

We note that LPVE (7),(8) is based on the assumption of smallness of the parameter (see e.g. [13], [14])

$$
\delta = \frac{1}{\rho_0 r_D^3} = \left(\frac{4\pi e^2 \rho_0^{1/3}}{T_0}\right)^{3/2} << 1. \tag{13}
$$

Hence,

$$
\begin{aligned}
\Lambda &= \log \frac{4\pi}{\delta} \approx \log \frac{1}{\delta} >> 1, \\
\frac{l}{r_D} &= 8\pi \frac{1}{\delta\Lambda} \approx 8\pi \left(\delta \log \frac{1}{\delta}\right)^{-1} >> 1,
\end{aligned}
$$

$$\frac{A}{B} = \frac{L_0 l}{r_D^2} \approx \frac{8\pi}{\delta \log \frac{1}{\delta}} \frac{L_0}{r_D} >> 1 \text{ if } L_0 \geq r_D.$$

Therefore the collision term in VPLE (7) is much smaller than the Vlasov force term for all practically interesting values of L_0. The case of moderately large values of L_0 is considered in more detail in the next section.

4 Long wave asymptotics for VPLE

The smallest important length scale for VPLE is obviously the Debye radius r_D. Therefore the long wave asymptotics can be defined as a formal limit

$$A = \left(\frac{L_0}{r_D}\right)^2 \to \infty, \quad B = \frac{L_0}{l} = \frac{1}{Kn} \text{ is bounded.} \tag{14}$$

Introducing a small parameter

$$\varepsilon = A^{-1/2} = \frac{r_D}{L_0} \to 0, \tag{15}$$

we study below this limit for a relatively simple problem. The same problem was previously studied in several works of physicists [3] − [8]. Our goal is to try to clarify some mathematical aspects of the problem.

We make one more simplification and assume that the initial data f_0 and the solution f of the problem (7), (8) depend only on one spatial variable. For brevity we keep the same notation x below, assuming that $x \in \mathbb{R}$, $v \in \mathbb{R}^3$, $t \in \mathbb{R}_+$. We also introduce the dimensionless electric field

$$E = A\varphi_x = \varphi_x(x,t)/\varepsilon^2.$$

and obtain from (7), (8)

$$f_t^\varepsilon + v_x f_x^\varepsilon - E^\varepsilon \cdot f_{v_x}^\varepsilon = \frac{1}{Kn} C(f^\varepsilon), \quad E_x^\varepsilon = \frac{1}{\varepsilon^2}(1 - \rho^\varepsilon), \ \varepsilon > 0. \tag{16}$$

The typical initial condition reads

$$f^\varepsilon|_{t=0} = [2\pi T_0(x)]^{-3/2} \exp[-|v|^2/2T_0(x)], \ T_0(|x|) \xrightarrow[|x|\to\infty]{} 1. \tag{17}$$

We denote

$$\rho^\varepsilon = \langle f^\varepsilon, 1\rangle, \; j^\varepsilon = \langle f^\varepsilon, v_x\rangle, \text{ where } \langle f, \psi\rangle = \int\limits_{\mathbb{R}^3} dv f(v)\psi(v). \tag{18}$$

Now we can consider a formal limit of VPLE for $\varepsilon = 0$ under some assumptions on "good behavior" of $f^\varepsilon(x, v, t)$ and $E^\varepsilon(x, t)$ for $\varepsilon \to 0$. The resulting equations have the following form.

Proposition 1 *Limiting functions $f^0(x, v, t)$ and $E^0(x, v, t)$ satisfy the equations*

$$f^0_t + v_x f^0_x - E^0 f^0_{v_x} = \frac{1}{Kn} C(f^0), \tag{19}$$

$$E^0 = \frac{1}{Kn}\langle C(f^0), v_x\rangle - \langle f^0, v_x^2\rangle_x, \quad f^0|_{t=0} = f_0, \tag{20}$$

which imply that

$$\rho^0(x, t) = \langle f^0, 1\rangle = 1, \; j^0(x, t) = \langle f^0, v_x\rangle = 0. \tag{21}$$

Proof. ▶ Formal proof of **Proposition** 1 is obvious. Eq. (16) for E leads to equality $\rho^0 = 1$. This in turn is possible only if $j^0 = 0$ because of the continuity equation and boundary conditions at infinity. Finally, the formula for E^0 follows from equation $j^0_t = 0$. ◀

The limiting equations (19)–(21) were studied by analytical and numerical methods in many publications, in particular, in [3] – [8]. One can say that in this case the Poisson equation for electric field is replaced by "explicit" formula (20) for E^0 which follows from the quasi neutrality conditions. Of course, a similar formal limit can be also defined for more complicated case of two-component plasma with ions having finite mass and arbitrary electric charge. We consider in this paper only a simplified model in order to avoid some less important details.

There are at least two mathematical questions which should be clarified.

1. Can Eqs. (19),(20) be justified more rigorously? In particular, what can be said about existence of limits $f^0(x, v, t)$ and $E^0(x, t)$ at $\varepsilon = 0$?

2. Are Eqs. (19),(20) well-posed for a wide class of initial data?

We shall see below that the answers to both questions are (at least partly) negative, though this does not mean that these equations are wrong. We just need to be careful while dealing with them. We consider both questions in the next sections.

5 Behavior of $E^\varepsilon(x,v,t)$ for $\varepsilon \to 0$

We begin with the first question and assume that there exists a ”nice” solution of the problem (16),(17) for small ε. Then we multiply Eq. (16) for $E^\varepsilon(x,t)$ by ε^2 and differentiate in t-variable. The continuity equation and boundary conditions at infinity lead to equation

$$\varepsilon^2\, E^\varepsilon_t = j^\varepsilon\,, \quad j^\varepsilon = \langle f^\varepsilon, v_x \rangle.$$

Then we differentiate the first equation once more and obtain after simple transformations based on Eqs. (16), (17)

$$\begin{aligned}\varepsilon^2 E^\varepsilon_{tt} + E^\varepsilon = S^\varepsilon(x,t) = \\ = -\langle f^\varepsilon, v_x^2\rangle_x + \frac{1}{Kn}\langle C(f^\varepsilon), v_x\rangle + \varepsilon^2 E^\varepsilon E^\varepsilon_x;\end{aligned} \tag{22}$$

$$E^\varepsilon|_{t=0} = E^\varepsilon_t|_{t=0} = 0. \tag{23}$$

The problem (22),(23) can be formally "solved" by Laplace transform in t. Then we obtain

$$\begin{aligned}E^\varepsilon(x,t) = \int_0^{t/\varepsilon} d\tau (\sin\tau) S^\varepsilon(x, t-\varepsilon\tau) = \\ = S^\varepsilon(x,t) - S^\varepsilon(x,0)\cos\frac{t}{\varepsilon} - \varepsilon \int_0^{t/\varepsilon} d\tau (\cos\tau) S^\varepsilon_t(x, t-\varepsilon\tau).\end{aligned} \tag{24}$$

Now we can prove the following estimate.

Proposition 2 *If*

$$|S^\varepsilon_{tt}(x,t)\,| \le C(x), \quad 0 \le t \le T,\ x \in \mathbb{R}, \tag{25}$$

where C does not depend on ε for $\varepsilon \to 0$, then the following asymptotic formula is valid

$$E^\varepsilon(x,t) = S^\varepsilon(x,t) - S^\varepsilon(x,0)\cos\frac{t}{\varepsilon} + \varepsilon\,\Delta(x,t), \tag{26}$$

$$|\Delta(x,t)| \le |S^\varepsilon_t(x,0)| + C(x)\,t, \quad 0 \le t \le T. \tag{27}$$

Proof ► It is sufficient to estimate the last integral in (24). Integrating by parts we obtain

$$\Delta(x,t) = -\int_0^{t/\varepsilon} d\tau (\sin\tau)' \, S_t^\varepsilon(x, t-\varepsilon\tau) =$$

$$= - S_t^\varepsilon(x,0)\sin\frac{t}{\varepsilon} - \varepsilon \int_0^{t/\varepsilon} d\tau (\sin\tau) S_{tt}^\varepsilon(x, t-\varepsilon\tau)$$

The estimate (27) follows directly from this equality and assumption (25) of **Proposition** 2. Hence the proof is completed. ◄

Note that $S^0(x,t) = E^0(x,t)$ in the notation of Eq. (20).

Thus, $E^\varepsilon(x,t)$ rapidly oscillates for $\varepsilon \to 0$ with frequency $\omega_\varepsilon = 1/\varepsilon$ near its average $E^0(x,t)$ with amplitude $|E^0(x,0)| = O(1)$. For example, $E^0(x,0) = -T_0^{'}(x)$ for the initial local Maxwellian with temperature $T_0(x)$.

Probably the extra term in the kinetic equation (19) leads only to a small perturbation of $f^0(x,v,t)$ because of fast oscilaltions. But it is important for understanding of behavior of $E^\varepsilon(x,t)$.

Of course, the assumption (25) of **Proposition** 2 remains unproved. However, it looks realistic for "nice" solution $f^0(x,v,t)$ of equation (19) with $E^\varepsilon(x,t)$ (without the error term) from **Proposition** 2. On the other hand, the asymptotic formula for $E^\varepsilon(x,t)$ can be verified numerically. In the next section we consider the second question related to well-posedness of Eqs. (19),(20).

6 Well-posedness of limiting kinetic equation

We consider the limiting (with $\varepsilon = 0$) equations (19), (20) and represent them in the form

$$f_t + v_x f_x + p_x f_{v_x} = \frac{1}{Kn}\,[C(f) + \langle C(f), v_x\rangle f_{v_x}],$$

$$p = p(f) = \langle f, v_x^2\rangle; \quad f|_{t=0} = f_0(x,v),\ x\in\mathbb{R}, v\in\mathbb{R}^3. \tag{28}$$

where upper zero indices are omitted. We will use in this section a bit weaker assumptions on the initial conditions:

$$f_0(x,v) \xrightarrow[|x|\to\infty]{} F_0(v),\ \langle f_0, 1\rangle = 1,\ \langle f_0, v_x\rangle = 0, \tag{29}$$

where F(v) is not necessary a Maxwellian. Note that this problem has a relatively simple hydrodynamics:

$$\begin{gathered} \rho = 1, \quad j = \langle f, v_x\rangle = 0, \quad T_t + Q_x = 0 \\ \text{for } T = \frac{1}{3}\langle f, |v|^2\rangle, \quad Q = \frac{1}{3}\langle f, |v|^2 v_x\rangle. \end{gathered} \tag{30}$$

The main result of this section can be formulated as follows.

Proposition 3 *The well-posedness of the Cauchy problem for indata*

$$f_0(x,v) = F_0(v) + \gamma h(x,v), \quad 0 < \gamma << 1,$$

$$\langle F_0, 1\rangle = 1,\ \langle F_0, v_x\rangle = 0,\ F_0(v) \geq 0,$$

satisfy the following criterion: the problem is well posed if and only if the equation for $z \in \mathcal{C}$

$$\int\limits_{\mathbb{R}^3} dv\, \frac{1}{z - v_x}\, \partial_{v_x} F_0(v) = 0 \tag{31}$$

does not have complex roots.

Idea of proof. ► We look for solutions in the form

$$f(x,v,t) = F(v,t) + \gamma h(x,v,t).$$

Obviously $F(v,t)$ is the spatially homogeneous solution such that $F|_{t=0} = F_0(v)$. Then for $\gamma \to 0$, h satisfies the linearized equation.

We pass to the Fourier-representation

$$\hat{h}(k) = \mathfrak{F}_{x\to k}(h) = \int\limits_{-\infty}^{\infty} dx\, h(x) e^{-ikx}, \quad k \in \mathbb{R},$$

and obtain

$$\hat{h}_t + ik\, \left(v_x \hat{h} + \langle \hat{h}, v_x^2\rangle\, \partial_{v_x} F\right) = L\hat{h}, \tag{32}$$

where the operator L does not depend on k.

Hence, if

$$0 < t << 1 \quad \text{and} \quad |k| >> 1,$$

we can neglect the term $L\hat{h}$ and reduce the problem to the Vlasov-type linearized equation (Eq. (32) with $L = 0$).

Then we look for solutions that behave like

$$\hat{h}(k, v, t) = \hat{h}(k, v, 0)e^{-izkt}, \quad |k| \to \infty, \tag{33}$$

and obtain the following k-independent equation:

$$(v_x - z)\,\hat{h} \,+\, \langle \hat{h}, v_x^2 \rangle\, \partial_{v_x} F_0(v) = 0. \tag{34}$$

Multiplying this equation by $v_x^2/(v_x - z)$ and integrating over $v \in \mathbb{R}^3$ we obtain

$$\begin{gathered} \langle \hat{h}, v_x^2 \rangle\, \Phi_G(z) = 0, \quad \Phi_G(z) = 1 + \int\limits_{-\infty}^{\infty} \frac{duG'(u)u^2}{u - z}\,, \\ G(u) = \int\limits_{\mathbb{R}^2} dv_y dv_z F_0(u, v_y, v_z). \end{gathered} \tag{35}$$

Note that

$$\begin{gathered} \Phi_G(z) = 1 \,+\, \int\limits_{-\infty}^{\infty} du(u + z)G'(u) \,+\, z^2 \Psi_G(z), \\ \Psi_G(z) = \int\limits_{-\infty}^{\infty} \frac{duG'(u)}{u - z}\,. \end{gathered} \tag{36}$$

Since

$$\int\limits_{-\infty}^{\infty} duG'(u) = 0, \quad \int\limits_{-\infty}^{\infty} duuG'(u) = -1,$$

we finally obtain

$$\langle \hat{h}, v_x^2 \rangle\, z^2\, \Psi_G(z) = 0.$$

It is easy to see that $z \neq 0$ and $\langle \hat{h}, v_x^2 \rangle \neq 0$ for any non-trivial solution of (34). Hence, we finally obtain the equation for z in the form (31). If there exists a solution z (with $\mathrm{Im}\, z \neq 0$) of this equation, then the corresponding function $\hat{h}(k, v, t)$ (33) grows like $\exp(|k|\,|\mathrm{Im}\, z|\, t)$ for all $k > 0$ or $k < 0$ (depending on the sign of Im z). This means that such equation is ill-posed, and explains the criterion of well-posedness from **Proposition** 3. ◀

We do not try to prove that the absence of complex roots of Eq. (31) is sufficient for well-posedness. Probably it can be done by methods developed in [9],[10] and related papers. We also did not try to prove in this paper that the "small"(for large $|k|$) Landau operator L in Eq. (32) can be really neglected. We plan to check it first by using the BGK model in the forcoming paper.

It is interesting to compare our results for Eq. (34) with well-known results for similar equation

$$(v_x - z)\,\hat{h} - \hat{U}(|k|)\,\langle \hat{h}, 1\rangle\,\partial_{v_x} F_0(v) = 0,$$

which follows from the linearized Vlasov equation

$$\hat{U}(|k|) = a^2\,|k|^{-2} \quad \text{for Coulomb potential}$$

and

$$\hat{U}(|k|) = b^2 \quad \text{for VDBE},$$

with some constant a and b (see e.g. [2],[9]). Then the eqution for z reads

$$\hat{U}^{-1}(|k|) = \Psi_G(z)$$

in the notation of Eqs. (35),(36). In the Coulomb (VDBE) case the existence of complex root $z = z(|k|)$ also implies the existence of unstable mode (ill-posedness for VDBE). Our condition (31) coincides with the limit $k = 0$ for Coulomb case.

We finish the paper with two elementary examples of functions $F_0(v)$ which correspond to different cases. For brevity we use the notation of Eq. (35).

1. Well-posedness

$$G_1(u) = \delta(u).$$

2. Ill-posedness (two-stream distribution)

$$G_2(u) = \frac{1}{2}[\delta(u-a) + \delta(u+a)], \quad a > 0.$$

Indeed the function (36) can be written as

$$\Psi_G(z) = \int_{-\infty}^{\infty} \frac{du G(u)}{(u-z)^2}\,.$$

Hence, $\Psi_{G_1}(z) = z^{-2}$ does not have zeros, whereas $\Psi_{G_2}(z) = \dfrac{z^2 + a^2}{(z^2 - a^2)^2}$

has two imaginary zeros $z_{\pm} = \pm\, i\, a$.

In the important Maxwellian case with

$$F_0(v) = \left(\frac{2}{\pi}\right)^{3/2} \exp\left(-\frac{|v|^2}{2}\right)$$

the integral (31) is studied in detail in literature, see e.g. the book [13]. It is known that it does not have complex zeros. The same is probably true for any F_0 such that the function $G_{F_0}(u)$ (see Eq. (35)) has only one maximum on the real line (*Nyquist's criterion*), it can be proved by standard methods [13].

7 Conclusions

We studied in this work a class of such solutions of VPLE, for which the typical length L_0 is much bigger that the Debye radius r_D.

Taking the formal limit

$$\varepsilon = \frac{r_D}{L_0} \to 0, \quad Kn = \frac{l}{L_0} \text{ remains bounded,}$$

we obtain the kinetic equation, which was studied numerically in several publications.

New facts found in this work at the formal level of mathematical rigour are the following.

♢ 1. The limiting equation yields probably correct results for distribution function, but not for electric field. The extra term proportional to $\cos(t/\varepsilon)$ is given in **Proposition** 2.

♢ 2. The limiting equation is well-posed for initial conditions closed to absolute Maxwellian. However there are classes of indata for which the equation is ill-posed (see a criterion in **Proposition** 3). It is not clear what happens with such indata.

We hope to clarify these and related questions in future work.

We are grateful to Claude Bardos for useful references and discussions.

References

[1] Feix M.R., Hohl F., Staton L.D. Nonlinear effects in plasmas, Eds. Kalman and Feix, Gordon and Breach, pp. 3–21, 1969.

[2] Bardos C., Besse N. The Cauchy problem for the Vlasov-Dirac-Benney equation and related issues in fluid mechanics and semi classical limits, Kin. Relat. Models 6 (4), pp. 893–917, 2013.

[3] Epperlein E.M., Short R.W. A practical nonlocal model for electron heat transport in laser plasmas. Phy.Fluids B, 3, p. 3092, 1991.

[4] Bychenkov V. Yu., Rozmus W., Tikhonchuk V. T., Brantov A. V. Nonlocal electron transport in a plasma. Phys. Rev. Lett., 75, p. 4405, 1995.

[5] Batishchev, O.V., Bychenkov, V.Yu., Detering, F., Rozmus, W., Sydora, R., Capjack, C.E., Novikov V.N. Heat transport and electron distribution function in laser produced with hot spots. Physics of Plasmas, 9, pp. 2302–2310, 2002.

[6] Brantov A.V., Bychenkov V.Yu., Batishchev O.V., Rozmus W. Nonlocal heat wave propagation due to skin layer plasma heating by short laser pulses. Computer Physics communications, 164, pp. 67–72, 2004.

[7] Guisset S., Brull S., Dubroca B., d' Humieres E., Karpov S. and Potapenko I. Asymptotic-preserving scheme for the M1-Maxwell system in the quasi-neutral regime, Communications in Computational Physics, 19(2), pp. 301–328, 2016.

[8] Bobylev A., Brantov A., Bychenkov V., Karpov S., Potapenko I. DSMC modeling of a single hot spot evolution using the Landau-Fokker-Planck equation, Acta Appl. Math., 8, 132(1), DOI 10.1007/s10440-014-9940-x, 2014.

[9] Bardos C., Besse N. Hamiltonian Structure, Fluid Representation and Stability for the Vlasov-Dirac-Benney equation, Fields Institute Communication 75 , Springer, pp. 1–31, 2015.

[10] Bardos C., Nouri A. Vlasov Equation with Dirac potential used in fusion plasma, J. Math. Phys 53 (11) pp. 115621–115673, 2012.

[11] Han-Kwan D., Rousset F. Quasi neutral limit for Vlasov-Poisson with Penrose stable data. Ann. Sci. École Norm. Sup., 49(6): pp. 1445–1495, 2016.

[12] Ландау Л.Д. Кинетическое уравнение в случае кулоновского взаимодействия, ЖЭТФ, № 7, с. 203, 1937.

[13] Ишимару С. Основные принциаы физики плазмы, Атомиздат, 1975.

[14] Lifshitz E.M., Pitaevskii L.P. Physical kinetics, Butterworth-Helnemann, 1981.

Contents

1 Introduction **3**

2 Statement of the problem **4**

3 Vlasov–Poisson–Landau Equation (VPLE) **5**

4 Long wave asymptotics for VPLE **7**

5 Behavior of $E^{\varepsilon}(x, v, t)$ for $\varepsilon \to 0$ **9**

6 Well-posedness of limiting kinetic equation **10**

7 Conclusions **14**