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Power geometry and expansions of solutions to the Painlevé equations

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Power geometry and expansions of solutions to the Painlevé equations.

We consider the complicated and exotic asymptotic expansions of solutions to a polynomial ordinary differential equation (ODE). They are such series on integral powers of the independent variable, which coefficients are the Laurent series on decreasing powers of the logarithm of the independent variable and on its pure imaginary power correspondingly. We propose an algorithm for writing ODEs for these coefficients. The first coefficient is a solution of a truncated equation. For some initial equations, it is a polynomial. Question: will the following coefficients be polynomials? Here the question is considered for the third (P_3), fifth (P_5) and sixth (P_6) Painlevé equations. We have found that second coefficients in six of eight families of complicated expansions are polynomials, as well in two of four families of exotic expansions, but in other four families, polynomiality of the second coefficient demands some conditions. We give a survey of these results.

Key words: expansions of solutions to ODE, complicated expansions, exotic expansions, polynomiality of coefficients, Painlevé equations.

Александр Дмитриевич Брюно

Степенная геометрия и разложения решений уравнений Пенлеве. Препринт Института прикладной математики им. М.В. Келдыша РАН, Москва, 2018.

Рассматриваются сложные и экзотические асимптотические разложения решений полиномиального обыкновенного дифференциального уравнения (ОДУ). Это такие ряды по целым степеням независимой переменной, коэффициенты которых суть ряды Лорана от логарифма этой переменной и её чисто мнимой степени соответственно. Предлагается алгоритм составления ОДУ для этих коэффициентов. Первый коэффициент является решением укороченного уравнения. Для некоторых исходных уравнений он является многочленом. Спрашивается: будут ли многочленами следующие коэффициенты? Здесь этот вопрос изучается для третьего, пятого и шестого уравнений Пенлеве. Оказалось, что в шести из восьми семейств сложных разложений и в двух из четырёх семейств экзотических разложений вторые коэффициенты — многочлены. Но в четырёх оставшихся семействах коэффициенты являются многочленами только при определённых условиях. Здесь дан обзор этих результатов.

Ключевые слова: разложения решений ОДУ, сложные разложения, экзотические разложения, полиномиальность коэффициентов, уравнения Пенлеве.

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1. Introduction

In 2004 I proposed a method for calculation of asymptotic expansions of solutions to a polynomial ordinary differential equation (ODE) [1]. It allowed to compute power expansions and power-logarithmic expansions (or Dulac series) of solutions, where coefficients of powers of the independent variable x are either constants or polynomials of logarithm of x. Later it is appeared that such equations have solutions with other expansions: they can have coefficients of powers of x as Laurent series either in increasing powers of $\log x$ or in increasing and decreasing imaginary powers of x. They are correspondingly complicated (psi-series) [2] or exotic [3] expansions. Methods from [1] are not suitable for their calculation. Now I have found a method to writing down ODE for each coefficient of such series (Section 2). The equations are linear and contain higher and low variations from some parts of the initial equation. The first coefficient is a solution of the truncated equation, and usually it is a Laurent series in $\log x$ or in $x^{i\gamma}$. But it is a polynomial or a Laurent polynomial for some equations.

Question: *Will be the following coefficients of the same structure?*

I consider this question for three Painlevé equations P_3, P_5 and P_6, because among 6 Painlevé equations P_1–P_6 there are 3 equations P_3, P_5, P_6 having complicated and exotic expansions of solutions ([4–6]). First coefficients for equations P_3, P_5 and P_6 are polynomials in $\log x$ in complicated expansions and usual or Laurent polynomials in $x^{i\gamma}$ in exotic expansions [4,6]. Each of the Painlevé equations P_3, P_5 and P_6 has 4 complex parameters a, b, c, d. Two of them are included into the truncated equation. These three Painlevé equations have 8 families of complicated expansions and 4 families of exotic expansions. I have calculated several first polynomial coefficients for all these 12 families, sometimes under some simplifications (Sections 3 and 4). Second coefficients in 6 of 8 families of complicated expansions are polynomials, as well in 2 families of exotic expansions, but two families of complicated and two families of exotic expansions demand some conditions for polynomiality of the second coefficient. The third coefficient is a polynomial ether always, either under some restrictions on parameters, or never. We give a survey of these results.

2. Writing ODEs for coefficients

2.1. Algebraic case. Let we have the polynomial

$$f(x, y) \tag{1}$$

and the series

$$y = \sum_{k=0}^{\infty} \varphi_k \, x^k \,, \tag{2}$$

where coefficients φ_k are functions of some quantities. Let we put the series (2) into the polynomial (1) and will select all addends with fixed power exponent of x. For that, we break up the polynomial (1) into the sum

$$f(x,y) = \sum_{i=0}^{m} f_i(y)\, x^i,$$

and we write the series (2) in the form

$$y = \varphi_0 + \sum_{k=1}^{\infty} \varphi_k\, x^k \stackrel{def}{=} \varphi_0 + \Delta.$$

Then

$$\Delta^j = \sum_{k=j}^{\infty} c_{jk}\, x^k,$$

where coefficients c_{jk} are definite sums of products of j coefficients φ_l and corresponding multinomial coefficients [7]. At last, each item $f_i(\varphi_0 + \Delta)$ can be expanded into the Taylor series

$$f_i = \sum_{j=0}^{\infty} \frac{1}{j!} \frac{d^j f_i}{dy^j}\bigg|_{y=\varphi_0} \Delta^j\,.$$

So the result of the substitution of series (2) into the polynomial (1) can be written as the sum

$$\sum_{i=0}^{m} x^i \left[f_i(\varphi_0) + \sum_{j=1}^{\infty} \frac{1}{j!} \frac{d^j f_i(\varphi_0)}{dy^j} \sum_{k=j}^{\infty} c_{jk}\, x^k \right]$$

of items of the form

$$x^i \frac{1}{j!} \frac{d^j f_i(\varphi_0)}{dy^j} c_{jk}\, x^k\,. \tag{3}$$

Here integral indexes $i, j, k \geqslant 0$ are such

$$k \geqslant j; \quad \text{if} \;\; j = 0, \;\; \text{then} \;\; k = 0\,. \tag{4}$$

Set of such points $(i,j,k) \in \mathbb{Z}^3$ will be denoted as $\mathbf{M}$. At last, all items (3) with fixed power exponent x^n are selected by the equation $i + k = n$. The set $\mathbf{M}$ can be considered as a part of the integer lattice $\mathbb{Z}^3$ in $\mathbb{R}^3$ with points (i,j,k), which satisfy (4).

If we look for expansion (2) as a solution of the equation $f(x,y)=0$ and want to use the method of indeterminate coefficients, then we obtain the equation $f_0(\varphi_0)=0$ for the coefficient φ_0, and equation

$$\frac{df_0(\varphi_0)}{dy}\varphi_n x^n + \sum_{(i,j,k)\in\mathbf{N}(n)} x^i \frac{1}{j!}\frac{d^j f_i(\varphi_0)}{dy^j} c_{jk}\, x^k + x^n f_n(\varphi_0) = 0\ , \qquad (5)$$

for the coefficient φ_n with $n>0$, where

$$\mathbf{N}(n) = \mathbf{M} \cap \{j>0,\ i+k=n \text{ and } j>1,\ \text{if } i=0\}.$$

That equation can be canceled by x^n and be written in the form

$$\frac{df_0(\varphi_0)}{dy}\varphi_n + \sum_{(i,j,k)\in\mathbf{N}(n)} \frac{1}{j!}\frac{d^j f_i(\varphi_0)}{dy^j} c_{jk} + f_n(\varphi_0) = 0\ . \qquad (6)$$

Theorem 1 ([8]). *If* $df_0(\varphi_0)/dy \neq 0$*, then coefficients* φ_n *can be found from equations* (6) *successfully with increasing* n*.*

2.2. Case of ODE. If $f(x,y)$ is a differential polynomial, i.e. it contains derivatives $d^l y/dx^l$, then the job of derivatives $\dfrac{d^j f_i}{dy^j}$ play variations $\dfrac{\delta^j f_i}{\delta y^j}$, which are derivatives of Frechet or Gateaux. Here the j-variation $\dfrac{\delta^j f}{\delta y^j} = \dfrac{d^j f}{dy^j}$, if the polynomial does not contain derivatives, and variation of a derivation is $\dfrac{\delta}{\delta y}\left(\dfrac{d^k y}{dx^k}\right) = \dfrac{d^k}{dx^k}$, and for products

$$\frac{\delta(f\cdot g)}{\delta y} = f\,\frac{\delta g}{\delta y} + \frac{\delta f}{\delta y}\cdot g\ , \quad \frac{\delta}{\delta y}\left(\frac{d^k y}{dx^k}\cdot\frac{d^l}{dx^l}\right) = \frac{d^{k+l}}{dx^{k+l}}\ .$$

Analog of the Taylor formula is correct for variations

$$f(y+\Delta) = \sum_{j=0}^{\infty} \frac{1}{j!}\frac{\delta^j f(y)}{\delta y^j}\Delta^j\ .$$

Let now we have the differential polynomial $f(x,y)$ and we look for solution of the equation $f(x,y)=0$ in the form of expansion (2). Here the technique, described above for algebraic equation, can be used, but with the following refinements.

1) According to [1], differential polynomial $f(x, y)$ is a sum of differential monomials $a(x, y)$, which are products of a usual monomial $\mathrm{const} \cdot x^r y^s$ and several derivatives $d^l y/dx^l$. Each monomial $a(x, y)$ corresponds to its vectorial power exponent $Q(a) = (q_1, q_2)$ under the following rules:

$$Q(\mathrm{const}) = 0, \quad Q(x^r y^s) = (r, s), \quad Q(d^l y/dx^l) = (-l, 1),$$

vectorial power exponent of a product of differential monomials is a vectorial sum of their vectorial power exponents $Q(ab) = Q(a) + Q(b)$. Set $\mathbf{S}(f)$ of all vectorial power exponents $Q(a)$ of all differential monomials $a(x, y)$ containing in $f(x, y)$ is called as *support* of f. Its convex hull $\Gamma(f)$ is a *Newton polygon* of f. Its boundary $\partial\Gamma$ consists of vertices $\Gamma_j^{(0)}$ and edges $\Gamma_j^{(1)}$. To each boundary element $\Gamma_j^{(d)}$ corresponds the *truncated equation* $\hat{f}_j^{(d)} = 0$, where $\hat{f}_j^{(d)}$ is a sum of all monomials with power exponents $Q \in \Gamma_j^{(d)}$. The first term of solution's expansion to the full equation is a solution to the corresponding truncated equation. Now the part $f_i(x, y)$ contains all such differential monomials $a(x, y)$, for which in $Q(a)$ the first coordinate $q_1 = i$. Besides, we assume that $f(x, y)$ has no monomials with $q_1 < 0$, and $f_0(y) \not\equiv 0$. Then all formula of the algebraic case with variations instead of derivations are correct.

2) Variations are operators, which are not commute with differential polynomials. So the formulae (5) takes the form

$$\frac{\delta f_0}{\delta y} x^n \varphi_n + \sum_{(i,j,k) \in \mathbf{N}(n)} x^i \frac{1}{j!} \frac{\delta^j f_i}{\delta y^j} x^k c_{jk} + x^n f_n = 0 \,, \tag{7}$$

but in it we cannot cancel by x^n and obtain an analog of formulae (6). In (7) all $\delta^j f_i / \delta y^j$ are taken for $y = \varphi_0$.

Theorem 2 ([8]). *In the expansion* (2) *coefficient* φ_n *satisfies equation* (7).

3) Rules of commutation of variations with functions of different classes exist. If φ_k is a series in $\log x$, then $\xi = \log x$ and $x^s = e^{s\xi}$.

Lemma 1 ([4]).

$$\frac{d^n}{d\xi^n}\left[e^{s\xi}\varphi(\xi)\right] = e^{s\xi} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} \varphi^{(k)}(\xi) \,,$$

where $\binom{n}{k}$ *are binomial coefficients and* $\varphi^{(k)}$ *is the* k*-th derivation of* $\varphi(\xi)$ *along* ξ.

If φ_k is a series in $x^{i\gamma}$, then $\xi = x^{i\gamma}$ and $x^s = \xi^{s/(i\gamma)}$.

Lemma 2 ([9]).

$$\frac{d^n}{d\xi^n}\left[\xi^{s/(i\gamma)}\varphi(\xi)\right] =$$

$$= \xi^{s/(i\gamma)}\left[\sum_{k=0}^{n-1}\binom{n}{k}\frac{s}{i\gamma}\left(\frac{s}{i\gamma}-1\right)\dots\left(\frac{s}{i\gamma}-n+k+1\right)\varphi^{(k)}(\xi)\frac{1}{\xi^{n-k}}+\varphi^{(n)}\right].$$

These Lemmas give rules of commutation of an operator with x^s. Applying them in equation (7), we can cancel the equation by x^n and obtain an equation without x, only with ξ. So the algorithm consists of the following steps.

Step 0. From the initial equation $f(x,y) = 0$, we select such truncated equation $\hat{f}_1^{(1)}(x,y) = 0$, which corresponds to edge $\Gamma_1^{(1)}$ of the polygon Γ of the differential sum $f(x,y)$ and has a complicated or exotic solution depending from $\log x$ or $x^{i\gamma}$, $\gamma \in \mathbb{R}$ correspondingly.

Step 1. We make a power transformation of the variables $y = x^l z$ to make the truncated equation correspond to the vertical edge.

Step 2. We divide the transformed equation $g(x,z) = 0$ into parts $g_i(x,y)x^i$, corresponding to different verticals of its support.

Step 3. In these parts $g_i(x,y)x^i$ we change the independent variable x by $\log x$ or by $x^{i\gamma}$.

Step 4. We write down equations for several first coefficients φ_k.

Step 5. Using the rules of commutation, we exclude powers of x from these equations and we obtain linear ODEs for coefficients with independent variable $\log x$ or $x^{i\gamma}$. Their solutions are power expansions and can be computed by known methods from [1].

3. Results for complicated expansions

3.1. The third Painlevé equation P_3. Written as differential polynomial, it is

$$f(x,y) \stackrel{def}{=} -xyy'' + xy'^2 - yy' + ay^3 + by + cxy^4 + dx = 0, \tag{8}$$

where a, b, c, d are complex parameters. Its support and polygon for $a, b, c, d \neq 0$ are shown on Fig. 1. The edge $\Gamma_1^{(1)}$ corresponds to the truncated equation

$$\hat{f}_1^{(1)} \stackrel{def}{=} -xyy'' + xy'^2 - yy' + by + dx = 0. \tag{9}$$

After the power transformation $y = xz$ and canceling by x, the full equation (8) became

$$g \stackrel{def}{=} -x^2zz'' + x^2z'^2 - xzz' + bz + d + ax^2z^3 + cx^4z^4 = 0. \tag{10}$$

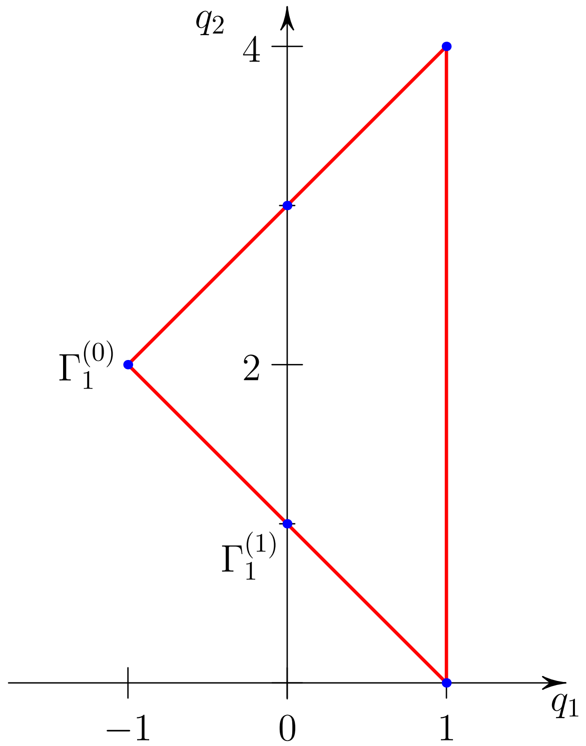

Figure 1. Support and polygon of the equation (8) for $a, b, c, d \neq 0$.

Here the truncated equation (9) takes the form

$$g_0 \stackrel{def}{=} -x^2 z z'' + x^2 z'^2 - x z z' + bz + d = 0\,. \tag{11}$$

Support and polygon of equation (10) are shown on Fig. 2. Here the truncated equation (11) corresponds to the vertical edge $\widetilde{\Gamma}_1^{(1)}$ at the axis $q_1 = 0$. Here $g_2 = az^3$, $g_4 = cz^4$. After the logarithmic transformation $\xi = \log x$, equation (11) takes the form

$$h_0 \stackrel{def}{=} -z\ddot{z} + \dot{z}^2 + bz + d = 0, \tag{12}$$

where $\dot{z} = dz/d\xi$. Support and polygon of equation (12) are shown on Fig. 3 in the case $bd \neq 0$. Here $h_2 = az^3$, $h_4 = cz^4$.

Let $b \neq 0$. The edge $\tilde{\Gamma}_1^{(1)}$ of Fig. 3 corresponds to the truncated equation

$$\hat{h}_1^{(1)} \stackrel{def}{=} -z\ddot{z} + \dot{z}^2 + bz = 0.$$

It has the power solution $z = -b\xi^2/2$. According to [1], extending it as expansion in decreasing powers of ξ, we obtain the solution of equation (11)

$$z = -\frac{b}{2}(\log x + \tilde{c})^2 - \frac{d}{2b} = \varphi_0, \tag{13}$$

where $\tilde{c}$ is arbitrary constant.

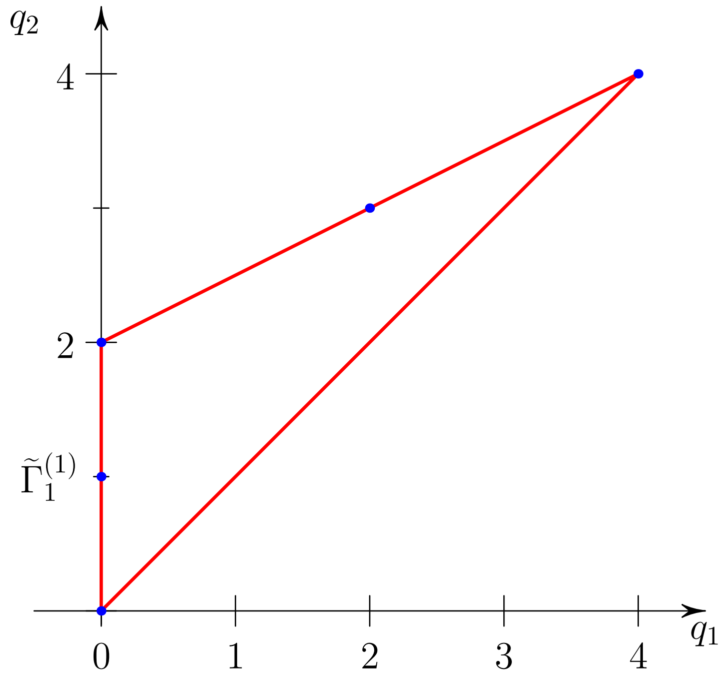

Figure 2. Support and polygon of the equation (10) for $a, b, c, d \neq 0$.

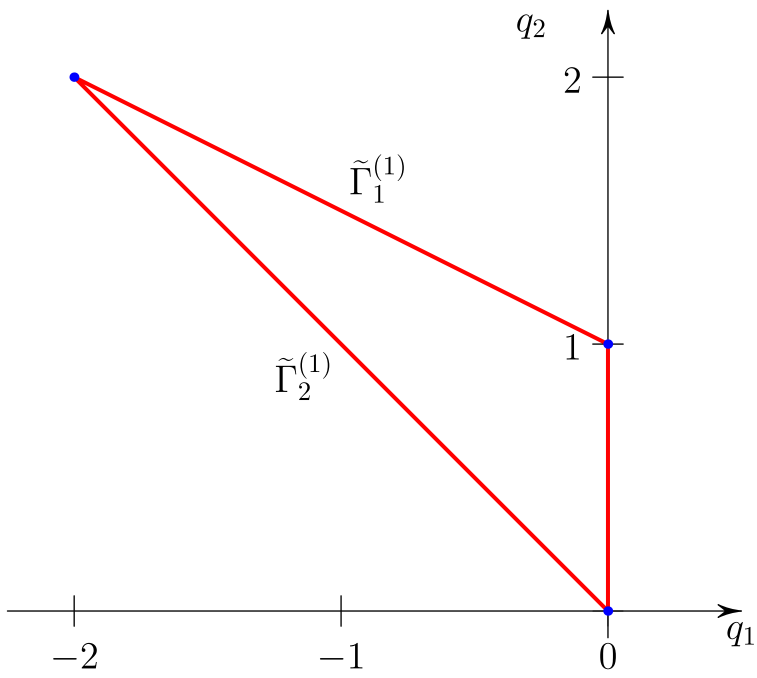

Figure 3. Support and polygon of the equation (12) with $bd \neq 0$.

Let us consider equation (11) in the case $b = 0$, $d \neq 0$. It has solution

$$z = \pm\sqrt{-d}\,(\log x + \tilde{c}) = \varphi_0. \tag{14}$$

Solutions to equation (10) have the form of expansion

$$z = \varphi_0(\xi) + \sum_{k=1}^{\infty} \varphi_{2k}(\xi) x^{2k}, \tag{15}$$

where φ_0 is given by (13) or (14).

In the first case $b \neq 0$, we call family of solutions (15) as **main**, and in the second case $b = 0,\ d \neq 0$, we call the family of solutions (15) as **additional**.

According to Theorem 2, equation for φ_2 is

$$\frac{\delta h_0}{\delta z}(x^2\varphi_2) + x^2 h_2(\varphi_0) = 0. \tag{16}$$

According to (12)

$$\frac{\delta h_0}{\delta z} = -\ddot{z} - z\frac{d^2}{d\xi^2} + 2\dot{z}\frac{d}{d\xi} + b.$$

According to (10) $h_2 = az^3$ and according to Lemma 1

$$\frac{d}{d\xi}x^2\varphi_2 = x^2\left[2\varphi_2 + \dot{\varphi}_2\right], \quad \frac{d^2}{d\xi^2}x^2\varphi_2 = x^2\left[4\varphi_2 + 4\dot{\varphi}_2 + \ddot{\varphi}_2\right].$$

So, equation (16), after cancelling x^2, takes the form

$$-z\left[4\varphi_2 + 4\dot{\varphi}_2 + \ddot{\varphi}_2\right] + 2\dot{z}\left[2\varphi_2 + \dot{\varphi}_2\right] + (b - \ddot{z})\varphi_2 + az^3 = 0,$$

where $z = \varphi_0$ from (13) or (14). In both cases that equation has a polynomial solution:

$$\varphi_2 = \frac{ab}{16}\left[\xi^4 - 2\xi^3 + (2+2\lambda)\xi^2 - (1+2\lambda)\xi + \lambda^2\right], \quad \varphi_2 = -\frac{ad}{4}\left(\xi^2 - \xi + \frac{1}{2}\right)$$

where $\lambda = d/b^2$, for the main family, and for the additional family correspondingly.

Hypothesis 1 ([8]). *Coefficients $\varphi_{2k}(\xi)$ in expansion (15) of the main family of the equation P_3 are polynomials in $\log x$, if the parameter of the equation $d = 0$.*

Theorem 3 ([8]). *Third φ_4 and fourth φ_6 coefficients in expansion (15) of the additional family of the equation P_3 are polynomials if the parameter of the equation $a = 0$. The fifth coefficient φ_8 never is a polynomial, if $|a|+|c| \neq 0$.*

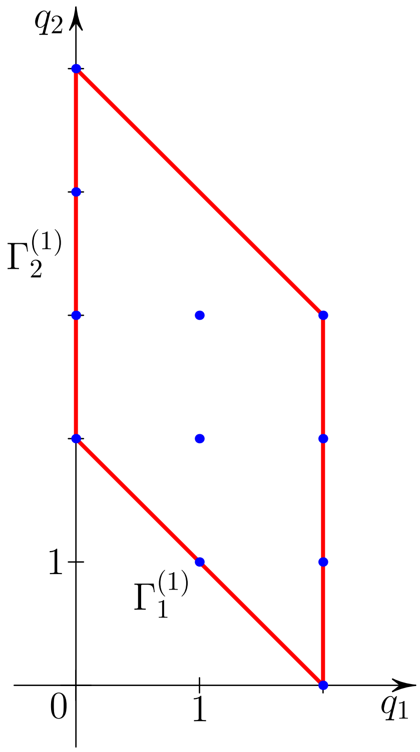

Figure 4. Support and Newton polygon of the equation P_5.

3.2. The fifth Painlevé equation P_5. It can be written as

$$-x^2zz''(z+1)+x^2z'^2\left(\frac{3}{2}z+1\right)-xzz'(z+1)+az^3(z+1)^2+bz^3+$$
$$+cxz(z+1)^2+dx^2(z+1)^2(2+z)=0\,. \quad (17)$$

It has two different cases of beginning of complicated expansions. Its Newton polygon Γ is on Fig. 4.

Two its edges $\Gamma_1^{(1)}$ (Case I) and $\Gamma_2^{(1)}$ (Case II) give truncated equations, which solutions can be continued as complicated expansions and as exotic expansions. The truncated equation, corresponding to the edge $\Gamma_1^{(1)}$, coincides with considered truncated equation for equation P_3 and contains parameters c, d. Let $v = z/x$.

To study Case II, in equation (17) we make transformation $z = 1/w$ and obtain equation

$$h(x,w) \stackrel{def}{=} x^2ww''(1+w)-x^2w'^2\left(\frac{1}{2}+w\right)+xww'(1+w)+a(1+w)^2+$$
$$+bw^2+cxw^2(w+1)^2+dx^2w^2(w+1)^2(1+2w)=0.$$

If write

$$h(x,w)=h_0(x,w)+xh_1(x,w)+x^2h_2(x,w)\,,$$

then

$$\begin{aligned} h_0(x,w) =& x^2ww''(w+1) - x^2w'^2\left(w+\frac{1}{2}\right) + xww'(w+1)+ \\ & + a(w+1)^2 + bw^2\,, \\ h_1(x,w) =& cw^2(1+w)^2\,, \\ h_2(x,w) =& dw^2(w+1)^2(2w+1)\,. \end{aligned} \tag{18}$$

Expansions of solutions to the full equation P_5 have the form

$$v \text{ or } w = \varphi_0(\xi) + \sum_{k=1}^{\infty} \varphi_k(\xi)x^k, \tag{19}$$

where φ_0 belongs to two families (main and additional) in each of both Cases I, II and are polynomials.

Theorem 4 ([10])**.** *For the equation P_5, the second coefficients $\varphi_1(\xi)$ are polynomials for 3 complicated expansions* (19)*, but for the main family in Case I, it is true iff the parameter $d = 0$.*

3.3. The sixth Painlevé equation P_6.

Its Newton polygon is on Fig. 5.

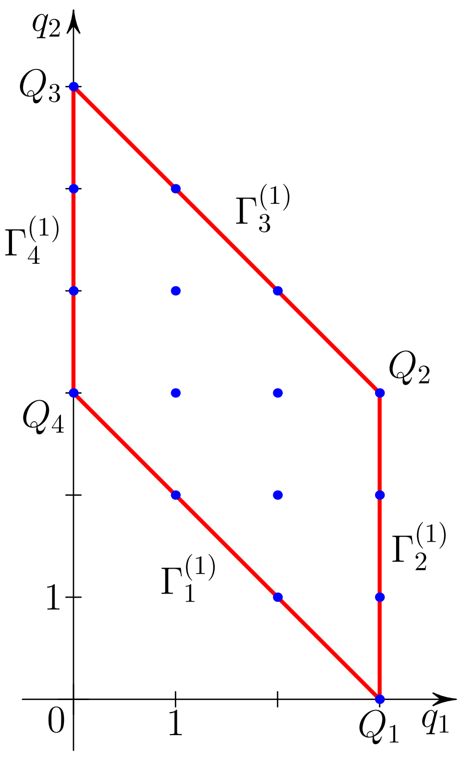

Figure 5. Support and Newton polygon of the equation P_6.

We consider the truncated equation corresponding to left vertical edge. It has 2 parameters a, c and after the power transformation $y = -1/w$ it coincides with the truncated equation of equation P_5 in the Case II, i. e. $h_0(x, w) = 0$ from (18) with $-c$ instead of b. If $\alpha \stackrel{\text{def}}{=} a - c \neq 0$, the truncated equation has solutions

$$w = \frac{\alpha}{2}(\xi + \tilde{c})^2 + \frac{a}{\alpha} = \varphi_0, \tag{20}$$

where $\tilde{c}$ is an arbitrary constant. If $\alpha = 0$, $a \neq 0$, then it has solutions

$$\varphi_0(\xi) = w = \pm\sqrt{2a}(\xi + \tilde{c}). \tag{21}$$

Here we look for expansions of solutions to the full equation P_6 in the form (19), where $\varphi_0(\xi)$ is either (20) or (21), then (19) forms the **main** family, or the **additional** family correspondingly.

Theorem 5. *In the complicated expansions* (19) *for the equation P_6, the second coefficient φ_1 is a polynomial for the additional family, but it is so for the main family iff $\alpha = 2a$.*

4. Results for exotic expansions

Exotic expansions can give real functions. For example, $x^i + x^{-i} = 2\cos\log x$. For beginning of exotic expansions, equations P_3, P_5 and P_6 have the same truncated equations as it was for complicated expansions. Each of the truncated equations of P_3, of P_5 in Case I, of P_5 in Case II and of P_6 has one big family of solutions in the form

$$\varphi_0(\xi) = A\xi + B + C\xi^{-1}, \tag{22}$$

where $A, B, C = \text{const} \in \mathbb{C}$, $\xi = x^{i\gamma}$, $\gamma = \text{const} \in \mathbb{R}$, $\gamma \neq 0$. Exotic expansions for equations P_3, P_5 and P_6 have the form (19), where all $\varphi_k(\xi)$ are convergent Laurent series, and k are even for equation P_3.

Theorem 6 ([9])**.** *In the exotic expansion* (19) *for equation P_3, the second coefficient $\varphi_2(\xi)$ is a Laurent polynomial.*

Theorem 7 ([10])**.** *In the exotic expansion* (19) *for the Case I of equation P_5, the second coefficient $\varphi_1(\xi)$ is always Laurent polynomial, but for the Case II of equation P_5, it is a Laurent polynomial only under two conditions*

$$2AC + B(B+1) = 0, \quad A(2B+1)C(\gamma^2 - 1) = 0$$

on parameters of the solution φ_0 in (22)*.*

Theorem 8. *In the exotic expansion* (19) *for equation P_6, the second coefficient $\varphi_1(\xi)$ is a Laurent polynomial only under three conditions:*

$$2AC + B(B+1) = 0, \quad A(2B+1)C(\gamma^2 - 1)(b - d) = 0,$$
$$AC\left[6B^2 - B - 3\right] = 0.$$

Usually the equation for $\varphi_k(\xi)$ has two solutions: with increasing and with decreasing powers of ξ. But they coincide if the solution is an usual or Laurent polynomial. If all coefficients $\varphi_k(\xi)$ are polynomials then there is one family of expansions (19). In another case there are two different families. Details see in [10].

5. Conclusion

In both cases: complicated and exotic expansions we have its own alternative. In complicated expansion the coefficient $\varphi_k(\xi)$ is either a polynomial or a divergent Laurent series. In exotic expansion the coefficient $\varphi_k(\xi)$ is either a Laurent polynomial, in that case it is unique, or a Laurent series, then there are two different coefficients both in form of convergent series.

In all considered cases, when coefficient $\varphi_k(\xi) = D\xi^m + E\xi^{m-1} + F\xi^{m-2} + \ldots$ of the complicated or exotic expansion is an usual or Laurent polynomial, its coefficients $D, E, F, \ldots$, satisfy to a system of linear algebraic equations. And number of equations is more then number of these coefficients. Such linear systems have solutions only in degenerated cases when rank of the extended matrix of the system is less then the maximal possible. Existence of such situations in the Painlevé equations shows their degeneracy or their inner symmetries.

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List of Figures

Contents