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Asymptotic solution of some nonlinear problems

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We propose algorithms that allow for nonlinear equations to obtain asymptotic expansions of solutions in the form of: (a) power series with constant coefficients, (b) power series with coefficients which are power series of logarithm and (c) power series of exponent of a power series with coefficients which are power series as well. These algorithms are applicable to nonlinear equations (A) algebraic, (B) ordinary differential and (C) partial differential, and to systems of such equations as well. We give the description of the method for one ordinary differential equation and we enumerate some applications of these algorithms.

Key words: expansions of solutions to ODE, power expansions, complicated expansions, exponential expansions.

Александр Дмитриевич Брюно

Асимптотическое решение некоторых нелинейных задач. Препринт Института прикладной математики им. М.В. Келдыша РАН, Москва, 2018.

Предлагаются алгоритмы, позволяющие получать для нелинейных уравнений асимптотические разложения решений в виде: (а) степенных рядов с постоянными коэффициентами, (б) степенных рядов с коэффициентами, являющимися степенными рядами от логарифма и (в) степенных рядов экспоненты степенного ряда, коэффициенты которых суть степенные ряды. Эти алгоритмы применимы к нелинейным уравнениям: (A) алгебраическим, (B) обыкновенным дифференциальным и (C) в частных производных, а также к системам таких уравнений. Изложение ведётся для одного обыкновенного дифференциального уравнения. Перечислены некоторые из приложений этих алгоритмов.

Ключевые слова: разложения решений ОДУ, степенные разложения, сложные разложения, экспоненциальные разложения.

1. Introduction

Tendency to solve the mathematical problems numerically increases in last time according to increasing of power of computers. And study of mathematicians is oriented to that instead of the study Mathematics itself. I.e. Mathematics is substituted by Arithmetic. That is especially true for problems, which cannot be solved by methods of Classic Analysis and Functional Analysis. Here I will describe a set of such problems, which can be solved by methods of Nonlinear Analysis, allowing to compute asymptotic forms and asymptotic expansions of solution of different classes of equations: algebraic, ordinary differential and partial differential. And of systems of such equations as well. One-year course of lectures on Nonlinear Analysis was given at the Mathematical Department of the Lomonosov Moscow State University. In the present lecture, I will explain main notions and methods of Nonlinear Analysis on examples of an ordinary differential equation

$$f(x, y, y', \dots, y^{(n)}) = 0,$$

where f is a polynomial of its arguments. These methods allow to obtain its solutions in the form of asymptotic expansions

$$y(x) = \sum_{k=0}^{\infty} \varphi_k(x) \tag{1}$$

when $x \to 0$ or $x \to \infty$. At the end, I will give a list of its applications.

For simplicity, here we consider the expansions with real power exponents only.

2. Selection of the leading terms

2.1. Order of a function [1]. Let put

$$\omega = \begin{cases} -1, & \text{if } x \to 0, \\ +1, & \text{if } x \to \infty. \end{cases}$$

The number

$$p_\omega(\varphi) = \omega \limsup_{x^\omega \to \infty} \frac{\log|\varphi(x)|}{\omega \log|x|}$$

calculated for the fixed $\arg x \in [0, 2\pi)$, is called as *order* of the function $\varphi(x)$. For the power function $\varphi(x) = \text{const} \cdot x^\alpha$ the order $p(\varphi) = \operatorname{Re} \alpha$ for any ω and $\arg x$. The expansion (1) is called as *asymptotic*, if

$$\omega p(\varphi_k) > \omega p(\varphi_{k+1}), \quad k = 0, 1, 2, \dots .$$

2.2. Truncated sums [2, 3]. Let x be independent and y be dependent variables, $x, y \in \mathbb{C}$. *The differential monomial* $a(x, y)$ is a product of an usual monomial $cx^{r_1}y^{r_2}$, where $c = \text{const} \in \mathbb{C}$, $(r_1, r_2) \in \mathbb{R}^2$, and a finite number or derivatives $d^l y/dx^l$, $l \in \mathbb{N}$. The sum of differential monomials

$$f(x, y) = \sum a_i(x, y) \tag{2}$$

is called as the *differential sum*. We want to select from it all such monomials $a_i(x, y)$, which have the biggest order after the substitution

$$y = \text{const } x^p, \quad p \in \mathbb{R} \tag{3}$$

Under the substitution

$$x^{q_1}y^{q_2} = \text{const } x^{q_1+pq_2} = \text{const } x^{\langle P,Q\rangle},$$

where $P = (1, p) = (p_1, p_2)$, $Q = (q_1, q_2)$, $\langle P, Q\rangle = p_1q_1 + p_2q_2$ is the scalar product. For fixed p and ω, the biggest order will give those monomial $\text{const } x^{q_1}y^{q_2}$, for which

$$\omega \langle P, Q\rangle$$

has the maximal value.

Analogously, the differential monomial $a(x, y)$ corresponds to its (vectorial) *power exponent* $Q(a) = (q_1, q_2) \in \mathbb{R}^2$ with the following rules:

$$Q(cx^{r_1}y^{r_2}) = (r_1,\ r_2); \quad Q(d^l y/dx^l) = (-l, 1);$$

power exponent of a product of monomials is a vectorial sum of their exponents:

$$Q(a_1a_2) = Q(a_1) + Q(a_2).$$

The set $\mathbf{S}(f)$ of vectorial power exponents $Q(a_i)$ of all differential monomials $a_i(x, y)$, containing in the differential sum (2), is called as the *support of the sum* $f(x, y)$. Evidently, $\mathbf{S}(f) \in \mathbb{R}^2$. The convex hull $\Gamma(f)$ of the support $\mathbf{S}(f)$ as called as the *polygon of the sum* $f(x, y)$. The boundary $\partial\Gamma(f)$ of the polygon $\Gamma(f)$ consists of vertices $\Gamma_j^{(0)}$ and edges $\Gamma_j^{(1)}$. We call them as *generalized faces* $\Gamma_j^{(d)}$, where the upper index shows the dimension of the face, and low index shows its number. Each face $\Gamma_j^{(d)}$ corresponds to the *truncated sum*

$$\hat{f}_j^{(d)}(x, y) = \sum a_i(x, y) \text{ along } Q(a_i) \in \Gamma_j^{(d)} \cap \mathbf{S}(f). \tag{4}$$

After substitution (3), all terms in (4) have the same order, which is $\langle P, Q\rangle$, if the vector $\omega P = \omega(1, p)$ is the exterior normal to the edge or vertex $\Gamma_j^{(d)}$. So the biggest value of $\omega \langle P, Q\rangle$ achieved on $Q \in \Gamma_j^{(d)}$.

Example. Let us consider the third Painlevé equation

$$f(x,y) \stackrel{\text{def}}{=} -xyy'' + xy'^2 - yy' + ay^3 + by + cxy^4 + dx = 0, \tag{5}$$

assuming that its complex parameters $a, b, c, d \neq 0$. Here the first three differential monomials have the same power exponent $Q_1 = (-1, 2)$, then $Q_2 = (0, 3)$, $Q_3 = (0, 1)$, $Q_4 = (1, 4)$, $Q_5 = (1, 0)$. They are shown in Fig. 1 in coordinates q_1, q_2. Their convex hull $\Gamma(f)$ is a triangle with three vertexes $\Gamma_1^{(0)} = Q_1$, $\Gamma_2^{(0)} = Q_4$, $\Gamma_3^{(0)} = Q_5$, and with three edges $\Gamma_1^{(1)}, \Gamma_2^{(1)}, \Gamma_3^{(1)}$. The vertex $\Gamma_1^{(0)} = Q_1$ corresponds to the truncated sum

$$\hat{f}_1^{(0)}(x,y) = -xyy'' + xy'^2 - yy',$$

and the edge $\Gamma_1^{(1)}$ corresponds to the truncated sum

$$\hat{f}_1^{(1)}(x,y) = \hat{f}_1^{(0)}(x,y) + by + dx. \quad \blacksquare$$

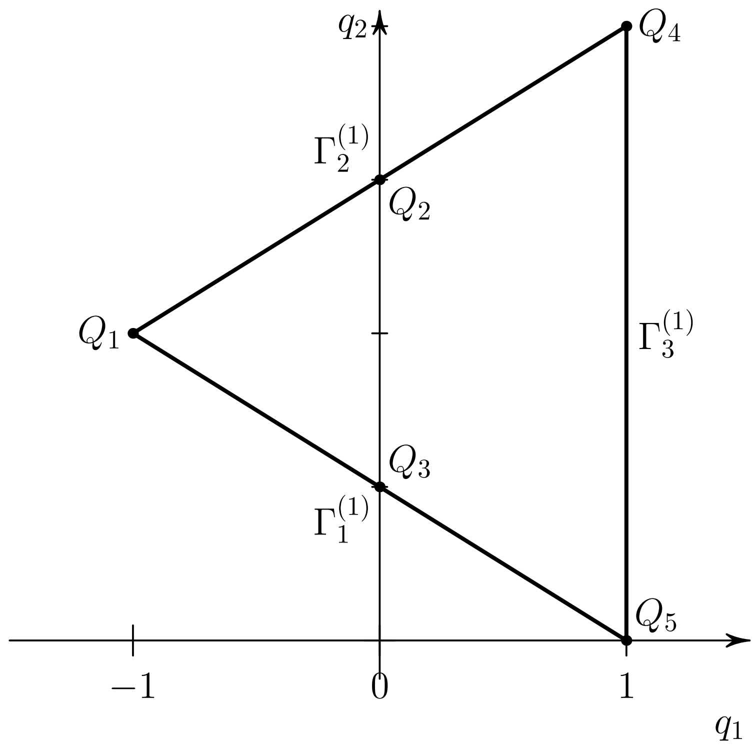

Figure 1. Support $\mathbf{S}(f)$, polygon $\Gamma(f)$ and its edges $\Gamma_j^{(1)}$ for the third Painlevé equation (5).

Let the plane $\mathbb{R}_*^2$ be such conjugate to the plane $\mathbb{R}^2$, that the scalar product

$$\langle P, Q \rangle \stackrel{\text{def}}{=} p_1 q_1 + p_2 q_2$$

be defined for $P = (p_1, p_2) \in \mathbb{R}_*^2$ and $Q = (q_1, q_2) \in \mathbb{R}^2$. Each face $\Gamma_j^{(d)}$ corresponds to its own *normal cone* $\mathbf{U}_j^{(d)} \subset \mathbb{R}_*^2$. It consists of the exterior normals P to the face

$\Gamma_j^{(d)}$. The normal cone $\mathbf{U}_j^{(1)}$ of the edge $\Gamma_j^{(1)}$ is a ray orthogonal to the edge $\Gamma_j^{(1)}$ and directed outside of the polygon $\Gamma(f)$. The normal cone $\mathbf{U}_j^{(0)}$ of the vertex $\Gamma_j^{(0)}$ is the open sector (angle) at the plane $\mathbb{R}_*^2$ with vertex in the origin $P = 0$ and restricted by rays, which are the normal cones of edges, adjoined to the vertex $\Gamma_j^{(0)}$. Generally

$$\mathbf{U}_j^{(d)} = \{P : \langle P, Q' \rangle = \langle P, Q'' \rangle > \langle P, Q''' \rangle\,,\ P', P'' \in \Gamma_j^{(d)},\ P''' \in \Gamma \backslash \Gamma_j^{(d)}\}\,.$$

Example. For the equation (5), normal cones $\mathbf{U}_j^{(d)}$ of faces $\Gamma_j^{(d)}$ are shown in Fig. 2.■

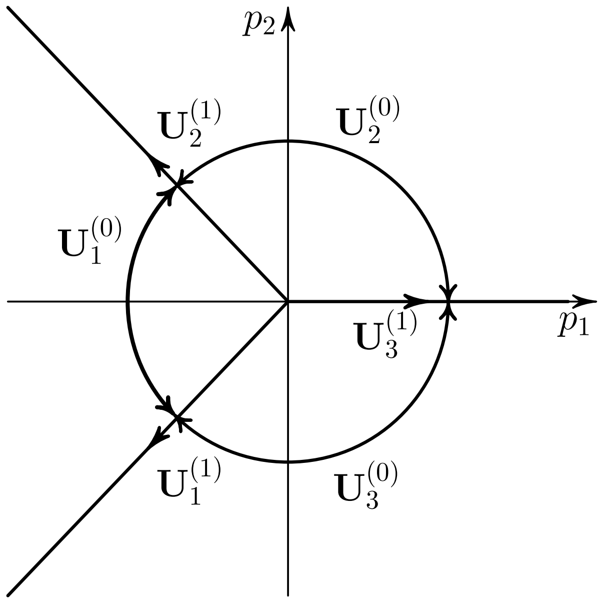

Figure 2. Normal cones $\mathbf{U}_j^{(d)}$ to vertices and edges $\Gamma_j^{(d)}$ of the polygon of Fig. 1.

So each face $\Gamma_j^{(d)}$ corresponds to the normal cone $\mathbf{U}_j^{(d)}$ in the plane $\mathbb{R}_*^2$ and the truncated sum (4).

2.3. Variations [3, 4]. In Classic Analysis, it is known the Taylor formula

$$f(x_0 + \Delta) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)\Delta^k.$$

In the Functional Analysis, there is it analog

$$f(x, y_0 + z) = \sum_{k=0}^{\infty} \frac{1}{k!}\, \frac{\delta^k}{\delta y^k} f \bigg|_{y=y_0} z^k, \tag{6}$$

where $f(x, y)$ is a differential sum, $\dfrac{\delta^k}{\delta y^k} f(x, y)$ is its k-*variation* along y (derivative of Frechet or Gateaux). It is taken on the function $y = y_0(x)$ and is an operator which is applied to the k-power of the small addendum z^k. All that is doing in the infinity-dimensional spaces. If $f(x, y)$ is an usual polynomial (without derivatives), then $\dfrac{\delta^k}{\delta y^k} f = \dfrac{\partial^k}{\partial y^k} f$. But variations are defined for differential polynomials containing derivatives.

Example. If $f = \dfrac{\partial^k y}{\partial x^k}$, then $\dfrac{\delta f}{\delta y} = \dfrac{\partial^k}{\partial x^k}$. ■

Theorem 1. *Let* $p(y_0) = p_0$, $p\left(y_0^{(k)}\right) = p_0 - k$, $k = 1, 2, \ldots$, $p(z) = p_1$, *along the curves* $y = \text{const}\ x^p$ *the order* $p(f) = \tilde{p}$ *and* $\omega(p_1 - p_0) < 0$, *then expansion* (6) *is asymptotic, where*

$$\omega p \left(\frac{\delta^k}{\delta y^k} f \bigg|_{y=y_0} z^k \right) \leqslant \omega \tilde{p} + k\omega(p_1 - p_0), \quad k = 0, 1, 2, \ldots$$

Corollary 1. *In the situation of the Theorem 1*

$$\omega p \left(\frac{\delta}{\delta y} f \bigg|_{y=y_0} z \right) < \omega p\left(f(y_0)\right).$$

i.e. the order of the first variation is less than the order of $f(y_0)$.

3. Power expansions of solutions [3, 5]

3.1. Statement of the problem.

Problem. *Let we have the ordinary differential equation*

$$f(x, y) = 0, \tag{7}$$

where $f(x, y)$ *is a differential sum. For solutions* $y = \varphi(x)$ *of the equation* (7) *for* $x \to 0$ *and* $x \to \infty$ *to find all expansions of the form*

$$y = c_r x^r + \sum c_s x^s, \quad c_r, c_s = \text{const} \in \mathbb{C}, \ \ c_r \neq 0, \tag{8}$$

where the power exponents $r, s \in \mathbb{R}$,

$$\omega r > \omega s. \quad ■$$

Computation of the expansions (8) consists of two steps: computation of the first term

$$y = c_r x^r, \quad c_r \neq 0$$

and computation of other terms in (8).

Theorem 2. *If the expansion* (8) *satisfies equation* (7) *and* $\omega(1, r) \in \mathbf{U}_j^{(d)}$*, then the truncation* $y = c_r x^r$ *of solution* (8) *is a solution of the truncated equation* $\hat{f}_j^{(d)}(x, y) = 0$.

So, to find all truncated solutions $y = c_r x^r$ of the equation (7), we must compute: the support $\mathbf{S}(f)$, polygon $\Gamma(f)$, all its faces $\Gamma_j^{(d)}$ and their normal cones $\mathbf{U}_j^{(d)}$. Then for each truncated equation $\hat{f}_j^{(d)}(x, y) = 0$, we must find all such its power solutions $y = c_r x^r$, for which one of two vectors $\pm(1, r)$ is in the normal cone $\mathbf{U}_j^{(d)}$.

3.2. Solving a truncated equation. The vertex $\Gamma_j^{(0)} = \{Q\}$ corresponds to the truncated equation $\hat{f}_j^{(0)}(x, y) = 0$ with the point support $Q = (q_1, q_2)$. Let put $g(x, y) = x^{-q_1} y^{-q_2} \hat{f}_j^{(0)}(x, y)$, then $g(x, cx^r)$ does not depend from x and c and is a polynomial of r. Hence, the power exponent r of the solution $y = c_r x^r$ to the equation $\hat{f}_j^{(0)}(x, y) = 0$ is a root of the *characteristic equation*

$$\chi(r) \stackrel{\text{def}}{=} g(x,\, x^r) = 0, \tag{9}$$

and the coefficient c_r is arbtrary. Among real roots r of the equation (9), we must take only such, for which the vector $\omega(1, r)$ is in the normal cone $\mathbf{U}_j^{(0)}$ of the vertex $\Gamma_j^{(0)}$.

Example. In equation (5), the vertex $\Gamma_1^{(0)} = Q_1 = (-1, 2)$ corresponds to the truncated equation

$$\hat{f}_1^{(0)}(x, y) \stackrel{\text{def}}{=} -xyy'' + xy'^2 - yy' = 0, \tag{10}$$

and $\hat{f}_1^{(0)}(x, x^r) = x^{2r-1}[-r(r-1) + r^2 - r] \equiv 0$, i.e. any expression $y = cx^r$ is a solution of the equation (10). Here $\omega = -1$ and we are interested only in such these solutions, for which the vecor $-(1, r) \in \mathbf{U}_1^{(0)}$. According to Fig. 2, it means that $r \in (-1, 1)$. So the vertex $\Gamma_2^{(0)}$ corresponds to two-parameter family of power asymptotic forms of solutions

$$y = cx^r, \text{ arbitrary } c \neq 0, \quad r \in (-1, 1). \ \blacksquare \tag{11}$$

The edge $\Gamma_j^{(1)}$ corresponds to the truncated equation $\hat{f}_j^{(1)}(x, y) = 0$. Its normal cone $\mathbf{U}_j^{(1)}$ is the ray $\{P = \lambda\omega'(1, r'),\ \lambda > 0\}$. Inclusion $\omega(1, r) \in \mathbf{U}_j^{(1)}$ means equalities $\omega = \omega'$ ans $r = r'$. They determine exponent r of the truncated solution $y = c_r x^r$ and value ω. To find the coefficient c_r, we must substitute the expression $y = c_r x^r$ into the truncated equation $\hat{f}_j^{(1)}(x, y) = 0$. After cancellation of some power of x, we obtain the algebraic *determining equation* for the coefficient c_r

$$\tilde{\tilde{f}}(c_r) \stackrel{\text{def}}{=} x^{-s} \hat{f}_j^{(1)}(x, c_r x^r) = 0.$$

Each its root $c_r \neq 0$ corresponds to its asymptotic form $y = c_r x^r$.

Example. In equation (5), the edge $\Gamma_1^{(1)}$ corresponds to the truncated equation

$$\hat{f}_1^{(1)}(x,y) \stackrel{\text{def}}{=} -xyy'' + xy'^2 - yy' + by + dx = 0. \qquad (12)$$

As $\mathbf{U}_1^{(1)} = \{P = -\lambda(1,1),\ \lambda > 0\}$, then $\omega = -1$ and $r = 1$. After substitution $y = c_1 x$ in the truncated equation (12) and cancel by x, we obtain for c_1 equation $bc_1 + d = 0$. Hence, $c_1 = -d/b$. So, the edge $\Gamma_1^{(1)}$ corresponds to unique power asymptotic form of solution

$$y = -(d/b)x, \quad x \to 0. \ \blacksquare \qquad (13)$$

3.3. Critical numbers of the truncated solution. If the truncated solution $y = c_r x^r$ is found, then the change $y = c_r x^r + z$ brings the equation $f(x,y) = 0$ to the form

$$f(x, cx^r + z) \stackrel{\text{def}}{=} \tilde{f}(x,z) \stackrel{\text{def}}{=} \mathscr{L}(x)z + h(x,z) = 0,$$

where $\mathscr{L}(x)$ is a liner differential operator and the support $\mathbf{S}(\mathscr{L}z)$ consists of one point $(\tilde{\nu}, 1)$, which is a vertex $\tilde{\Gamma}_1^{(0)}$ of the polygon $\Gamma(\tilde{f})$, and the support $\mathbf{S}(h)$ has not the point $(\tilde{\nu}, 1)$. The operator $\mathscr{L}(x)$ is the first variation $\delta \hat{f}_j^{(d)}/\delta y$ on the curve $y = c_r x^r$. Let $\nu(k)$ be the characteristic polynomial of the differential sum $\mathscr{L}(x)z$, i.e.

$$\nu(k) = x^{-v-k}\mathscr{L}(x)x^k.$$

The real roots $k_1, \ldots, k_\varkappa$ of the polynomial $\nu(k)$, which satisfy the inequality $\omega r > \omega k_i$, are called as *cririical numbers of the truncated solution* $y = c_r x^r$.

Example. The first variation for the truncated equation (10) is

$$\frac{\delta \hat{f}_1^{(0)}}{\delta y} = -xy'' - xy\frac{d^2}{dx^2} + 2xy'\frac{d}{dx} - y' - y\frac{d}{dx}.$$

At the curve $y = c_r x^r$, that variation gives the operator

$$\mathscr{L}(x) = c_r x^{r-1}\left[-r(r-1) - x^2\frac{d^2}{dx^2} + 2rx\frac{d}{dx} - r - x\frac{d}{dx}\right].$$

The characteristic polynomial of the sum $\mathscr{L}(x)z$, i.e. $\mathscr{L}(x)x^k$, is

$$\nu(k) = c_r[-r(r-1) - k(k-1) + 2rk - r - k] = -c_r(k-r)^2.$$

It has one double root $k_1 = r$, which is not a critical number, because it does not satisfy the inequality $\omega r > \omega k_1$. Hence, the truncated solution (11) has not critical numbers.

The first variation for the truncated equation (12) is

$$\frac{\delta \hat{f}_1^{(1)}}{\delta y} = \frac{\delta \hat{f}_1^{(0)}}{\delta y} + b.$$

At the curve (13), i.e. $y = c_1 x$, $c_1 = -d/b$, the variation gives the operator

$$\mathscr{L}(x) = c_1 \left[-x^2 \frac{d^2}{dx^2} + 2x\frac{d}{dx} - 1 - x\frac{d}{dx} - \frac{b^2}{d} \right]$$

and the characteristic polynomial

$$\nu(k) = -c_1[k^2 - 2k + 1 + b^2/d].$$

Its roots are $k_{1,2} = 1 \pm b/\sqrt{-d}$. If $\mathrm{Im}\,(b/\sqrt{-d}) \neq 0$, then the critical numbers are absent. If $\mathrm{Im}\,(b/\sqrt{-d}) = 0$, then only one root $k_1 = 1 + |b/\sqrt{-d}|$ satisfies the inequality $\omega r > \omega k_i$, and it is the unique critical number of the power asymptotic form (13). ■

3.4. Computation of the power expansion of a solution [3, § 3]. Let $\Gamma_j^{(0)}$ be a vertex of the polygon $\Gamma(f)$ and vectors M_1 and M_2 are directed from the vertex along the adjoint edges, and all points of the shifted support $\mathbf{S}(f) - \Gamma_j^{(0)}$ have the form $l_1 M_1 + l_2 M_2$ with integers $l_1, l_2 \geqslant 0$. Then the set

$$\mathbf{K}_j^{(0)}(r) \stackrel{\text{def}}{=} \{s = r + l_1 r_1 + l_2 r_2,\ \text{целые}\ l_i \geqslant 0, l_1 + l_2 > 0\},$$

where $r_i = \langle (1, r), M_i \rangle$, $i = 1, 2$.

Let $\Gamma_j^{(1)}$ be an edge of the polygon $\Gamma(f)$ with vertexes $\Gamma_k^{(0)}$, $\Gamma_l^{(0)}$ and with the normal $\omega(1, r)$. Then

$$\mathbf{K}_j^{(1)} \stackrel{\text{def}}{=} \mathbf{K}_k^{(0)}(r) \cap \mathbf{K}_l^{(0)}(r).$$

Theorem 3. *If the truncated solution $y = c_r x^r$ corresponds to the vertex $\Gamma_j^{(0)}$ with $\omega(1, r) \subset \mathbf{U}_j^{(0)}$ or to the edge $\Gamma_j^{(1)}$ with $\omega(1, r) \subset \mathbf{U}_j^{(1)}$ and all critical numbers of the truncated solution does not ly in the set $\mathbf{K} = \mathbf{K}_j^{(0)}(r)$ or $\mathbf{K}_j^{(1)}$, then the initial equation has a solution in the form of expansion* (8)*, where s runs the set $\mathbf{K}_j^{(0)}(r)$ or $\mathbf{K}_j^{(1)}$ correspondingly.*

Proof is based on the asymptotic expansions

$$f(x, y) = \hat{f}_j^{(d)}(x, y) + \hat{\hat{f}}(x, y) + \ldots, \quad y = c_r x^r + c_s x^s + \ldots$$

Substituting one into another and using Corollary 1, we obtain the equation

$$f(x, c_r x^r + c_s x^s + \dots) = \hat{f}_j^{(d)}(x, c_r x^r) + \left.\frac{\delta \hat{f}_j^{(d)}}{\delta y}\right|_{y=c_r x^r} \cdot c_s x^s + \\ + \hat{\hat{f}}(x, c_r x^r) + \dots = 0.$$

But here $\hat{f}_j^{(d)}(x, c_r x^r) = 0$ and the leading terms are next two. Hence, the equation

$$\left.\frac{\delta \hat{f}_j^{(d)}}{\delta y}\right|_{y=c_r x^r} \cdot c_s x^s + \hat{\hat{f}}(x, c_r x^r) = 0,$$

must be satisfied. It gives the equation of the form

$$\nu(s) c_s + b_s = 0, \quad b_s = \mathrm{const} \in \mathbb{C}.$$

As $s \in \mathbf{K}$ and according to condition of the Theorem $\nu(s) \neq 0$, then moving along $s \in \mathbf{K}$ with decreasing ωs we successfully compute coefficients c_s of expansion (8).

Example. The vertex $\Gamma_1^{(0)} = Q_1$ for the equation (5) corresponds to vectors $M_1 = (1, 1)$, $M_2 = (1, -1)$, so $r_1 = 1 + r$, $r_2 = 1 - r$, where $|r| < 1$ and the set

$$\mathbf{K}_1^{(0)}(r) = \{s = r + l_1(1 + r) + l_2(1 - r), \text{ целые } l_1, l_2 \geqslant 0, l_1 + l_2 > 0\}. \quad (14)$$

As there are no critical numbers, then according to Theorem 3, each truncated solution (11) corresponds to the solution (8) with $s \in \mathbf{K}_1^{(0)}(r)$.

The edge $\Gamma_1^{(1)}$ has two vertices Q_1 and $Q_5 = (1, 0)$, $r = 1$. Here, according to (14), $\mathbf{K}_1^{(0)}(1) = \{1 + 2l_1\}$ for the vertex Q_1. For the vertex $\Gamma_3^{(0)} = Q_5$, we have $M_1 = (-1, 1)$, $M_2 = (0, 2)$. So $r_1 = 0$, $r_2 = 2$, and $\mathbf{K}_3^{(0)}(1) = \mathbf{K}_1^{(0)}(1) = \{1 + 2l_1, \text{ integral } l_1 > 0\}$. If $\operatorname{Im}(b/\sqrt{-d}) \neq 0$, then the truncated solution (13) has no critical numbers and, in the expansion (8) all power exponents s are odd integral numbers more then 1, and coefficients c_s are unique constants. If $\operatorname{Im}(b/\sqrt{-d}) = 0$, then there is only one critical number $k_1 = 1 + |b/\sqrt{-d}|$. Hence, if the number k_1 is not odd, then there is the expansion (8). ■

4. Complicated expansions of solutions [3, § 5], [7,8]

Truncated equations can have nonpower solutions, which can be continued into asymptotic expansions. Here we will look for solutions of the full equation $f(x, y) = 0$ in the form of the *complicated asymptotic expansions*

$$y = \varphi_r(\log x) x^r + \sum \varphi_s(\log x) x^s, \quad \omega s < \omega r, \quad (15)$$

where $\varphi_r(\log x)$ and $\varphi_s(\log x)$ are series on decreasing powers of logarithm.

Theorem 4. *If the series* (15) *is a solution of the full equation* (7) *and* $\omega(1,r) \subset \mathbf{U}_j^{(d)}$, *then* $y = \varphi_r x^r$ *is a solution of the corresponding truncated equation* $\hat{f}_j^{(d)}(x,y) = 0$.

A truncated equation, corresponding to a vertex, has a nonpower solution only in very degenerate cases [3, § 5]. So, here we will consider only truncated equations, corresponding to edges $\Gamma_j^{(1)}$.

4.1. Case of the vertical edge $\Gamma_j^{(1)}$. If the edge $\Gamma_j^{(1)}$ is vertical, then its normal cone is

$$\mathbf{U}_j^{(1)} = \lambda\omega(1,0), \quad \lambda > 0,$$

and all points $Q = (q_1, q_2) \in \Gamma_j^{(1)}$ have the same coordinate q_1. Let put

$$g(x,y) = x^{-q_1} \hat{f}_j^{(d)}(x,y),$$

then the support $\mathbf{S}(g)$ lies at the coordinate axis $q_1 = 0$. For the truncated equation, all power solutions with $\omega(1,r) \in \mathbf{U}_j^{(1)}$ are constants $y = y^0 = \text{const}$, where y^0 is a root of the determining equation

$$\tilde{g}(y) \stackrel{\text{def}}{=} g(0,y) = 0.$$

To find nonpower solutions of the equation $g(x,y) = 0$ we make the *logarithmic transformation*

$$\xi \stackrel{\text{def}}{=} \ln x. \tag{16}$$

According to Theorem 2.4 from [2, Ch. VI], here the differential sum $g(x,y)$ comes to the differential sum $h(\xi,y) \stackrel{\text{def}}{=} g(x,y)$ and the equation $g = 0$ takes the form

$$h(\xi,y) = 0. \tag{17}$$

From (16), we see that $\xi \to \infty$ as $x \to 0$ and as $x \to \infty$, because ξ and x are complex, i.e. for the equation (17) we obtain the problem with

$$p \geqslant 0.$$

Applying the described above technique to the equation (17), we select truncated equations $\hat{h}_l^{(d)}(\xi,y) = 0$ with $\omega = 1$ and find their power solutions $y = c_\rho \xi^\rho$. Each of them corresponds to its characteristic polynomial $\nu^*(k^*)$, its own critical numbers k_j^* and its own set $\mathbf{K}^*$. Under conditions of Theorem 3 on k_j^* and $\mathbf{K}^*$, we find the power expansion of solution to equation $h(\xi,y) = 0$ in the form

$$y = c_\rho \xi^\rho + \sum c_\sigma \xi^\sigma, \quad \sigma \in \mathbf{K}^*, \quad \sigma < \rho, \quad c_\rho, c_\sigma = \text{const} \in \mathbb{C}.$$

Besides, the solution $y = c\xi^\rho$ to the truncated equation $\hat{h}_l^{(d)}(\xi, y) = 0$ corresponds to its own *complicated characteristic equation* $\mu(\varkappa) = 0$. It is formed by the following way. We have the variation

$$\frac{\delta \hat{h}_l^{(d)}}{\delta y} = \sum_{i=1}^{M} b_i(\xi, y)\mu_i\left(\frac{d}{d\xi}\right),$$

where b_i are differential monomials and μ_i are differential operators with constant coefficients

$$\mu_i\left(\frac{d}{d\xi}\right) = \sum_{k=0}^{l_i} \alpha_{ik}\frac{d^k}{d\xi^k}, \quad \alpha_{ik} = \mathrm{const} \in \mathbb{C}.$$

Among all monomials $b_i(\xi, y)$, we select such, which give the maximal power of ξ after the substitution $y = \xi^\rho$: $b_i = \beta_i\xi^n + \ldots$, $i = 0, \ldots, M$, where n is the maximal power of ξ in all b_i and $\beta_i = 0$ or const. *Polynomial*

$$\mu(\varkappa) = \sum_{i=0}^{M} \beta_i\mu_i(\varkappa),$$

where $d^k/d\xi^k$ are changed by $\varkappa^k$, is called as *complicated-characteristic* for the double truncated solution $y = c_\rho\xi^\rho$.

Theorem 5. *If root of polynomials $\nu^*(k^*)$ and $\mu(\varkappa)$ for a vertical edge does not ly in sets $\mathbf{K}^*$ and $\mathbf{K}$ correspondingly, then the double truncated solution $y = c_\rho\xi^\rho$ corresponds to a solution to the full equation in the form of complicated expansion* (15).

Proof is similar to the proof of Theorem 3.

4.2. Inclined edge.

Theorem 6. *The power transformation*

$$y = x^\alpha z \tag{18}$$

transforms the differential sum $f(x, y)$ into the differential sum $g(x, z) = f(x, y)$. Here their supports and normal cones are connected by the affine transformations

$$\mathbf{S}(g) = \mathbf{S}(f)A, \quad \mathbf{U}_g = A^{*-1}\mathbf{U}_f\,,$$

where matrices are $A = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}, \quad A^{*-1} = \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}.$

The case of the inclined edge $\Gamma_j^{(1)}$ with the normal vector $(1, r)$ is reduced to the case of the vertical edge $\widetilde{\Gamma}_j^{(1)}$ by means of the power transformation (18) with $\alpha = r$. After computation of Sections 3 and 4.1 for the transformed equation, we obtain a double truncated solution $z = c_\rho \xi^\rho$ together with characteristic polynomial $\nu^*(k^*)$ and complicated-characteristic polynomial $\mu(\varkappa)$. From Theorems 5 and 6 we obtain

Corollary 2. *For an inclined edge with normal* $(1, r)$*, if roots of polynomials* $\nu^*(k^*)$ *and* $\mu(\varkappa)$ *do not belong to sets* $\mathbf{K}^*$ *and* $\mathbf{K} - r$ *correspondingly, then the double truncated solution* $z = c_\rho \xi^\rho$ *corresponds to a solution to the full equation in the form of complicated expansion.*

Example. In the truncated equation (12), corresponding to the edge $\Gamma_1^{(1)}$ with normal $-(1, 1)$, we make the power transformation

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

As $y' = xz' + z$, $y'' = xz'' + 2z'$, then equation (12) after canceling by x and grouping takes the form

$$\hat{g}(x, z) \stackrel{\text{def}}{=} -x^2 zz'' + x^2 z'^2 - xzz' + bz + d = 0. \tag{19}$$

Its support consists of three points $\tilde{Q}_2 = (0, 2)$, $\tilde{Q}_4 = (0, 1)$, $\tilde{Q}_1 = 0$, lying at the axis $q_1 = 0$. Now we make the logarithmic transformation $\xi = \ln x$. As $z' = \dot{z}/x$, $z'' = (\ddot{z} - \dot{z})/x^2$, where $\dot{} = d/d\xi$, then equation (19) after grouping takes the form

$$h \stackrel{\text{def}}{=} -z\ddot{z} + \dot{z}^2 + bz + d = 0. \tag{20}$$

Its support and polygon are shown in Fig. 3 in the case $bd \neq 0$.

Let us consider case $b \neq 0$. The edge $\tilde{\Gamma}_1^{(1)}$ of Fig. 3 corresponds to the truncated equation

$$\hat{h}_1^{(1)} \stackrel{\text{def}}{=} -z\ddot{z} + \dot{z}^2 + bz = 0. \tag{21}$$

It has power solution $z = -b\xi^2/2$. The edge $\widetilde{\Gamma}_1^{(1)}$ has 2 vertices $(-2, 2)$ and $(0, 1)$. For the vertex $(-2, 2)$, vectors $M_1 = (2, -1)$ and $M_2 = (2, -2)$. Here $r = 2$. So $r_1 = 0$, $r_2 = -2$, $\mathbf{K}^* = \{2 - 2l_1\}$. For the vertex $(0, 1)$, vectors $M_1 = (-2, 1)$ and $M_2 = (0, -1)$. So $r_1 = 0$, $r_2 = -2$, $\mathbf{K}^* = \{2 - 2l_1\}$, integral $l_1 > 0$. The characteristic polynomial of the solution $z = -b\xi^2/2$ is $\nu^*(k^*) = (b/2)\left(k^{*2} - 5k^* + 4\right) = (b/2)(k^* - 1)(k^* - 4)$. As here $r^* = 2$, then there is only one critical number $k_1^* = 1 < r^*$. As it does not belong to the set $\mathbf{K}^*$, then according to Theorem 3, the equation (20) has a solution of the form

$$z = -\frac{b}{2}\xi^2 + \sum_{k=0}^{\infty} c_{-2k}\xi^{-2k}. \tag{22}$$

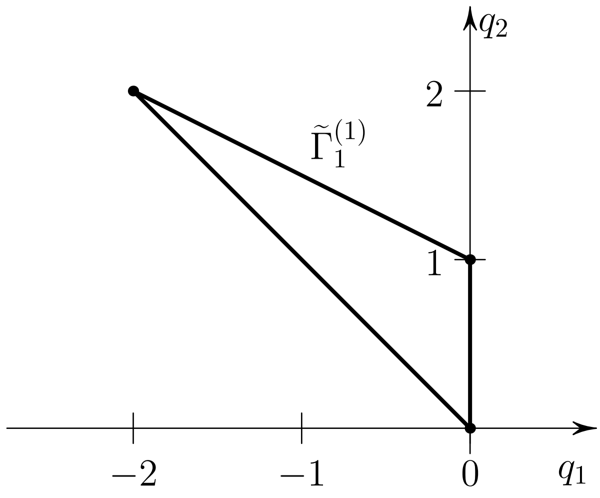

Figure 3. Support and polygon of equation (20).

Indeed solutions to equation (20) have the form

$$z = -\frac{b}{2}(\xi + \tilde{c})^2 - \frac{d}{2b}, \tag{23}$$

where $\tilde{c}$ is arbitrary constant. The solution (22) corresponds to the case $\tilde{c} = 0$. According to (21), the first variation is

$$\frac{\delta \hat{h}_1^{(1)}}{\delta z} = -\ddot{z} - z\frac{d^2}{d\xi^2} + 2\dot{z}\frac{d}{d\xi} + b.$$

So

$$\frac{\delta \hat{h}_1^{(1)}}{\delta z} = b_1\mu_1\left(\frac{d}{d\xi}\right) + b_2\mu_2\left(\frac{d}{d\xi}\right) + b_3\mu_3,$$

where $b_1 = -z$, $b_2 = 2\dot{z}$, $b_3 = -\ddot{z} + b$, $\mu_1 = \dfrac{d^2}{d\xi^2}$, $\mu_2 = \dfrac{d}{d\xi}$, $\mu_3 = 1$. As $\rho = 2$, then the leading term is $b_1\mu_1$. It gives the characteristic polynomial $\mu_1(\varkappa) = \varkappa^2$ without nonzero roots. Hence, there are no complicated critical roots, and we can apply 5. After the power transformation $y = xz$ and cancel by x, the full equation (5) takes the form

$$g \stackrel{\text{def}}{=} -x^2zz'' + x^2z'^2 - xzz' + bz + d + ax^2z^3 + cx^4z^4 = 0. \tag{24}$$

The set $\mathbf{K}$ consists of all even natural numbers. According to Theorem 3, solution to (24) has the form

$$z = \varphi_0(\xi) + \sum_{k=1}^{\infty} \varphi_{2k}(\xi)x^{2k},$$

where φ_0 is given by (23) and $x \to 0$. ■

5. Exponential expansions of solutions [9, 10]

Let the truncated equation $\hat{f}_j^{(1)}(x,y) = 0$ correspond to the horizontal edge $\Gamma_j^{(1)}$ of the polygon $\Gamma(f)$. Hence, at the edge $q_2 = m \in \mathbb{N}$. According to [3, § 5], we make the logarithmic transformation

$$\zeta = d \log y / dx \,, \tag{25}$$

and from the truncated equation $\hat{f}_j^{(1)} = 0$, we obtain the equation

$$h(x,\zeta) y^m \stackrel{\text{def}}{=} \hat{f}_j^{(1)}(x,y) = 0$$

where $h(x,\zeta)$ is a differential sum [2, Ch. VI]. Let $\Gamma(h) = \tilde{\Gamma}$ be its polygon and $\tilde{\Gamma}_i^{(1)}$ is its edge with outside normal $\tilde{N} = (1,\rho)$, lying in the cone of the problem $\mathscr{K}_\omega = \{\tilde{P} = (\tilde{p}_1, \tilde{p}_2) : \tilde{p}_1 + \tilde{p}_2 > 0,\ \operatorname{sgn} \tilde{p}_1 = \omega\}$. That determines the sign of ω and direction of tendency of x (to zero or to infinity). Th edge $\tilde{\Gamma}_i^{(1)}$ corresponds to the truncated equation $\hat{h}_i^{(1)}(x,\zeta) = 0$, which is algebraic and has several power solutions $\zeta = \gamma^* x^\rho$, where $\gamma = \gamma^* = \text{const}$ is one of the roots of the determining equation $\hat{h}_i^{(1)}(1,\gamma) = 0$. Each power solution $\zeta = \gamma^* x^\rho$ to the truncated equation $\hat{h}_i^{(1)}(x,\zeta) = 0$ is continued by the unique manner into power expansion

$$\zeta = \gamma^* x^\rho + \sum \gamma_\sigma x^\sigma \stackrel{\text{def}}{=} \varphi'(x) \tag{26}$$

of a solution to the full equation $h(x,\zeta) = 0$. The first variation can be written as

$$\frac{\delta \hat{f}_j^{(1)}}{\delta y} = y^{m-1} g\left(x, \zeta, \frac{d}{dx}\right),$$

where g is a polynomial of its arguments, if $\left(\frac{d}{dx}\right)^l$ means $\frac{d^l}{dx^l}$. Its order in $\zeta, \zeta', \dots, \zeta^{(n-1)}$ is less than m. Now in the operator g, we change $\frac{d^l}{dx^l}$ by $k^l \zeta^l$ and ζ by $\gamma^* x^\rho$. Then we select the leading term $\lambda(\gamma^*, k) x^\tau$ in x. Coefficient $\lambda(\gamma^*, k)$ is the *exponential characteristic polynomial*, corresponding to the truncated solution $\zeta = \gamma^* x^\rho$.

If the equation $h(x,\zeta) = 0$ has a solution of form (26), then the truncated equation $\hat{f}_j^{(1)}(x,y) = 0$ has the family of solutions

$$y = c \exp \varphi(x) \,, \tag{27}$$

where c is arbitrary constant and $\varphi(x)$ is an integral of the power expansion (26).

Now we come to the full equation $f(x,y) = 0$. Let the set Σ be the projection of the support $\mathbf{S}(f)$ on axis q_2 parallel to axis q_1. Let put $\Sigma' = \Sigma - m$, i.e. Σ' is a shifted on m set Σ. Finally, Σ'_+ is a set of all possible sums of numbers of the set Σ'.

Theorem 7. *Let $\hat{f}_j^{(d)}(x,y)=0$ be a truncated equation of $f(x,y)=0$, corresponding to a horizontal edge of height m. If no one of numbers $k\in\Sigma'_{+}+1$, $k\neq 1$ is not a root of the exponential characteristic polynomial $\lambda(\gamma^*,k)$, then solutions* (27) *of the truncated equation $\hat{f}_j^{(d)}(x,y)=0$ are continued in the form of the exponential expansions*

$$y=c\exp\varphi(x)+\sum b_k(x)c^k\exp(k\varphi(x))\quad \textit{no } k\in\Sigma'_{+}+1,\quad k\neq 1, \tag{28}$$

of solutions to the full equation $f(x,y)=0$, where $b_k(x)$ are power expansions.

Let $\varphi(x)=\alpha x^\beta+\cdots$, where α and $\beta=\mathrm{const}\in\mathbb{C}$. Then for $x^\beta\to\infty$

$$\exp\varphi(x)\to\begin{cases}0, \text{ if } \mathrm{Re}(\alpha x^\beta)<0,\\ \infty, \text{ if } \mathrm{Re}(\alpha x^\beta)>0\end{cases}$$

If $\Gamma_j^{(1)}$ is the lower edge, then it corresponds to values of y near 0 and values of $\exp\varphi(x)$ near 0 corresponds to solutions of the initial equation, but values of $\exp\varphi(x)$ near infinity do not corresponds to solutions of the initial equation. Thus, expansion (28) gives only parts of solutions for sectors of complex plane x with $\mathrm{Re}\,\alpha x^\beta<0$ and it does not give information about solutions outside these sectors.

If $\Gamma_1^{(1)}$ is the upper edge, then expansion (28) gives only parts of solutions in sectors with $\mathrm{Re}\,\alpha x^\beta>0$.

Example. Let us consider the fourth Painlevé equation

$$f(x,y)\stackrel{\text{def}}{=}-2yy''+y'^2+3y^4+8xy^3+4(x^2-a)y^2+2b=0, \tag{29}$$

where a and b are complex parameters. If $b=0$, its polygon $\Gamma(f)$ has a horizontal edge $\Gamma_1^{(1)}$ of height $m=2$ (Fig. 4), which corresponds to the truncated equation

$$\hat{f}_1^{(1)}\stackrel{\text{def}}{=}-2yy''+(y')^2+4\left(x^2-a\right)y^2=0.$$

After the logarithmic transformation (25), we obtain $y'=\zeta y$, $y''=y\left(\zeta'+\zeta^2\right)$ and

$$h(x,\zeta)=-2(\zeta'+\zeta^2)+\zeta^2+4x^2-4a.$$

Support $\mathbf{S}(h)$ and polygon $\Gamma(h)$ are shown in Fig. 5.

Polygon $\Gamma(h)$ has the inclined edge $\widetilde{\Gamma}_1^{(1)}$ corresponding to $\omega=1$ with the truncated equation

$$\hat{h}_1^{(1)}(x,\zeta)\stackrel{\text{def}}{=}-\zeta^2+4x^2=0.$$

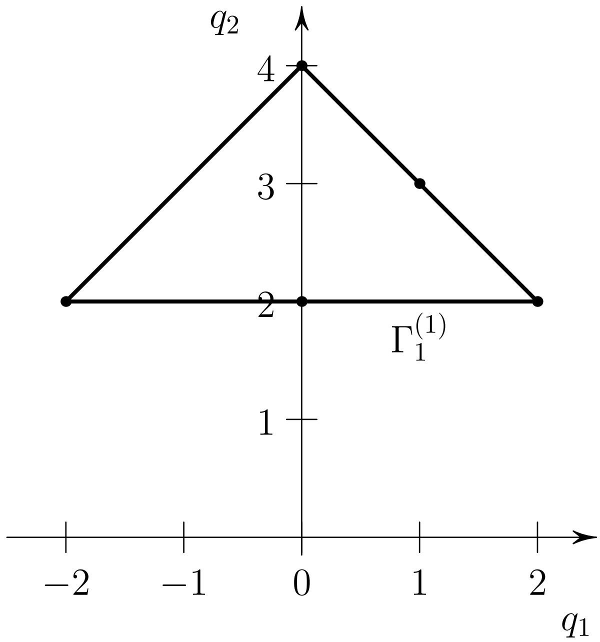

Figure 4. Support and polygon of the equation (29) с $b = 0$.

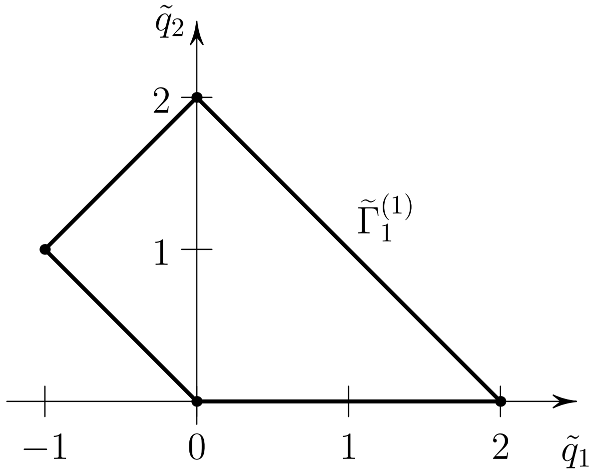

Figure 5. Support and polygon for $h(x, \zeta)$.

Hence, $\zeta = \pm 2x$, i. e. $\gamma^* = \pm 2$ and $\rho = 1$. According to Theorem 3, equation $h(x, \zeta) = 0$ has two solutions

$$\zeta_i = (-1)^i 2x + \alpha_i x^{-1} + \beta_i x^{-3} + x^{-1} \sum_{l=2}^{\infty} c_{i,l} x^{-2l} \stackrel{\text{def}}{=} \varphi_i'(x), \quad i = 1, 2,$$
$$\alpha_i = (-1)^i a - 1, \quad \beta_i = (-1)^{i+1}(a^2 + 3) + 4a.$$

If one of these numbers α_i, β_i is zero, then the corresponding expansion $\varphi_i'(x)$ is finite.

Let us compute the exponentially characteristic polynomial

$$\frac{\delta \hat{f}_1^{(1)}}{\delta y} = -2y'' - 2y\frac{d^2}{dx^2} + 2y'\frac{d}{dx} + 8(x^2 - a)y =$$
$$= y\left[-2(\zeta' + \zeta^2) - 2\frac{d^2}{dx^2} + 2\zeta\frac{d}{dx} + 8(x^2 - a)\right].$$

We change $\dfrac{d^2}{dx^2}$ and $\dfrac{d}{dx}$ by $k^2\zeta^2$ and $k\zeta$ correspondingly and ζ by $\gamma^* x$. Then the leading term for $x \to \infty$ is

$$-2\zeta^2 - 2k^2\zeta^2 + 2k\zeta^2 + 8x^2 = -2\zeta^2(k^2 - k).$$

Hence, the exponentially characteristic polynomial is $\lambda(\gamma^*, k) = -2(k^2 - k)$ for both values $\gamma^* = \pm 2$. The set Σ consists of numbers $2, 3, 4$; so $\Sigma' = \Sigma - 2 = \{0, 1, 2\}$ and the set Σ'_+ consists of all nonnegative integral numbers, but $\Sigma'_+ + 1$ is all the set of all natural numbers. Roots of the polynomial $\lambda(\gamma^*, k)$ are $k = 0$ and $k = 1$. As the root $k = 0$ does not ly in the set $\Sigma'_+ + 1$ and $k = 1$ was excluded, then according to Theorem 7, for $x \to \infty$ solutions to equation (29) with $b = 0$ are expanded in series

$$y = c \exp \varphi_i(x) + \sum_{k=2}^{\infty} b_{ik}(x) c^k \exp k\varphi_i(x), \quad i = 1, 2, \tag{30}$$

where

$$\varphi_i = (-1)^i x^2 + \alpha_i \ln x - \beta_i x^{-2}/2 - \sum_{l=2}^{\infty} c_{i,l} x^{-2l}/(2l), \quad i = 1, 2. \tag{31}$$

Here $\Gamma_1^{(1)}$ is the lower edge and $x \to \infty$, i. e. $\omega = 1$. So, expansions (30) describe families of solutions for $(-1)^i \operatorname{Re} x^2 < 0$, $i = 1, 2$. In the complex plane x, equality $\operatorname{Re} x^2 = 0$ corresponds to two bissectrices $\operatorname{Re} x = \pm \operatorname{Im} x$, dividing the plane into 4 domains $\mathscr{D}_1$, $\mathscr{D}_2$, $\mathscr{D}_3$, $\mathscr{D}_4$ (Fig. 6). So, the expansion (30) with $i = 1$ represents two families of solutions in domains $\mathscr{D}_1$ and $\mathscr{D}_3$, and the expansions (30) with $i = 2$ represents two families of solutions in domains $\mathscr{D}_2$ and $\mathscr{D}_4$. Series (31) diverge, but they are summable in some sectors of the complex plane [32]

More complicated examples of computation of domains of existence of solutions, described by expansions of type (30), see in [6].

Exponential expansions were proposed in [11].

6. Generalizations

1. The technique was used for algebraic equations [1, 2, 28, 29], for equations in partial derivations [1, 2, 18] and for systems [1, 2, 15].

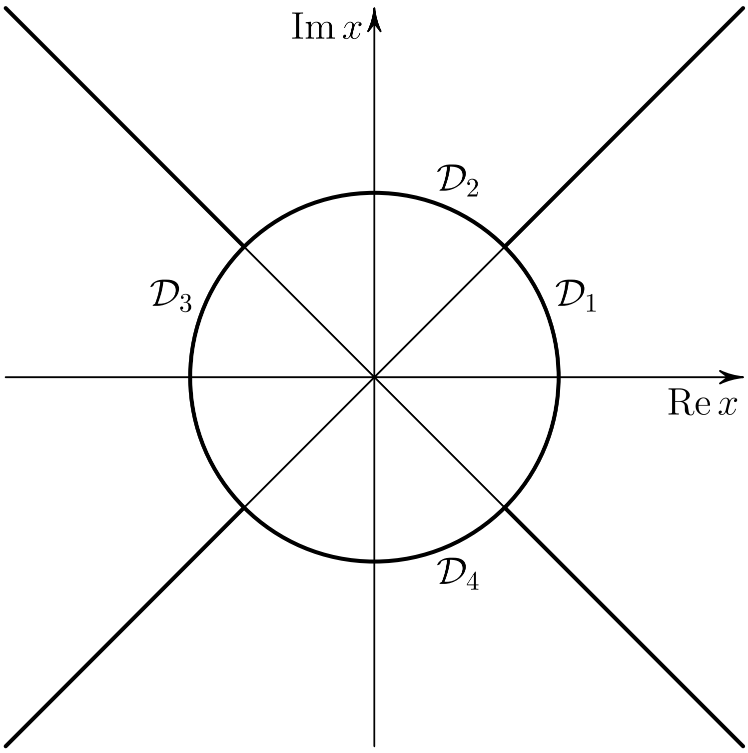

Figure 6. Domains, where expansion (30) describe solutions of initial equation (29) with $b = 0$.

2. Solutions in the form of power expansions with complex power exponents were studied in [3, 13]. Then we have the new type of expansions: exotic [5, 12].
3. We have studied asymptotic expansions of such solutions, for which difference of orders of two next derivatives is different from one [1, 16, 17, 20].
4. In Theorem 3 one can reject condition that critical numbers do not lie in the set $\mathbf{K}$. Then there are the power-logarithmic expansions [3], or Dulac series. Similarly, in Theorem 5 we obtain expansions with multiple logarithm [8].
5. Comparison with other approaches [31].

7. Applications

- Solutions to the Painleve equations [5, 14, 16–20].
- The Beletsky equation [21, 22].
- The Euler-Poisson equations [23].
- The restricted three-body problem [24–26].
- Integrability of an ODE system [27, 30].
- The boundary layer on a needle [1, 14, 15, 18].
- Evolution of a turbulent flow [18].
- Sets of stability of a multiparameter ODE system [28].
- Waves on water [2, Ch. V].

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