Proceedings of the International Conference

Modern Achievements in Symmetries of Differential Equations (Symmetry 2022)

A Hybrid Conference

December 13-16, 2022

Suranaree University of Technology
Nakhon Ratchasima, Thailand

Foreword

On December 13-16, 2022, the School of Mathematics at the Institute of Science of Suranaree University of Technology in Nakhon Ratchasima, Thailand, hosted the International Conference "Modern Achievements in Symmetries of Differential Equations" (Symmetry 2022). This conference was a collaborative effort between the School of Mathematics at Suranaree University of Technology (Thailand), the Keldysh Institute of Applied Mathematics (Russia), the Department of Mathematical Sciences at the University of Stellenbosch (South Africa), and the Moscow Center for Fundamental and Applied Mathematics (Russia). The conference was included in the list of events of the Russian Federation dedicated to the Russia-ASEAN Year and received approval from the Government of the Russian Federation. Symmetry 2022 marked the third international conference jointly organized by Suranaree University of Technology (Thailand), Russian Institutes, and South African Universities. The first conference took place with the Institute of Hydrodynamics (Novosibirsk) in 2019, and the second conference with the Institute of Theoretical and Applied Mechanics (Novosibirsk) in 2021.

The primary objective of these conferences was to focus on the latest advancements in the applications of Lie groups, encompassing a wide range of topics in interdisciplinary studies within theoretical and applied sciences. We aimed to bring together researchers who apply Lie groups to address a variety of problems. The "Modern Achievements in Symmetries of Differential Equations and Applications" conference was a resounding success.

The main topics of the conference revolved around Symmetry Methods and their Applications. The conference was conducted in a virtual format, enabling the participation of a diverse audience, including experts in the subject area, young scientists, and graduate students interested in the analysis of nonlinear equations and group theoretical methods. In total, there were 66 registered participants from 21 countries, representing all inhabited continents of the world. Talks were scheduled from late morning until night local time, accommodating the differences in time zones. While most presentations focused on the analysis of wave motion in continuum mechanics, numerous other topics were also addressed, such as the stability of fluid flows, qualitative analysis and integrability of partial differential equations, methods for constructing exact solutions to continuum mechanics problems, symmetries and conservation laws of differential equations with nonlocal terms, the method of differential constraints, numerical analysis, construction of numerical schemes, and applications of mathematics in medicine.

These Proceedings contain eight papers from among the 55 conference presentations.

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Approximate Analytical Solutions to Delay Fractional Differential Equations and Application to Economic Models

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Abstract

In recent years, considerable attention is being paid to Fractional Differential Equations (FDEs) due to their ability to model complex phenomena. Many numerical methods have been proposed for FDEs. This study investigates the application of a blend of variational iteration method with Sumudu Transform for solving nonlinear delay FDEs. A genuine model in science, engineering or economics, must comprise of time delays as they are natural components of the dynamic processes. Under the assumption that the market is in equilibrium, special emphasis is given to the formulation and construction of delayed fractional economic models such as price adjustment equations involving Caputo derivative. This study presents the qualitative solutions of the proposed economic models and suitable parameters are chosen to analyze the models for different fractional orders.

Keywords: Sumudu Transform, Delay, Fractional derivatives, Demand and supply, Market equilibrium.

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1 Introduction

Abbreviations:
FDEs: Fractional Differential Equations
ST: Sumudu Transform
SVI: Sumudu Variational Iteration

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Unlike the integer-order derivative operators, fractional operators consider the global correlation, and not only local characteristics in the study of evolution of a system [17]. In some cases, discrepancy occurs between the results of the integer-order calculus and the experimental results. Fractional derivatives are suitable in describing several physical phenomena (see, e.g, [9]). A large class of dynamical systems constitutes delay and they are best described by the delay differential equations. Dynamic processes such as in engineering, science and economics, constitute delays as their natural components. Introducing delays into a model can improve its vitality and suitability in describing several phenomena. A challenge with models that are associated with delays and fractional operators is how to obtain their solutions (see, e.g, [2, 3, 12]). There is no any precise analytical method for solving such problems to obtain their exact solutions (see, e.g, [4, 5, 15, 16]). The research efforts abound in the literature on the proposition of the numerical methods for solving such equations. The construction of efficient analytical and numerical methods for the solutions of ordinary and partial Fractional Differential Equations (FDEs) with delay, is an active research area and it is of great interest to the researchers (see, e.g., [1]).

The economic models can help to understand and predict the economic behaviour (see, e.g, [10]). The economy concerning a commodity determines the trend of its price, which may increase or decrease rapidly. Through the economic models, the economists can predict the optimal profit to show the link between demand and supply. Mathematical models of economic processes give insight into the interaction that exists between the price, demand and supply, dependence of supply and demand on price and how to estimate the equilibrium point on the supply and demand curves (see, e.g, [18]). Market equilibrium refers to a state in which the quantity demand and the quantity supply of a commodity are equivalent. Both the market equilibrium and economic growth occupy important positions in the description of the real world problems. By the demand and supply functions, we shall refer to the quantity demand and quantity supply as functions of price, respectively. These functions are respectively given as:

$$f_d(t) = d_0 - d_1 p(t) \text{ and } f_s(t) = -s_0 + s_1 p(t), \tag{1.1}$$

where $p(t)$ is the price of the commodities, t is the time, while d_0, d_1, s_0 and s_1 are positive constants (see, e.g, [13]). At equilibrium, $f_d(t) = f_s(t)$, which means that the quantity demand and quantity supply are equal and the equilibrium price is obtained as

$$p^* = \frac{d_0 + s_0}{d_1 + s_1}.$$

The price tends to be invariant at equilibrium and there is neither surplus nor shortage. Consider the price adjustment equation which is given as

$$p'(t) = q\left(f_d - f_s\right), \tag{1.2}$$

where $q > 0$ denotes the speed of adjustment constant. Substitute (1.1) into (1.2) to get

$$p'(t) + q\left(d_1 + s_1\right) p(t) = q\left(d_0 + s_0\right). \tag{1.3}$$

The solution of linear differential equation (1.3) is obtained as

$$p(t) = p^* + \left[p(0) - p^*\right] e^{-q(d_1+s_1)t},$$

where $p(0)$ denotes the price at the time $t = 0$. An increase in the price of a commodity will urge the buyers to buy more before prices increase further while the suppliers tend to offer less for the hoping of more earning from higher prices in future (see, e.g, [6, 13]). In addition, when

$p'(t) = 0$ for all $t \geq 0$, this describes equilibrium in a changing economy, which implies that market is in dynamic equilibrium.

This study presents a blend of variational iteration method with Sumudu Transform (ST) for solving nonlinear delay FDEs. Under the assumption that the market is in equilibrium, we propose the formulation and construction of delayed fractional economic models such as price adjustment equations involving Caputo derivative. Then, a blend of variational iteration method with ST which is presented in this study is applied to obtain the solutions of the newly introduced economic models. Suitable parameters are chosen for different fractional orders to analyze the solutions obtained for the newly introduced economic models.

2 Preliminaries

In this section, we give some definitions and a proposition which are essential in establishing the main results of this paper. Throughout this paper, $\mathbb{C}$ and $\mathbb{R}$ will denote the sets of complex and real numbers, respectively. In addition, $\mathbb{N}$ will denote the set of natural number.

Definition 2.1. Consider a set of functions A defined as (see, e.g, [7])

$$A = \left\{ g(t) : \exists\, Q, \tau_1, \tau_2 > 0, |g(t)| < Qe^{|t|/\tau_j}, \text{if } t \in (-1)^j \times [0, \infty) \right\}.$$

For all real $t \geq 0$ and $g(t) \in A$, the ST of $g(t)$ is denoted by $\mathcal{S}[g(t)] = G(u)$ and it is defined as

$$G(u) = \mathcal{S}[g(t)] = \int_0^\infty g(tu)e^{-t}dt, \;\; u \in (-\tau_1, \tau_2). \tag{2.1}$$

The function $g(t)$ in equation (2.1) is the inverse ST of $G(u)$ and the relation is denoted by $g(t) = \mathcal{S}^{-1}[G(u)]$. Recall that the Laplace transform of $g(t)$, denoted by $\mathcal{L}[g(t)] = L(u)$ is defined as

$$L(u) = \mathcal{L}[g(t)] = \int_0^\infty g(t)e^{-st}dt, \;\; s > 0. \tag{2.2}$$

By considering (2.1) and (2.2), one can express a duality relation which the Sumudu and Laplace transforms exhibit as follows:

$$G\left(1/s\right) = sL(s), \quad L\left(1/u\right) = uG(u).$$

Like the well known Laplace transform, the ST is an integral method. Using ST technique is appealing as it yields an accurate result quickly and it does not impose any restricting assumptions about the results. It is a simple, effective and universal way by which one can obtain the Lagrange multiplier. Linearity property of ST is well known (see, e.g, [7, 8, 12, 19]), that is, for two given functions $f(t)$ and $g(t)$, and for arbitrary constants α and β,

$$\mathcal{S}\left[\alpha f(t) + \beta g(t)\right] = \alpha \mathcal{S}\left[f(t)\right] + \beta \mathcal{S}\left[g(t)\right].$$

The ST for the integer order derivatives is expressed as

$$S\left[\frac{dg(t)}{dt}\right] = \frac{1}{u}\left[G(u) - g(0)\right]. \tag{2.3}$$

For the n-order derivative, the ST is given as

$$S\left[\frac{d^n g(t)}{dt^n}\right] = \frac{1}{u^n}\left[G(u) - \sum_{k=0}^{n-1} u^k \frac{d^k g(t)}{dt^k}|_{t=0}\right]. \tag{2.4}$$

Table 1 gives some special STs.

Table 1: Special Sumudu Transforms

$g(t)$	$G(u)=\mathcal{S}\left[g(t)\right]$
1	1
t	u
$\frac{t^n}{n!}=\frac{t^n}{\Gamma(n+1)}$	u^n
e^{at}	$\frac{1}{1-au}$
$\frac{\sin\ at}{a}$	$\frac{u}{1+a^2u^2}$
$\cos\ at$	$\frac{1}{1+a^2u^2}$
$\frac{e^{bt}-e^{at}}{b-a}, b\neq a$	$\frac{1}{(1-bu)(1-au)}$

Definition 2.2. Let $a>0, b>0$ be positive real numbers. The left and right sided Caputo-fractional derivatives of order μ are defined respectively as

$$ {}_{a}^{C}D_t^{\mu}g(t)=\frac{1}{\Gamma(1-\mu)}\int_a^t (t-\tau)^{-\mu}g'(\tau)d\tau $$

and

$$ {}^{C}D_{b,t}^{\mu}g(t)=\frac{-1}{\Gamma(1-\mu)}\int_t^b (\tau-t)^{\mu}g'(\tau)d\tau, $$

where $0<\mu<1$. The Caputo-fractional derivatives of order μ admit the ST in the form (see, e.g, [9])

$$ \mathcal{S}\left[{}_{0}^{C}D_t^{\mu}g(t)\right]=u^{-\mu}\left(\mathcal{S}\left[g(t)\right]-\omega(0)\right). \tag{2.5} $$

In general, let $n\in\mathbb{N}$ and $\mu>0$ be such that $n-1\leq\mu<n$ and $G(u)$ be the ST of the function $g(t)$, then the ST of the Riemann-Liouville fractional derivative of $g(t)$ of order μ is given by

$$ \mathcal{S}\left[{}_{0}^{C}D_t^{\mu}g(t)\right]=u^{-\mu}\left[G(u)-\sum_{k=0}^{n-1}u^{-(k+1)}\left[{}_{0}D_t^{\mu-k-1}g(t)\right]|_{t=0}\right]. $$

Proposition 2.3. *Let $\phi,\varphi:[0,\infty)\to\mathbb{R}$, then the classical convolution product is given by*

$$ (\phi\times\varphi)(t)=\int_0^t \phi(t-x)\varphi(x)dx. $$

The ST for the convolution product is given by

$$ \begin{aligned} \mathcal{S}\left[(\phi\times\varphi)(t)\right] &= u\mathcal{S}[\phi(t)]\mathcal{S}[\varphi(t)] \\ &= u\phi(u)\varphi(u). \end{aligned} $$

Definition 2.4. One-parameter Mittag-Leffler function $E_{\mu}(t)$ is defined as

$$ E_{\mu}(t)=\sum_{n=0}^{\infty}\frac{t^n}{\Gamma(\mu n+1)}, \mu>0. $$

The following results about Mittag-Leffler functions and ST are well known (see, e.g, [13]):

(i) $\mathcal{S}\left[E_{\mu}\left(-at^{\mu}\right)\right]=\frac{1}{1+au^{\mu}}$;

(ii) $\mathcal{S}\left[1-E_{\mu}\left(-at^{\mu}\right)\right]=\frac{au^{\mu}}{1+au^{\mu}}$.

3 Main results

We present a blend of variational iterative method with ST for solving nonlinear FDEs with delay. Then we apply the results to study economic models.

3.1 Sumudu variational iteration method

We shall refer to a blend of ST with variational iterative method as Sumudu Variational Iteration (SVI) method. When compared with other well-known methods, the flexibility, consistency and effectiveness of the variational iterative method (see e.g, [20, 21] and references therein), motivated its choice for amalgamation with the ST.

3.1.1 Presentation of SVI method

The Caputo-fractional derivatives are of great use in the modeling of phenomena where consideration is given to the interactions within the past and also problems with nonlocal properties (see, e.g, [17]). The SVI method is presented for solving nonlinear problems which involve delay and Caputo-fractional derivatives of order μ,

$$ {}_{a}^{C}D^{\mu}g(t) + R\left[g(t)\right] + N\left[g(t-\tau)\right] = \omega(t), \tag{3.1} $$

subject to the initial conditions

$$ g(0) = g_0, \tag{3.2} $$

where $\tau > 0, R$ is a linear operator, N is a nonlinear operator and $\omega(t)$ is a given continuous function. The ST of (3.1) takes the form

$$ \mathcal{S}\left[{}_{a}^{C}D^{\mu}g(t)\right] = \mathcal{S}\left[\omega(t) - R\left[g(t)\right] - N\left[g(t-\tau)\right]\right]. $$

Apply (2.5) with $a = 0$, to obtain

$$ u^{-\mu}\left(\mathcal{S}\left[g(t)\right] - g(0)\right) = \mathcal{S}\left[\omega(t) - R\left[g(t)\right] - N\left[g(t-\tau)\right]\right], $$

which leads to

$$ u^{-\mu}\left(G(u) - g_0\right) = \mathcal{S}\left[\omega(t) - R\left[g(t)\right] - N\left[g(t-\tau)\right]\right], $$

since by (3.2), $g(0) = g_0$. Therefore, the SVI formula is given as (see, e.g., [11]),

$$ G_{n+1}(u) = G_n(u) + \alpha(u)\left(\frac{G_n(u) - g_0}{u^{\mu}} - \mathcal{S}\left[\omega(t) - R\left[g(t)\right] - N\left[g(t-\tau)\right]\right]\right), n \in \mathbb{N}. \tag{3.3} $$

In taking the classical variation operator on both sides of (3.3), considering $\mathcal{S}\left[R\left[g(t)\right] + N\left[g(t-\tau)\right]\right]$ as the restricted term leads to

$$ \delta G_{n+1}(u) = \delta G_n(u) + \alpha(u)\frac{1}{u^{\mu}}\delta G_n(u), $$

which gives a Lagrange multiplier as

$$ \alpha(u) = -u^{\mu}. $$

Taking the inverse ST of (3.3) gives the explicit iteration formula as

$$ \begin{aligned} g_{n+1}(t) &= g_n(t) + \mathcal{S}^{-1}\left[-u^{\mu}\left(\frac{G_n(u) - g_0}{u^{\mu}} - \mathcal{S}\left[\omega(t) - R\left[g_n(t)\right] - N\left[g_n(t-\tau)\right]\right]\right)\right] \\ &= g_1(t) + \mathcal{S}^{-1}\left[u^{\mu}\mathcal{S}\left[\omega(t) - R\left[g_n(t)\right] - N\left[g_n(t-\tau)\right]\right]\right], \end{aligned} \tag{3.4} $$

with the initial approximation which is given as $g_1(t) = \mathcal{S}^{-1}\left[-u^{\mu}\left(\frac{-g_0}{u^{\mu}}\right)\right] = g_0\mathcal{S}^{-1}\left[1\right] = g_0$.

3.1.2 Variable coefficients nonlinear FDEs with delay

Suppose the given general nonlinear problem (3.1) contains variable coefficients such that the equation has the form

$$ {}_{a}^{C}D^{\mu}g(t) + \lambda R_1[g(t)] + \gamma(t)R_2[g(t)] + N\left[g(t-\tau)\right] = \omega(t), \tag{3.5} $$

where λ is a constant, $\gamma(t)$ is a variable coefficient, R_1 and R_2 denote linear operators and other terms remain as defined in (3.1). Taking the ST of (3.5) and further computations give the SVI formula

$$ \begin{aligned} G_{n+1}(u) \quad = \quad & G_n(u) + \alpha(u)\Bigg(\frac{G_n(u)-g_0}{u^{\mu}} - \mathcal{S}\Big[\omega(t) - \lambda R_1[g(t)] \\ & -\gamma(t)R_2[g(t)] - N\left[g(t-\tau)\right]\Big]\Bigg), n \in \mathbb{N}. \end{aligned} $$

$$ \begin{aligned} G_{n+1}(u) = G_n(u) + \alpha(u)\Bigg(\frac{G_n(u)-g_0}{u^{\mu}} - \mathcal{S}\Big[\omega(t) - \lambda R_1[g(t)] \\ - \gamma(t)R_2[g(t)] - N\left[g(t-\tau)\right]\Big]\Bigg), n \in \mathbb{N}. \end{aligned} \tag{3.6} $$

In taking the classical variation operator on both sides of (3.6), considering

$$ \mathcal{S}\left[\gamma(t)R_2[g(t)] + N\left[g(t-\tau)\right]\right] $$

as the restricted term leads to

$$ \delta G_{n+1}(u) = \delta G_n(u) + \alpha(u)\frac{1}{u^{\mu}}\delta G_n(u), $$

which gives a Lagrange multiplier as

$$ \alpha(u) = -u^{\mu}. $$

Substitute for $\alpha(u)$ in (3.6) and take its inverse ST to obtain the explicit iteration formula

$$ \begin{aligned} g_{n+1}(t) &= g_n(t) + \mathcal{S}^{-1}\Bigg[-u^{\mu}\Bigg(\frac{G_n(u)-g_0}{u^{\mu}} - \mathcal{S}\left[\omega(t) - \lambda R_1[g_n(t)] - \gamma(t)R_2[g_n(t)] - N[g_n(t-\tau)]\right]\Bigg)\Bigg] \\ &= g_1(t) + \mathcal{S}^{-1}\left[u^{\mu}\mathcal{S}\left[\omega(t) - \lambda R_1[g_n(t)] - \gamma(t)R_2[g_n(t)] - N\left[g_n(t-\tau)\right]\right]\right], \end{aligned} $$

with the initial approximation which is given as $g_1(t) = \mathcal{S}^{-1}\left[-u^{\mu}\left(\frac{-g_0}{u^{\mu}}\right)\right] = g_0\mathcal{S}^{-1}\left[1\right] = g_0$.

3.2 Economic models

In this section, we present the solution of price adjustment equations with Caputo-fractional derivative by using ST method. A special class of delay differential equations with a proportional delay is referred to as pantograph differential equations and they have the form

$$ p'(t) = \Omega p(t) + \Phi p(\lambda t), $$

where Ω, Φ, λ are real constants and $0 < \lambda < 1$ (see [14]). We propose a new economic model by introducing the delay, which is in the form of a pantograph type into the formulation and construction of price adjustment equations with Caputo-fractional derivative. In addition, we present the precepts for obtaining the qualitative solutions of the newly proposed economic models and choose suitable parameters to analyze the models for different fractional orders. Moreover, we use the graphs to show the comparison between the solutions of the price adjustment equation with Caputo-fractional derivative and the newly introduced price adjustment equation with Caputo-fractional derivative that involves delays.

3.2.1 Price adjustment equations with Caputo-fractional derivative

Consider the price adjustment equation with Caputo-fractional derivative which is given as

$$ {}_a^C D^{\mu} p(t) + q\left(d_1 + s_1\right) p(t) = q\left(d_0 + s_0\right), \ p(0) = p_0. \tag{3.7}$$

Take the ST of (3.7) as

$$ \mathcal{S}\left[{}_a^C D^{\mu} p(t)\right] + q\left(d_1 + s_1\right) \mathcal{S}\left[p(t)\right] = q\mathcal{S}\left[\left(d_0 + s_0\right)\right], $$

to obtain

$$ u^{-\mu}\left(\mathcal{S}\left[p(t)\right] - p(0)\right) + q\left(d_1 + s_1\right) \mathcal{S}\left[p(t)\right] = q\left(d_0 + s_0\right), $$

and then

$$ u^{-\mu}\left(\mathcal{S}\left[p(t)\right] - p_0\right) + q\left(d_1 + s_1\right) \mathcal{S}\left[p(t)\right] = q\left(d_0 + s_0\right), \tag{3.8}$$

since $p(0) = p_0$. Factorise and simplify (3.8) to obtain

$$ \begin{aligned} \mathcal{S}\left[p(t)\right] &= \frac{u^{-\mu} p_0}{u^{-\mu} + q\left(d_1 + s_1\right)} + \frac{q\left(d_0 + s_0\right)}{u^{-\mu} + q\left(d_1 + s_1\right)} \\ &= \frac{p_0}{1 + q\left(d_1 + s_1\right) u^{\mu}} + \frac{q\left(d_0 + s_0\right) u^{\mu}}{1 + q\left(d_1 + s_1\right) u^{\mu}} \\ &= \frac{1}{1 + q\left(d_1 + s_1\right) u^{\mu}} p_0 + \frac{\left(d_0 + s_0\right)}{\left(d_1 + s_1\right)} \frac{q\left(d_1 + s_1\right) u^{\mu}}{1 + q\left(d_1 + s_1\right) u^{\mu}}. \end{aligned} \tag{3.9}$$

Taking inverse ST of both sides of (3.9) and applying Definition 2.4 (ii) gives

$$ p(t) = p_0 E_{\mu}\left(-q\left(d_1 + s_1\right) t^{\mu}\right) + \frac{\left(d_0 + s_0\right)}{\left(d_1 + s_1\right)}\left(1 - E_{\mu}\left(-q\left(d_1 + s_1\right) t^{\mu}\right)\right). \tag{3.10}$$

To display the solution (3.10) graphically, real values are assigned to the constants as follows: $p_0 = 1, d_0 = 15, s_0 = 100, d_1 = 14, s_1 = 97$ and $q = 0.2$. For $\mu = 0.76$, Figure 1 displays graph of the solution of price adjustment equation (3.7). It is the curve for the (3.10). Figure 2 shows the effect of adjusting the values of the fractional order of the Caputo-fractional derivative on the solutions of (3.7). It displays the trend for (3.10) as μ varies.

3.2.2 Price adjustment equations with Caputo-fractional derivative that involve delays

We propose a model with delay for the price adjustment equation with the Caputo-fractional derivative and it is given as

$$ {}_a^C D^{\mu} p(t) + q\left(d_1 + s_1\right) p\left(\frac{t}{2}\right) = q\left(d_0 + s_0\right), \tag{3.11}$$

where $\mu \in (0,1)$ and $p(0) = p_0$.

Remark 3.1. We shall solve equation (3.11) by applying the newly introduced SVI method. It is impossible to use ordinary ST to solve equation (3.11) due to the presence of delay.

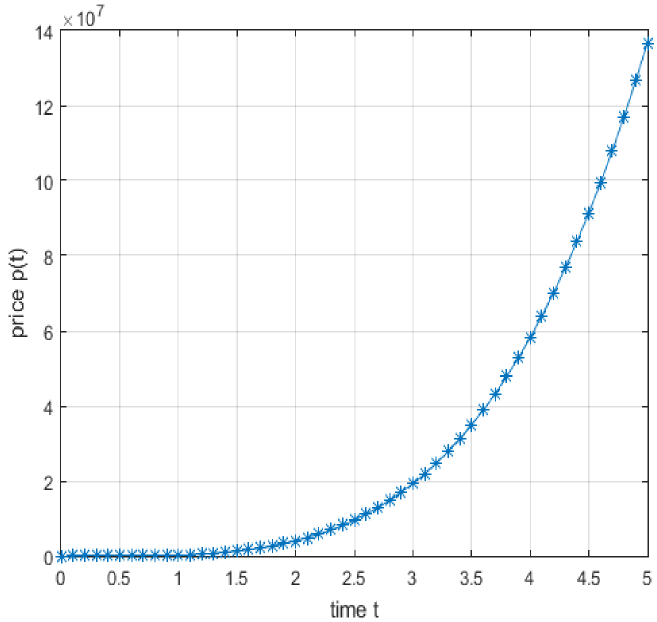

Figure 1: Price adjustment equations without a delay.

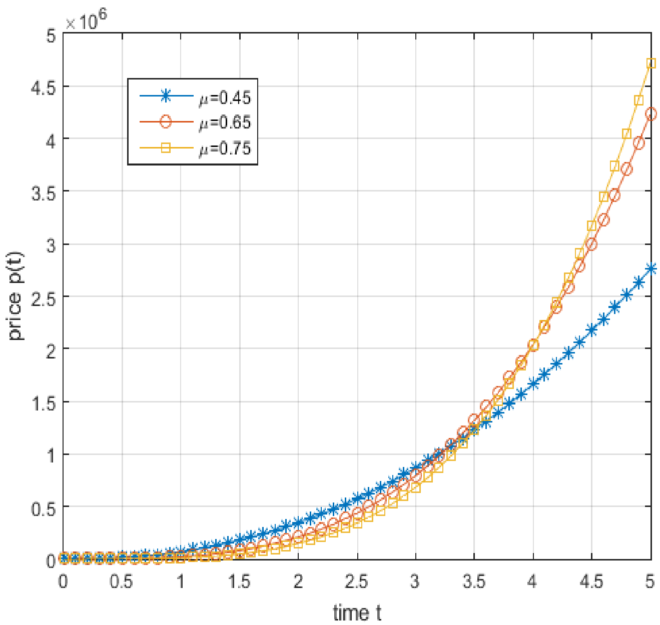

Figure 2: Variation as μ varies in price adjustment equations without a delay.

Therefore, we start by taking the ST of (3.11), which takes the form

$$\mathcal{S}\left[{}_a^C D^{\mu} p(t)\right] + q\left(d_1 + s_1\right)\mathcal{S}\left[p\left(\frac{t}{2}\right)\right] = q\mathcal{S}\left[\left(d_0 + s_0\right)\right].$$

Apply (2.5) with $a = 0$, to obtain

$$u^{-\mu}\left(\mathcal{S}\left[p(t)\right] - p(0)\right) + q\left(d_1 + s_1\right)\mathcal{S}\left[p\left(\frac{t}{2}\right)\right] = q\left(d_0 + s_0\right),$$

which leads to

$$u^{-\mu}\left(P(u) - p_0\right) + q\left(d_1 + s_1\right)\mathcal{S}\left[p\left(\frac{t}{2}\right)\right] = q\left(d_0 + s_0\right),$$

since $p(0) = p_0$. Therefore, the Sumudu variational iteration formula is given as

$$P_{n+1}(u) = P_n(u) + \alpha(u)\left(\frac{P_n(u) - p_0}{u^{\mu}} + q\left(d_1 + s_1\right)\mathcal{S}\left[p_n\left(\frac{t}{2}\right)\right] - q\left(d_0 + s_0\right)\right), n \in \mathbb{N}. \quad (3.12)$$

Taking the classical variation operator on both sides of (3.12) with $p_n(\frac{t}{2})$ as the restricted term makes the Lagrange multiplier to be obtained as

$$\alpha(u) = -u^{\mu}.$$

Taking the inverse-ST of (3.12) gives

$$\begin{aligned} p_{n+1}(t) &= p_n(t) + \mathcal{S}^{-1}\left[-u^{\mu}\left(\frac{P_n(u) - p_0}{u^{\mu}} + q\left(d_1 + s_1\right)\mathcal{S}\left[p_n\left(\frac{t}{2}\right)\right] - q\left(d_0 + s_0\right)\right)\right] \\ &= p_1(t) + \mathcal{S}^{-1}\left[-u^{\mu}\left(q\left(d_1 + s_1\right)\mathcal{S}\left[p_n\left(\frac{t}{2}\right)\right] - q\left(d_0 + s_0\right)\right)\right] \\ &= p_1(t) + q\mathcal{S}^{-1}\left[u^{\mu}\left(\left(d_0 + s_0\right) - \left(d_1 + s_1\right)\mathcal{S}\left[p_n\left(\frac{t}{2}\right)\right]\right)\right], \end{aligned}$$

with the initial approximation which is given as $p_1(t) = \mathcal{S}^{-1}\left[u^{\mu}\left(\frac{p_0}{u^{\mu}}\right)\right] = p_0\mathcal{S}^{-1}\left[1\right] = p_0$. Therefore, the explicit iteration formula is obtained as

$$p_{n+1}(t) = p_0 + q\mathcal{S}^{-1}\left[u^{\mu}\left(\left(d_0 + s_0\right) - \left(d_1 + s_1\right)\mathcal{S}\left[p_n\left(\frac{t}{2}\right)\right]\right)\right]. \quad (3.13)$$

Observe that since $p_1(t) = p_0$, then $p_1(\frac{t}{2}) = p_0$. Therefore,

$$\begin{aligned} p_2(t) &= p_0 + q\mathcal{S}^{-1}\left[u^{\mu}\left(\left(d_0 + s_0\right) - \left(d_1 + s_1\right)\mathcal{S}\left[p_1\left(\frac{t}{2}\right)\right]\right)\right] \\ &= p_0 + q\Bigg(\left(d_0 + s_0\right) - p_0\left(d_1 + s_1\right)\Bigg)\mathcal{S}^{-1}\left[u^{\mu}\right] \\ &= p_0 + q\Bigg(\left(d_0 + s_0\right) - p_0\left(d_1 + s_1\right)\Bigg)\frac{t^{\mu}}{\Gamma(\mu + 1)}. \end{aligned}$$

Notice that $p_2(\frac{t}{2}) = p_0 + q\Big((d_0+s_0) - p_0\,(d_1+s_1)\Big)\frac{t^\mu}{2^\mu\Gamma(\mu+1)}$, therefore

$$\begin{aligned}
p_3(t) &= p_0 + q\mathcal{S}^{-1}\left[u^\mu\left((d_0+s_0)-(d_1+s_1)\,\mathcal{S}\left[p_2\left(\frac{t}{2}\right)\right]\right)\right]\\
&= p_0 + q\mathcal{S}^{-1}\left[u^\mu\left((d_0+s_0)-(d_1+s_1)\,\mathcal{S}\left[p_0+q\Big((d_0+s_0) - p_0\,(d_1+s_1)\Big)\frac{t^\mu}{2^\mu\Gamma(\mu+1)}\right]\right)\right]\\
&= p_0 + q\mathcal{S}^{-1}\left[u^\mu\left((d_0+s_0)-(d_1+s_1)\left(p_0+q\Big((d_0+s_0) - p_0\,(d_1+s_1)\Big)\frac{u^\mu}{2^\mu}\right)\right)\right]\\
&= p_0 + q\mathcal{S}^{-1}\left[u^\mu\left((d_0+s_0)-p_0\,(d_1+s_1)-q\,(d_1+s_1)\Big((d_0+s_0) - p_0\,(d_1+s_1)\Big)\frac{u^\mu}{2^\mu}\right)\right]\\
&= p_0 + q\Big((d_0+s_0) - p_0\,(d_1+s_1)\Big)\mathcal{S}^{-1}\left[u^\mu - q\,(d_1+s_1)\,\frac{u^{2\mu}}{2^\mu}\right]\\
&= p_0 + q\Big((d_0+s_0) - p_0\,(d_1+s_1)\Big)\left(\frac{t^\mu}{\Gamma(\mu+1)} - q\,(d_1+s_1)\,\frac{t^{2\mu}}{2^\mu\Gamma(2\mu+1)}\right).
\end{aligned}$$

Notice that $p_3(\frac{t}{2}) = p_0 + q\Big((d_0+s_0) - p_0\,(d_1+s_1)\Big)\left(\frac{t^\mu}{2^\mu\Gamma(\mu+1)} - q\,(d_1+s_1)\,\frac{t^{2\mu}}{2^{3\mu}\Gamma(2\mu+1)}\right)$, therefore

$$\begin{aligned}
p_4(t) &= p_0 + q\mathcal{S}^{-1}\left[u^\mu\left((d_0+s_0)-(d_1+s_1)\,\mathcal{S}\left[p_3\left(\frac{t}{2}\right)\right]\right)\right]\\
&= p_0 + q\mathcal{S}^{-1}\Bigg[u^\mu\Bigg((d_0+s_0)-(d_1+s_1)\\
&\quad\times\mathcal{S}\left[p_0+q\Big((d_0+s_0) - p_0\,(d_1+s_1)\Big)\left(\frac{t^\mu}{2^\mu\Gamma(\mu+1)} - q\,(d_1+s_1)\,\frac{t^{2\mu}}{2^{3\mu}\Gamma(2\mu+1)}\right)\right]\Bigg)\Bigg]\\
&= p_0 + q\mathcal{S}^{-1}\Bigg[u^\mu\Bigg((d_0+s_0)-(d_1+s_1)\\
&\quad\times\left\{p_0+q\Big((d_0+s_0) - p_0\,(d_1+s_1)\Big)\left(\frac{u^\mu}{2^\mu} - q\,(d_1+s_1)\,\frac{u^{2\mu}}{2^{3\mu}}\right)\right\}\Bigg)\Bigg]\\
&= p_0 + q\Big((d_0+s_0) - p_0\,(d_1+s_1)\Big)\mathcal{S}^{-1}\left[u^\mu\left\{1 - q\,(d_1+s_1)\left(\frac{u^\mu}{2^\mu} - q\,(d_1+s_1)\,\frac{u^{2\mu}}{2^{3\mu}}\right)\right\}\right]\\
&= p_0 + q\Big((d_0+s_0) - p_0\,(d_1+s_1)\Big)\mathcal{S}^{-1}\left[u^\mu - q\,(d_1+s_1)\left(\frac{u^{2\mu}}{2^\mu} - q\,(d_1+s_1)\,\frac{u^{3\mu}}{2^{3\mu}}\right)\right]\\
&= p_0 + q\Big((d_0+s_0) - p_0\,(d_1+s_1)\Big)\Bigg(\frac{t^\mu}{\Gamma(\mu+1)}\\
&\quad-q\,(d_1+s_1)\left(\frac{t^{2\mu}}{2^\mu\Gamma(2\mu+1)} - q\,(d_1+s_1)\,\frac{t^{3\mu}}{2^{3\mu}\Gamma(3\mu+1)}\right)\Bigg)\\
&= p_0 + q\Big((d_0+s_0) - p_0\,(d_1+s_1)\Big)\Bigg(\frac{t^\mu}{\Gamma(\mu+1)}\\
&\quad-q\,(d_1+s_1)\,\frac{t^{2\mu}}{2^\mu\Gamma(2\mu+1)} + q^2\,(d_1+s_1)^2\,\frac{t^{3\mu}}{2^{3\mu}\Gamma(3\mu+1)}\Bigg).
\end{aligned}$$

Notice that

$$\begin{aligned}
p_4(\frac{t}{2}) &= p_0 + q\Big((d_0+s_0) - p_0\,(d_1+s_1)\Big)\Bigg(\frac{t^\mu}{2^\mu\Gamma(\mu+1)}\\
&\quad-q\,(d_1+s_1)\,\frac{t^{2\mu}}{2^{3\mu}\Gamma(2\mu+1)} + q^2\,(d_1+s_1)^2\,\frac{t^{3\mu}}{2^{6\mu}\Gamma(3\mu+1)}\Bigg),
\end{aligned}$$

therefore

$$\begin{aligned}
p_5(t) &= p_0 + q\mathcal{S}^{-1}\left[u^{\mu}\left((d_0+s_0)-(d_1+s_1)\,\mathcal{S}\left[p_4\left(\frac{t}{2}\right)\right]\right)\right]\\
&= p_0 + q\mathcal{S}^{-1}\Bigg[u^{\mu}\Bigg(\,(d_0+s_0)-(d_1+s_1)\\
&\quad\times\mathcal{S}\Bigg[p_0+q\Bigg(\,(d_0+s_0)-p_0\,(d_1+s_1)\Bigg)\Bigg(\frac{t^{\mu}}{2^{\mu}\Gamma(\mu+1)}\\
&\quad -q\,(d_1+s_1)\,\frac{t^{2\mu}}{2^{3\mu}\Gamma(2\mu+1)}+q^2\,(d_1+s_1)^2\,\frac{t^{3\mu}}{2^{6\mu}\Gamma(3\mu+1)}\Bigg)\Bigg]\Bigg)\Bigg]\\
&= p_0 + q\mathcal{S}^{-1}\Bigg[u^{\mu}\Bigg(\,(d_0+s_0)-(d_1+s_1)\\
&\quad\times\Bigg\{p_0+q\Bigg(\,(d_0+s_0)-p_0\,(d_1+s_1)\Bigg)\Bigg(\frac{u^{\mu}}{2^{\mu}}-q\,(d_1+s_1)\,\frac{u^{2\mu}}{2^{3\mu}}+q^2\,(d_1+s_1)^2\,\frac{u^{3\mu}}{2^{6\mu}}\Bigg)\Bigg\}\Bigg)\Bigg]\\
&= p_0+q\Bigg(\,(d_0+s_0)-p_0\,(d_1+s_1)\Bigg)\\
&\quad\times\mathcal{S}^{-1}\left[u^{\mu}\left\{1-q\,(d_1+s_1)\left(\frac{u^{\mu}}{2^{\mu}}-q\,(d_1+s_1)\,\frac{u^{2\mu}}{2^{3\mu}}+q^2\,(d_1+s_1)^2\,\frac{u^{3\mu}}{2^{6\mu}}\right)\right\}\right]\\
&= p_0+q\Bigg(\,(d_0+s_0)-p_0\,(d_1+s_1)\Bigg)\\
&\quad\times\mathcal{S}^{-1}\left[\left(u^{\mu}-q\,(d_1+s_1)\left(\frac{u^{2\mu}}{2^{\mu}}-q\,(d_1+s_1)\,\frac{u^{3\mu}}{2^{3\mu}}+q^2\,(d_1+s_1)^2\,\frac{u^{4\mu}}{2^{6\mu}}\right)\right)\right]\\
&= p_0+q\Bigg(\,(d_0+s_0)-p_0\,(d_1+s_1)\Bigg)\Bigg(\frac{t^{\mu}}{\Gamma(\mu+1)}-q\,(d_1+s_1)\Bigg(\frac{t^{2\mu}}{2^{\mu}\Gamma(2\mu+1)}\\
&\quad -q\,(d_1+s_1)\,\frac{t^{3\mu}}{2^{3\mu}\Gamma(3\mu+1)}+q^2\,(d_1+s_1)^2\,\frac{t^{4\mu}}{2^{6\mu}\Gamma(4\mu+1)}\Bigg)\Bigg)\\
&= p_0+q\Bigg(\,(d_0+s_0)-p_0\,(d_1+s_1)\Bigg)\Bigg(\frac{t^{\mu}}{\Gamma(\mu+1)}-q\,(d_1+s_1)\,\frac{t^{2\mu}}{2^{\mu}\Gamma(2\mu+1)}\\
&\quad +q^2\,(d_1+s_1)^2\,\frac{t^{3\mu}}{2^{3\mu}\Gamma(3\mu+1)}-q^3\,(d_1+s_1)^3\,\frac{t^{4\mu}}{2^{6\mu}\Gamma(4\mu+1)}\Bigg).
\end{aligned}$$

Hence, it can be deduced that

$$\begin{cases}
p_1(t)=p_0,\\
p_n(t)=p_0+q\Bigg(\,(d_0+s_0)-p_0\,(d_1+s_1)\Bigg)\displaystyle\sum_{k=1}^{n-1}\left(-q\,(d_1+s_1)\right)^{k-1}\frac{t^{k\mu}}{2^{\frac{k}{2}(k-1)\mu}\Gamma(k\mu+1)},\, n>1,\\
p(t)=\displaystyle\lim_{n\to\infty}p_n(t),\, n\in\mathbb{N}.
\end{cases}\tag{3.14}$$

To display the solution (3.14) graphically, real values are assigned to the constants as follows: $p_0 = 1, d_0 = 15, s_0 = 100, d_1 = 14, s_1 = 97$ and $q = 0.2$. For $\mu = 0.76$, Figure 3 displays the iterations of the solutions of price adjustment equation with a delay (3.11). It shows how the iterations progress to give the solution of (3.11). Figure 4 shows the effect of changing the values of the fractional order of the Caputo-fractional derivative on the solution of (3.11). It displays the trend for (3.14) as μ varies.

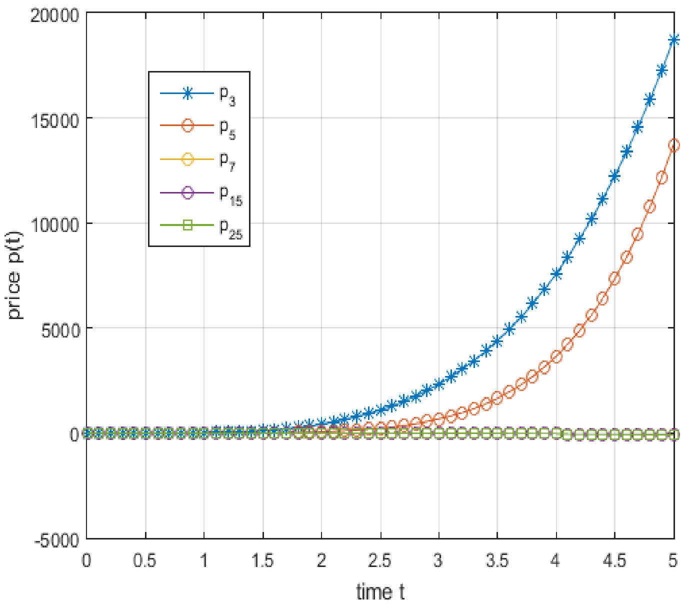

Figure 3: Price adjustment equations with a delay.

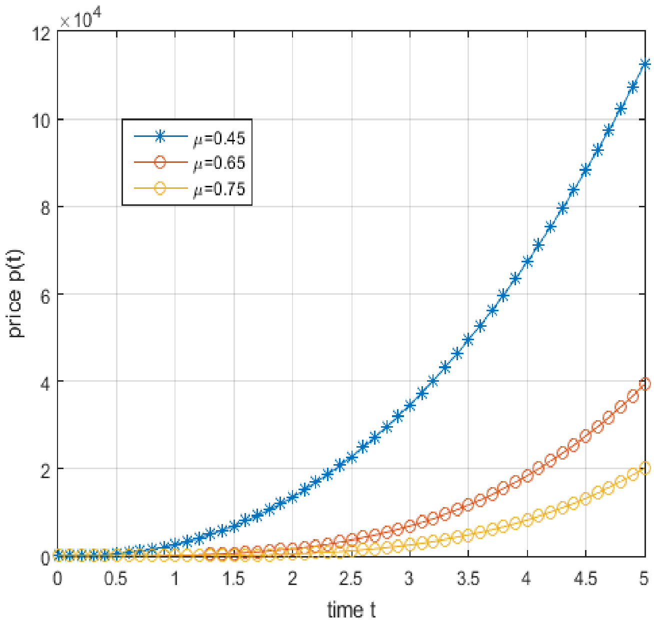

Figure 4: Variation as μ varies in the price adjustment equations with a delay.

4 Conclusion

This study considers how to obtain the solutions of FDEs. The study presents SVI method, which is a blend of ST with variational iterative method, for solving nonlinear FDEs with delay. The study displays the craft of SVI method in obtaining the Lagrange multiplier. A genuine model in science, engineering or economics, must comprise of time delays as they are natural components of the dynamic processes. Therefore, we propose a new economic model by introducing the delay into the formulation and construction of price adjustment equations with Caputo-fractional derivatives. We follow the systematic steps in the newly introduced method that we presented in this study, to obtain the solutions of the newly proposed economic models with delay. Choosing suitable parameters, we use Matlab to display the curves for the models of different fractional orders. Furthermore, the graphs show the comparison between the solutions of the price adjustment equations with Caputo-fractional derivatives and the newly introduced price adjustment equations with Caputo-fractional derivatives that involve a delay.

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Dynamical and Integrability Properties of Epidemic Models

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Abstract

This paper investigates the dynamical and integrability properties and the complete analytical solutions of the well-known SEIR and the SIRV models utilized for the COVID-19 pandemic by employing the partial Hamiltonian method based on Lie group theory. Regarding the model's parameters, two distinct cases are evaluated for each model. The closed-form solutions for the SIRV model are examined by using a single-phase space. Furthermore, the graphical representations of the dynamical behavior of the analytical solutions are discussed.

Keywords: SEIR model, SIRV model, first integrals, analytical solutions, artificial Hamiltonian, epidemic models, Lie groups, Covid-19.

1 Introduction

One of the most important approaches for analyzing differential equations is Lie group theory [1–3]. There are several examples of Lie group applications to ordinary, partial, and integrodifferential equations in the literature [12–14] and [24–26]. In addition, the artificial Hamiltonian technique, which offers a mechanism to obtain the exact solutions of a number of coupled nonlinear systems of ordinary differential equations (ODEs), may be used to apply the Lie groups theory to nonlinear dynamical systems as well. Additionally, since every first-order system of ODEs can be written as an artificial Hamiltonian system, the idea of an artificial Hamiltonian offers an efficient method for figuring out how to solve these dynamical systems of first-order ODEs. Some examples of systems where this method is particularly useful are given in the studies [18–23].

The SIR model, which only considers the susceptible, infectious, and recovered populations, is a well-known model for analyzing epidemic disease in the literature [4] & [5–8]. However, the current COVID-19 epidemic outbreak demonstrates that the effect of vaccines (mRNA, viral vectors, or subunit vaccines) on the other population fractions suspected S, infected I, and recovered R populations is significant and that this effect should be incorporated into the model to produce more accurate results from the studies on epidemics. As a result, a new

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model known as the SIRV model is developed as a generalized version of the SIR model to demonstrate the effect of vaccines. The SIRV model is a system of coupled first-order nonlinear ordinary differential equations from the mathematical perspective, and it is a fourth-dimensional nonlinear dynamical system from the standpoint of mechanics.

The remainder of the study is divided into the following sections: In Section 2, the artificial Hamiltonian definition is presented. The SEIR and SIRV model's mathematical and physical characteristics are described in Section 3. And then the SIRV model's first integral and analytical solutions are represented along with a graphical depiction of the dynamics of the model.

2 Biological models in epidemiology

Bio-mathematical models can be based on ordinary differential equations (ODEs) or partial differential equations (PDEs). ODE models are the most commonly used bio-mathematical models since they are effective to formulate and solve and can reveal system behavior over time. ODE models can range from simple systems with two variables to complex systems with a large number of variables. For example, the Lotka-Volterra predator-prey model, which describes predator-prey populations, is a popular ODE model [23]. This model describes the growth rates and interactions of the two populations using coupled ODEs. The SIR model is a standard mathematical model for describing the spread of infectious diseases within a population. [4] The model classifies the population into three groups: susceptible (S), infectious (I), and recovered (R) individuals. A four-variable ODE model of neuron action potential dynamics is the Hodgkin-Huxley model. [9] The membrane potential and ion channel opening and closing are described by four non-linear ODEs in this model. The SEIRD model extends the SIR model by adding a compartment for exposed non-infectious individuals [10,15]. Five-dimensional non-linear models include susceptible, exposed, infectious, recovered, and deceased individuals. A well-known example of a six-dimensional model that describes the dynamics of a gene regulatory network is the Goodwin oscillator [11]. The six-dimensional non-linear model describes the interactions between genes and proteins in a straightforward feedback loop.

2.1 The SEIR model

The SEIR model is a mathematical model that is used to examine the transmission of infectious illnesses within a community. The model classifies the population as susceptible (S), exposed (E), infected (I), and recovered (R). According to the model, individuals progress from being susceptible to exposure to infection to recovering. A set of differential equations governs the transition between categories by describing the rate of change of each category. The SEIR model has been extensively utilized to analyze the transmission of infectious diseases such as COVID-19, influenza, and Ebola, as well as the effectiveness of interventions such as vaccination, quarantine, and social isolation. [15–17]. Figure 1 depicts the flowchart for the SEIR model.

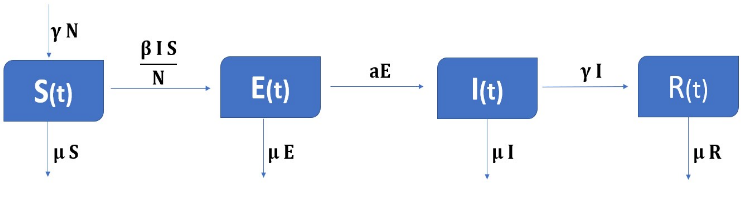

Figure 1: The flow diagram of epidemic SEIR-model

Based on the flow diagram in Figure 1, the SEIR model can be expressed in the literature as a system of coupled nonlinear first-order ODEs

$$\begin{aligned}\dot{S}(t) &= \mu N - \mu S - \frac{\beta IS}{N},\\ \dot{E}(t) &= \frac{\beta IS}{N} - (\mu + a)E,\\ \dot{I}(t) &= aE - (\gamma + \mu)I,\\ \dot{R}(t) &= \gamma I - \mu R,\end{aligned} \tag{2.1}$$

where a^{-1} represents the typical latency period and μ, γ, and β are the ratios of death, recovery, and infection. and N is the total population, constant.

2.1.1 General case of the SEIR model

In this section, we investigate the first integrals and their associated analytical solutions using the artificial Hamiltonian approach for the SEIR model (2.1) without considering any constraints on the model parameters.

Remark 2.1. Assume that time t is an independent variable for i= 1,2,...,n, and that the phase space parameters are (p_i, q_i) as dependent variables. As a result, the systems of the first-order ODEs listed below can be defined

$$\dot{q}_i = W_1^i(t, p_i, q_i), \tag{2.2}$$
$$\dot{p}_i = W_2^i(t, p_i, q_i), \tag{2.3}$$

where $W_1^i(t, p_i, q_i)$ and $W_2^i(t, p_i, q_i)$ are continuous differentiable functions of the form

$$\dot{p}_i = D_t(p_i), \quad \dot{q}_i = D_t(q_i), \tag{2.4}$$

in which D_t represents the total derivative operator defined by

$$D_t = \frac{\partial}{\partial t} + \dot{q}_i\frac{\partial}{\partial q_i} + \dot{p}_i\frac{\partial}{\partial p_i} + \dots. \tag{2.5}$$

In order to achieve this, we first define the phase space of the form

$$S = p_1, \quad E = p_2, \quad I = q_1, \quad R = q_2. \tag{2.6}$$

In terms of Hamiltonian functions H, the following differential expressions are written as

$$\begin{aligned}\dot{q}_1 &= \frac{\partial H}{\partial p_1} \longrightarrow \dot{I} = \frac{\partial H}{\partial S},\\ \dot{q}_2 &= \frac{\partial H}{\partial p_2} \longrightarrow \dot{R} = \frac{\partial H}{\partial E},\\ \dot{p}_1 &= -\frac{\partial H}{\partial q_1} + \Gamma_1 \longrightarrow \dot{S} = -\frac{\partial H}{\partial I} + \Gamma_1,\\ \dot{p}_2 &= -\frac{\partial H}{\partial q_2} + \Gamma_2 \longrightarrow \dot{E} = -\frac{\partial H}{\partial R} + \Gamma_2.\end{aligned} \tag{2.7}$$

Hence, the direct integration (2.7) yields

$$\Gamma_1 = \mu N - \frac{\beta p_1 q_1}{N} - p_1(\gamma + 2\mu) + \gamma p_2, \quad \Gamma_2 = -p_2(a + \mu) + \frac{\beta p_1 q_1}{N} - \mu p_2, \tag{2.8}$$

and

$$H = ap_1p_2 - p_1q_1(\gamma + \mu) + \gamma p_2 q_1 - \mu p_2 q_2. \tag{2.9}$$

Definition 2.2. The function $W_2^i(t, p_i, q_i)$ can be written as

$$W_2^i(t, q_i, p_i) = -\int \frac{\partial W_1^i}{\partial q_i}\, dp_i + g_i(t, q_i) + \Gamma^i(t, q_i, p_i), \tag{2.10}$$

where $\Gamma^i(t, q_i, p_i)$ and $g_i(t, q_i)$ are integrable functions. Therefore, the system (2.2) and (2.3) can be written in the following form

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \tag{2.11}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} + \Gamma_i(t, p_i, q_i), \tag{2.12}$$

in which H called as artificial Hamiltonian [19–21] satisfies the relation

$$H(t, q_i, p_i) = \int W_1^i(t, q_i, p_i)\, dp_i - \int g_i(t, q_i)\, dq_i. \tag{2.13}$$

Remark 2.3. It can be shown that both the functions $g^2(t, q_1, q_2)$ and $g^1(t, q_1, q_2)$, for the phase space (2.6) in the equation (2.10) are zero and then the equation (2.13) is satisfied.

Definition 2.4. The generator of Lie point symmetries for phase space (t, p_i, q_i) is a differential operator:

$$X = \xi(t, p_i, q_i)\frac{\partial}{\partial t} + \eta_i(t, p_i, q_i)\frac{\partial}{\partial q_i} + \zeta_t\frac{\partial}{\partial p_i}, \tag{2.14}$$

called the partial Hamiltonian operator. If there exists a gauge function $B(t, p_i, q_i)$ satisfying the equation

$$\zeta_t\frac{\partial H}{\partial p_i} + p_i D_t(\eta_i) - X(H) - HD_t(\xi) = D_t(B) - \left(\eta_i - \xi\frac{\partial H}{\partial p_i}\right)(\Gamma_i), \tag{2.15}$$

and then the first integrals of system (2.2)-(2.3) are determined by

$$I : p_i\eta_i - \xi H - B. \tag{2.16}$$

The differential operator (2.15) for $n = 2$ is written as below

$$X = \xi(t, q_1, q_2)\frac{\partial}{\partial t} + \eta^1(t, q_1, q_2)\frac{\partial}{\partial q_1} + \eta^2(t, q_1, q_2)\frac{\partial}{\partial q_2}. \tag{2.17}$$

It can be verified that the first term at (2.15) is zero. As a result, by using the partial Hamiltonian operator (2.14) on the equations (2.1), the determining equation (2.15) in terms of $x(t, q_1, q_2)$, $\eta^1(t, q_1, q_2)$, $\eta^2(t, q_1, q_2)$ and $B(t, q_1, q_2)$ is written of the form

$$\begin{aligned}
&B_{q_1}(q_1(\gamma+\mu) - ap_2) + \frac{(N(\mu N + \gamma p_2) - p_1(N(\gamma+2\mu) + \beta q_1))(\xi(q_1(\gamma+\mu) - ap_2) + \eta^1)}{N} \\
&\quad + \left(\frac{\beta p_1 q_1}{N} - p_2(a+2\mu)\right)(\eta^2 - \xi(ap_1 + \gamma q_1 - \mu q_2)) \\
&\quad + p_1\left(\eta^1_{q_1}(ap_2 - q_1(\gamma+\mu)) + (\gamma q_1 - \mu q_2)\eta^1_{q_2} + \eta^1_t\right) \\
&\quad - (ap_1p_2 - p_1q_1(\gamma+\mu) + \gamma p_2 q_1 - \mu p_2 q_2)\left(\xi_{q_1}(ap2 - q_1(\gamma+\mu)) + (\gamma q_1 - \mu q_2)\xi_{q_2} + \xi_t\right) \\
&\quad + p_2\left(\eta^2_{q_1}(ap_2 - q_1(\gamma+\mu)) + (\gamma q_1 - \mu q_2)\eta^2_{q_2} + \eta^2_t\right) + (\mu q_2 - \gamma q_1)B_{q_2} - B_t \\
&\quad + (p_1(\gamma+\mu) - \gamma p_2)\eta^1 + \mu p_2\eta^2 = 0.
\end{aligned} \tag{2.18}$$

For the general case, it is straightforward to demonstrate that the solution of the system (2.18) is

$$\xi = 0, \quad \eta^1 = 0, \quad \eta^2 = 0, \quad B = c_1. \tag{2.19}$$

where c_1 is a constant. As a result, it gives only a trivial first integral. Thus, we consider some constraint relations between the model's parameters.

2.1.2 Sub-case of the SEIR model

Remark 2.5. One can take into account the case $a = -\gamma$ and $\beta = \frac{\gamma^2-\mu^2}{\gamma}$ for the integrability requirements of the SEIR model in order to find nontrivial closed-form solutions.

Taking into account this constraint, the solution of the determining equations (2.18) is found as

$$
\begin{aligned}
&\xi = 0,\\
&\eta^1 = e^{\mu t}\left(q_2 c_2 e^{\mu t} + c_3\right),\\
&\eta^2 = e^{\mu t}\left(c_2 e^{\mu t}\left(q_2 - \frac{\gamma^2 N}{\gamma^2-\mu^2}\right) + c_3\right),\\
&B = c_4 + \frac{c_2 e^{2\mu t}\left(-2\gamma N(q_1+q_2) + 2\mu N q_2 + (\gamma-\mu)(q_1+q_2)^2\right)}{2(\mu-\gamma)} + c_3 e^{\mu t}(N - q_1 - q_2),
\end{aligned}
\tag{2.20}
$$

where c_2, c_3 and c_4 are constants. Using the formula (2.16), the following first integrals

$$
\begin{aligned}
&I_1 : \frac{(\gamma+\mu)\left(2(\gamma-\mu)q_2(N-p_1-q_1) + q_1(2\gamma N + (\mu-\gamma)q_1) + (\mu-\gamma)q_2^2\right) + 2p_2\left(\gamma^2 N + \left(\mu^2-\gamma^2\right)q_2\right)}{2e^{-2\mu t}(\mu-\gamma)(\gamma+\mu)}\\
&\quad - c_5 = 0,\\
&I_2 : e^{\mu t}(-N + p_1 + p_2 + q_1 + q_2) - c_6 = 0,\\
&I_3 : -1 - c_7 = 0,
\end{aligned}
\tag{2.21}
$$

are obtained. We can take the two preceding constants, namely c_5 and c_6 as zero without losing generality. From I_2, we can calculate p_1:

$$
p_1 = N - p_2 - q_1 - q_2. \tag{2.22}
$$

The first integral I_1 yields

$$
p_1 = \frac{(\gamma+\mu)\left(-2\gamma N q_1 + (\gamma-\mu)q_1^2 + (\mu-\gamma)q_2^2\right)}{2\gamma^2 N}, \tag{2.23}
$$

and the fourth equation of the system (2.1) gives

$$
q_1 = \frac{\dot{q}_2 + \mu q_2}{\gamma}. \tag{2.24}
$$

By taking $\gamma = 2\mu$, the third equation of system (2.1) yields the following analytical solution

$$
q_2 = e^{-\frac{1}{3}t\sqrt{2\mu^2+6\sqrt{3\mu^2(N+8)}}}\left(c_8 e^{\frac{2}{3}t\sqrt{2\mu^2+6\sqrt{3\mu^2(N+8)}}} + c_9\right), \tag{2.25}
$$

where c_8 and c_9 are integration constants. It can be shown that the solutions (2.25), (2.24), (2.23), and (2.22) satisfies the SEIR model given by (2.1).

2.2 The SIRV model

The SIRV model is one of the most widely used models for describing epidemiology and classifying diseases in the community. The vaccinated population variable as a function of time is added to the system in the SIRV model, which is connected to the SIR model. Studies looking at diseases like COVID-19 and Ebola have employed this methodology [22, 23]. In the recent study [22], by using the partial Hamiltonian approach based on the theory of Lie groups, the

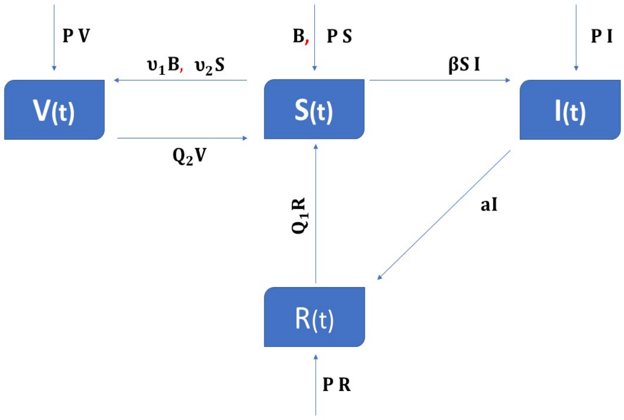

Figure 2: The flow diagram of epidemic SIRV-model

integrability conditions and exact analytical solutions of the initial-value problem defined for the prominent SIRV model used for the pandemic Covid-19 are investigated. However, in this study, a distinct SIRV model different from the model considered in the study [22] will be analyzed. Figure 2 depicts the flowchart for the modified SIRV model.

Based on the flow diagram in Figure 1, the SIRV model can be expressed in the literature as a system of coupled nonlinear first-order ODEs.

$$\begin{aligned} \dot{S}(t) &= B(1-\nu_1) - \beta SI - \nu_2 S + Q_1 R + Q_2 V - pS, \\ \dot{I}(t) &= \beta SI - aI - pI, \\ \dot{R}(t) &= aI - Q_1 R - pR, \\ \dot{V}(t) &= \nu_1 B + \nu_2 S - Q_2 V - pV, \end{aligned} \tag{2.26}$$

where the dot represents the time derivative. From the perspective of mechanics, the system can be categorized as a fourth-dimensional nonlinear dynamical system. Four variables make up the model's structure in this illustration: $S(t)$ represents the susceptible individual, $I(t)$ represents the infected individual, $R(t)$ represents the recovered individual, and $V(t)$ represents the population that has received vaccinations. Based on the behavior of the disease described, the structural components of the model might be improved. Additionally, B and p stand for the typical birth and death rates. ν_1 and ν_2, which stand for newborns and questionable individuals are two different vaccination rates. The parameters β and a show the infection and recovery rates. Additionally, in the model, there are two separate susceptibility ratios, Q_1 and Q_2, which are defined for the susceptible people following recovery and vaccination.

2.2.1 General case of the SIRV model

In this section, we investigate the first integrals and their associated analytical solutions for the SIRV model given by (2.26) without putting any constraints on the model parameters. In order to achieve this, we similarly define the phase space:

$$S = q_2, \quad I = p_1, \quad R = q_2, \quad V = p_2. \tag{2.27}$$

In terms of Hamiltonian functions H, the following differential expressions can be written

$$\begin{aligned} \dot{q_1} &= \frac{\partial H}{\partial p_1} \longrightarrow \dot{S} = \frac{\partial H}{\partial I}, \\ \dot{q_2} &= \frac{\partial H}{\partial p_2} \longrightarrow \dot{R} = \frac{\partial H}{\partial V}, \\ \dot{p_1} &= -\frac{\partial H}{\partial q_1} + \Gamma_1 \longrightarrow \dot{I} = -\frac{\partial H}{\partial S} + \Gamma_1, \\ \dot{p_2} &= -\frac{\partial H}{\partial q_2} + \Gamma_2 \longrightarrow \dot{V} = -\frac{\partial H}{\partial R} + \Gamma_2. \end{aligned} \tag{2.28}$$

The direct integration of the system (2.28) produces

$$\Gamma_1 = -\frac{1}{2}p_1(2a+4p+2\nu_2+\beta p_1-2\beta q_1), \quad \Gamma_2 = -p_2(Q_1+Q_2+2p)+Q_1p_1+B\nu_1+\nu_2 q_1, \tag{2.29}$$

and

$$H = ap_1p_2+Q_1p_1q_2-Q_1p_2q_2+Q_2p_1p_2-pp_1q_1-pp_2q_2+B(p_1-\nu_1p_1)-\frac{1}{2}\beta p_1^2q_1-\nu_2p_1q_1. \tag{2.30}$$

Remark 2.6. Similarly, it can be shown that the functions $g^2(t,q_1,q_2)$ and $g^1(t,q_1,q_2)$ are zero.

The differential operator (2.15) for $n=2$ is provided by

$$X = \xi(t,q_1,q_2)\frac{\partial}{\partial t} + \eta^1(t,q_1,q_2)\frac{\partial}{\partial q_1} + \eta^2(t,q_1,q_2)\frac{\partial}{\partial q_2}. \tag{2.31}$$

Additionally, it can be verified that the first term at (2.15) equals zero. As a result, by using the partial Hamiltonian operator (2.15) on the equations (2.26), the determining equation is written in the form:

$$\begin{aligned} &B_{q_2}(q_2(Q_1+p)-ap_1) - \frac{1}{2}(-2ap_1p_2 - 2Q_1p_1q_2 + 2Q_1p_2q_2 - 2Q_2p_1p_2 + 2pp_1q_1 + 2pp_2q_2 + 2B(\nu_1-1)p_1 \\ &+ \beta p_1^2 q_1 + 2\nu_2p_1q_1)(\xi_{q_2}(q_2(Q_1+p)-ap_1) + \xi_{q_1}(-Q_1q_2 - Q_2p_2 + q_1(p+\nu_2+\beta p_1) + B(\nu_1-1)) - \xi_t) \\ &+ p_1(\eta^1_{q_2}(ap_1 - q_2(Q_1+p)) + \eta^1_{q_1}(Q_1q_2 + Q_2p_2 - q_1(p+\nu_2+\beta p_1) + B(-\nu_1) + B) + \eta^1_t) \\ &- \frac{1}{2}p_1(2a+4p+2\nu_2+\beta p_1 - 2\beta q_1)(\xi(p_2(-(a+Q_2)) - Q_1q_2 + q_1(p+\nu_2+\beta p_1) + B(\nu_1-1)) \\ &+ \eta^1) + p_2(\eta^2_{q_2}(ap_1 - q_2(Q_1+p)) + \eta^2_{q_1}(Q_1q_2 + Q_2p_2 - q_1(p+\nu_2+\beta p_1) \\ &+ B(-\nu_1) + B) + \eta^2_t) + (-p_2(Q_1+Q_2+2p) + Q_1p_1 + B\nu_1 + \nu_2q_1)(\xi(q_2(Q_1+p) - p_1(a+Q_2)) \\ &+ \eta^2) + B_{q_1}(-Q_1q_2 - Q_2p_2 + q_1(p+\nu_2+\beta p_1) + B(\nu_1-1)) - B_t + (p_2(Q_1+p) - Q_1p_1)\eta^2 \\ &+ \frac{1}{2}p_1(2p+2\nu_2+\beta p_1)\eta^1 = 0. \end{aligned} \tag{2.32}$$

For the general situation, it is straightforward to show that the solution of the system (2.32) is

$$\xi = 0, \quad \eta^1 = c_1e^{pt}, \quad \eta^2 = c_1e^{pt}, \quad B = \frac{c_1e^{pt}(B - p(q_1+q_2))}{p} + c_2. \tag{2.33}$$

where c_1 and c_2 are constants of integration. Using the formula (2.16), the following first integrals

$$\begin{aligned} &I_1 : \frac{e^{pt}(p(p_1+p_2+q_1+q_2) - B)}{p} - c_3 = 0, \\ &I_2 : 1 - c_4 = 0, \end{aligned} \tag{2.34}$$

are determined, where c_3 and c_4 are constants, in which the first integral obtained above is a novel first integral for the SIRV model, however, it is not sufficient to determine the complete analytical solutions of the system.

2.2.2 Sub-case of the SIRV model

Remark 2.7. One can take into account the constraint $a = -Q_1 + Q_2 + \nu_2$ for the complete integrability of the SIRV model in order to obtain the nontrivial closed-form solutions.

Taking into account the constraint between the parameters of the model, the solution of the determining equations (2.32) gives the solution for the infinitesimals as below

$$\begin{aligned}
&\xi = 0,\\
&\eta^1 = c_6 e^{pt} - \frac{c_5 \nu_2 e^{t(Q_2+p+\nu_2)}}{Q_2},\\
&\eta^2 = c_5 e^{t(Q_2+p+\nu_2)} + c_6 e^{pt},\\
&B = \frac{c_5 e^{t(Q_2+p+\nu_2)} \left(Q_1(-Q_2(B\nu_1 + \nu_2(q_1+q_2)) - \nu_2((p+\nu_2)(q_1+q_2) + B(\nu_1-1))) + (Q_2+\nu_2)(\Omega)\right)}{Q_2(-Q_1+Q_2+\nu_2)(Q_2+p+\nu_2)}\\
&+ \frac{c_6 e^{pt}(B - p(q_1+q_2))}{p} + c_7,
\end{aligned} \tag{2.35}$$

and

$$\Omega = \nu_2 q_1(Q_2+p) + Q_2 B\nu_1 + B(\nu_1-1)\nu_2 + \nu_2^2 q_1, \tag{2.36}$$

where c_5, c_6 and c_7 are constants. Using the formula (2.16), the following first integrals

$$\begin{aligned}
&I_1 : \frac{e^{t(Q_2+p+\nu_2)}\left(\frac{Q_1\nu_2 q_2}{-Q_1+Q_2+\nu_2} + \frac{B(\nu_2 - \nu_1(Q_2+\nu_2))}{Q_2+p+\nu_2} + Q_2 p_2 - \nu_2(p_1+q_1)\right)}{Q_2} - c_8 = 0,\\
&I_2 : e^{pt}\left(-\frac{B}{p} + p_1 + p_2 + q_1 + q_2\right) - c_9 = 0,\\
&I_3 : -1 - c_{10} = 0,
\end{aligned} \tag{2.37}$$

are obtained, where c_8, c_9 and c_{10} are constants. First, from I_1, we can calculate p_2:

$$p_2 = \frac{\nu_2\left(-\frac{Q_1 q_2}{-Q_1+Q_2+\nu_2} + p_1 + q_1\right) + \frac{B(Q_2\nu_1 + (\nu_1-1)\nu_2)}{Q_2+p+\nu_2}}{Q_2}. \tag{2.38}$$

From I_2, we have

$$p_1 = -\frac{(Q_2-Q_1)q_2}{-Q_1+Q_2+\nu_2} + \frac{B(Q_2 - p\nu_1 + p)}{p(Q_2+p+\nu_2)} - q_1, \tag{2.39}$$

and from the third equation of the system (2.26), one can write

$$q_1 = \frac{(Q_2+p)q_2 + \dot{q}_2}{Q_1 - Q_2 - \nu_2} + \frac{B(Q_2 - p\nu_1 + p)}{p(Q_2+p+\nu_2)}. \tag{2.40}$$

Remark 2.8. By considering the constraint $Q_1 = Q_2 = -p$ between the parameters of the model, we can obtain the analytical solution of the system.

From the first equation in the system (2.26), the differential equation

$$\nu_2\dot{q}_2\left(p\left(-\left(\frac{\beta B}{p\nu_2} - 1\right)\right) - 2p - \frac{\beta B(\nu_1-1)}{\nu_2} - \nu_2\right) - \nu_2\ddot{q}_2 - \beta\dot{q}_2^2, \tag{2.41}$$

is obtained, which has the solution of the form

$$q_2(t) = \frac{\nu_2\left(\ln\left(e^{t\left(p+\frac{\beta B\nu_1}{\nu_2}+\nu_2\right)} - \beta e^{c_{11}(\nu_2(p+\nu_2)+\beta B\nu_1)}\right) - \ln\left(e^{t\left(p+\frac{\beta B\nu_1}{\nu_2}+\nu_2\right)}\right)\right)}{\beta} + c_{12}, \tag{2.42}$$

where c_{11} and c_{12} are constants. By applying (2.42) to (2.38), (2.39), and (2.40), the exact analytical solutions for the SIRV model

$$
\begin{aligned}
S(t) &= -\frac{\nu_2(p+\nu_2)e^{c_{11}(\nu_2(p+\nu_2)+\beta B\nu_1)} + B\nu_1 e^{t\left(p+\frac{\beta B\nu_1}{\nu_2}+\nu_2\right)}}{\nu_2\left(e^{t\left(p+\frac{\beta B\nu_1}{\nu_2}+\nu_2\right)} - \beta e^{c_{11}(\nu_2(p+\nu_2)+\beta B\nu_1)}\right)},\\
I(t) &= \frac{(\nu_2(p+\nu_2)+\beta B\nu_1)e^{c_{11}(\nu_2(p+\nu_2)+\beta B\nu_1)}}{\nu_2\left(e^{t\left(p+\frac{\beta B\nu_1}{\nu_2}+\nu_2\right)} - \beta e^{c_{11}(\nu_2(p+\nu_2)+\beta B\nu_1)}\right)},\\
R(t) &= \frac{\nu_2\left(\ln\left(e^{t\left(p+\frac{\beta B\nu_1}{\nu_2}+\nu_2\right)} - \beta e^{c_{11}(\nu_2(p+\nu_2)+\beta B\nu_1)}\right) - \ln\left(e^{t\left(p+\frac{\beta B\nu_1}{\nu_2}+\nu_2\right)}\right)\right)}{\beta} + c_{12},\\
V(t) &= \frac{\nu_2\left(\ln\left(e^{t\left(p+\frac{\beta B\nu_1}{\nu_2}+\nu_2\right)}\right) - \ln\left(e^{t\left(p+\frac{\beta B\nu_1}{\nu_2}+\nu_2\right)} - \beta e^{c_{11}(\nu_2(p+\nu_2)+\beta B\nu_1)}\right)\right)}{\beta} + B\left(\frac{1}{p}+\frac{\nu_1}{\nu_2}\right) - c_{12},
\end{aligned}
\tag{2.43}
$$

are determined. It can be shown that the analytical solutions satisfy the SIRV model(2.26). Additionally, by taking into account the numerical values of the parameters in Table 2.1, the graphical representations for the analytical solutions are provided in Figure 2 and Figure 3.

Parameters	ν_1	ν_2	p	B	β
Values	0.50	0.70	0.55	0.9	2.5×10^{-5}, 2.5×10^{0}, $2.5\times10^{+1}$

Table 2.1: Parameter values considered for the subcase

As seen from the graphs, they reveal that as time passes, the population's estimated number of suspects and infected people gradually declines, while the number of people who have successfully recovered from the disease and received the vaccine rises. These developments are directly related to the vaccination drive, which gives people immunity and aids in halting the disease's spread.

The proportion of susceptible people in the population declines as vaccination rates rise, making it more difficult for the disease to spread. Because of this, fewer infections are acquired, which lowers the number of active cases. Concurrently, those who have recovered from the illness or received the vaccine offer an additional layer of protection to the populace, lowering the overall impact of the disease. All of these patterns suggest that vaccination can be a useful tool for halting the spread of infectious diseases and lessening their negative effects on society.

3 Conclusions

In this study, we have presented an analytical analysis of two different epidemic models namely the SEIR and the SIRV models. The SEIR model is a prominent epidemiological model used to track the transmission of infectious diseases. The SEIR model splits the population into four compartments: susceptible (S), exposed (E), infected (I), and recovered (R). The rate of movement from one compartment to another depends on many things, such as the mortality rate, the infection rate, and the recovery rate.

For a specific subcase, an exact solution to the SEIR model is obtained. This solution offers a closed-form formula for the number of organisms in each compartment as a function of time.

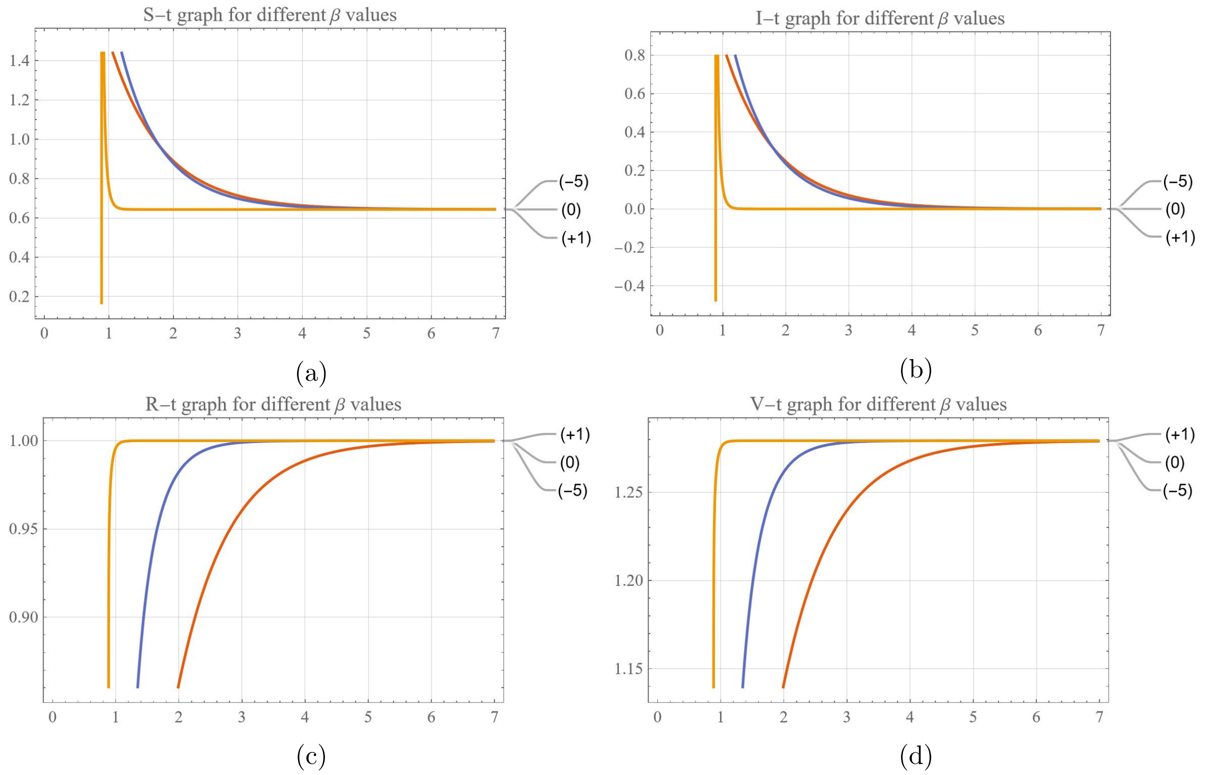

Figure 3: Graphics showing how SIRV population variables have changed over time for the subcase, For each figure, three different situations are considered for the different values of β changing from 2.5×10^1 to 2.5×10^{-5}

Our results reveal that the behavior of the SEIR model is highly sensitive to the values of the model parameters and that changes in these parameters can have a major impact on the overall course of an outbreak. This study offers valuable insights into the dynamics of infectious disease transmission and can inform public health policies and activities targeted at preventing the spread of infectious illnesses. Under one subbase, two unique first integrals for the study of the SEIR model are obtained, and we show these two first integrals are sufficient to obtain the exact analytic solution for the SEIR model.

In addition, this study examines the impact of vaccination on the course of the disease using the SIRV (Susceptible-Infected-Recovered-Vaccinated) epidemic model, which is used to analyze the COVID-19 pandemic disease. The integrability characteristics of a COVID-19 model known as the SIRV model are examined in this paper based on the theory of Lie groups and the Hamiltonian technique in order to investigate the exact analytical solutions. The SIRV model generalizes the well-known SIR model (Susceptible-Infected-Recovered), one of the most popular epidemic models for comprehending the mathematical and dynamic components of an epidemic outbreak. The SIR model does not, however, take into account the populations of vaccinated people and their impacts, which are critical in situations like the COVID-19 or SARS epidemic outbreaks.

In the first place, two nontrivial first integrals are produced for the specific subcase. It is demonstrated that these two first integrals render the system fully integrable. After obtaining the analytical solutions, the innovative, exact analytical solutions of the model are derived. In addition, the graphical representations of the solutions' evaluations are provided. It is proven that the results are compatible with the outcomes expected. If we compare the results acquired by numerical methods with the analytical results obtained using Lie group analysis, one can find that the analytical solutions are consistent with the results produced by numerical methods. In

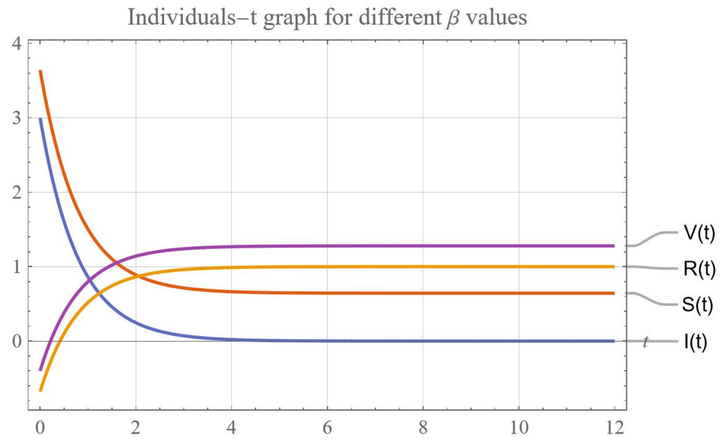

Figure 4: Graphical representations of all population fractions, $S(t)$, $I(t)$, $R(t)$, and $V(t)$, evaluated by time t.

addition, the outcomes of this investigation reflect genuine physical conditions in the real world. For instance, it is evident from Figure 3 that the number of suspected and infected individuals reduces as the number of vaccines grows over time. From graphs Figure 3 and Figure 4, it can be seen that the number of infected and susceptible people will decrease to almost zero, while the number of recovered people will increase in the future as a result of vaccination when the number of vaccinated people will first increase and then become nearly constant as a steady-state case in the distant future. The analytical results and graphs in this study show that vaccine use influences the number of infected and recovered populations over time. These results can be taken as an acceptable and fair forecast from the results acquired in this study concerning the end of the COVID-19 pandemic, as one of the novel contributions to the study.

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Inviscid Rayleigh Criteria for Vibrational Excited Dissociated Gas

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Abstract

For a plane flow of a vibrationally excited dissociating diatomic gas the necessary conditions of the existence of growing (neutral) inviscid disturbances, similar to the Rayleigh criterion of a "generalized" inflection point, are obtained. The corresponding formulas are presented for cases having a certain physical interpretation. In particular, the model of a vibrationally excited one-component gas is considered as the initial stage of thermal dissociation, as well as a wide spread model with one dissociation-recombination reaction. The case of a binary molecular-atomic mixture with a vibrationally excited molecular component and a "frozen" gas-phase dissociation-recombination reaction is considered as an intermediate one. Comparative numerical calculations were carried out, which showed, in particular, that under conditions of developed dissociation, the use of the criterion of the "generalized" inflection point does not take into account the specifics of the process. The wave numbers and phase velocities of the I and II inviscid modes calculated on its basis may differ significantly from the results obtained using the new necessary condition.

Keywords: inviscid disturbances, Rayleigh criterion of "generalized" inflection point, vibrational excitation, dissociation-recombination reaction, I and II inviscid modes.

1 Introduction

The problem of finding growing (neutral) inviscid disturbances is a part of the general problem of linear stability of flows. From the spectrum of inviscid disturbances, the most growing modes are distinguished, which are reproduced in the viscous problem. For an ideal incompressible fluid, the necessary condition for the existence of such disturbances is known as the Rayleigh criterion [1]. A compressible ideal gas has a similar condition for a "generalized" inflection point [2]. For a supersonic boundary layer at Mach numbers $\mathrm{M} > 2.2$, the appearance of more unstable high-frequency modes [3] has been established. The manifestation of the real properties

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of the gas, such as vibrational excitation, dissociation-recombination, and other physical and chemical processes, necessarily affects the physical picture of the occurrence and development of disturbances. From a mathematical point of view, taking into account these processes leads to the appearance of source terms in the equations of continuity and energies of the initial steady flow and the corresponding equations for disturbances of the linear theory of stability. In this case, a consistent consideration of the conditions for the growth of inviscid disturbances should lead to further generalizations of the inviscid criterion.

In paper [4], an expression for the growth criterion for inviscid disturbances was obtained for a gas model with a simple first-order chemical reaction. However, when deriving the energy equation, the term corresponding to the work of pressure forces during volumetric deformation of the medium was omitted. Such an unjustified simplification casts doubt on the adequacy of the result obtained, especially not confirmed by numerical calculations. The article considers the necessary conditions for the existence of neutral (growth) inviscid perturbations in a vibrationally excited dissociating gas for the case of single-mode vibrational relaxation and dissociation-recombination of a diatomic gas according to the scheme

$$\mathrm{A_2} + \mathrm{M} \leftrightarrows \mathrm{A} + \mathrm{A} + \mathrm{M}.$$

Here $\mathrm{A_2}$ is a molecule, A is an atom, and M is the collision partner (the third body in recombination), which can be either a molecule or an atom. Thus, a binary reacting gas mixture is considered.

As a first approximation, the model of a vibrationally excited one-component gas is considered as the initial stage of thermal dissociation, when the concentration of atoms is insignificant. As another approximation, the case of a binary molecular-atomic mixture with a vibrationally excited molecular component and a "frozen" gas-phase dissociation-recombination reaction is analyzed. This approximation corresponds to the experimental conditions in a high-enthalpy wind tunnel. The practical interest is the criterion of neutral (growth) inviscid disturbances obtained for the widely used model of a dissociating gas without taking into account vibrational excitation, which is substantiated by a significant difference in the characteristic times of the processes. To check the significance of the criteria obtained, numerical calculations were performed for the conditions of developed dissociation.

2 Statement of problem and basic equations

A model of a diatomic gas is considered with allowance for vibrational relaxation and the dissociation-recombination reaction [5]. The current distance $x = L$ along the plate and parameters of the unperturbed flow outside the boundary layer, marked by the index "∞" were chosen for nondimensionalization — the velocity, U_∞, the density, ρ_∞, and the temperature, T_∞, the coefficients of the shear and bulk viscosities, η_∞ and $\eta_{b\,\infty}$, correspondingly, the thermal conductivity coefficient due to the energy transfer in translational and rotational degrees of freedom, $\lambda_\infty = \lambda_{t\,\infty} + \lambda_{r\,\infty}$, the coefficient of thermal conductivity describing the diffusion transfer of the energy of vibrational quanta, $\lambda_{v\,\infty}$. For nondimensionalization of the pressure and time, the combined values of $\rho_\infty U_\infty^2$ and L/U_∞, respectively were used. Energies and enthalpies are dimensionless by the value $\rho_\infty T_\infty R/(2M_a)$, where R is the universal gas constant, M_a is the molecular weight of the atom. The production (death) rate $\dot{w}$ of the atomic component is scaled by the complex $k_d\rho_\infty^2/(2M_a)$, where k_d is the dissociation constant. The problem is characterized by dimensionless criteria — the Reynolds number, $\mathrm{Re}_\infty = \rho_\infty L U_\infty/\eta_\infty$, the Mach number, $\mathrm{M}_\infty = U_\infty/\sqrt{\gamma_\infty T_\infty R/(2M_a)}$, the Damköhler number, $Da_d = k_d\rho_\infty^2 L/(2M_a U_\infty)$, the Schmidt number, $\mathrm{Sc} = \eta_\infty/(\rho_\infty D_{12\infty})$, where $D_{12\infty}$ is binary the coefficient of diffusion.

In the inviscid approximation, it is assumed that

$$\eta \sim \lambda \sim \lambda_v \sim D_{12} \sim o(1).$$

As a consequence, in the limit $\mathrm{Re}_\infty \to \infty$, the terms describing the processes of transfer of momentum, heat, and mass are excluded from the initial equations [5]. As a result, the system of equations takes the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial u\rho}{\partial x} + \frac{\partial v\rho}{\partial y} = 0, \quad \rho\frac{\partial c}{\partial t} + \rho u\frac{\partial c}{\partial x} + \rho v\frac{\partial c}{\partial y} = Da_d\dot{w},$$
$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x}, \quad \rho\frac{\partial v}{\partial t} + \rho u\frac{\partial v}{\partial x} + \rho v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y},$$
$$\rho\frac{\partial e_i}{\partial t} + \rho u\frac{\partial e_i}{\partial x} + \rho v\frac{\partial e_i}{\partial y} + \gamma p \mathrm{M}_\infty^2\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\right) = Da_d\left(J - \frac{1}{2}e_v\dot{w}\right) - (1-c)Q_{tr-v}, \tag{2.1}$$
$$\rho\frac{\partial(1-c)e_v}{\partial t} + \rho u\frac{\partial(1-c)e_v}{\partial x} + \rho v\frac{\partial(1-c)e_v}{\partial y} = \frac{1}{2}Da_d e_v\dot{w} + (1-c)Q_{tr-v},$$
$$p = \frac{1}{\gamma \mathrm{M}_\infty^2}(1+c)\rho T.$$

Since the pressure is constant across the layer, then $\rho T = (1+c_\infty)/(1+c)$.

Here

$$e_i = \left(\frac{5}{2} + \frac{1}{2}c\right)T, \quad e_v = \theta_h\left[\exp\left(\frac{\theta_h}{T_v}\right) - 1\right]^{-1}, \quad Q_{tr-v} = \rho\frac{e_v(T) - e_v(T_v)}{\tau},$$

θ_h is the characteristic temperature of gas molecules, $J = h_a^0\dot{w}$ is the dimensionless thermal effect of dissociation-recombination reactions, h_a^0 is the dimensionless enthalpy of formation of atoms, τ is the relaxation time.

The production of atoms is calculated by the formula [5]

$$\dot{w} = R_1 = k_d^{(1)}\frac{(1-c)^2\rho^2}{4M_a} - k_r^{(1)}\frac{(1-c)c^2\rho^3}{2M_a^2}, \quad k_d^{(1)} = a_1 T^{-1/2}\exp\left(-\frac{E_d}{kT}\right), \quad k_r^{(1)} = b_1 T^{-1/2},$$

where E_d is the dissociation energy of a gas molecule, k is the Boltzmann constant.

The system (2.1) was linearized on a stationary boundary layer solution for a plate. In deriving the linearized equations for disturbances the instantaneous values of the hydrodynamic variables were represented in the form

$$u = U_S + u',\ v = v',\ \rho = \rho_S + \rho',\ c = c_S + c',\ T = T_S + T',\ p = p_S + p',\ T_v = T_{vS} + T_v'.$$

Here the subscript "S" denotes the values of the variables related to the stationary flow, and the primed quantities are the disturbances of these variables. The disturbances of quantities that are functionally dependent on the main variables were expressed in terms of their first-order total differentials. In this case, the derivatives included in them were calculated on a stationary solution. As a result we get following expressions

$$e_i = e_{iS} + e_i' = e_{iS} + e_{iT}T' + e_{ic}c', \quad e_v = e_{vS} + e_v' = e_{vS} + e_{vT_v}T_v',$$
$$\dot{w} = \dot{w_S} + \dot{w}' = \dot{w_S} + \dot{w}_\rho\rho' + \dot{w_T}T' + \dot{w}_c c'. \tag{2.2}$$

Here

$$e_{iT} = \frac{5}{2} + \frac{1}{2}c_S, \quad e_{ic} = \frac{1}{2}T_S, \quad e_{vT_v} = \theta_h\left[\exp\left(\frac{\theta_h}{T_{vS}}\right) - 1\right]^{-2}\exp\left(\frac{\theta_h}{T_{vS}}\right)\frac{\theta_h}{T_{vS}^2},$$
$$w_\rho = \left[k_{dS}^{(1)}\frac{(1-c_S)^2\rho_S}{2M_a} - k_{rS}^{(1)}\frac{(1-c_S)c_S^2 3\rho_S^2}{2M_a^2}\right],$$

$$\dot{w}_T = \left[a_1 \exp\left(-\frac{\theta_d}{T_S}\right) \frac{(\theta_d/T_S - 1/2)}{T_S^{3/2}} \frac{(1-c_S)^2\rho_S^2}{4M_a} + \frac{1}{2}\frac{b_1}{T_S^{3/2}} \frac{(1-c_S)c_S^2 3\rho_S^2}{2M_a^2}\right]$$

$$\dot{w}_c = -\left[k_{dS}^{(1)} \frac{(1-c_S)\rho_S}{M_a} + k_r^{(1)} \frac{(2c_S - 3c_S^2)\rho^3}{2M_a^2}\right].$$

We considered the stability of disturbances periodic in the longitudinal coordinate x in the form of traveling plane waves

$$\mathbf{q}'(x,y,t) = \mathbf{q}(y) \exp\left[i\alpha\,(x - Vt)\right],$$

$$\mathbf{q}'(x,y,t) = (u', v', \rho', T', T_v', p', c', \dot{w}')^*, \quad \mathbf{q}(y) = (u, \alpha v, \rho, \theta, \theta_v, p, C, \Omega)^*,$$

Here α is the wave number along the periodic variable x,$V = V_r + iV_i$ is the complex phase velocity, i is the imaginary unit, the asterisk "$*$" denotes transposition. For part of the amplitude functions, the descriptions of the corresponding initial variables are used. Substituting $\mathbf{q}'(x,y,t)$ into the system of equations for disturbances gives a system of equations for their amplitudes

$$D\rho + \alpha\rho_S\sigma + \alpha v\frac{d\rho_S}{dy} = 0, \quad D\rho_S + \rho_S\alpha v\frac{dc_S}{dy} = Da_d\Omega',$$

$$D\rho_S u + \rho_S\alpha v\frac{dU_S}{dy} = -i\alpha p, \quad D\rho_S\alpha v = -\frac{dp}{dy},$$

$$D\rho_S e_{ia}' + \rho_S\alpha v\frac{de_{iS}}{dy} + \gamma_\infty \mathrm{M}_\infty^2 p_S\alpha\sigma = Da_d\left(J_a' - \frac{1}{2}\left(e_{va}'\dot{w}_S + e_{vS}\Omega'\right)\right) - Q_{tr-v,a}',$$

$$(1-c_S)D\rho_S e_{va}' - \rho_S e_{vS} DC + \rho_S\alpha v\frac{d(1-c_S)e_{vS}}{dy} = \frac{1}{2}Da_d\left(e_{va}'\dot{w}_S + e_{vS}\Omega'\right) + Q_{tr-v,a}',$$

$$p = p_S\left[C/\left(1+c_S\right) + \rho/\rho_S + \theta/T_S\right], \quad p_S = \frac{\rho_S T_S\left(1+c_S\right)}{\gamma\mathrm{M}_\infty^2}. \tag{2.3}$$

Here we introduced the notation

$$D = i\alpha W, \quad W = U_S - V, \quad \sigma = iu + \frac{dv}{dy}, \quad \Omega' = \dot{w}_\rho\rho + \dot{w}_T\theta + \dot{w}_c C.$$

The subscript "a" in expressions of disturbances means that in formulas (2.2) values of corresponding amplitudes are substituted. For example

$$e_{ia}' = e_{iT}\theta + e_{ic}C,$$

$$Q_{tr-v,a}' = (1-c_S)\rho_S\frac{e_v'(\theta) - e_v'(\theta_v)}{\tau} - C\rho_S\frac{e_{vS}(T) - e_{vS}(T_v)}{\tau} + (1-c_S)\rho\frac{e_{vS}(T) - e_{vS}(T_v)}{\tau}.$$

The system (2.3) reduces to two first-order equations for a pair of functions $(v,\,p)$ or to a second-order equation for one of them [3, 4].

3 Criteria for inviscid instability

In all considered cases, criteria for inviscid instability are derived within the framework of a single calculation scheme generalizing the approach used in [6]. At the first stage, the system (2.3) is reduced to a second-order equation for the transverse velocity perturbation amplitude v. The equation can be written in the following universal form

$$\frac{1}{W}\frac{d}{dy}\left(\frac{v'W - vU_S'}{\chi} + vWS\right) = \alpha^2\rho_S v, \quad v = v_r + iv_i. \tag{3.1}$$

Equation (3.1) is complemented by homogeneous boundary conditions

$$v(0) = v(\delta) = 0, \tag{3.2}$$

where δ is the conditional upper boundary of the boundary layer. Here and below, the primes denote derivatives with respect to the coordinate y, the functions χ and S are defined in each particular case. In order to go to (3.1), (3.2) to a well-posed spectral problem and in the derivation of the criterion, D complexes entering in a complicated way into the expressions χ, S, containing the spectral parameter α, are omitted.

The equation (3.1) is multiplied by the complex conjugate function v^*. The complex conjugate to it is subtracted from the resulting equation. After a series of transformations, we arrive at a differential identity

$$\frac{v^* W^*}{|W|^2} \frac{d}{dy} \left(\frac{W v' - v U_S'}{\chi} + v W S \right) = \frac{v W}{|W|^2} \frac{d}{dy} \left(\frac{W^* v'^* - v^* U_S'}{\chi^*} + v^* W^* S^* \right). \tag{3.3}$$

Regrouping the terms in (3.3), we obtain the expression

$$v^* \frac{d}{dy} \left(\frac{v'}{\chi} \right) - v \frac{d}{dy} \left(\frac{v^{*\prime}}{\chi^*} \right) = \frac{v v^*}{|W|^2} \left[W \frac{d}{dy} \left(\frac{U_S'}{\chi^*} \right) - W^* \frac{d}{dy} \left(\frac{U_S'}{\chi} \right) + \left(\frac{W}{\chi^*} - \frac{W^*}{\chi} \right) \frac{dU_S}{dy} + \right.$$

$$\left. + \left(W S^* \frac{dW^*}{dy} - W^* S \frac{dW}{dy} \right) + |W|^2 \frac{d}{dy} (S^* - S) \right] + (v^{*\prime} v S^* - v^* v' S).$$

For disturbances close to neutral, one can set $|V_i| \ll |V_r|$, $|U_S|$ and neglect the imaginary component in W. As a result, we get $W = W^*$, $\chi = \chi^*$, $S = S^*$. After that, the expression is converted to the form

$$\frac{d}{dy} \left(\frac{1}{\chi} \frac{d}{dy} (v_r v_i) - S v_r v_i \right) = \frac{V_i |v|^2}{|W|^2} \left[\frac{d}{dy} \left(\frac{1}{\chi} \frac{dU_S}{dy} \right) + S \frac{dU_S}{dy} \right] - v_r v_i \frac{dS}{dy}. \tag{3.4}$$

Here the expressions on both sides of the identity (3.4) are purely real.

The necessary condition (criterion) for growing inviscid perturbations at $V_i > 0$ is obtained from (3.4) based on the following reasoning. On the left side (3.4) is the derivative of the differentiable function

$$F(y) = \frac{1}{\chi} \frac{d}{dy} (v_r v_i) - S v_r v_i.$$

Due to (3.2), this function vanishes at the ends of the interval $[0, \delta]$. By Rolle's theorem, its derivative must vanish at least at one interior point of the interval. By virtue of the structure of the right-hand side, such a point should be the coordinate of the critical layer $y = y_c$, where $W = 0$. Indeed, at this point the first term contains a singularity, which must be compensated in order to preserve the boundedness of the derivative of the differentiable function $F(y)$ on the left side. With positive increments $V_i > 0$ this is possible if and only if the expression in square brackets is zeroed out

$$\frac{d}{dy} \left(\frac{1}{\chi} \frac{dU_S}{dy} \right) + S \frac{dU_S}{dy} = 0. \tag{3.5}$$

For this point to be the zero of the derivative of $F(y)$, one must also require

$$\frac{dS}{dy} = 0. \tag{3.6}$$

Thus, the system of equations can serve as a criterion for inviscid instability

$$‘\frac{d}{dy} \left(\frac{1}{\chi_1} \frac{dU_S}{dy} \right) + S_1 \frac{dU_S}{dy} = 0, \quad \frac{dS_1}{dy} = 0. \tag{3.7}$$

Here χ_1, S_1 are expressions of the functions χ, S for $W = 0$, whose joint solution determines the coordinate y_c.

The resulting system (3.7) is too complicated for practical use as a criterion, in particular, in the case of bulk dissociation-recombination. Assuming the leading role of the first equation, which compensates for the singularity $U_S - c_r = 0$, possible simplifications were considered. As a result of evaluation calculations, an approximation was chosen in the form of equation

$$\frac{d}{dy}\left(\frac{1}{\chi_1}\frac{dU_S}{dy}\right) + \left(\max_y S_1\right)\frac{dU_S}{dy} = 0, \tag{3.8}$$

the first roots of which differ from the roots of the system (3.7) by no more than the third decimal place.

3.1 Vibrationally excited gas without dissociation-recombination reaction

This case corresponds to the initial stage of thermal dissociation, which is always preceded by vibrational excitation. In this case, the concentration of atoms is negligible and the gas remains one-component. In this case, when deriving the equation (3.1) in the system (2.3), it is necessary to vanish all quantities associated with dissociation-recombination and use the equation of state for the boundary layer of an ideal gas [5]

$$p = \frac{1}{\gamma_\infty \mathrm{M}_\infty^2}\left(\frac{\rho}{\rho_S} + \frac{\theta}{T_S}\right), \quad \rho_S T_S = 1. \tag{3.9}$$

Let us rewrite the energy equations in terms of temperature perturbations

$$D\theta + \alpha v T_S'' + \alpha\sigma\frac{1}{\rho_S C_{trv}} = -\gamma_v\frac{\theta-\theta_v}{\tau}, \quad D\theta_v + \alpha v T_{vS}'' = \frac{\theta-\theta_v}{\tau},$$

$$\gamma_v = \frac{C_{Vv}}{C_{Vtrv}}, \quad C_{Vtrv} = \frac{de_i}{dT} = \frac{5}{2}, \quad C_{Vv} = \frac{de_v}{dT_v} = \frac{\theta_h}{[\exp(\theta_h/T_v) - 1]^2}\exp\left(\frac{\theta_h}{T_v}\right)\frac{\theta_h}{T_v^2}.$$

After transformations of the resulting system, we arrive at an equation of the form (3.1) for the perturbation of the transverse velocity v, in which the coefficient functions are defined as

$$\chi = T_S - \mathrm{M}_\infty^{*2}W^2, \quad S = \frac{1}{\chi}\frac{\gamma_v(T_S' - T_{vS}')}{(\tau D + 1)\gamma + \gamma_v}. \tag{3.10}$$

Here

$$\mathrm{M}_\infty^{*2} = m^2\mathrm{M}_\infty^2, \quad m^2 = m_r^2 + i m_i^2, \quad m_r^2 = \frac{R_1(1+\gamma_v+\alpha\tau V_i) + \Delta^2}{R_1^2 + \Delta^2}, \quad m_i^2 = -\frac{\gamma_v}{\gamma}\frac{(\gamma-1)\Delta}{R_1^2+\Delta^2},$$

$$R_1 = 1 + \frac{\gamma_v}{\gamma} + \alpha\tau V_i, \quad \Delta = \alpha\tau V_r.$$

In accordance with equalities (2), their expressions for $W = 0$ are

$$\chi_1 = T_S, \quad S_1 = \frac{1}{T_S}\frac{\gamma_v(T_S' - T_{vS}')}{\gamma + \gamma_v}.$$

As a result, the inviscid instability criterion for a vibrationally excited gas is expressed as

$$\frac{d}{dy}\left(\frac{1}{T_S}\frac{dU_S}{dy}\right) + \max_y\left[\frac{\gamma_v(T_S' - T_{vS}')}{T_S(\gamma+\gamma_v)}\right]\frac{dU_S}{dy} = 0. \tag{3.11}$$

It can be seen that in the absence of vibrational excitation, the equality (3.11) is transformed into well-known condition of the “generalized” inflection point [2]

$$\frac{d}{dy}\left(\frac{1}{T_S}\frac{dU_S}{dy}\right) = 0. \tag{3.12}$$

3.2 "Frozen" dissociation-recombination reaction with an accounting for vibrational excitation

In this case, it is assumed that the gas-phase dissociation-recombination reaction is "frozen" in volume, which corresponds to $\dot{w} = 0$ in (2.1) Ω' in (2.3), leaving a heterogeneous reaction on the solid surface. In supersonic flows, such a model corresponds to the rapid expansion of a thermally dissociated gas when recombination slows down [7]. In this case, in fact, there is a mixture of two non-reacting gases, one component of which is vibrationally excited. For each of the components, it is convenient to consider separate continuity equations and energy equations written in terms of temperatures. In this case, the system of equations for amplitudes of disturbances (2.3) in a two-component gas is rewritten in the form

$$D\rho_1 + \alpha\rho_{1S}\sigma + \alpha v\frac{d\rho_{1S}}{dy} = 0, \quad D\rho_2 + \alpha\rho_{2S}\sigma + \alpha v\frac{d\rho_{2S}}{dy} = 0, \quad D\rho_S u + \rho_S\alpha v\frac{dU_S}{dy} = -i\alpha p,$$

$$D\rho_S\alpha v = -\frac{dp}{dy}, \quad D\rho_{1S}\theta + \alpha v\rho_{1S}T_S' + \alpha\sigma\frac{1}{c_{Va}}\frac{\rho_{1S}}{\rho_S} = 0,$$

$$D\rho_{2S}\theta + \alpha v\rho_{2S}T_S' + \alpha\sigma\frac{1}{c_{Vm}}\frac{\rho_{2S}}{\rho_S} = -\gamma_v\rho_{2S}\frac{\theta-\theta_v}{\tau}, \quad D\rho_{2S}\theta_v + \alpha v\rho_{2S}T_v' = \rho_{2S}\frac{\theta-\theta_v}{\tau},$$

$$p = \frac{1}{\gamma M_\infty^2}\left[(2\rho_{1S}+\rho_{2S})\theta + (2\rho_1+\rho_2)T_S\right], \quad \rho_S = \rho_{1S}+\rho_{2S}. \tag{3.13}$$

Having excluded from the system (3.13) all dependent variables, except for the perturbation of the transverse velocity v,, we pass to a second-order equation of the form (3.1). In the resulting equation, the coefficient functions have the form

$$\chi = T_S\left(\frac{1+c_S}{1+c_\infty}\right)\left[1-\frac{\gamma M_\infty^2 W^2}{(1+c_S)T_S A}\right], \quad A = \frac{(\tau D+1)\gamma+\gamma_v}{\tau D+1+\gamma_v} \quad S = \frac{1}{\chi}\frac{\gamma_v(T_S'-T_{vS}')}{(\tau D+1)\gamma+\gamma_v}. \tag{3.14}$$

The expressions (3.14) for $W = 0$ are

$$\chi_1 = T_S\frac{1+c_S}{1+c_\infty}, \quad S_1 = \frac{1}{\chi_1}\frac{\gamma_v(T_S'-T_{vS}')}{\gamma+\gamma_v}.$$

The inviscid instability criterion for a vibrationally excited gas with a "frozen" dissociation-recombination reaction is given by formula

$$\frac{d}{dy}\left(\frac{1}{T_S}\frac{1+c_\infty}{1+c_S}\frac{dU_S}{dy}\right) + \max_y\left[\frac{1}{T_S}\frac{1+c_\infty}{1+c_S}\frac{\gamma_v(T_S'-T_{vS}')}{T_S(\gamma+\gamma_v)}\right]\frac{dU_S}{dy} = 0. \tag{3.15}$$

Neglecting vibrational excitation, one can obtain the inviscid instability criterion for a gas with a "frozen" exchange reaction in the form

$$\frac{d}{dy}\left(\frac{1}{T_S}\frac{1+c_\infty}{1+c_S}\frac{dU_S}{dy}\right) = 0. \tag{3.16}$$

The transition from the formulas (3.15), (3.16) to the "generalized" inflection point criterion (3.12) is obvious.

3.3 Dissociating gas without vibrational excitation

In this case, the equation for vibrational energy is excluded from the system (2.3) and $\gamma_v = 0$ is set. Due to the cumbersomeness of the expressions, the coefficient functions in the equation of the form (3.1) for the perturbation of the transverse velocity v are not presented here. For $W = 0$ they have the form

$$\chi_1 = \frac{1}{\rho_S}, \quad S_1 = \Lambda\left[\rho_S\left(\frac{dc_S}{dy}h_a^0 - \frac{de_{iS}}{dy}\right) + N\frac{d\rho_S}{dy}\right], \quad \Lambda = \frac{\rho_S}{\rho_S N - \gamma \mathrm{M}_\infty^2 p_S},$$

$$N = \left(h_a^0 - e_{iSc}\right)(1 + c_S) + \left[\left(h_a^0 - e_{iSc}\right)(1 + c_S) + e_{iST}T_S\right]\frac{\dot{w}_c(1 + c_S) - \dot{w}_\rho \rho_S}{\dot{w}_c(1 + c_S) - \dot{w}_T T_S}. \tag{3.17}$$

The inviscid criterion is given by the expression

$$\frac{d}{dy}\left(\rho_S\frac{dU_S}{dy}\right) + \max_y\left\{\Lambda\left[\rho_S\left(\frac{dc_S}{dy}h_a^0 - \frac{de_{iS}}{dy}\right) + N\frac{d\rho_S}{dy}\right]\right\}\frac{dU_S}{dy} = 0. \tag{3.18}$$

Disregarding dissociation-recombination, the criterion (3.18) transforms into the classical condition (3.12). Indeed, the fractional expression in (3.17) can be converted to the equivalent expression

$$-\frac{(1 + c_S) + \rho_S\dot{w}_\rho/\dot{w}_c}{(1 + c_S) - T_S\dot{w}_T/\dot{w}_c}.$$

This expression turns into -1 when passing to an ideal gas. Accordingly, the content of the square brackets in (3.18) goes into the expression

$$-\rho_S\frac{dT_S}{dy} - T_S\frac{d\rho_S}{dy} = -\frac{d}{dy}\left(\rho_S T_S\right).$$

This expression is vanished due to (3.9). This gives the desired transition. Similarly, it is shown that in the case of a "frozen" reaction, the criterion (3.18) passes into the corresponding criterion (3.16).

4 Numerical calculations

For a preliminary assessment of the significance of the obtained criteria in identifying the most growing inviscid modes, we chose the mode of hypersonic flight in the undisturbed terrestrial atmosphere. The boundary layer on the plate was considered. As boundary conditions at the upper boundary of the boundary layer, we used the flow parameters behind an oblique shock wave on the head part in the form of a semi-wedge with an angle $\alpha = 20°$, flying with a Mach number $\mathrm{M}_0 = 15$ at an altitude of $h = 30$ km, where the temperature is $T_0 \simeq 227°$K and the pressure is $p_0 \simeq 1197$ Pa. The values of the flow parameters behind the oblique shock wave were obtained based on the formulas of the oblique shock theory [8]: $\mathrm{M}_\infty = 6.337$, $T_\infty = T_{v\infty} = 1986.3°$K, $p_\infty = 57688.2$ Pa.

In the calculations, all physical characteristics of the gas were taken from the data for nitrogen. The profiles of hydrodynamic quantities in the carrier flow were calculated on the basis of locally self-similar equations [5]. At the upper boundary of the boundary layer, the dimensionless boundary conditions had the form

$$U_S(\delta) = 1, \quad T_S(\delta) = T_{vS}(\delta) = 1, \quad c_S(\delta) = c_\infty = 0.01.$$

The boundary conditions for an adiabatic absolutely non-catalytic wall were set on the surface of the plate [7]

$$U_S(0) = 0, \quad T_S'(0) = 0, \quad T_{vS}(0) = T(0).$$

Chosen condition for the vibrational temperature is due to the fact that at hypersonic Mach numbers the temperature of the adiabatic wall is sufficient to excite the vibrational degrees of freedom.

Table 1: Wave numbers $\alpha_k^{(c)}$ and perturbation frequencies $\omega_k^{(c)}$ of the first four inviscid Mack modes.

Mode I		Mode II		Mode III		Mode IV	
			Criterion (3.12)				
$\alpha_k^{(c)}$	$\omega_k^{(c)} \times 10^3$	$\alpha_k^{(c)}$	$\omega_k^{(c)} \times 10^3$	$\alpha_k^{(c)}$	$\omega_k^{(c)} \times 10^3$	$\alpha_k^{(c)}$	$\omega_k^{(c)} \times 10^3$
0.0959	0.9452	0.2726	3.8041	0.3233	0.8199	0.5548	-0.3999
			Criterion (3.11)				
0.0957	0.8309	0.2721	3.3474	0.3226	0.7209	0.5537	-0.3515
			Criterion (3.15)				
0.0843	0.7329	0.2392	2.9424	0.2849	0.6366	0.4878	-0.3104
			Criterion (3.18)				
0.0821	0.7136	0.2329	2.8641	0.2773	0.6198	0.4748	-0.3022

Using the obtained distributions, based on the formulas for the criteria (3.12), (3.11), (3.15), (3.18) were the coordinates of the critical layer y_c are calculated, where the phase velocities of the perturbation are equal to the velocity of the carrier flow $V_r = U_{Sc}$. Then, for each value of the phase velocity at $M_\infty = 6.337$, based on the equation (3.1) with homogeneous boundary conditions (3.2), real wave numbers $\alpha_k^{(c)}$ for first four the inviscid Mack modes [3], where $k = \mathrm{I}, \ldots, \mathrm{IV}$, and the superscript fixes the criterion by which they were calculated. Finally, for each wave number $\alpha_k^{(c)}$, we solved the spectral problem (3.1), (3.2), whose eigenvalues are the complex phase velocities $V_k^{(c)} = V_{rk}^{(c)} + iV_{ik}^{(c)}$, and the perturbation frequencies were calculated $\omega_k^{(c)} = \alpha_k^{(c)} V_{ik}^{(c)}$. All spectral problems were solved by the "shooting" method. To do this, the problem (3.1), (3.2) was replaced by a normal system of first-order equations with homogeneous boundary conditions. The system thus obtained was integrated numerically using the fourth-order Runge — Kutta procedure on the intervals $y \in (0,\, 0.5\delta)$ and $y \in (0.5\delta,\, \delta)$ with step $\Delta y = 10^{-3}$. The "aiming" point was the middle of the boundary layer $y = 0.5\delta$, where it was required that the calculated "left" and "right" at the point $y = 0.5\delta$ the solution values coincided up to 10^{-6}. The calculated data for first four the inviscid Mack modes are summarized in Table 1. As follows from Table 1, modes I and II are growing for all the introduced criteria. This is important from the point of view that, in contrast to the classical criterion (3.12), the obtained conditions are only necessary according to the inference logic and do not formally guarantee the growth of perturbations allocated on their basis. All criteria define mode II as the most growing, which also confirms their physicality. The results calculated on the same profiles of hydrodynamic variables, taking into account joint vibrational excitation and dissociation-recombination reactions, are relative. They make it possible to estimate the error associated with using the "generalized" inflection point criterion (3.12) under non-ideal gas conditions. Comparing the data obtained by the criteria (3.11) and (3.12), one can see that, under the calculated conditions, vibrational excitation has almost no effect on the wave numbers of inviscid growing modes. This conclusion is indirectly confirmed by comparing the corresponding results obtained on the basis of the criteria (3.15) and (3.18). This suggests that if the vibrational excitation, far from the onset of dissociation, is the only deviation from an ideal gas, we can restrict ourselves to using the "generalized" inflection point condition, as was done in [9]. The relative deviation of the wave numbers of modes I and II, calculated by the criterion (3.12) for an ideal gas and the criterion (3.18) for a dissociating gas without vibrational excitation, is about 14.5 %. At the same time, the relative deviation of imaginary frequencies (growth increments) lies within 24 % — 25 %, which is significant. In this case, the criterion (3.18) gives smaller growth increments, which directly corresponds to the damping effect of the dissociation process noted in [7]. The analysis performed shows that the obtained criteria, in particular, (3.18) for a dissociating gas, allow us to take into account the influence of the real properties of the gas on the characteristics of growing inviscid disturbances, and through them on the results of calculations of the stability of the corresponding flows.

5 Conclusion

Necessary conditions for the existence of growing (neutral) inviscid disturbances were obtained for a plane flow of a vibrationally excited dissociating diatomic gas. These conditions generalize the inviscid Rayleigh criterion, which determines the presence of a "generalized" inflection point on the velocity profile of an unperturbed flow. The equations for the amplitudes of sinusoidal disturbances were used as initial ones. The derivation of the corresponding dependences is a natural generalization of the well-known calculations for obtaining the condition of a "generalized" inflection point in a compressible gas. Criteria are obtained for a vibrationally excited one-component gas as the initial stage of thermal dissociation, as well as for a gas with a single dissociation-recombination reaction. The case of a binary molecular-atomic mixture with a vibrationally excited molecular component and a "frozen" gas-phase dissociation-recombination reaction is considered as an intermediate one. It is shown that all relations, when vibrational excitation and dissociation are neglected, go over into the classical condition of a "generalized" inflection point. The performed comparative numerical calculations for the conditions of developed dissociation showed that the use of the "generalized" inflection point criterion does not take into account the specifics of the process. The wave numbers and phase velocities of I and II inviscid modes calculated on its basis may differ significantly from the results obtained using the new necessary condition.

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On Conservative Finite-Difference Schemes for the One-Dimensional MHD Equations in Cylindrical Geometry Possessing Additional Conservation Laws

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Abstract

Based on recent results in the construction of conservative finite-difference schemes, a new scheme for one-dimensional isentropic flows of polytropic gas in cylindrical geometry in the presence of a radial magnetic field is presented. The proposed scheme adds to the list of recently constructed schemes, and completes the consideration of conservative schemes for the case of isentropic flows in cylindrical geometry.

Keywords: Magnetohydrodynamics, conservation laws, numerical scheme.

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1 Introduction

Magnetohydrodynamics (MHD) describes flows of electrically conductive fluids and is important in a wide range of natural phenomena and technological applications, including astrophysics, fusion energy, and geophysics. The MHD equations are nonlinear, and even obtaining particular solutions for them encounters significant difficulties. In this regard, it is necessary to use various methods of numerical simulation. The most common numerical simulation methods in fluid dynamics are based on the finite-difference method. Classical results on modeling one-dimensional MHD flows for the case of finite conductivity were obtained by Samarskiy and Popov in [1, 2], where they used mass Lagrangian coordinates to simplify the formulation of boundary value problems in plasma physics. Following their approach, here we also consider the equations in mass Lagrangian coordinates.

In recent studies [3, 4], the Samarskiy–Popov schemes have been analyzed from the perspective of Group Analysis [5–7]. This included examining the symmetries they admit and the difference conservation laws they possess. A specific set of additional conservation laws in the

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case of infinite conductivity is determined by the form of the entropy and magnetic flux and is established as a result of group classification [8, 9].

It has been found that the results of [1, 2] can be extended to the case of infinite conductivity, however, there are certain limitations. Specifically, it is extremely difficult to construct difference schemes that have an extended set of conservation laws in this case. Instead of simply extending existing schemes, a more productive approach is to construct conservative schemes by approximating additional conservation laws that arise for entropy or magnetic fluxes of a specific form. During this process, it is common to find that some terms included in the difference scheme are not fully determined, meaning they can be chosen as various approximations for the corresponding terms of the differential equations. By refining the form of these approximations, one can obtain schemes with a wider set of conservation laws.

The results of group classifications [8, 9] and set of conservation laws for the equations vary significantly depending on the presence of the longitudinal (radial) component of the magnetic field. The present study examines the case of cylindrical spatial symmetry (cylindrical geometry) and isentropic flows in the presence of a radial magnetic field H^r. In [4], the case when the radial component of the magnetic field is absent ($H^r = 0$) was studied in detail, and difference schemes were constructed that possess a number of conservation laws, including the conservation of entropy, mass, and magnetic fluxes. The present study focuses on the more complex case of $H^r \neq 0$.

2 The Studied Equations

The following system in mass Lagrangian coordinates is considered [2, 4].

$$\rho_t = -\rho^2 (ru)_s, \tag{2.1a}$$

$$u_t - \frac{v^2}{r} = -r p_s - \frac{1}{2r}\left(r^2 (H^\theta)^2\right)_s - \frac{r}{2}\left((H^z)^2\right)_s, \tag{2.1b}$$

$$v_t + \frac{uv}{r} = H^r \left(rH^\theta\right)_s, \qquad \theta_t = \frac{v}{r}, \tag{2.1c}$$

$$w_t = rH^r H^z_s, \qquad z_t = w, \tag{2.1d}$$

$$p_t = -\gamma\rho p (ru)_s, \tag{2.1e}$$

$$H^\theta_t = r\rho((vH^r)_s - H^\theta u_s), \tag{2.1f}$$

$$H^z_t = \rho((rwH^r)_s - H^z (ru)_s), \tag{2.1g}$$

$$r_t = u, \qquad r_s = \frac{1}{r\rho}, \tag{2.1h}$$

where

$$H^r = \frac{A}{r}, \qquad A = \text{const} \neq 0,$$

t is time, s is Lagrangian mass coordinate, ρ is density, p is pressure, $\mathbf{u} = (u, v, w)$ is the velocity vector, $\mathbf{H} = (H^r, H^\theta, H^z)$ is the magnetic field vector, ε is internal energy, r, θ, z are cylindrical (Eulerian) spatial coordinates.

The energy evolution equation (2.1e) for the polytropic gas with the standard state equation

$$\varepsilon = \frac{1}{\gamma - 1}\frac{p}{\rho}, \tag{2.2}$$

taking into account (2.1a), is simplified to

$$\left(\frac{p}{\rho^\gamma}\right)_t = S_t = 0. \tag{2.3}$$

The latter means the preservation of the entropy $S = S(s)$ along the paths of motion.

Further we restrict ourselves to the isentropic case $S = S_0 = \text{const}$. In this case the conservation laws possessed by system (2.1) are the following [4].

- mass

$$\left(\frac{1}{\rho}\right)_t - (ru)_s = 0; \tag{2.4}$$

- momentum along z-axis

$$(w)_t - (rH^r H^z)_s = 0; \tag{2.5}$$

- motion of the center of mass along z-axis

$$(tw - z)_t - (rtH^r H^z)_s = 0; \tag{2.6}$$

- angular momentum in (r, θ)-plane

$$(rv)_t - \left(r^2 H^r H^\theta\right)_s = 0; \tag{2.7}$$

- magnetic fluxes

$$\left(\frac{H^\theta}{r\rho}\right)_t - (vH^r)_s = 0, \tag{2.8}$$

$$\left(\frac{H^z}{\rho}\right)_t - (rwH^r)_s = 0; \tag{2.9}$$

- energy

$$\left\{\frac{1}{\gamma - 1}\frac{p}{\rho} + \frac{1}{2}(u^2 + v^2 + w^2) + \frac{(H^\theta)^2 + (H^z)^2}{2\rho}\right\}_t + \left\{ru\left(p + \frac{(H^\theta)^2 + (H^z)^2}{2}\right) - rH^r(vH^\theta + wH^z)\right\}_s = 0; \tag{2.10}$$

- entropy along trajectories of motion

$$\left(\frac{p}{\rho^\gamma}\right)_t = 0; \tag{2.11}$$

- the additional conservation law (for $S = S_0 = \text{const}$)

$$\left(\frac{u}{r\rho} + \frac{vH^\theta + wH^z}{A\rho}\right)_t - \left(\frac{1}{2}(u^2 + v^2 + w^2) - \frac{\gamma S_0}{\gamma - 1}\rho^{\gamma - 1}\right)_s = 0. \tag{2.12}$$

In the next section, we construct finite-difference schemes possessing finite-difference analogues of the above conservation laws.

3 Construction of Conservative Finite-Difference Schemes

Following [3, 4], we start with an approximation for the additional conservation law (2.12). By approximating this conservation law, we are able to derive the equations of the scheme in a more or less general form, leaving some terms unspecified whenever possible. Next, we further refine the form of the unspecified terms of the scheme by considering approximations for the remaining conservation laws.

Recall [5] that any finite-difference conservation law of a system of N difference equations

$$F^j = 0, \qquad j = 1, 2, \ldots, N,$$

can be represented as a sum

$$\sum_{j=1}^{N} \Lambda_j F^j = 0,$$

where the quantities Λ_j, $j = 1, \ldots, N$, are referred to as finite-difference conservation law multipliers, analogous to differential equations.

In terms of equations (2.1) and conservation law multipliers, (2.12) can be written as

$$\begin{aligned}
&\frac{u}{\rho r^2}(r_t - u) + \left(\frac{v^2 - u^2}{r} - \frac{uvH^\theta}{A}\right)\left(r_s - \frac{1}{r\rho}\right) + \left(\frac{vH^\theta + wH^z}{A\rho^2} + \frac{u}{r\rho^2}\right)\left(\rho_t + \rho^2 (ru)_s\right)\\
&\quad - \frac{1}{r\rho}\left(u_t - \frac{v^2}{r} + rH^z H^z_s + H^\theta (rH^\theta)_s + \frac{\gamma S_0}{\gamma - 1} r\rho(\rho^{\gamma-1})_s\right)\\
&\quad - \frac{H^\theta}{rA}\left(v_t + \frac{uv}{r} - \frac{A}{r}(rH^\theta)_s\right) - \frac{H^z}{rA}(w_t - AH^z_s)\\
&\quad - \frac{v}{rA}\left(H^\theta_t - r\rho\left(\frac{vA}{r}\right)_s + r\rho H^\theta u_s\right) - \frac{w}{rA}\left(H^z - \rho\left(Aw_s - H^z (ru)_s\right)\right) = 0. \qquad (3.1)
\end{aligned}$$

The scheme is constructed on the basis of the following finite-difference analogue of (3.1)

$$\begin{aligned}
&\frac{u^+_*}{\rho r\hat{r}}(r_t - u_+) + \left(\frac{v_*^2}{r_{(1)}} - \frac{u^+ u^+_*}{\hat{r}} - \frac{uv_* H^\theta I_{(2)}}{A}\right)\left(\hat{r}_s - \frac{1}{r\rho}\right) + \left(\frac{v_* H^\theta + w_* H^z}{A\rho\hat{\rho}} + \frac{u^+_*}{\hat{r}\rho\hat{\rho}}\right)\left(\rho_t + \rho\hat{\rho}(\hat{r}u)_s\right)\\
&\quad - \frac{1}{\hat{r}\hat{\rho}}\left((u^+_*)_t - \frac{\hat{r}\hat{\rho}}{r\rho}\frac{v_*^2}{r_{(1)}} + \hat{r}\hat{H}^\theta \Xi + \hat{r}\hat{H}^z (H^z_{(1)})_s + \frac{\gamma S_0}{\gamma - 1}\hat{r}\hat{\rho}(\rho^{\gamma-1})_s\right)\\
&\quad - \frac{\hat{H}^\theta}{A\hat{\rho}}\left(v^*_t + \frac{\hat{\rho}}{\rho}\frac{H^\theta}{\hat{H}^\theta}\frac{uv_*}{r_{(2)}} - A\Xi\right) - \frac{\hat{H}^z}{A\hat{\rho}}\left(w^*_t - A(H^z_{(1)})_s\right)\\
&\quad - \frac{v_*}{A\hat{\rho}}\left(H^\theta_t + \hat{r}_+\hat{\rho}\left(H^\theta u_s + \frac{Av_*\hat{r}_s}{\hat{r}_+ r_{(1)}} - \frac{Av_{\bar{s}}}{\hat{r}_+}\right)\right)\\
&\quad - \frac{w_*}{A\hat{\rho}}\left(H^z_t + \hat{\rho}\left(H^z (\hat{r}u)_s - Aw_{\bar{s}}\right)\right)\\
&= -\left(\frac{u^+_*}{r\rho} + \frac{v_* H^\theta + w_* H^z}{A\rho}\right)_t + \left(\frac{u^2 + v_-^2 + w_-^2}{2} - \frac{\gamma S_0}{\gamma - 1}\rho^{\gamma-1}\right)_s = 0, \qquad (3.2)
\end{aligned}$$

where

$$I_{(2)} = \frac{r(1 - \rho r_{(2)}\hat{r}_s)}{r_{(2)}(1 - \rho r\hat{r}_s)},$$

$r_{(1)}$ and $r_{(2)}$ are some approximations for r, $H^z_{(1)}$ is an approximation for H^z, and Ξ approximates the term $\frac{1}{r}(rH^\theta)_s$. Here and further, we use the standard notation introduced in [2], and f_t and f_s denote the finite-difference derivatives of f with respect to t and s.

Hence, (3.2) determines the family of schemes for system (2.1)

$$\rho_t + \rho\hat{\rho}(\hat{r}u)_s = 0, \tag{3.3a}$$

$$(u_*^+)_t - \frac{\hat{r}\hat{\rho}}{r\rho}\frac{v_*^2}{r_{(1)}} + \hat{r}\hat{H}^\theta\Xi + \hat{r}\hat{H}^z(H^z_{(1)})_s + \frac{\gamma S_0}{\gamma-1}\hat{r}\hat{\rho}(\rho^{\gamma-1})_s = 0, \tag{3.3b}$$

$$v_t^* + \frac{\hat{\rho}}{\rho}\frac{H^\theta}{\hat{H}^\theta}\frac{uv_*}{r_{(2)}} - A\Xi = 0, \tag{3.3c}$$

$$w_t^* - A(H^z_{(1)})_s = 0, \tag{3.3d}$$

$$r_t = u_+, \qquad \hat{r}_s = \frac{1}{r\rho}, \qquad z_t = w_*, \tag{3.3e}$$

$$H_t^\theta + \hat{r}_+\hat{\rho}\left(H^\theta u_s + \frac{Av^*\hat{r}_s}{\hat{r}_+ r_{(1)}} - \frac{Av_{\bar{s}}}{\hat{r}_+}\right) = 0, \tag{3.3f}$$

$$H_t^z + \hat{\rho}\left(H^z(\hat{r}u)_s - Aw_{\bar{s}}\right) = 0, \tag{3.3g}$$

$$\frac{p}{\rho^\gamma} = S_0. \tag{3.3h}$$

Remark 3.1. For any natural value of $\gamma > 1$, one can also derive the evolutionary equation for pressure from (3.3h):

$$p_t = -\rho\hat{p}\left(\hat{r}u\right)_s \sum_{k=0}^{\gamma-1}\left(\frac{\rho}{\hat{\rho}}\right)^k.$$

A more general formula can be established for any rational $\gamma > 1$. See [4] for the details.

Thus, by choosing free parameters (indefinite approximations of some terms of the equations and conservation law multipliers for (3.2)), one arrives at the scheme with the following conservation laws:

- mass

$$\left(\frac{1}{\rho}\right)_t - (\hat{r}u)_s = 0; \tag{3.4}$$

Remark 3.2. By means of the equations of system (3.3) one can also write

$$\left(\frac{1}{\rho r}\right)_t - u_s = 0; \tag{3.5}$$

- magnetic flux along θ-axis

$$\left(\frac{H^\theta}{\rho r}\right)_t - \left(\frac{Av_-}{\hat{r}}\right)_s = 0, \tag{3.6}$$

provided

$$r_{(1)} = \frac{Av_*}{Av - h\hat{r}_+H^\theta u_s}\hat{r}_+ = r + O(h+\tau), \qquad A \neq 0.$$

Remark 3.3. Here, the desired form of the term $r_{(1)}$ has been refined by considering approximations for the conservation law of magnetic flux. In contrast, the use of simpler forms of $r_{(1)}$ results in conservation laws with source terms. For example, given $r_{(1)} = \dfrac{v_*}{v_-}\hat{r}$, one derives

$$\left(\frac{H^\theta}{\rho r_+}\right)_t - \left(\frac{Av_-}{\hat{r}}\right)_s = H^\theta\left(\left(\frac{1}{\rho r_+}\right)_t - u_s\right).$$

- magnetic flux along z-axis
$$\left(\frac{H^z}{\rho}\right)_t - (Aw_-)_s = 0; \tag{3.7}$$

- momentum along z-axis
$$w_t^* - (AH^z_{(1)})_s = 0; \tag{3.8}$$

- motion of the center of mass along z-axis
$$(tw_* - z)_t - (A\hat{t}H^z_{(1)})_s = 0; \tag{3.9}$$

- angular momentum in (r, θ)-plane
$$(v_* r_-)_t - (ArH^\theta)_s = 0, \tag{3.10}$$

  provided
$$r_{(2)} = \frac{\hat{\rho}}{\rho}\frac{H^\theta}{\hat{H}^\theta}\hat{r}_-, \qquad \Xi = \frac{1}{\hat{r}_-}(rH^\theta)_s;$$
  Thus, the desired forms of the terms $r_{(2)}$ and Ξ have been established.

- entropy along trajectories of motion
$$\left(\frac{p}{\rho^\gamma}\right)_t = 0; \tag{3.11}$$

- additional conservation law (3.2)
$$\left(\frac{u_*^+}{r\rho} + \frac{v_* H^\theta + w_* H^z}{A\rho}\right)_t - \left(\frac{u^2 + v_-^2 + w_-^2}{2} - \frac{\gamma S_0}{\gamma - 1}\rho^{\gamma-1}\right)_s = 0.$$

Notice that the conservation laws do not impose any restrictions on the choice of approximation $H^z_{(1)}$, so we still get a *family* of schemes parameterized by this approximation.

While the schemes constructed do not have the conservation of energy property, they do preserve entropy at a single point, as opposed to the preservation of entropy at two points by the schemes presented in [4].

4 Conclusion

A family of conservative finite-difference schemes is constructed for one-dimensional MHD flows in cylindrical geometry in the presence of a radial magnetic field, in the case of constant entropy. These schemes possess finite-difference analogues of most of the conservation laws of the original differential model.

The method used for constructing new conservative schemes involves approximating additional conservation laws and refining indefinite terms. This approach has proven to be effective in a number of specific examples, as seen in previous studies [3, 4]. While it is relatively labor-intensive and relies on certain assumptions and observations, further research could focus on developing an algorithmic approach to this method.

The results obtained complete our study [4] of conservative finite-difference schemes for one-dimensional isentropic MHD flows ($S = S_0 = \text{const}$) for an arbitrary adiabatic exponent in cylindrical geometry.

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Iteratively Split Symmetry Generators of Complex Scalar Ordinary Differential Equations

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Abstract

Complex symmetry methods were developed to enhance the power of Lie symmetry analysis by splitting the variables into their real and imaginary parts. These methods managed to not only solve systems of ordinary differential equations with an inadequate number of symmetries for the purpose, but even those with none. To explore if that success could be extended, iterative application of the complex splitting was investigated. On splitting, these methods produced many operators that did not form a Lie algebra. The dimension of the system normally obtained doubled the original dimension, but could be constrained to include odd dimensions. While the original investigation had been limited to one application of the splitting, here we extend those methods to repeated applications and study how the number of operators and the dimensions develop more generally. It is hoped that this study will either enhance the range of the complex methods further, or give the limitations for such enhancements.

Keywords: Half-integer splitting, Iterative complex splitting, Lie-like operators. **2020 MSC:** Primary 34M30; Secondary 76M60.

1 Introduction

Lie symmetries of ordinary differential equations (ODEs) are widely discussed in the literature (see, e.g. [8, 16]). In [1] it was noted that complex analyticity had not been used by Lie, even though he assumed differentiable complex functions of complex variables, and in the complex domain that automatically implies complex analyticity. This means that the *Cauchy-Riemann equations* (CREs) must hold. The CREs are an extra system of equations. *It is not the system of equations one thought one had as there are additional equations*. Powerful methods of complex symmetry analysis (CSA) were developed, which allowed one to solve systems not only with not enough symmetries, but even with *no* symmetry [12]. Following Ali [4], we call the differential equations (DEs) in which dependent variables are complex functions, (CODEs), and

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those of them in which the independent variable is real, *r-CODEs*. To use analyticity the dependent and independent variables are split into two real dependent and independent variables each. We call the split system of symmetry generators *Lie-like operators*, which are not generally Lie symmetries, as they may not form a Lie algebra. On splitting, a scalar CODE yields a 4-d system of PDEs with associated CREs [1,4], which can be reduced to a 2-d system of ODEs by restricting the independent variable to be real. This procedure can easily be extended for n iterations to yield 2^n-d systems.

We can access odd-dimensional systems of ODEs by inserting an algebraic constraint in the iteratively split systems. Thus, after double splitting one gets 3-d systems, after the next level 5-d systems, and so on [11]. We call such a splitting *half-integer splitting*. We will apply the half-integer splitting on a scalar "Emden-Fowler" equation. At any stage of the iterative splitting according to the required odd-dimensional system, we can consider one of the dependent variables as pure imaginary and the others as complex. The processes of splitting and applying algebraic constraints do not commute, e.g. the result of first splitting a doubly split system and then applying an algebraic constraint, or first applying the algebraic constraint and then splitting are different. As such, the procedure is not unique. Notice that by splitting a 3-d system we can get a 6-d system and so on. Thus we can get systems of *all* dimensions in principle, and are not limited to 2^n-d systems.

The plan of the paper is as follows. In section 2, the ideas of iterative splitting of scalar 2^{nd} order r-CODE and theorems are given. In section 3, the procedure for obtaining higher odd-dimensional systems of ODEs by half-integer splitting is presented. An example is constructed for several odd-dimensional systems. The final section, 4, consists of the conclusion and discussion.

2 Iterative Splitting of a Scalar CODE

Though the original paper [13] presented the first iterative step, i.e. *double splitting*, we will repeat that to obtain the number of Lie-like operators and compare it with the number of Lie symmetries. The equations become cumbersome beyond the first iteration and are not displayed in detail here, merely quoting the values for the third and then the n^{th}. We use the procedure outlined above for the the general 2^{nd} order scalar ODE,

$$v''(s) = h(s, v(s), v'(s)), \tag{2.1}$$

and put $v = f_1 + \imath f_2$ to obtain the 2-d system of ODEs

$$f_1'' = h_1(s, f_1, f_2, f_1', f_2'), \quad f_2'' = h_2(s, f_1, f_2, f_1', f_2'), \tag{2.2}$$

with the CR-conditions

$$\begin{aligned} h_{1,f_1} &= h_{2,f_2}, \quad h_{1,f_2} = -h_{2,f_1}; \\ h_{1,f_1'} &= h_{2,f_2'}, \quad h_{1,f_2'} = -h_{2,f_1'}. \end{aligned} \tag{2.3}$$

For the 2^{nd} splitting write $f_j = f_{j_r} + \imath f_{j_i}$, $h_j = h_{j_r}(s, \mathbf{a}, \mathbf{a}') + \imath h_{j_i}(s, \mathbf{a}, \mathbf{a}')$; $(j = 1, 2)$, where $\mathbf{a} = (f_{1_r}, f_{1_i}, f_{2_r}, f_{2_i})$ and $\mathbf{a}' = (f_{1_r}', f_{1_i}', f_{2_r}', f_{2_i}')$. Splitting yields the 4-d system

$$f_{j_r}'' = h_{j_r}(s, \mathbf{a}, \mathbf{a}'), \; f_{j_i}'' \quad = h_{j_i}(s, \mathbf{a}, \mathbf{a}'), \tag{2.4}$$

with the CR-conditions

$$
\begin{aligned}
&h_{1_{r},f_{1_r}} + h_{1_{i},f_{1_i}} = h_{2_{r},f_{2_r}} + h_{2_{i},f_{2_i}}, && h_{1_{r},f_{1_i}} - h_{1_{i},f_{1_r}} = h_{2_{r},f_{2_i}} - h_{2_{i},f_{2_r}},\\
&h_{1_{r},f_{2_r}} + h_{1_{i},f_{2_i}} = -h_{2_{r},f_{1_r}} - h_{2_{i},f_{1_i}}, && h_{1_{i},f_{2_r}} - h_{1_{r},f_{2_i}} = h_{2_{i},f_{1_r}} + h_{2_{r},f_{1_i}},\\
&h_{1_{r},f'_{1_r}} + h_{1_{i},f'_{1_i}} = h_{2_{r},f'_{2_r}} + h_{2_{i},f'_{2_i}}, && h_{1_{r},f,_{1_i}} - h_{1_{i},f'_{1_r}} = h_{2_{r},f'_{2_i}} - h_{2_{i},f'_{2_r}},\\
&h_{1_{r},f'_{2_r}} + h_{1_{i},f'_{2_i}} = -h_{2_{r},f'_{1_r}} - h_{2_{i},f'_{1_i}}, && h_{1_{i},f'_{2_r}} - h_{1_{r},f'_{2_i}} = h_{2_{i},f'_{1_r}} + h_{2_{r},f'_{1_i}}.
\end{aligned}
\tag{2.5}
$$

This can be continued to the n^{th} iteration to obtain a 2^n-d system of ODEs.

We now present five examples for iterative splitting to all iterations, but will give the algebras associated with the split systems up to only the 2^{nd} iteration.

Example 2.1. As a first example we take the 2^{nd} order r-CODE

$$v'' = h(s), \tag{2.6}$$

with the symmetry generators [16]

$$\mathbf{Z}_1 = \frac{\partial}{\partial v}, \quad \mathbf{Z}_2 = s\frac{\partial}{\partial v}. \tag{2.7}$$

The first splitting yields the 2-d system of ODEs:

$$f_1'' = h(s), \quad f_2'' \quad = 0, \tag{2.8}$$

with symmetries

$$
\begin{aligned}
&\mathbf{Z}_1 = \frac{\partial}{\partial f_1}, && \mathbf{Z}_2 = \frac{\partial}{\partial f_2}, && \mathbf{Z}_3 = s\frac{\partial}{\partial f_1},\\
&\mathbf{Z}_4 = s\frac{\partial}{\partial f_2}, && \mathbf{Z}_5 = f_2\frac{\partial}{\partial f_1}, && \mathbf{Z}_6 = f_2\frac{\partial}{\partial f_2}.
\end{aligned}
\tag{2.9}
$$

They form a 6-d algebra $\mathfrak{h}_6$ with non-zero commutators $[\mathbf{Z}_2,\mathbf{Z}_5] = \mathbf{Z}_1$, $[\mathbf{Z}_2,\mathbf{Z}_6] = \mathbf{Z}_2$, $[\mathbf{Z}_4,\mathbf{Z}_5] = \mathbf{Z}_3$, $[\mathbf{Z}_4,\mathbf{Z}_6] = \mathbf{Z}_4$, $[\mathbf{Z}_5,\mathbf{Z}_6] = -\mathbf{Z}_5$. The largest normal subalgebra of $\mathfrak{h}_6$ is $\mathfrak{g}_4 = \langle \mathbf{Z}_c \rangle$ (c=1,...,4) and $\mathfrak{h}_6/\mathfrak{g}_4 = \mathfrak{I}_2$, where $\mathfrak{I}_2 = \langle \mathbf{Z}_5, \mathbf{Z}_6 \rangle$ is the dilation algebra. Putting $\mathbf{Z_j} = \mathbf{X_j} + \iota \mathbf{Y_j}$, the split system of symmetry generators yields the four Lie-like operators

$$\mathbf{X}_1 = \frac{\partial}{\partial f_1}, \quad \mathbf{Y}_1 = \frac{\partial}{\partial f_2}, \quad \mathbf{X}_2 = s\frac{\partial}{\partial f_1}, \quad \mathbf{Y}_2 = s\frac{\partial}{\partial f_2}, \tag{2.10}$$

which are also symmetries of the split system and form an abelian algebra.

The next splitting gives

$$f_{1_r}'' = h(s), \qquad f_{1_i}'' = 0, \qquad f_{2_r}'' = 0, \qquad f_{2_i}'' = 0, \tag{2.11}$$

with symmetries

$$\begin{aligned}
&\mathbf{Z}_1=\frac{\partial}{\partial f_{1_r}}, && \mathbf{Z}_2=\frac{\partial}{\partial f_{1_i}}, && \mathbf{Z}_3=\frac{\partial}{\partial f_{2_r}}, && \mathbf{Z}_4=\frac{\partial}{\partial f_{2_i}},\\
&\mathbf{Z}_5=s\frac{\partial}{\partial f_{1_r}}, && \mathbf{Z}_6=s\frac{\partial}{\partial f_{1_i}}, && \mathbf{Z}_7=s\frac{\partial}{\partial f_{2_r}}, && \mathbf{Z}_8=s\frac{\partial}{\partial f_{2_i}},\\
&\mathbf{Z}_9=f_{1_i}\frac{\partial}{\partial f_{1_r}}, && \mathbf{Z}_{10}=f_{2_r}\frac{\partial}{\partial f_{1_r}}, && \mathbf{Z}_{11}=f_{2_i}\frac{\partial}{\partial f_{1_r}}, && \mathbf{Z}_{12}=f_{1_i}\frac{\partial}{\partial f_{1_i}},\\
&\mathbf{Z}_{13}=f_{1_i}\frac{\partial}{\partial f_{2_r}}, && \mathbf{Z}_{14}=f_{1_i}\frac{\partial}{\partial f_{2_i}}, && \mathbf{Z}_{15}=f_{2_r}\frac{\partial}{\partial f_{1_i}}, && \mathbf{Z}_{16}=f_{2_i}\frac{\partial}{\partial f_{1_i}},\\
&\mathbf{Z}_{17}=f_{2_r}\frac{\partial}{\partial f_{2_r}}, && \mathbf{Z}_{18}=f_{2_r}\frac{\partial}{\partial f_{2_i}}, && \mathbf{Z}_{19}=f_{2_i}\frac{\partial}{\partial f_{2_r}}, && \mathbf{Z}_{20}=f_{2_i}\frac{\partial}{\partial f_{2_i}}.
\end{aligned} \tag{2.12}$$

Writing $\mathbf{X}_j=\mathbf{X}_{j_1}+\imath\mathbf{X}_{j_2}$, $\mathbf{Y}_j=\mathbf{Y}_{j_1}+\imath\mathbf{Y}_{j_2}$, gives the eight Lie-like operators

$$\begin{aligned}
&\mathbf{X}_{1_1}=\frac{\partial}{\partial f_{1_r}}, && \mathbf{X}_{1_2}=\frac{\partial}{\partial f_{1_i}}, && \mathbf{Y}_{1_1}=\frac{\partial}{\partial f_{2_r}}, && \mathbf{Y}_{1_2}=\frac{\partial}{\partial f_{2_i}},\\
&\mathbf{X}_{2_1}=s\frac{\partial}{\partial f_{1_r}}, && \mathbf{X}_{2_2}=s\frac{\partial}{\partial f_{1_i}}, && \mathbf{Y}_{2_1}=s\frac{\partial}{\partial f_{2_r}}, && \mathbf{Y}_{2_2}=s\frac{\partial}{\partial f_{2_i}},
\end{aligned} \tag{2.13}$$

which are also the symmetries of the system (2.11) and form an abelian algebra.

The 3$^{\text{rd}}$ splitting gives an 8-d system of ODEs, with 72 symmetries and 16 Lie-like operators. The sequences of numbers for the iterative procedure up to the n^{th} splitting are given in Table 1.

n	d_n	l_n	$m_n=2d_n-l_n$	L_n	$e_n=L_n-l_n$
1	2	4	0	6	2
2	4	8	0	20	12
3	8	16	0	72	56
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	2^n	$2^{(n+1)}$	0	$2^n(2^n+1)$	$2^n(2^n-1)$

Table 1: Here n is the number of splitting iterations, d_n the dimension of the split system, l_n the number of Lie-like operators, m_n the number of missing Lie-like operators, L_n the number of symmetries of the corresponding split system, and e_n the number of extra symmetries apart from the Lie-like operators.

The general result can be presented as:

Theorem 2.2. *For (2.6) with symmetries given by (2.7), the iteratively split system is of 2^n-dimension, has $2^{(n+1)}$ Lie-like operators with no missing operator, $2^n(2^n+1)$ symmetries and $2^n(2^n-1)$ extra symmetries.*

Proof. Let the theorem hold true for some n. Obviously, d_n doubles on each splitting, so it will be true for $(n+1)$. Since there is no derivative with respect to the independent variables among the symmetry generators, l_n also doubles at each splitting. Hence it holds for $l_{(n+1)}$. (Notice that this would not hold if there was no explicit dependence of the equation on the independent variable.) Since, by definition, $l_n=2d_n$, there are no missing operators, i.e. $m_{(n+1)}$ remains zero. It remains to prove if the algebra of split operators closes for the n^{th} iteration it will also hold for the $(n+1)^{\text{st}}$ iteration. Since the new split operators are only of the type of a derivative with respect to the dependent variable and the independent variable times a derivative with respect to a dependent variable, it is obvious that they must all commute. Hence for the next iteration the algebra will be closed. Hence the algebra closes for all n. □

Example 2.3. As a second example we take the 2^{nd} order r-CODE

$$v'' = h(v), \tag{2.14}$$

with symmetries

$$\mathbf{Z}_1 = \frac{\partial}{\partial s}, \quad \mathbf{Z}_2 = v\frac{\partial}{\partial s}. \tag{2.15}$$

The first splitting yields the 2-d system of ODEs:

$$f_1'' = h_1(f_1, f_2), \qquad f_2'' = h_2(f_1, f_2), \tag{2.16}$$

with symmetries

$$\mathbf{Z}_1 = \frac{\partial}{\partial s}, \qquad \mathbf{Z}_2 = f_1\frac{\partial}{\partial s}, \qquad \mathbf{Z}_3 = f_2\frac{\partial}{\partial s}, \tag{2.17}$$

which form a 3-d abelian algebra $\mathfrak{h}_3$, and the split system remains unchanged.

The next splitting gives a 4-d system of ODEs

$$f_{j_r}'' = h_{j_r}(f_{1_r}, f_{1_i}, f_{2_r}, f_{2_i}), \; f_{j_i}'' \quad = h_{j_i}(f_{1_r}, f_{1_i}, f_{2_r}, f_{2_i}), \tag{2.18}$$

with symmetries

$$\mathbf{Z}_1 = \frac{\partial}{\partial s}, \quad \mathbf{Z}_2 = f_{1_r}\frac{\partial}{\partial s}, \qquad \mathbf{Z}_3 = f_{1_i}\frac{\partial}{\partial s}, \quad \mathbf{Z}_4 = f_{2_r}\frac{\partial}{\partial s}, \qquad \mathbf{Z}_5 = f_{2_i}\frac{\partial}{\partial s}, \tag{2.19}$$

that form a 5-d abelian algebra, $\mathfrak{h}_5$, and again the split system remains unchanged.

The 3^{rd} splitting gives an 8-d system of ODEs, with 9 symmetries and identical Lie-like operators. Looking at the sequences of numbers the iterative procedure up to the n^{th} splitting should be as given in Table 2.

n	d_n	l_n	$m_n = 2d_n - l_n$	L_n	$e_n = L_n - l_n$
1	2	3	1	3	0
2	4	5	3	5	0
3	8	9	7	9	0
⋮	⋮	⋮	⋮	⋮	⋮
n	2^n	2^n+1	2^n-1	2^n+1	0

Table 2: In this case missing Lie-like operators are non-zero with zero extra symmetries.

The general result can be presented as:

Theorem 2.4. *For (2.14) with symmetries given by (2.15), the iteratively split system, has 2^n-dimension, 2^n+1 Lie-like operators with 2^n-1 missing operators, 2^n+1 symmetries and no extra symmetry.*

Proof. The result for the dimension of the system is the same as before. Let the theorem hold true for some n. Since there is a derivative with respect to the independent variables among the symmetry generators and in one generator the coefficient is a dependent variable, which doubles after splitting, while the other one remains the same throughout the splitting, $l_n = d_n + 1$ at each splitting. Hence, it holds for $l_{(n+1)}$. There are missing operators m_n at each splitting because one operator does not split into two. Also $m_{(n+1)}$ remains non-zero. It remains to prove that the algebra of the split operators closes all iterations. Because, as before, the operators only involve derivatives with respect to the independent variable, so the algebra always closes. □

Example 2.5. As a third example we take the 2$^{\text{nd}}$ order r-CODE for nonzero function of one variable

$$v'' = h(v'), \tag{2.20}$$

with symmetries

$$\mathbf{Z}_1 = \frac{\partial}{\partial s}, \quad \mathbf{Z}_2 = \frac{\partial}{\partial v}. \tag{2.21}$$

The first splitting yields the 2-d system of ODEs:

$$f_1'' = h_1(f_1', f_2'), \qquad f_2'' = h_2(f_1', f_2'), \tag{2.22}$$

with symmetries

$$\begin{aligned} &\mathbf{Z}_1 = \frac{\partial}{\partial s}, \qquad \mathbf{Z}_2 = \frac{\partial}{\partial f_1}, \qquad \mathbf{Z}_3 = \frac{\partial}{\partial f_2}, \\ &\mathbf{Z}_4 = s\frac{\partial}{\partial s}, \qquad \mathbf{Z}_5 = f_1\frac{\partial}{\partial s}, \qquad \mathbf{Z}_6 = f_2\frac{\partial}{\partial s}. \end{aligned} \tag{2.23}$$

They form a 6-d algebra $\mathfrak{h}_6$ with the only non-zero commutators $[\mathbf{Z}_1,\mathbf{Z}_4] = \mathbf{Z}_1$, $[\mathbf{Z}_2,\mathbf{Z}_5] = \mathbf{Z}_1$, $[\mathbf{Z}_3,\mathbf{Z}_6] = \mathbf{Z}_1$. The maximal normal subalgebra of $\mathfrak{h}_6$ is $\mathfrak{g}_3$=$\langle \mathbf{Z}_c \rangle$, where $c = 1,2,3$ and $\mathfrak{h}_6/\mathfrak{g}_3$ =$\langle \mathbf{Z}_4,\mathbf{Z}_5,\mathbf{Z}_6 \rangle$ which is an abelian algebra. Thus $\mathfrak{h}_6 = \mathfrak{g}_3 \otimes_s \mathfrak{I}_3$. The split system of symmetry generators yields the three Lie-like operators

$$\mathbf{X}_1 = \frac{\partial}{\partial s}, \quad \mathbf{X}_2 = \frac{\partial}{\partial f_1}, \quad \mathbf{Y}_2 = \frac{\partial}{\partial f_2}, \tag{2.24}$$

which are also symmetries of the split system and form an abelian algebra.

The next splitting gives

$$f_{j_r}'' = h_{j_r}(f_{1_r}', f_{1_i}', f_{2_r}', f_{2_i}'), \; f_{j_i}'' \quad = h_{j_i}(f_{1_r}', f_{1_i}', f_{2_r}', f_{2_i}'), \tag{2.25}$$

with symmetries

$$\begin{aligned} &\mathbf{Z}_1 = \frac{\partial}{\partial s}, \qquad \mathbf{Z}_2 = \frac{\partial}{\partial f_{1_r}}, \qquad \mathbf{Z}_3 = \frac{\partial}{\partial f_{1_i}}, \qquad \mathbf{Z}_4 = \frac{\partial}{\partial f_{2_r}}, \qquad \mathbf{Z}_5 = \frac{\partial}{\partial f_{2_i}}, \\ &\mathbf{Z}_6 = s\frac{\partial}{\partial s}, \qquad \mathbf{Z}_7 = f_{1_r}\frac{\partial}{\partial s}, \qquad \mathbf{Z}_8 = f_{1_i}\frac{\partial}{\partial s}, \qquad \mathbf{Z}_9 = f_{2_r}\frac{\partial}{\partial s}, \qquad \mathbf{Z}_{10} = f_{2_i}\frac{\partial}{\partial s}. \end{aligned} \tag{2.26}$$

They form a 10-d algebra $\mathfrak{h}_{10}$ with non-zero commutators $[\mathbf{Z}_1,\mathbf{Z}_6] = \mathbf{Z}_1$, $[\mathbf{Z}_2,\mathbf{Z}_7] = \mathbf{Z}_1$, $[\mathbf{Z}_3,\mathbf{Z}_8] = \mathbf{Z}_1$, $[\mathbf{Z}_4,\mathbf{Z}_9] = \mathbf{Z}_1$, $[\mathbf{Z}_5,\mathbf{Z}_{10}] = \mathbf{Z}_1$, $[\mathbf{Z}_6,\mathbf{Z}_7] = -\mathbf{Z}_7$, $[\mathbf{Z}_6,\mathbf{Z}_8] = -\mathbf{Z}_8$, $[\mathbf{Z}_6,\mathbf{Z}_9] = -\mathbf{Z}_9$, $[\mathbf{Z}_6,\mathbf{Z}_{10}] = -\mathbf{Z}_{10}$. The largest normal subalgebra is $\mathfrak{g}_5$ =$\langle \mathbf{Z}_c \rangle$; (c=1,2,...,5) and $\mathfrak{h}_{10}/\mathfrak{g}_5$=$\mathfrak{I}_5$, where $\mathfrak{I}_5$ =$\langle \mathbf{Z}_d \rangle$; (d=6,7,...,10) is the dilation algebra. The split system of symmetry generators yields the five Lie-like operators

$$\mathbf{X}_{1_1} = \frac{\partial}{\partial s}, \quad \mathbf{X}_{2_1} = \frac{\partial}{\partial f_{1_r}}, \qquad \mathbf{X}_{2_2} = \frac{\partial}{\partial f_{1_i}}, \quad \mathbf{Y}_{2_1} = \frac{\partial}{\partial f_{2_r}}, \qquad \mathbf{Y}_{2_2} = \frac{\partial}{\partial f_{2_i}}, \tag{2.27}$$

which are also the symmetries of the split system and form an abelian algebra.

n	d_n	l_n	$m_n = 2d_n - l_n$	L_n	$e_n = L_n - l_n$
1	2	3	1	6	3
2	4	5	3	10	5
3	8	9	7	18	9
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	2^n	2^n+1	2^n-1	$2(2^n+1)$	2^n+1

Table 3: In this case the number of missing Lie-like operators are non-zero and number of symmetries at each step is more than that of the Lie-like operators.

The 3^{rd} splitting gives an 8-d system of ODEs, with 18 symmetries and 9 Lie-like operators. Looking at the sequences of numbers the iterative procedure up to the n^{th} splitting should be as given in Table 3.

The general result can be presented as:

Theorem 2.6. *For (2.20) with symmetries given by (2.21), the iteratively split system, has 2^n-dimension, 2^n+1 Lie-like operators with 2^n-1 missing operators, $2(2^n+1)$ symmetries and 2^n+1 extra symmetry.*

Proof. The first part of the proof is trivial as before and we only give the argument for the closure of the algebra. Since the split operators again have no coefficients for the derivatives, they must all commute and form an abelian Lie algebra. □

Example 2.7. As a fourth example we take the 2^{nd} order r-CODE

$$v'' = h(s, v'), \tag{2.28}$$

with symmetry

$$\mathbf{Z}_1 = \frac{\partial}{\partial v}. \tag{2.29}$$

The first splitting yields the 2-d system of ODEs:

$$f_1'' = h_1(s, f_1', f_2'), \quad f_2'' = h_2(s, f_1', f_2'), \tag{2.30}$$

with symmetries

$$\mathbf{Z}_1 = \frac{\partial}{\partial f_1}, \quad \mathbf{Z}_2 = \frac{\partial}{\partial f_2} \tag{2.31}$$

which form a 2-d algebra $\mathfrak{h}_2$, and the split system remains unchanged.

The next splitting gives

$$f_{j_r}'' = h_{j_r}(s, f_{1_r}', f_{1_i}', f_{2_r}', f_{2_i}'), f_{j_i}'' = h_{j_i}(s, f_{1_r}', f_{1_i}', f_{2_r}', f_{2_i}'), \tag{2.32}$$

with symmetries

$$\mathbf{Z}_1 = \frac{\partial}{\partial f_{1_r}}, \quad \mathbf{Z}_2 = \frac{\partial}{\partial f_{1_i}}, \quad \mathbf{Z}_3 = \frac{\partial}{\partial f_{2_r}}, \quad \mathbf{Z}_4 = \frac{\partial}{\partial f_{2_i}}, \tag{2.33}$$

that form a 4-d abelian algebra, $\mathfrak{h}_4$, and again the split system remains unchanged.

The 3^{rd} splitting gives an 8-d system of ODEs, with 8 symmetries and identical Lie-like operators. Looking at the sequences of numbers the iterative procedure up to the n^{th} splitting should be as given in Table 4.
The general result can be presented as:

n	d_n	l_n	$m_n = d_n - l_n$	L_n	$e_n = L_n - l_n$
1	2	2	0	2	0
2	4	4	0	4	0
3	8	8	0	8	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	2^n	2^n	0	2^n	0

Table 4: In this case the number of obtained Lie-like operators is equal to the number of symmetries, with no missing Lie-like operators or extra symmetries.

Theorem 2.8. *For (2.28) with symmetries given by (2.29), the iteratively split system, has 2^n-dimension, 2^n Lie-like operators with no missing operator, 2^n symmetries and no extra symmetry.*

Proof. The first part of the proof is trivial as before and we only give the argument for the closure of the algebra. Since the split operators again have no coefficients for the derivatives, they must all commute and form an abelian algebra. □

Example 2.9. As a fifth example we take the 2^{nd} order r-CODE

$$v'' = h(v, v'), \tag{2.34}$$

with symmetry

$$\mathbf{Z} = \frac{\partial}{\partial s}. \tag{2.35}$$

The first splitting yields the 2-d system of ODEs:

$$f_1'' = h_1(f_1, f_2, f_1', f_2'), \qquad f_2'' = h_2(f_1, f_2, f_1', f_2'), \tag{2.36}$$

with symmetry (2.35) and identical Lie-like operator.

The next splitting gives

$$f_{j_r}'' = h_{j_r}(\mathbf{a}, \mathbf{a}'), f_{j_i}'' \quad = h_{j_i}(\mathbf{a}, \mathbf{a}'); \tag{2.37}$$

where $\mathbf{a} = (f_{1_r}, f_{1_i}, f_{2_r}, f_{2_i})$ and $\mathbf{a}' = (f_{1_r}', f_{1_i}', f_{2_r}', f_{2_i}')$ with unchanged symmetry and Lie-like operator.

The 3^{rd} splitting gives an 8-d system of ODEs, again identical symmetry and Lie-like operator. Looking at the sequences of numbers the iterative procedure up to the n^{th} splitting should be as given in Table 5.

n	d_n	l_n	$m_n = d_n - l_n$	L_n	$e_n = L_n - l_n$
1	2	1	1	1	0
2	4	1	3	1	0
3	8	1	7	1	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	2^n	1	$2^n - 1$	1	0

Table 5: Since the symmetry is only of the type of a derivative with respect to the independent variable, so it remains same after splitting with missing Lie-like operators and no extra symmetries.

The general result can be presented as:

Theorem 2.10. *For (2.34) with symmetries given by (2.35), the iteratively split system, has 2^n-dimension, 1 Lie-like operator with 2^n-1 missing operators, 1 symmetry and no extra symmetry.*

Proof. As before the proof is trivial because there is only one split operator always. □

3 Half-integer Splitting

Now we provide the iterative splitting procedure for obtaining odd-dimensional split systems of ODEs, which cannot be obtained by direct splitting. In the second splitting we have two options: either we obtain a 3-d system of ODEs by considering one variable as pure imaginary and the other as complex, or a 4-d system of ODEs by retaining all variables as complex. Similarly, we have four options for the third splitting. Two of the options will come from a 3-d system which will be either 5 or 6-d systems or from a 4-d system which will be 7 or 8-d and so on.

For a 2nd order r-CODE in the second splitting, we consider $f_1 = \imath f_{1_i}$ and $f_2 = f_{2_r} + \imath f_{2_i}$ which implies

$$\imath f_{1_i}'' = h_{1_r}(s,\mathbf{a},\mathbf{a}') + \imath h_{1_i}(s,\mathbf{a},\mathbf{a}'), \quad f_{2_r} + \imath f_{2_i} \quad = h_{1_r}(s,\mathbf{a},\mathbf{a}') + \imath h_{1_i}(s,\mathbf{a},\mathbf{a}'). \tag{3.1}$$

Quadratic algebraic constraints between the dependent variables, of the split systems, are required (which are illustrated in the examples below). Then we are left with only three ODEs.

$$f_{1_i}'' = h_{1_i}(s,\mathbf{a},\mathbf{a}'), \qquad f_{2_r}'' = h_{2_r}(s,\mathbf{a},\mathbf{a}'), \qquad f_{2_i}'' = h_{2_i}(s,\mathbf{a},\mathbf{a}'), \tag{3.2}$$

with the CR-conditions

$$h_{2_{r,f_{2r}}} = h_{2_{i,f_{2i}}}, \qquad h_{2_{r,f_{2i}}} = -h_{2_{i,f_{2r}}}, \qquad h_{2_{r,f'_{2r}}} = h_{2_{i,f'_{2i}}}, \qquad h_{2_{r,f'_{2i}}} = -h_{2_{i,f'_{2r}}}. \tag{3.3}$$

Instead, we can take $f_1 = f_{1_r} + \imath f_{1_i}$ and $f_2 = \imath f_{2_i}$, which can be regarded as "*dual*" to the above. After applying this procedure for the r-CODE, we always have an *algebraic constraint* for the odd number of ODEs. Iterative use of this procedure yields higher odd-dimensional systems with an algebraic constraint. We call this procedure *half-integer splitting.*

Example 3.1. As an example we take the 2nd order scalar Emden-Fowler equation

$$v''(s) + \frac{5}{s}v'(s) + v^2(s) = 0, \tag{3.4}$$

which has one scaling symmetry generator

$$\mathbf{Z}_1 = s\frac{\partial}{\partial s} - 2v\frac{\partial}{\partial v}. \tag{3.5}$$

By applying half-integer splitting on this equation we obtain 2, 3, 4, 5, 6 and 7-d systems of ODEs. We focus on the two directions at the second splitting. First consider one variable pure imaginary and the other complex. Then proceed to the third splitting.

CASE: 1 For this, put $v = f_1 + \imath f_2$ to obtain the 2-d system of ODEs

$$\begin{aligned} f_1'' + \frac{5}{s}f_1' + (f_1^2 - f_2^2) = 0,\\ f_2'' + \frac{5}{s}f_2' + 2f_1f_2 = 0, \end{aligned} \tag{3.6}$$

with symmetry

$$\mathbf{Z}=s\frac{\partial}{\partial s}-2f_1\frac{\partial}{\partial f_1}-2f_2\frac{\partial}{\partial f_2}. \tag{3.7}$$

The split system yields the two Lie-like operators

$$\begin{aligned}\mathbf{X}_1&=s\frac{\partial}{\partial s}-f_1\frac{\partial}{\partial f_1}-f_2\frac{\partial}{\partial f_2},\\ \mathbf{Y}_1&=f_2\frac{\partial}{\partial f_1}-f_1\frac{\partial}{\partial f_2}.\end{aligned} \tag{3.8}$$

The next splitting, $f_1=\iota f_{1_1},\ f_2=f_{2_1}+\iota f_{2_2}$, gives 3-d system of ODEs

$$\begin{aligned}&f''_{1_1}+\frac{5}{s}f'_{1_1}-f_{2_1}f_{2_2}=0,\\ &f''_{2_1}+\frac{5}{s}f'_{2_1}-f_{1_1}f_{2_2}=0,\\ &f''_{2_2}+\frac{5}{s}f'_{2_2}+f_{1_1}f_{2_1}=0,\end{aligned} \tag{3.9}$$

with constraint

$$f_{2_2}^2=f_{1_1}^2+f_{2_1}^2. \tag{3.10}$$

The symmetry of the system is

$$\mathbf{Z}=s\frac{\partial}{\partial s}-2f_{1_1}\frac{\partial}{\partial f_{1_1}}-2f_{2_1}\frac{\partial}{\partial f_{2_1}}-2f_{2_2}\frac{\partial}{\partial f_{2_2}} \tag{3.11}$$

with four Lie-like operators

$$\begin{aligned}&\mathbf{X}_{1_1}=2s\frac{\partial}{\partial s}-2f_{1_1}\frac{\partial}{\partial f_{1_1}}-f_{2_1}\frac{\partial}{\partial f_{2_1}}-f_{2_2}\frac{\partial}{\partial f_{2_2}}, &&\mathbf{X}_{1_2}=-f_{2_2}\frac{\partial}{\partial f_{2_1}}+f_{2_1}\frac{\partial}{\partial f_{2_2}},\\ &\mathbf{Y}_{1_1}=2f_{2_2}\frac{\partial}{\partial f_{1_1}}-f_{1_1}\frac{\partial}{\partial f_{2_2}}, &&\mathbf{Y}_{1_2}=-2f_{2_1}\frac{\partial}{\partial f_{1_1}}-f_{1_1}\frac{\partial}{\partial f_{2_1}},\end{aligned} \tag{3.12}$$

which are not the symmetries of the split system. Further for 5-d system of ODEs consider $f_{1_1}=\iota f_{1_{1_1}}$, $f_{2_1}=f_{2_{1_1}}+\iota f_{2_{1_2}}$ and $f_{2_2}=f_{2_{2_1}}+\iota f_{2_{2_2}}$

$$\begin{aligned}f''_{1_{1_1}}+\frac{5}{s}f'_{1_{1_1}}-2(f_{2_{1_1}}f_{2_{2_2}}-f_{2_{1_2}}f_{2_{2_1}})&=0,\\ f''_{2_{1_1}}+\frac{5}{s}f'_{2_{1_1}}+2f_{1_{1_1}}f_{2_{2_2}}&=0,\\ f''_{2_{1_2}}+\frac{5}{s}f'_{2_{1_2}}-2f_{1_{1_1}}f_{2_{2_1}}&=0,\\ f''_{2_{2_1}}+\frac{5}{s}f'_{2_{2_1}}-2f_{1_{1_1}}f_{2_{1_2}}&=0,\\ f''_{2_{2_2}}+\frac{5}{s}f'_{2_{2_2}}+2f_{1_{1_1}}f_{2_{1_1}}&=0,\end{aligned} \tag{3.13}$$

with constraint

$$f_{2_{1_1}}f_{2_{2_1}}=f_{2_{1_2}}f_{2_{2_2}}. \tag{3.14}$$

The symmetries of the system are

$$\begin{aligned}\mathbf{Z}_1&=f_{2_{1_2}}\frac{\partial}{\partial f_{2_{1_1}}}+f_{2_{1_1}}\frac{\partial}{\partial f_{2_{1_2}}}-f_{2_{2_2}}\frac{\partial}{\partial f_{2_{2_1}}}-f_{2_{2_1}}\frac{\partial}{\partial f_{2_{2_2}}},\\ \mathbf{Z}_2&=f_{2_{2_1}}\frac{\partial}{\partial f_{2_{1_1}}}-f_{2_{2_2}}\frac{\partial}{\partial f_{2_{1_2}}}+f_{2_{1_1}}\frac{\partial}{\partial f_{2_{2_1}}}-f_{2_{1_2}}\frac{\partial}{\partial f_{2_{2_2}}},\\ \mathbf{Z}_3&=s\frac{\partial}{\partial s}-2f_{1_{1_1}}\frac{\partial}{\partial f_{1_{1_1}}}-2f_{2_{1_1}}\frac{\partial}{\partial f_{2_{1_1}}}-2f_{2_{1_2}}\frac{\partial}{\partial f_{2_{1_2}}}-2f_{2_{2_1}}\frac{\partial}{\partial f_{2_{2_1}}}-2f_{2_{2_2}}\frac{\partial}{\partial f_{2_{2_2}}},\end{aligned} \tag{3.15}$$

with eight Lie-like operators

$$\begin{aligned}
\mathbf{X}_{1_{1_1}} &= 4s\frac{\partial}{\partial s} - 4f_{1_{1_1}}\frac{\partial}{\partial f_{1_{1_1}}} - f_{2_{1_1}}\frac{\partial}{\partial f_{2_{1_1}}} & \mathbf{X}_{1_{1_2}} &= -f_{2_{1_2}}\frac{\partial}{\partial f_{2_{1_1}}} + f_{2_{1_1}}\frac{\partial}{\partial f_{2_{1_2}}},\\
&\quad - f_{2_{1_2}}\frac{\partial}{\partial f_{2_{1_2}}}, & &\\
\mathbf{X}_{1_{2_1}} &= -f_{2_{2_1}}\frac{\partial}{\partial f_{2_{1_1}}} - f_{2_{2_2}}\frac{\partial}{\partial f_{2_{1_2}}}, & \mathbf{X}_{1_{2_2}} &= -f_{2_{2_2}}\frac{\partial}{\partial f_{2_{1_1}}} + f_{2_{2_1}}\frac{\partial}{\partial f_{2_{1_2}}},\\
\mathbf{Y}_{1_{1_1}} &= 4f_{2_{2_2}}\frac{\partial}{\partial f_{1_{1_1}}} + f_{1_{1_1}}\frac{\partial}{\partial f_{2_{2_2}}}, & \mathbf{Y}_{1_{1_2}} &= -4f_{2_{2_1}}\frac{\partial}{\partial f_{1_{1_1}}} - f_{1_{1_1}}\frac{\partial}{\partial f_{2_{2_1}}},\\
\mathbf{Y}_{1_{2_1}} &= -4f_{2_{1_2}}\frac{\partial}{\partial f_{1_{1_1}}} - f_{1_{1_1}}\frac{\partial}{\partial f_{2_{1_2}}}, & \mathbf{Y}_{1_{2_2}} &= 4f_{2_{1_1}}\frac{\partial}{\partial f_{1_{1_1}}} - f_{1_{1_1}}\frac{\partial}{\partial f_{2_{1_1}}}.
\end{aligned} \tag{3.16}$$

For the 6-d system we take the variables of the form $f_{1_1} = f_{1_{1_1}} + \iota f_{1_{1_2}}$, $f_{2_1} = f_{2_{1_1}} + \iota f_{2_{1_2}}$ and $f_{2_2} = f_{2_{2_1}} + \iota f_{2_{2_2}}$

$$\begin{aligned}
f''_{2_{1_1}} + \frac{5}{s}f'_{2_{1_1}} - 2f_{1_{1_1}}f_{2_{2_1}} + 2f_{1_{1_2}}f_{2_{2_2}} &= 0,\\
f''_{1_{1_2}} + \frac{5}{s}f'_{1_{1_2}} - 2f_{2_{1_1}}f_{2_{2_2}} - 2f_{2_{1_2}}f_{2_{2_1}} &= 0,\\
f''_{2_{1_1}} + \frac{5}{s}f'_{2_{1_1}} - 2f_{1_{1_1}}f_{2_{2_1}} + 2f_{1_{1_2}}f_{2_{2_2}} &= 0,\\
f''_{2_{1_2}} + \frac{5}{s}f'_{2_{1_2}} - 2f_{1_{1_1}}f_{2_{2_2}} - 2f_{1_{1_2}}f_{2_{2_1}} &= 0,\\
f''_{2_{2_1}} + \frac{5}{s}f'_{2_{2_1}} + 2f_{1_{1_1}}f_{2_{1_1}} - 2f_{1_{1_2}}f_{2_{1_2}} &= 0,\\
f''_{2_{2_2}} + \frac{5}{s}f'_{2_{2_2}} + 2f_{1_{1_1}}f_{2_{1_2}} + 2f_{1_{1_2}}f_{2_{1_1}} &= 0,
\end{aligned} \tag{3.17}$$

with symmetry

$$\begin{aligned}
\mathbf{Z} = s\frac{\partial}{\partial s} - 2f_{1_{1_1}}\frac{\partial}{\partial f_{1_{1_1}}} - 2f_{1_{1_2}}\frac{\partial}{\partial f_{1_{1_2}}} - 2f_{2_{1_1}}\frac{\partial}{\partial f_{2_{1_1}}} - 2f_{2_{1_2}}\frac{\partial}{\partial f_{2_{1_2}}}&\\
-2f_{2_{2_1}}\frac{\partial}{\partial f_{2_{2_1}}} - 2f_{2_{2_2}}\frac{\partial}{\partial f_{2_{2_2}}} = 0.&
\end{aligned} \tag{3.18}$$

This yields the eight Lie-like operators,

$$\begin{aligned}
\mathbf{Y}_{1_{1_1}} &= 2f_{2_{2_1}}\frac{\partial}{\partial f_{1_{1_1}}} + 2f_{2_{2_2}}\frac{\partial}{\partial f_{1_{1_2}}} - f_{1_{1_1}}\frac{\partial}{\partial f_{2_{2_1}}} - f_{1_{1_2}}\frac{\partial}{\partial f_{2_{2_2}}},\\
\mathbf{Y}_{1_{1_2}} &= 2f_{2_{2_2}}\frac{\partial}{\partial f_{1_{1_1}}} - f_{2_{2_1}}\frac{\partial}{\partial f_{1_{1_2}}} - f_{1_{1_2}}\frac{\partial}{\partial f_{2_{2_1}}} + f_{1_{1_1}}\frac{\partial}{\partial f_{2_{2_2}}},\\
\mathbf{Y}_{1_{2_1}} &= -2f_{2_{1_1}}\frac{\partial}{\partial f_{1_{1_1}}} - 2f_{2_{1_2}}\frac{\partial}{\partial f_{1_{1_2}}} - f_{1_{1_1}}\frac{\partial}{\partial f_{2_{1_1}}} - f_{1_{1_2}}\frac{\partial}{\partial f_{2_{1_2}}},\\
\mathbf{Y}_{1_{2_2}} &= -2f_{2_{1_2}}\frac{\partial}{\partial f_{1_{1_1}}} - f_{2_{1_1}}\frac{\partial}{\partial f_{1_{1_2}}} - f_{1_{1_2}}\frac{\partial}{\partial f_{2_{1_1}}} - f_{1_{1_1}}\frac{\partial}{\partial f_{2_{1_2}}}.
\end{aligned} \tag{3.19}$$

We summarize the results in Table 6.

CASE: 2 Following the second branch, at the stage of second splitting consider both variables

n	Variables Form	Constraint	d_n	l_n	m_n	L_n	e_n
1	$v = f_1 + \imath f_2$.		2	2	0	1	0
2	$f_1 = \imath f_{1_1}$, $f_2 = f_{2_1} + \imath f_{2_2}$.	$f_{2_2}^2 = f_{1_1}^2 + f_{2_1}^2$	3	4	0	1	0
3	$f_{1_1} = \imath f_{1_{1_1}}$, $f_{2_1} = f_{2_{1_1}} + \imath f_{2_{1_2}}$, $f_{2_2} = f_{2_{2_1}} + \imath f_{2_{2_2}}$.	$f_{2_{1_1}} f_{2_{2_1}} = f_{2_{1_2}} f_{2_{2_2}}$	5	8	0	3	0
	$f_{1_1} = f_{1_{1_1}} + \imath f_{1_{1_2}}$, $f_{2_1} = f_{2_{1_1}} + \imath f_{2_{1_2}}$, $f_{2_2} = f_{2_{2_1}} + \imath f_{2_{2_2}}$.		6	8	0	1	0

Table 6: In the first case of half-integer splitting the number of Lie-like operators is more than the symmetries, with no missing operators and extra symmetries. Note the algebraic constraints because of the odd dimensions of the system.

as complex then proceed to the third splitting with one variable pure imaginary and the other complex. At the first step we have a 4-d system of ODEs

$$\begin{aligned} f_{1_1}'' + \frac{5}{s} f_{1_1}' + f_{1_1}^2 - f_{1_2}^2 - f_{2_1}^2 + f_{2_2}^2 &= 0, \\ f_{1_2}'' + \frac{5}{s} f_{1_2}' + 2 f_{1_1} f_{1_2} - 2 f_{2_1} f_{2_2} &= 0, \\ f_{2_1}'' + \frac{5}{s} f_{2_1}' + 2 f_{1_1} f_{2_1} - 2 f_{1_2} f_{2_2} &= 0, \\ f_{2_2}'' + \frac{5}{s} f_{2_2}' + 2 f_{1_1} f_{2_2} + 2 f_{1_2} f_{2_1} &= 0, \end{aligned} \tag{3.20}$$

with the single symmetry generator

$$\mathbf{Z} = s \frac{\partial}{\partial s} - 2 f_{1_1} \frac{\partial}{\partial f_{1_1}} - 2 f_{1_2} \frac{\partial}{\partial f_{1_2}} - 2 f_{2_1} \frac{\partial}{\partial f_{2_1}} - 2 f_{2_2} \frac{\partial}{\partial f_{2_2}}, \tag{3.21}$$

and the corresponding four Lie-like operators

$$\begin{aligned} \mathbf{X}_{1_1} &= 2s \frac{\partial}{\partial s} - f_{1_1} \frac{\partial}{\partial f_{1_1}} - f_{1_2} \frac{\partial}{\partial f_{1_2}} - f_{2_1} \frac{\partial}{\partial f_{2_1}} - f_{2_2} \frac{\partial}{\partial f_{2_2}}, \\ \mathbf{X}_{1_2} &= -f_{1_2} \frac{\partial}{\partial f_{1_1}} + f_{1_1} \frac{\partial}{\partial f_{1_2}} - f_{2_2} \frac{\partial}{\partial f_{2_1}} + f_{2_1} \frac{\partial}{\partial f_{2_2}}, \\ \mathbf{Y}_{1_1} &= f_{2_1} \frac{\partial}{\partial f_{1_1}} + f_{2_2} \frac{\partial}{\partial f_{1_2}} - f_{1_1} \frac{\partial}{\partial f_{2_1}} - f_{1_2} \frac{\partial}{\partial f_{2_2}}, \\ \mathbf{Y}_{1_2} &= f_{2_2} \frac{\partial}{\partial f_{1_1}} - f_{2_1} \frac{\partial}{\partial f_{1_2}} - f_{1_2} \frac{\partial}{\partial f_{2_1}} + f_{1_1} \frac{\partial}{\partial f_{2_2}}. \end{aligned} \tag{3.22}$$

For the next splitting take the first dependent variable to be imaginary and the others complex; $f_{1_1} = \imath f_{1_{1_1}}$, $f_{1_2} = f_{1_{2_1}} + \imath f_{1_{2_2}}$, $f_{2_1} = f_{2_{1_1}} + \imath f_{2_{1_2}}$ and $f_{2_2} = f_{2_{2_1}} + \imath f_{2_{2_2}}$, which gives a 7-d system of

ODEs

$$
\begin{aligned}
&f''_{1_{1_1}} + \frac{5}{s} f'_{1_{1_1}} - 2 f_{1_{2_1}} f_{1_{2_2}} - 2 f_{2_{1_1}} f_{1_{1_2}} + 2 f_{2_{2_1}} f_{2_{2_2}} = 0,\\
&f''_{1_{2_1}} + \frac{5}{s} f'_{1_{2_1}} - 2 f_{1_{1_1}} f_{1_{2_2}} - 2 f_{2_{1_1}} f_{2_{2_1}} + 2 f_{2_{1_2}} f_{2_{2_2}} = 0,\\
&f''_{1_{2_2}} + \frac{5}{s} f'_{1_{2_2}} + 2 f_{1_{1_1}} f_{1_{2_1}} - 2 f_{2_{1_1}} f_{2_{2_2}} - 2 f_{2_{1_2}} f_{2_{2_1}} = 0,\\
&f''_{2_{1_1}} + \frac{5}{s} f'_{2_{1_1}} - 2 f_{1_{1_1}} f_{2_{1_2}} - 2 f_{1_{2_1}} f_{2_{2_1}} + 2 f_{1_{2_2}} f_{2_{2_2}} = 0,\\
&f''_{2_{1_2}} + \frac{5}{s} f'_{2_{1_2}} + 2 f_{1_{1_1}} f_{2_{1_1}} - 2 f_{1_{2_1}} f_{2_{2_2}} - 2 f_{1_{2_2}} f_{2_{2_1}} = 0,\\
&f''_{2_{2_1}} + \frac{5}{s} f'_{2_{2_1}} - 2 f_{1_{1_1}} f_{2_{2_2}} - 2 f_{1_{2_2}} f_{2_{1_2}} + 2 f_{1_{2_1}} f_{2_{1_1}} = 0,\\
&f''_{2_{2_2}} + \frac{5}{s} f'_{2_{2_2}} + 2 f_{1_{1_1}} f_{2_{2_1}} + 2 f_{1_{2_1}} f_{2_{1_2}} + 2 f_{1_{2_2}} f_{2_{1_1}} = 0,
\end{aligned}
\tag{3.23}
$$

with constraint

$$
f^2_{1_{1_1}} + f^2_{1_{2_1}} + f^2_{2_{1_1}} + f^2_{2_{2_2}} = f^2_{1_{2_2}} + f^2_{2_{1_2}} + f^2_{2_{2_1}}. \tag{3.24}
$$

The symmetry generator of the split system is

$$
\begin{aligned}
\mathbf{Z} = s\frac{\partial}{\partial s} - 2 f_{1_{1_1}} \frac{\partial}{\partial f_{1_{1_1}}} - 2 f_{1_{2_1}} \frac{\partial}{\partial f_{1_{2_1}}} - 2 f_{1_{2_2}} \frac{\partial}{\partial f_{1_{2_2}}} - 2 f_{2_{1_1}} \frac{\partial}{\partial f_{2_{1_1}}} - 2 f_{2_{1_2}} \frac{\partial}{\partial f_{2_{1_2}}}\\
-2 f_{2_{2_1}} \frac{\partial}{\partial f_{2_{2_1}}} - 2 f_{2_{2_2}} \frac{\partial}{\partial f_{2_{2_2}}},
\end{aligned}
\tag{3.25}
$$

with eight Lie-like operators

$$
\begin{aligned}
\mathbf{X}_{1_1} &= 4s\frac{\partial}{\partial s} - 4 f_{1_{1_1}} \frac{\partial}{\partial f_{1_{1_1}}} - f_{1_{2_1}} \frac{\partial}{\partial f_{1_{2_1}}} - f_{1_{2_2}} \frac{\partial}{\partial f_{1_{2_2}}} - f_{2_{1_1}} \frac{\partial}{\partial f_{2_{1_1}}} - f_{2_{1_2}} \frac{\partial}{\partial f_{2_{1_2}}}\\
&\quad - f_{2_{2_1}} \frac{\partial}{\partial f_{2_{2_1}}} - f_{2_{2_2}} \frac{\partial}{\partial f_{2_{2_2}}},\\
\mathbf{X}_{1_2} &= -f_{1_{2_2}} \frac{\partial}{\partial f_{1_{2_1}}} + f_{1_{2_1}} \frac{\partial}{\partial f_{1_{2_2}}} - f_{2_{1_2}} \frac{\partial}{\partial f_{2_{1_1}}} + f_{2_{1_1}} \frac{\partial}{\partial f_{2_{1_2}}} - f_{2_{2_2}} \frac{\partial}{\partial f_{2_{2_1}}} + f_{2_{2_1}} \frac{\partial}{\partial f_{2_{2_2}}},\\
\mathbf{X}_{2_1} &= -2 f_{1_{2_2}} \frac{\partial}{\partial f_{1_{1_1}}} + f_{1_{1_1}} \frac{\partial}{\partial f_{1_{2_2}}} - f_{2_{2_1}} \frac{\partial}{\partial f_{2_{1_1}}} - f_{2_{2_2}} \frac{\partial}{\partial f_{2_{1_2}}} + f_{2_{1_1}} \frac{\partial}{\partial f_{2_{2_1}}} + f_{2_{1_2}} \frac{\partial}{\partial f_{2_{2_2}}},\\
\mathbf{X}_{2_2} &= 2 f_{1_{2_1}} \frac{\partial}{\partial f_{1_{1_1}}} + f_{1_{1_1}} \frac{\partial}{\partial f_{1_{2_1}}} - f_{2_{2_2}} \frac{\partial}{\partial f_{2_{1_1}}} + f_{2_{2_1}} \frac{\partial}{\partial f_{2_{1_2}}} + f_{2_{1_2}} \frac{\partial}{\partial f_{2_{2_1}}} - f_{2_{1_1}} \frac{\partial}{\partial f_{2_{2_2}}},\\
\mathbf{Y}_{1_1} &= -2 f_{2_{1_2}} \frac{\partial}{\partial f_{1_{1_1}}} + f_{2_{2_1}} \frac{\partial}{\partial f_{1_{2_1}}} + f_{2_{2_2}} \frac{\partial}{\partial f_{1_{2_2}}} - f_{1_{1_1}} \frac{\partial}{\partial f_{2_{1_2}}} - f_{1_{2_1}} \frac{\partial}{\partial f_{2_{2_1}}} - f_{1_{2_2}} \frac{\partial}{\partial f_{2_{2_2}}},\\
\mathbf{Y}_{1_2} &= 2 f_{2_{1_1}} \frac{\partial}{\partial f_{1_{1_1}}} + f_{2_{2_2}} \frac{\partial}{\partial f_{1_{2_1}}} - f_{2_{2_1}} \frac{\partial}{\partial f_{1_{2_2}}} - f_{1_{1_1}} \frac{\partial}{\partial f_{2_{1_1}}} + f_{1_{2_2}} \frac{\partial}{\partial f_{2_{2_1}}} - f_{1_{2_1}} \frac{\partial}{\partial f_{2_{2_2}}},\\
\mathbf{Y}_{2_1} &= -2 f_{2_{2_2}} \frac{\partial}{\partial f_{1_{1_1}}} - f_{2_{1_1}} \frac{\partial}{\partial f_{1_{2_1}}} - f_{2_{1_2}} \frac{\partial}{\partial f_{1_{2_2}}} - f_{1_{2_1}} \frac{\partial}{\partial f_{2_{1_1}}} - f_{1_{2_2}} \frac{\partial}{\partial f_{2_{1_2}}} + f_{1_{1_1}} \frac{\partial}{\partial f_{2_{2_2}}},\\
\mathbf{Y}_{2_2} &= 2 f_{2_{2_1}} \frac{\partial}{\partial f_{1_{1_1}}} - f_{2_{1_2}} \frac{\partial}{\partial f_{1_{2_1}}} + f_{2_{1_1}} \frac{\partial}{\partial f_{1_{2_2}}} - f_{1_{2_2}} \frac{\partial}{\partial f_{2_{1_1}}} + f_{1_{2_1}} \frac{\partial}{\partial f_{2_{1_2}}} + f_{1_{1_1}} \frac{\partial}{\partial f_{2_{2_1}}}.
\end{aligned}
$$

The table for this case is

n	Variables Form	Constraint	d_n	l_n	m_n	L_n	e_n
2	$f_1 = f_{1_1} + \iota f_{1_2}$, $f_2 = f_{2_1} + \iota f_{2_2}$.		4	4	0	1	0
3	$f_{1_1} = \iota f_{1_{1_1}}$, $f_{1_2} = f_{1_{2_1}} + \iota f_{1_{2_2}}$, $f_{2_1} = f_{2_{1_1}} + \iota f_{2_{1_2}}$, $f_{2_2} = f_{2_{2_1}} + \iota f_{2_{2_2}}$.	$f_{1_{1_1}}^2 + f_{1_{2_1}}^2 + f_{2_{1_1}}^2 + f_{2_{2_2}}^2 = f_{1_{2_2}}^2 + f_{2_{1_2}}^2 + f_{2_{2_1}}^2$	7	8	0	1	0

Table 7: The same outcomes as in the previous case are obtained since there are more Lie-like operators than symmetries, there are no missing operators, and there are no extra symmetries. Also it is worth noting the algebraic constraint for odd dimension.

4 Conclusion

The aim of the last section was to obtain odd-dimensional systems of ODEs by iterative complex splitting. We observed that iterative splitting with an algebraic constraint can be used to obtain higher odd-dimensional systems of ODEs. For this purpose we provided a new procedure, which we called half integer splitting. We applied the half-integer splitting to the scalar Emden-Fowler equation, for which we obtained several odd-dimensional systems of ODEs. The algebraic constraint that arises in the system of odd-dimensional ODEs was not apparent in setting up the system but was made explicit in the examples. The iterative splitting for obtaining a 2^n-d system of second order ODEs was provided. This splitting can be done for third and higher order ODEs [5–7, 9–11, 15]. The procedure of iterative splitting and half-integer splitting can also be performed for second or higher order PDEs for obtaining 2^n-d systems of PDEs of the corresponding order. However, the examples show that we can lose all the Lie symmetry generators and be left only with Lie-like ones. For the Emden-Fowler equation, we are left with no Lie symmetries from the Lie-like operators, though the equation has a scaling symmetry and so does the split system. In general, we obtain Lie-like operators and not Lie symmetry generators that would form an algebra [13]. The Lie-like operators somehow encode the symmetries of the base equation. It would be most important to learn how they do so. It may be that the CR-conditions will enable us to re-construct the Lie symmetries from the Lie-like operators. It is of interest to note that not only for the ODEs but also for the systems of PDEs, we get Lie-like operators arising and lose some Lie-symmetry generators [2, 14]. We hope that in the future, it would lead to interesting and useful insights.

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Rate Type Hypoplastic Differential Equations under Mixed Stress-Strain Control in Biaxial Test

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Abstract

The hypoplastic constitutive equations stemming from soil mechanics, which are under mixed stress-strain control carried out in a biaxial test, are studied with respect to its well-posedness and non-uniqueness of a solution. The result of theoretical investigation of the strongly nonlinear dynamic system is supported by computer simulation of numerical tests. Besides investigating the existence of global solutions, the simulations give an insight about the numerical convergence and validate the physical consistency of the system of equations when data is chosen in a domain of parameters satisfying feasibility conditions.

Keywords: Hypoplasticity, rate-independent system, implicit differential equation, non-uniqueness.

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1 Introduction

We study constitutive relations between stress and strain rate describing granular materials, like cohesionless soils or broken rocks, within the hypoplastic theory proposed first by Kolymbas [15], further continued by [26, 27], and extended to barodesy in the recent books [16, 17]. Unlike hyper- and hypoelastic material laws, the hypoplastic response differs for loading and unloading, thus corresponds to inelastic materials. While in classical elastoplastic models the strain is decomposed into elastic and plastic parts, e.g. [1, 8, 14], our approach relies on hypoplastic models of the rate type which are incrementally nonlinear. The interested reader is referred to [25] for a survey on rate-independent processes and hysteresis problems, and to [10–12, 22–24] for an account on mathematical modeling of granular and multiphase media.

Our study considers a simplified version of the hypoplastic constitutive relation that was originally introduced by Bauer [2] and Gudehus [13]. In the previous works [5, 6, 18, 20, 21] we considered the stress-strain rate law as a nonlinear differential equation for the stress under a given proportional strain rate, that we call strain control. Recently, the case of unknown strain rate that should be derived from a given proportional stress, called stress control, was investigated within implicit differential equations in [7, 19]. In the current study, we investigate the case of mixed stress-strain control in a so-called plane strain biaxial test.

In particular, for plane strain conditions we study the response of a hypoplastic material element under constant lateral stress and a monotonic vertical compression/extension. Such tests are of interest in various fields of applied mechanics to study the onset and evolution of shear strain localization [3, 4, 9, 29]. Shear strain localization may occur under a particular stress state where the constitutive equations describe not only continuous homogeneous deformations, but also non-homogeneous deformations. Thus, the system of constitutive equations exhibits non-unique solutions and in the case of a shear band bifurcation two symmetric shear bands may appear. While for the investigation of the onset of strain localization usually the theory of shear-band localization [28] can be applied, the focus of the present paper is based on possible solutions of the system of differential equations under the specified plane strain conditions considered.

2 Plan strain biaxial problem

For coaxial and homogeneous deformation, the tensors of Cauchy stress $\boldsymbol{\sigma}$ and strain rate $\dot{\boldsymbol{\varepsilon}}$ have the diagonal form

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}, \quad \dot{\boldsymbol{\varepsilon}} = \begin{pmatrix} \dot{\varepsilon}_1 & 0 & 0 \\ 0 & \dot{\varepsilon}_2 & 0 \\ 0 & 0 & \dot{\varepsilon}_3 \end{pmatrix}. \tag{2.1}$$

The hypoplastic constitutive equations due to Bauer et al. [6] under the assumption (2.1) are

$$\frac{d\sigma_1}{dt} = f_{\mathrm{s}}\mathrm{tr}\boldsymbol{\sigma}\Big\{a^2\dot{\varepsilon}_1 + \Big(\frac{\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}}{\mathrm{tr}\boldsymbol{\sigma}}\Big)\frac{\sigma_1}{\mathrm{tr}\boldsymbol{\sigma}} + af_{\mathrm{d}}\Big(\frac{2\sigma_1}{\mathrm{tr}\boldsymbol{\sigma}} - \frac{1}{3}\Big)\|\dot{\boldsymbol{\varepsilon}}\|\Big\}, \tag{2.2}$$

$$\frac{d\sigma_2}{dt} = f_{\mathrm{s}}\mathrm{tr}\boldsymbol{\sigma}\Big\{a^2\dot{\varepsilon}_2 + \Big(\frac{\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}}{\mathrm{tr}\boldsymbol{\sigma}}\Big)\frac{\sigma_2}{\mathrm{tr}\boldsymbol{\sigma}} + af_{\mathrm{d}}\Big(\frac{2\sigma_2}{\mathrm{tr}\boldsymbol{\sigma}} - \frac{1}{3}\Big)\|\dot{\boldsymbol{\varepsilon}}\|\Big\}, \tag{2.3}$$

$$\frac{d\sigma_3}{dt} = f_{\mathrm{s}}\mathrm{tr}\boldsymbol{\sigma}\Big\{a^2\dot{\varepsilon}_3 + \Big(\frac{\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}}{\mathrm{tr}\boldsymbol{\sigma}}\Big)\frac{\sigma_3}{\mathrm{tr}\boldsymbol{\sigma}} + af_{\mathrm{d}}\Big(\frac{2\sigma_3}{\mathrm{tr}\boldsymbol{\sigma}} - \frac{1}{3}\Big)\|\dot{\boldsymbol{\varepsilon}}\|\Big\}, \tag{2.4}$$

where the scalar product $\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}} = \sigma_1\dot{\varepsilon}_1 + \sigma_2\dot{\varepsilon}_2 + \sigma_3\dot{\varepsilon}_3$, and the Frobenius norm $\|\dot{\boldsymbol{\varepsilon}}\| = \sqrt{\dot{\varepsilon}_1^2 + \dot{\varepsilon}_2^2 + \dot{\varepsilon}_3^2}$. Herein $f_{\mathrm{s}}(t) < 0$ and $f_{\mathrm{d}}(t) > 0$ represent state dependent parameters of the model, and the constant $a > 0$ is related to the yield strength.

In a biaxial test, the following three quantities are prescribed:

$$\dot{\varepsilon}_1 = D_1, \quad \sigma_2 = \sigma_2^0, \quad \dot{\varepsilon}_3 = 0, \tag{2.5}$$

with constant D_1 and $\sigma_2^0 < 0$. In the constitutive relations $\sigma_1(t) < 0$, $\sigma_3(t) < 0$, and $\dot\varepsilon_2(t)$ are three unknown functions of time $t \geq 0$. We insert the assumption (2.5) into the hypoplastic equations such that (2.2)–(2.4) become

$$\frac{d\sigma_1}{dt} = f_s \mathrm{tr}\boldsymbol{\sigma}\Big\{a^2 D_1 + \Big(\frac{\sigma_1 D_1 + \sigma_2^0\dot\varepsilon_2}{\mathrm{tr}\boldsymbol{\sigma}}\Big)\frac{\sigma_1}{\mathrm{tr}\boldsymbol{\sigma}} + a f_d\Big(\frac{2\sigma_1}{\mathrm{tr}\boldsymbol{\sigma}} - \frac{1}{3}\Big)\sqrt{D_1^2 + \dot\varepsilon_2^2}\Big\}, \tag{2.6}$$

$$0 = f_s \mathrm{tr}\boldsymbol{\sigma}\Big\{a^2 \dot\varepsilon_2 + \Big(\frac{\sigma_1 D_1 + \sigma_2^0\dot\varepsilon_2}{\mathrm{tr}\boldsymbol{\sigma}}\Big)\frac{\sigma_2^0}{\mathrm{tr}\boldsymbol{\sigma}} + a f_d\Big(\frac{2\sigma_2^0}{\mathrm{tr}\boldsymbol{\sigma}} - \frac{1}{3}\Big)\sqrt{D_1^2 + \dot\varepsilon_2^2}\Big\}, \tag{2.7}$$

$$\frac{d\sigma_3}{dt} = f_s \mathrm{tr}\boldsymbol{\sigma}\Big\{\Big(\frac{\sigma_1 D_1 + \sigma_2^0\dot\varepsilon_2}{\mathrm{tr}\boldsymbol{\sigma}}\Big)\frac{\sigma_3}{\mathrm{tr}\boldsymbol{\sigma}} + a f_d\Big(\frac{2\sigma_3}{\mathrm{tr}\boldsymbol{\sigma}} - \frac{1}{3}\Big)\sqrt{D_1^2 + \dot\varepsilon_2^2}\Big\}. \tag{2.8}$$

The sum of (2.6)–(2.8) implies the following differential equation for the trace

$$\frac{d(\mathrm{tr}\boldsymbol{\sigma})}{dt} = f_s \mathrm{tr}\boldsymbol{\sigma}\Big\{a^2(D_1 + \dot\varepsilon_2) + \frac{\sigma_1 D_1 + \sigma_2^0\dot\varepsilon_2}{\mathrm{tr}\boldsymbol{\sigma}} + a f_d\sqrt{D_1^2 + \dot\varepsilon_2^2}\Big\}. \tag{2.9}$$

Denoting for brevity the ratio of the stress tensor scaled with its trace by

$$\hat\sigma_1 = \frac{\sigma_1}{\mathrm{tr}\boldsymbol{\sigma}}, \quad \hat\sigma_2 = \frac{\sigma_2^0}{\mathrm{tr}\boldsymbol{\sigma}}, \quad \hat\sigma_3 = \frac{\sigma_3}{\mathrm{tr}\boldsymbol{\sigma}}, \quad \mathrm{tr}\boldsymbol{\sigma} = \sigma_1 + \sigma_2^0 + \sigma_3, \tag{2.10}$$

we get the following expression for its derivative

$$\frac{d\hat\sigma_n}{dt} = \frac{1}{\mathrm{tr}\boldsymbol{\sigma}}\frac{d\sigma_n}{dt} - \frac{\hat\sigma_n}{\mathrm{tr}\boldsymbol{\sigma}}\frac{d(\mathrm{tr}\boldsymbol{\sigma})}{dt}, \qquad n = 1, 3.$$

Hence, from equations (2.9), (2.6) and (2.8) we infer that

$$\frac{d\hat\sigma_1}{dt} = f_s\Big\{a^2 D_1 - a^2(D_1 + \dot\varepsilon_2)\hat\sigma_1 + a f_d\Big(\hat\sigma_1 - \frac{1}{3}\Big)\sqrt{D_1^2 + \dot\varepsilon_2^2}\Big\}, \tag{2.11}$$

$$\frac{d\hat\sigma_3}{dt} = f_s\Big\{-a^2(D_1 + \dot\varepsilon_2)\hat\sigma_3 + a f_d\Big(\hat\sigma_3 - \frac{1}{3}\Big)\sqrt{D_1^2 + \dot\varepsilon_2^2}\Big\}. \tag{2.12}$$

After summation of (2.11) and (2.12), the identity $\hat\sigma_1 + \hat\sigma_2 + \hat\sigma_3 = 1$ leads to

$$\frac{d\hat\sigma_2}{dt} = f_s\Big\{a^2\dot\varepsilon_2 - a^2(D_1 + \dot\varepsilon_2)\hat\sigma_2 + a f_d\Big(\hat\sigma_2 - \frac{1}{3}\Big)\sqrt{D_1^2 + \dot\varepsilon_2^2}\Big\}. \tag{2.13}$$

Whereas the algebraic equation (2.7) can be rewritten using (2.10) as

$$D_1\hat\sigma_1\hat\sigma_2 + (a^2 + \hat\sigma_2^2)\dot\varepsilon_2 + a f_d\Big(2\hat\sigma_2 - \frac{1}{3}\Big)\sqrt{D_1^2 + \dot\varepsilon_2^2} = 0. \tag{2.14}$$

Note that $\hat\sigma_3$ does not enter (2.14), and (2.12) can be deduced from the governing equations. The coupled system (2.11), (2.13), and (2.14) has to be solve with respect to three unknowns $\hat\sigma_1$, $\hat\sigma_2$, and $\dot\varepsilon_2$, endowed with the initial conditions:

$$\hat\sigma_1(0) = \frac{\sigma_1^0}{\sigma_1^0 + \sigma_2^0 + \sigma_3^0}, \quad \hat\sigma_2(0) = \frac{\sigma_2^0}{\sigma_1^0 + \sigma_2^0 + \sigma_3^0}, \tag{2.15}$$

for prescribed $\sigma_1^0 < 0$, $\sigma_3^0 < 0$, and σ_2^0 from (2.5).

Theorem 2.1 (Solution)**.** *A solution to the linear Cauchy system* (2.11) *and* (2.13) *under initial conditions* (2.15) *and constrained by* (2.14) *can be written in the integral form:*

$$\hat\sigma_1(t) - \frac{1}{3} = e^{\int_0^t \phi_3(\tau)\,d\tau}\Big[\hat\sigma_1(0) - \frac{1}{3} + \int_0^t \phi_1(\xi)e^{-\int_0^\xi \phi_3(\tau)\,d\tau}d\xi\Big], \tag{2.16}$$

$$\hat\sigma_2(t) - \frac{1}{3} = e^{\int_0^t \phi_3(\tau)\,d\tau}\Big[\hat\sigma_2(0) - \frac{1}{3} + \int_0^t \phi_2(\xi)e^{-\int_0^\xi \phi_3(\tau)\,d\tau}d\xi\Big], \tag{2.17}$$

where the integrands are

$$\phi_1 = a^2 f_{\rm s}\frac{2D_1-\dot\varepsilon_2}{3},\quad \phi_2 = a^2 f_{\rm s}\frac{2\dot\varepsilon_2-D_1}{3},\quad \phi_3 = a^2 f_{\rm s}\Big\{-(D_1+\dot\varepsilon_2)+\frac{f_{\rm d}}{a}\sqrt{D_1^2+\dot\varepsilon_2^2}\Big\}. \quad (2.18)$$

Moreover, under the solvability condition

$$\mathcal{D} := a^2 f_{\rm d}^2 D_1^2\Big(2\hat\sigma_2-\frac{1}{3}\Big)^2\Big\{\hat\sigma_1^2\hat\sigma_2^2+\big(a^2+\hat\sigma_2^2\big)^2-a^2 f_{\rm d}^2\Big(2\hat\sigma_2-\frac{1}{3}\Big)^2\Big\}\geq 0 \quad (2.19)$$

(where the discriminant $\mathcal{D}$ becomes zero for $\hat\sigma_2 = 1/6$), from the algebraic equation (2.14) *we deduce two possible expressions for $\dot\varepsilon_2$, namely*

$$(\dot\varepsilon_2)_\pm = \frac{D_1\hat\sigma_1\hat\sigma_2(a^2+\hat\sigma_2^2)\pm\sqrt{\mathcal{D}}}{a^2 f_{\rm d}^2\big(2\hat\sigma_2-\frac{1}{3}\big)^2-\big(a^2+\hat\sigma_2^2\big)^2}. \quad (2.20)$$

Proof. Using the notation (2.18) we can rewrite (2.11) and (2.13) in a unified way as

$$\frac{d}{dt}\Big(\hat\sigma_n-\frac{1}{3}\Big) = \phi_n+\phi_3\Big(\hat\sigma_n-\frac{1}{3}\Big),\qquad n=1,2. \quad (2.21)$$

The multiplication of (2.21) by the factor $\exp(-\int_0^t\phi_3(\tau)\,d\tau)$ yields the equivalent equation

$$\frac{d}{dt}\Big[\Big(\hat\sigma_n(t)-\frac{1}{3}\Big)e^{-\int_0^t\phi_3(\tau)\,d\tau}\Big] = \phi_n(t)e^{-\int_0^t\phi_3(\tau)\,d\tau},\qquad n=1,2.$$

Thus, formulas (2.16) and (2.17) can be obtained by simple integration over the interval $[0,t]$ and taking into account the initial values given in (2.15).

Considering the aforementioned $\hat\sigma_1$ and $\hat\sigma_2$, from (2.14) we deduce a quadratic equation for the unknown $\dot\varepsilon_2$ as follows:

$$\Big[a^2 f_{\rm d}^2\Big(2\hat\sigma_2-\frac{1}{3}\Big)^2-\big(a^2+\hat\sigma_2^2\big)^2\Big]\dot\varepsilon_2^2-2D_1\hat\sigma_1\hat\sigma_2(a^2+\hat\sigma_2^2)\dot\varepsilon_2 + D_1^2\Big[a^2 f_{\rm d}^2\Big(2\hat\sigma_2-\frac{1}{3}\Big)^2-\hat\sigma_1^2\hat\sigma_2^2\Big]=0, \quad (2.22)$$

whose discriminant $\mathcal{D}$ is given in (2.19), and consequently we have two possible solutions $(\dot\varepsilon_2)_\pm$ as in (2.20). Note that (2.19) ensures that $\mathcal{D}$ is non-negative independently of the sign of D_1. The proof is completed. □

For a physically consistent model relevant for cohesionless granular materials, only negative normal stresses are admissible. Therefore, $\mathrm{tr}\boldsymbol{\sigma}<0$, and formula (2.10) leads to the restriction $\hat\sigma_n = \sigma_n/\mathrm{tr}\boldsymbol{\sigma}>0$ for $n=1,2,3$.

3 Numerical simulations

Based on Theorem 2.1, we analyze the existence of numerical solutions for two systems with unknowns $\hat\sigma_1$, $\hat\sigma_2$, and $\dot\varepsilon_2$ and accounting for the plain strain biaxial model in the form of Cauchy problem (2.11), (2.13)–(2.15). Note that the normalized stresses $\hat\sigma_1,\hat\sigma_2\in(0,1)$ as unknown variables in the linear equations (2.11), (2.13) are numerically advantageous over the stresses σ_1,σ_3 in the nonlinear equations (2.6), (2.8), which are unbounded in general. After finding the solution, we can determine the quantities $\hat\sigma_3$ and $\mathrm{tr}\boldsymbol{\sigma}$ in (2.9) and (2.12) which are implicitly given by

$$\hat\sigma_3 = 1-\hat\sigma_1-\hat\sigma_2,\qquad \mathrm{tr}\boldsymbol{\sigma}=\frac{\sigma_2^0}{\hat\sigma_2}. \quad (3.1)$$

To make our presentation of the numerical scheme more precise, we distinguish the two systems comprehending equations (2.11), (2.13), and (2.20) taking into account the discriminant as in (2.19), which are gathered as follows:

$$\begin{aligned}
\mathcal{D} &= a^2 f_{\mathrm{d}}^2 D_1^2\Big(2\hat\sigma_2 - \frac{1}{3}\Big)^2\Big\{\hat\sigma_1^2\hat\sigma_2^2 + \big(a^2+\hat\sigma_2^2\big)^2 - a^2 f_{\mathrm{d}}^2\Big(2\hat\sigma_2 - \frac{1}{3}\Big)^2\Big\},\\
\dot\varepsilon_2 &= \frac{D_1\hat\sigma_1\hat\sigma_2(a^2+\hat\sigma_2^2) - \sqrt{\mathcal{D}}}{a^2 f_{\mathrm{d}}^2\big(2\hat\sigma_2 - \frac{1}{3}\big)^2 - \big(a^2+\hat\sigma_2^2\big)^2},\\
\frac{d\hat\sigma_1}{dt} &= f_{\mathrm{s}}\Big\{a^2 D_1 - a^2(D_1+\dot\varepsilon_2)\hat\sigma_1 + a f_{\mathrm{d}}\Big(\hat\sigma_1 - \frac{1}{3}\Big)\sqrt{D_1^2+\dot\varepsilon_2^2}\Big\},\\
\frac{d\hat\sigma_2}{dt} &= f_{\mathrm{s}}\Big\{a^2 \dot\varepsilon_2 - a^2(D_1+\dot\varepsilon_2)\hat\sigma_2 + a f_{\mathrm{d}}\Big(\hat\sigma_2 - \frac{1}{3}\Big)\sqrt{D_1^2+\dot\varepsilon_2^2}\Big\},
\end{aligned} \qquad (-\sqrt{\mathrm{D}})$$

and

$$\begin{aligned}
\mathcal{D} &= a^2 f_{\mathrm{d}}^2 D_1^2\Big(2\hat\sigma_2 - \frac{1}{3}\Big)^2\Big\{\hat\sigma_1^2\hat\sigma_2^2 + \big(a^2+\hat\sigma_2^2\big)^2 - a^2 f_{\mathrm{d}}^2\Big(2\hat\sigma_2 - \frac{1}{3}\Big)^2\Big\},\\
\dot\varepsilon_2 &= \frac{D_1\hat\sigma_1\hat\sigma_2(a^2+\hat\sigma_2^2) + \sqrt{\mathcal{D}}}{a^2 f_{\mathrm{d}}^2\big(2\hat\sigma_2 - \frac{1}{3}\big)^2 - \big(a^2+\hat\sigma_2^2\big)^2},\\
\frac{d\hat\sigma_1}{dt} &= f_{\mathrm{s}}\Big\{a^2 D_1 - a^2(D_1+\dot\varepsilon_2)\hat\sigma_1 + a f_{\mathrm{d}}\Big(\hat\sigma_1 - \frac{1}{3}\Big)\sqrt{D_1^2+\dot\varepsilon_2^2}\Big\},\\
\frac{d\hat\sigma_2}{dt} &= f_{\mathrm{s}}\Big\{a^2 \dot\varepsilon_2 - a^2(D_1+\dot\varepsilon_2)\hat\sigma_2 + a f_{\mathrm{d}}\Big(\hat\sigma_2 - \frac{1}{3}\Big)\sqrt{D_1^2+\dot\varepsilon_2^2}\Big\}.
\end{aligned} \qquad (+\sqrt{\mathrm{D}})$$

In both cases we consider the initial conditions in (2.15) with the constant parameters yet to be prescribed.

The local solutions to $(-\sqrt{\mathrm{D}})$ and $(+\sqrt{\mathrm{D}})$ might be ensured for $t \in [0, t_0]$ with small $t_0 > 0$. However, our interest concerns global solutions for arbitrary $t \geq 0$, and, if a global solution to either $(-\sqrt{\mathrm{D}})$ or $(+\sqrt{\mathrm{D}})$ exists, in its asymptotic behavior for growing t.

To measure an error of numerical schemes applied, we suggest to estimate the residual of the algebraic equation (2.7):

$$\mathrm{Res} := f_{\mathrm{s}}\mathrm{tr}\boldsymbol{\sigma}\Big\{a^2\dot\varepsilon_2 + \Big(\frac{\sigma_1 D_1 + \sigma_2^0\dot\varepsilon_2}{\mathrm{tr}\boldsymbol{\sigma}}\Big)\frac{\sigma_2^0}{\mathrm{tr}\boldsymbol{\sigma}} + a f_{\mathrm{d}}\Big(\frac{2\sigma_2^0}{\mathrm{tr}\boldsymbol{\sigma}} - \frac{1}{3}\Big)\sqrt{D_1^2+\dot\varepsilon_2^2}\Big\}. \qquad (3.2)$$

From our numerical tests we can report the following features. Refining the uniform time mesh, numerical iterations may diverge or leave a region of the physical consistency, the residual error may remain large or converge very slow, thus theoretical solution be numerically unattainable. Typically we observe only one numerically reasonable and physically consistent solution.

3.1 Biaxial extension test

For a first simulation test, we prescribe the initial stresses and the constant strain rate $\dot\varepsilon_1$ to be

$$\sigma_1^0 = \sigma_2^0 = \sigma_3^0 = -100, \quad \text{and} \quad D_1 = 1, \qquad (3.3)$$

which implies an isotropic state in a domain of parameters satisfying the feasibility conditions. In view of (2.5) and taking into account the sign convention of rational mechanics, the choice $D_1 = 1$ describes a state of vertical extension with a constant strain rate 1. The material parameters for the hypoplastic equations (2.2)–(2.4) are taken from [18]:

$$a = 0.33, \quad f_{\mathrm{d}} = 1, \quad f_{\mathrm{s}} = -550.$$

These data satisfy the solvability condition (2.19) at $t = 0$ and, by continuity, the condition also holds true, at least, in an interval $[0, t_0]$, with some $t_0 > 0$.

Since the constitutive equations are rate independent, we opted for the representation of numerical results with respect to the vertical strain rather than the time evolution. The numerical result for solution $(\sigma_1(t), \varepsilon_2(t), \sigma_3(t))$ of $(+\sqrt{\mathrm{D}})$ under data set (3.3) and a constant lateral stress σ_2 obtained with a MAPLE code is depicted versus the vertical strain $\varepsilon_1(t) = D_1 t$ from (2.5) for $t \in (0, 0.04)$ in the four plots of Figure 1.

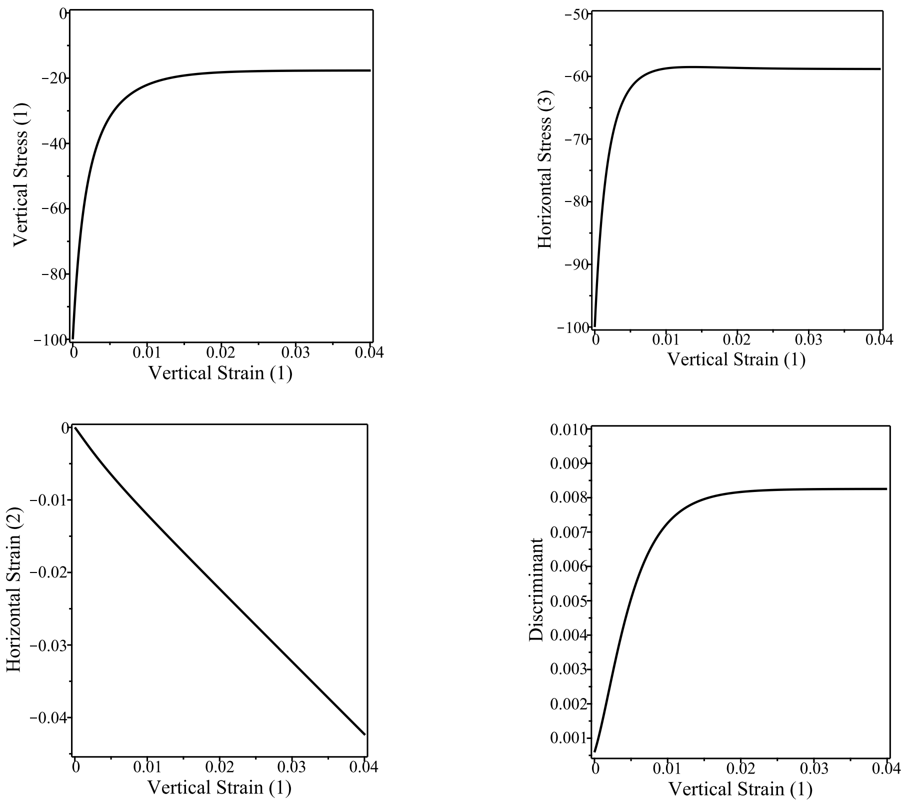

Figure 1: The simulation test of biaxial extension under data set (3.3).

In the upper left and right plots of Figure 1 the first (vertical) and the third (horizontal) stress components are depicted. We observe that the evolution is physically consistent: $\sigma_1 < 0$ and $\sigma_3 < 0$ stress components remain negative during extension and tend from below towards asymptotic values which are related to the parameter a for the stress limit state. In the lower left plot of Figure 1, the horizontal strain component $\varepsilon_2(t)$ is calculated from its rate $\dot{\varepsilon}_2(t)$ and the initial value $\varepsilon_2(0) = 0$. We remark that the strain $\varepsilon_2 \leq 0$ in this case, that is, when extension takes place. Evolution of the discriminant $\mathcal{D}$ is presented in the lower right plot of Figure 1. We can observe that the discriminant is strictly positive and tends towards an asymptotic value.

To check if the solutions converge or diverge by the time discretization, the system of differential algebraic equation is solved in MATLAB using the standard solver RK4. In Figure 2 we show in the log-log scale the absolute value of the residual for the numerical solution of the system $(+\sqrt{\mathrm{D}})$ when decreasing the time mesh size as $\{10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}\}$. For this we calculate the average of $|\mathrm{Res}|$ in (3.2) over time for the current states under equdistant meshing by 101, 1001, 10001, 100001 time points, respectively. In Figure 2 a high rate of convergence of the solution $(+\sqrt{\mathrm{D}})$ when refining the mesh is clearly observed. The evolution of the system $(-\sqrt{\mathrm{D}})$ is not presented here, since its residual is large of order 10^4 and converges very slowly such that the limit (if exists) is numerically unattainable.

Figure 2: Log-log plot of the residual for $(+\sqrt{\mathrm{D}})$ versus time step size.

3.2 Biaxial compression test

Now let us consider the initial values as defined above but changing the sign of the strain rate, that is,

$$\sigma_1^0 = \sigma_2^0 = \sigma_3^0 = -100, \quad \text{and} \quad D_1 = -1, \tag{3.4}$$

which describes a state of vertical compression with a constant strain rate -1.

In this test, the standard numerical schemes are not well behaved. Therefore, from the two possible solutions $\dot{\varepsilon}_2$ we select the one which minimizes the value of $|\mathrm{Res}|$ in (3.2). The general idea of the such selection procedure is commonly used in many numerical methods in which the solution is based on the minimum of a defined residual. Indeed, if we look at the evolution of the residual Res calculated from $(+\sqrt{\mathrm{D}})$ and $(-\sqrt{\mathrm{D}})$ as drawn in the left and right plots of Figure 3, respectively, we observe a point approximately $\varepsilon_1 = -0.01324$, where the zero residual switches from $(+\sqrt{\mathrm{D}})$ to $(-\sqrt{\mathrm{D}})$.

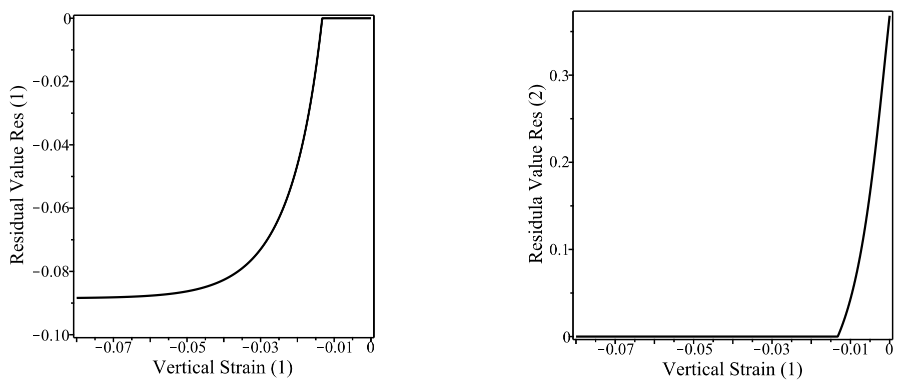

Figure 3: The residual Res in (3.2) for $(+\sqrt{\mathrm{D}})$ and $(-\sqrt{\mathrm{D}})$.

The corresponding numerical solution is depicted versus the vertical strain $\varepsilon_1 = D_1 t$ for $t \in (0, 0.08)$ in Figure 4. We can see in the upper left and right plots that $\sigma_1 < 0$ and $\sigma_3 < 0$. Under the axial compression, the amount of both the vertical stress and the horizontal stress increases, which is physically consistent. Moreover, the stress components show an asymptotical

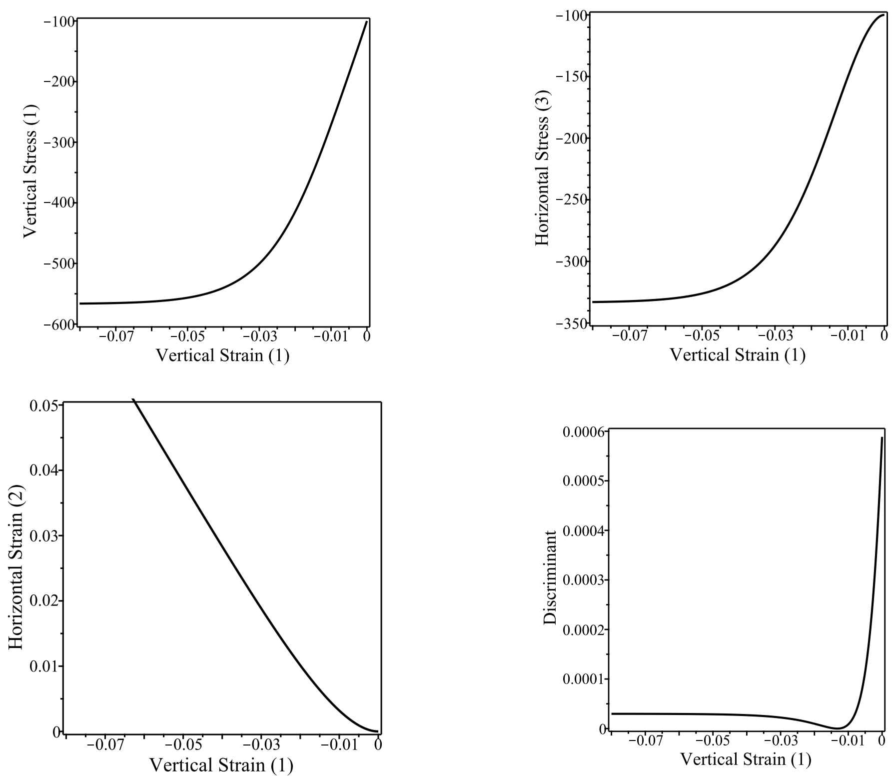

Figure 4: The simulation test of biaxial compression under data set (3.4).

behaviour with continued vertical compression. Note that in the lower left plot $\varepsilon_2 \geq 0$ during the whole evolution. In contrast to the extension test, in the lower right plot the discriminant $\mathcal{D}$ decreases at the beginning of vertical compression, becomes zero at the vertical strain of approximately $\varepsilon_1 = -0.01324$, and afterwards it slightly increases and reaches an almost constant value. Exactly this state is relevant to switch for the solution of $\dot{\varepsilon}_2$ from $(+\sqrt{\mathrm{D}})$ to $(-\sqrt{\mathrm{D}})$.

4 Conclusion

We have studied well-posedness of the hypoplastic constitutive equations carried out in a plane strain biaxial test. Under mixed stress-strain control, we construct two systems of differential algebraic equations, corresponding to the strain rate obtained by solving a quadratic equation. In numerical simulations we find a single feasible solution, provided that the data are chosen in a domain of parameters satisfying the proposed solvability conditions. More detailed investigations are still under way and the results are the topic of an future publication.

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Optimal System of One-Dimensional Subalgebras of the Ramani Equation

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Abstract

This article is a continuation of our previous work under the title "A Study on the exact solutions of the Ramani equation by using Lie symmetry analysis". We studied about the Lie point symmetries, associated similarity reductions and the Painleve analysis of certain symmetry-reduced equations. Furthemore, we computed the travelling wave solutions using an improved $(G'/G)-$ expansion method. This paper focus on the study of the classification of all group-invariant solutions. We therefore derive the one-dimensional optimal system of subalgebras of the Ramani equation using the method outlined in [4] and [3]. The algorithm requires the computation of the commutator table, adjoint representation table, adjoint transformation matrix and calculation of invariants. Finally, we derive the mutually inequivalent one-dimensional subalgebras and present new reductions and solutions with respect to the optimal system. Also, we study here the classification of the admitted four-dimensional Lie algebra.

Keywords: Lie symmetries, optimal system, similarity-reductions, invariant solutions.

1 Introduction

The Lie point symmetries, associated similarity reductions and the Painleve analysis of certain symmetry-reduced equations of the sixth-order nonlinear Ramani equation

$$u_{xxxxxx} + 15(u_x u_{xxxx} + u_{xx}u_{xxx}) + 45u_x^2 u_{xx} - 5(u_{xxxt} + 3u_x u_{xt} + 3u_t u_{xx}) - 5u_{tt} = 0. \quad (1.1)$$

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were studied by the authors in [1]. The travelling-wave solutions were also computed using an improved $(G'/G)-$ expansion method. In this paper we focus on the study of the classification of all group-invariant solutions of (1.1). Because there are an infinite number of subalgebras of a given dimension, a classification is achieved by constructing the optimal system of subalgebras (here one-dimensional). We employ the method outlined in [4] and [3]. The algorithm requires the computation of the commutator table, adjoint representation table, adjoint transformation matrix and calculation of invariants. Finally, we derive the mutually inequivalent one-dimensional subalgebras and present new reductions and solutions with respect to the optimal system. Also, we present the classification of the admitted four-dimensional Lie algebra. It is to be noted that in [2] the authors have computed the optimal system of the Ramani equation. However each member in their system can be seen to be equivalent to a member in the optimal system of one-dimensional subalgebras derived in this study.

2 Construction of the optimal system

According to the authors' findings in [1], the four-dimensional Lie algebra $\mathcal{L}^4$ admitted by (1.1) is spanned by

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= \frac{\partial}{\partial u}, \\ X_4 &= t\frac{\partial}{\partial t} + \frac{x}{3}\frac{\partial}{\partial x} - \frac{u}{3}\frac{\partial}{\partial u}. \end{aligned} \tag{2.1}$$

The commutator Table 1 is obtained by using the Lie bracket $[X_i, X_j] = X_iX_j - X_jX_i$.

Table 1: **Commutator table**

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	0	0	X_1
X_2	0	0	0	$\frac{X_2}{3}$
X_3	0	0	0	$-\frac{X_3}{3}$
X_4	$-X_1$	$-\frac{X_2}{3}$	$\frac{X_3}{3}$	0

This admitted Lie algebra $\mathcal{L}^4$ is the $A_{4,5}^{-\frac{1}{3},\frac{1}{3}}$ algebra in the Patera et al. classification.

Now any element of $\mathcal{L}^4$ can be written as

$$X = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4, a_i \in \mathbb{R}. \tag{2.2}$$

As there is an infinite number of one-dimensional subalgebras for various values of a_i, it is important to identify all the equivalent subalgebras (here one-dimensional) in one class and choose a representative for each class. This results in the optimal system constuting of the representative from each class which are mutually inequivalent.

Two subalgebras $\mathcal{L}_i$ and $\mathcal{L}_j$ of $\mathcal{L}^4$ are quivalent under the adjoint representation if

$$\mathcal{L}_i = Ad\ g(\mathcal{L}_j), g \in \mathcal{G}$$

where (1.1) is invariant under the Lie group of point transformations $\mathcal{G}$. The adjoint represen-atation of the underlying group $\mathcal{G}$ given in Table 2 is constructed using the Lie series

$$Ad(e^{\epsilon X_i})(X_j) = X_j - \epsilon[X_i, X_j] + \frac{1}{2!}\epsilon^2[X_i, [X_i, X_j]] + ... \tag{2.3}$$

Table 2: Adjoint representation table

Ad	X_1	X_2	X_3	X_4
X_1	X_1	X_2	X_3	$X_4 - \epsilon X_1$
X_2	X_1	X_2	X_3	$X_4 - \epsilon \frac{X_2}{3}$
X_3	X_1	X_2	X_3	$X_4 + \epsilon \frac{X_3}{3}$
X_4	$e^{\epsilon} X_1$	$e^{\frac{\epsilon}{3}} X_2$	$e^{-\frac{\epsilon}{3}} X_3$	X_4

2.1 Construction of adjoint transformation matrix A

Now we calculate the general adjoint transformation matrix A. It is the product of the matrices A_1, A_2, A_3, A_4 (taken in any order) which represents the seperate adjoint actions of $X_1, X_2, X_3,$ and X_4 to X respectively. The adjoint action of X_1 to X is given by

$$\begin{aligned} Ad(e^{\epsilon_1 X_1})(X) &= (a_1 - a_4\epsilon_1)X_1 + a_2X_2 + a_3X_3 + a_4X_4 \\ &= [(a_1 - a_4\epsilon_1), a_2, a_3, a_4] \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} \\ &= [a_1, a_2, a_3, a_4] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\epsilon_1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} \\ &= [a_1, a_2, a_3, a_4] A_1 [X_1, X_2, X_3, X_4]^T. \end{aligned} \tag{2.4}$$

In a similar manner, we calculate the matrices A_2, A_3 and A_4, which are given by

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{\epsilon_2}{3} & 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{\epsilon_3}{3} & 1 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} e^{\epsilon_4} & 0 & 0 & 0 \\ 0 & e^{\frac{\epsilon_4}{3}} & 0 & 0 \\ 0 & 0 & e^{-\frac{\epsilon_4}{3}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore the general adjoint transformation matrix A, which is the product of the matrices of the separate adjoint actions A_1, A_2, A_3, A_4 is given by

$$A = \begin{bmatrix} e^{\epsilon_4} & 0 & 0 & 0 \\ 0 & e^{\frac{\epsilon_4}{3}} & 0 & 0 \\ 0 & 0 & e^{-\frac{\epsilon_4}{3}} & 0 \\ -\epsilon_1 e^{\epsilon_4} & \frac{-\epsilon_2 e^{\frac{\epsilon_4}{3}}}{3} & \frac{\epsilon_3 e^{\frac{-\epsilon_4}{3}}}{3} & 1 \end{bmatrix} \tag{2.5}$$

The adjoint transformation equation for (1.1) is,

$$(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4) = (a_1, a_2, a_3, a_4)A$$

where A is the the general adjoint transformation matrix A. This results in a system of equations:

$$\begin{cases} \tilde{a}_1 = a_1 e^{\epsilon_4} - a_4 \epsilon_1 e^{\epsilon_4} \\ \tilde{a}_2 = a_2 e^{\frac{\epsilon_4}{3}} - a_4 \dfrac{\epsilon_2 e^{\frac{\epsilon_4}{3}}}{3} \\ \tilde{a}_3 = a_3 e^{\frac{-\epsilon_4}{3}} + a_4 \dfrac{\epsilon_3 e^{\frac{-\epsilon_4}{3}}}{3} \\ \tilde{a}_4 = a_4 \end{cases} \tag{2.6}$$

If the above system has a solution, then X is equivalent to $\tilde{X} = \tilde{a}_1 X_1 + \tilde{a}_2 X_2 + \tilde{a}_3 X_3 + \tilde{a}_4 X_4$.

2.2 Calculation of the invariants

A real function ϕ on the Lie algebra $\mathcal{L}$ is called an invariant [4] if

$$\phi(Adg(v)) = \phi(v), \forall v \in \mathcal{L} \text{ and } \forall g \in \mathcal{G}.$$

Finding such an invariant, as *Olver* [4] stated, is essential because it restricts the amount of simplification that may be expected for X. Now the adjoint action of $w = \sum_{i=1}^{n} b_i X_i$ to $v = \sum_{j=1}^{n} a_j X_j$ is given by

$$\begin{aligned} Ad(e^{\epsilon w})(v) &= v - \epsilon[w, v] + \frac{\epsilon^2}{2!}[w, [w, v]] + ... \\ &= (a_1 v_1 + a_2 v_2 + ... + a_n v_n) - \epsilon(\Theta_1 v_1 + \Theta_2 v_2 + ... + \Theta_n v_n) + \mathcal{O}(\epsilon^2). \end{aligned} \tag{2.7}$$

To determine the invariant ϕ, expanding the RHS of (2.7), we get

$$\begin{aligned} &\phi(a_1 - \epsilon\Theta_1, a_2 - \epsilon\Theta_2, ...a_n - \epsilon\Theta_n + \mathcal{O}(\epsilon^2)) \\ &\quad = \phi(a_1, a_2, ..., a_n) - \epsilon(\Theta_1 \frac{\partial \phi}{\partial a_1} + ... + \Theta_n \frac{\partial \phi}{\partial a_n}) + \mathcal{O}(\epsilon^2) \end{aligned} \tag{2.8}$$

and we require,

$$\Theta_1 \frac{\partial \phi}{\partial a_1} + ... + \Theta_n \frac{\partial \phi}{\partial a_n} = 0 \tag{2.9}$$

for any b_i. Extracting the coefficients of all b_i, $N(\leq n)$ linear differential equations of ϕ are obtained. Now (2.9) can also be written as

$$\sum_{j=1}^{4} c_{i,j}^{k} a_j \frac{\partial \phi}{\partial a_k} = 0, i = 1, 2, 3, 4 \tag{2.10}$$

where $[X_i, X_j] = c_{i,j}^{k} X_k$. For i = 1, the L.H.S. of (2.10) is given by,

$$\sum_{j=1}^{4} c_{1,j}^{k} a_j \frac{\partial \phi}{\partial a_k} = c_{1,1}^{k} a_1 \frac{\partial \phi}{\partial a_k} + c_{1,2}^{k} a_2 \frac{\partial \phi}{\partial a_k} + c_{1,3}^{k} a_3 \frac{\partial \phi}{\partial a_k} + c_{1,4}^{k} a_4 \frac{\partial \phi}{\partial a_k}$$

$$\therefore a_4 \frac{\partial \phi}{\partial a_1} = 0. \tag{2.11}$$

Therefore we have a system of PDEs,

$$\begin{cases} a_4\dfrac{\partial\phi}{\partial a_1} = 0, \\ \dfrac{a_4}{3}\dfrac{\partial\phi}{\partial a_2} = 0, \\ -\dfrac{a_4}{3}\dfrac{\partial\phi}{\partial a_3} = 0, \\ -a_1\dfrac{\partial\phi}{\partial a_1} - \dfrac{a_2}{3}\dfrac{\partial\phi}{\partial a_2} + \dfrac{a_3}{3}\dfrac{\partial\phi}{\partial a_3} = 0. \end{cases} \tag{2.12}$$

Solving the above equations, we get the invariant function

$$\phi(a_1, a_2, a_3, a_4) = F(a_4). \tag{2.13}$$

We discuss two cases $a_4 \neq 0$ and $a_4 = 0$. Next, we consider an element of $\mathcal{L}$ given by

$$X = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4$$

and simplify as many as coefficients by the application of adjoint maps.

Case 1. $a_4 \neq 0$. Assume $a_4 = 1$. Consider

$$\begin{aligned} \tilde{X} &= Ad(e^{\epsilon_4X_4})Ad(e^{\epsilon_3X_3})Ad(e^{\epsilon_2X_2})Ad(e^{\epsilon_1X_1})X \\ &= Ad(e^{\epsilon_4X_4})Ad(e^{\epsilon_3X_3})Ad(e^{\epsilon_2X_2})(Ad(e^{\epsilon_1X_1})(a_1X_1 + a_2X_2 + a_3X_3 + X_4)) \\ &= Ad(e^{\epsilon_4X_4})Ad(e^{\epsilon_3X_3})Ad(e^{\epsilon_2X_2})\left((a_1 - \epsilon_1)X_1 + a_2X_2 + a_3X_3 + X_4\right) \\ \therefore \tilde{X} &= \Big((a_1 - \epsilon_1)X_1 + (a_2 - \frac{\epsilon_2}{3})X_2 + (a_3 + \frac{\epsilon_3}{3})X_3 + X_4\Big) \end{aligned} \tag{2.14}$$

We choose $\tilde{a_1} = 0, \tilde{a_2} = 0, \tilde{a_3} = 0$ and $\epsilon_4 = 0$. So the first representative element is X_4.

Case 2. $a_4 = 0, a_3 \neq 0$. Let $a_3 = 1$ Then,

$$\begin{aligned} X &= a_1X_1 + a_2X_2 + X_3 \\ \tilde{X} &= Ad(e^{\epsilon_4X_4})Ad(e^{\epsilon_3X_3})Ad(e^{\epsilon_2X_2})Ad(e^{\epsilon_1X_1})X \\ &= a_1e^{\epsilon_4}X_1 + a_2e^{\frac{\epsilon_4}{3}}X_2 + e^{\frac{-\epsilon_4}{3}}X_3. \end{aligned} \tag{2.15}$$

With $\epsilon_4 = 0$, we have the next representative element,

$$a_1X_1 + a_2X_2 + X_3.$$

Case 3. $a_4 = 0, a_3 = 0, a_2 \neq 0$. Let $a_2 = 1$ Then,

$$\begin{aligned} X &= a_1X_1 + X_2 \\ \tilde{X} &= Ad(e^{\epsilon_4X_4})Ad(e^{\epsilon_3X_3})Ad(e^{\epsilon_2X_2})Ad(e^{\epsilon_1X_1})X \\ &= a_1e^{\epsilon_4}X_1 + e^{\frac{\epsilon_4}{3}}X_2. \end{aligned} \tag{2.16}$$

With $\epsilon_4 = 0$, we have the next representative element,

$$a_1X_1 + X_2.$$

Case 4. $a_4 = 0, a_3 = 0, a_2 = 0, a_1 \neq 0$. Let $a_1 = 1$ Then we have the next representative element,

$$X_1.$$

Thus the one-dimensional optimal system of subalgebras is given by,

$$X_4, a_1X_1 + a_2X_2 + X_3, a_1X_1 + X_2, X_1.$$

3 Similarity-reductions

The similarity variable and the corresponding solution form with respect to each symmetry in the optimal system are presented below.

Symmetry	Similarity Variable	Similarity Solution
X_4	$\gamma = \frac{x}{t^{\frac{1}{3}}}$	$u(x,t) = \frac{v(\gamma)}{t^{\frac{1}{3}}}$
$aX_1 + bX_2 + X_3$	$\gamma = x - \frac{b}{a}t$	$u(x,t) = v(\gamma) + t$
$cX_1 + X_2, , c \neq 0$	$\gamma = x - kt,\ k = \frac{1}{c}$	$u(x,t) = v(\gamma)$
X_1	$\gamma = x$	$u(x,t) = v(\gamma)$

It is to be noted that the reductions using the symmetries X_4, cX_1+X_2, X_1 and the corresponding solutions has been given by the authors in [1]. So we discuss only the following case:

Reduction using $aX_1 + bX_2 + X_3$:

$$a(av'''''' + 5(3av' + b)v'''') + 5(3a^2v''' + 9a^2v'^2 + 6abv' - b^2 - 3a)v'' = 0 \tag{3.1}$$

For $a \neq 0$ and $b = 0$, we get the following reduction using $aX_1 + X_3$:

$$a(v'''''' + 15v''''v') + 15(av''' + 3av'^2 - 1)v'' = 0, \tag{3.2}$$

where $u(x,t) = v(x) + \dfrac{t}{a}$.

Equations (3.1) and (3.2) have two symmetries namely $\frac{\partial}{\partial v}$ and $\frac{\partial}{\partial \nu}$. Using these symmetries, the subsequent reduction leads to a nonlinear fourth-order ODE which has no point symmetries.

However equation (3.1) and (3.2) is analogous to the symmetry-reduced equation obtained using cX_1+X_2 in which the solutions is analysed in [1] using improved $(\frac{G'}{G})-$expansion method.

For $b \neq 0$ and $a = 0$, we get the following reduction using $bX_2 + X_3$:

$$v'' = 0, \tag{3.3}$$

where $u(x,t) = v(t) + \dfrac{x}{b}$.

4 Conclusion

We have derived the one-dimensional optimal system of subalgebras of the Ramani equation. It is seen that the symmetries in the optimal system derived by the authors in [2] to be equivalent to one of the symmetry in the system derived in this paper. Thus all details corresponding to the symmetry-reductions with respect to each member of the optimal system and possible solution have been presented. This includes some new reductions and also explicit invariant solution in one of the cases which was not reported in [1].

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Study of Instability for Spherically Symmetrical Dynamic Equilibrium States of Self-Gravitating Vlasov-Poisson Gas

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Abstract

This work considers the linear stability problem for spherically symmetrical states of dynamic equilibrium of a boundless collisionless self-gravitating Vlasov-Poisson gas with respect to perturbations of the same symmetry by the direct Lyapunov method. Using two changes of independent variables, the transition from kinetic equations to two infinite systems of gas-dynamic equations of the "vortex shallow water" type in the Boussinesq approximation was carried out, and absolute linear instability for the studied stationary solutions in the gas-dynamic description was proved. With the help of the first version for change of independent variables, the formal nature of well-known Antonov criterion for linear stability of dynamic equilibrium states of self-gravitating stellar systems was discovered, so that this criterion is valid only with regard to some incomplete unclosed subclass of small perturbations. The same fundamental differential inequalities for the Lyapunov functionals were deduced in each case of independent variables replacement. Also, along with them, the constructive sufficient conditions for linear practical instability of the considered states of dynamic equilibrium with respect to spherically symmetrical perturbations were obtained. Eventually, for both changes of independent variables, the a priori exponential estimates from below were found, and initial data was described for the studied small perturbations increasing in time. To confirm the results obtained, for the second version of independent variables replacement, analytical examples of the considered dynamic equilibrium states and small spherically symmetrical perturbations superimposed on them, which grow in time according to the found estimates, were constructed.

Keywords: Vlasov-Poisson equations, spherical symmetry, stationary solutions, small perturbations, direct Lyapunov method, Antonov criterion, hydrodynamic substitution, gas-dynamic equations, Lyapunov functional, differential inequality, a priori estimate, instability, analytical example.

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1 Introduction

The Vlasov equation is a differential equation describing the time evolution of the distribution function of a plasma composed of charged particles with long-range interactions such as Coulomb ones. A. Vlasov first proposed this equation to describe the plasma in 1938 [1] and later discussed it in detail in his monograph [2].

The Vlasov equation has many connections with other equations. It is an important type of dynamical equations which can be used to describe physical phenomena, such as the motion of nebulae and the evolution of plasma. Because this equation has a very important position in dynamics, it not only attracted the attention of many physicists, but also prompted a large number of mathematicians to engage in research in this area.

The sufficient stability conditions for exact stationary solutions to the kinetic Vlasov-Poisson equations have already been found previously, as indicated in the following publications [3–7]. As far as we are aware, these conditions have not been reversed until now (neither for small perturbations, nor, in particular, for finite ones) [4]. A new transition from the kinetic Vlasov-Poisson equations to the gas-dynamic equations is found in [8]. For the last equations, there are methods to reverse sufficient stability conditions (at least, in the linear approximation) [9]. By utilizing the direct Lyapunov method in the present paper, we demonstrate that the spherically symmetrical states of dynamic equilibrium of the boundless collisionless self-gravitating Vlasov-Poisson gas are absolutely unstable with regard to small perturbations of the same symmetry [9].

V. Antonov conducted in-depth study on self-gravitating systems. In [6], he studied the stellar system with an isotropic velocity distribution and Emden's polytropic density by applying a criterion previously derived by him in [5]. He assumed too that the velocity diagram is spherically symmetrical at any distance from the center. As a result, V. Antonov proved that the stellar system, in which the phase density is a decreasing function of the energy integral, is stable.

Unexpectedly, Antonov's conclusion [6] runs counter to the basic findings in our study on instability. The explanation for this is that his conclusion is conditional and valid for some incomplete unclosed subclass of small spherically symmetrical perturbations. More consideration will also be given to Antonov's article [5].

2 The kinetic Vlasov-Poisson equations

In the spherically symmetrical space, the self-gravitating Vlasov-Poisson gas is described by kinetic equations shown below [10, 11]

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial r} - \frac{\partial \varphi}{\partial r}\frac{\partial f}{\partial v} = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \varphi}{\partial r}\right) = 16\pi^2\left(\int_0^\infty f(r,v,t)v^2\,dv - n_g\right) \tag{2.1}$$

$$f(r,v,t) = f_0(r,v)$$

where $f = f(r,v,t) \geq 0$ is the distribution function of gas particles (for simplicity of being used, the mass of particles is set equal to unity); $t \in [0,\infty)$ denotes time; $r, v \in [0,\infty)$ are coordinates and velocities of gas particles respectively; $\varphi = \varphi(r,t)$ denotes the potential of a self-consistent gravitating field; $n_g \equiv const > 0$ is the gas density in some spherically symmetrical static state of global thermodynamic equilibrium; $f_0(r,v)$ denotes the initial data for distribution function f. We suppose that $f \to 0$ when $v \to \infty$, and the functions $f, \varphi \to 0$ or are periodic when $r \to \infty$.

The following integrals are preserved on the exact evolutionary solutions to system (2.1)

$$E \equiv 8\pi^2 \iint_0^\infty f v^4 r^2 \, dr dv - \frac{1}{2} \int_0^\infty \left(r \frac{\partial \varphi}{\partial r} \right)^2 dr = const \tag{2.2}$$

$$C \equiv 16\pi^2 \int_0^\infty \Phi(f) v^2 r^2 \, dr dv = const.$$

Here, E denotes the functional of full energy, C is the integral of motion, $\Phi = \Phi(f)$ denotes an arbitrary function of its argument.

It is assumed that kinetic system (2.1) has the exact stationary solutions

$$f = f^0(v), \ \varphi = \varphi^0 \equiv const \tag{2.3}$$

which satisfy the following stationary equation

$$\int_0^\infty f^0(v) v^2 \, dv = n_g. \tag{2.4}$$

Solutions (2.3), (2.4) correspond to some spherically symmetrical dynamic state of local thermodynamic equilibria.

These exact stationary solutions will be investigated for stability with respect to small spherically symmetrical perturbations. For such purpose, system (2.1) is linearized near solutions (2.3), (2.4) and has the form

$$\frac{\partial f'}{\partial t} + v \frac{\partial f'}{\partial r} - \frac{\partial \varphi'}{\partial r} \frac{df^0}{dv} = 0$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \varphi'}{\partial r} \right) = 16\pi^2 \int_0^\infty f'(r, v, t) v^2 \, dv \tag{2.5}$$

$$f'(r, v, 0) = f_0'(r, v)$$

where, according to the linearization procedure, the sought functions are represented in the form $f(r, v, t) = f^0(v) + f'(r, v, t)$ and $\varphi(r, t) = \varphi^0 + \varphi'(r, t)$; f', φ' denote the small perturbations; $f_0'(r, v)$ is the initial data for perturbed distribution function f'.

Let $I \equiv E + C$ (see (2.2)). The first variation of functional I is calculated as

$$\delta I = 16\pi^2 \iint_0^\infty \left[\frac{v^2 + \varphi^0}{2} + \frac{d\Phi}{df} \left(f^0 \right) \right] \delta f v^2 r^2 \, dr dv.$$

Here, δf denotes the first variation of distribution function f. The condition is written out when the integral δI is equal to zero:

$$\frac{v^2 + \varphi^0}{2} = -\frac{d\Phi}{df} \left(f^0 \right).$$

The second variation of functional I is calculated as

$$\delta^2 I = 8\pi^2 \iint_0^\infty \frac{d^2\Phi}{df^2} \left(f^0 \right) \left(\delta f \right)^2 v^2 r^2 \, dr dv - \frac{1}{2} \int_0^\infty \left(r \frac{\partial \delta\varphi}{\partial r} \right)^2 dr \tag{2.6}$$

where $\delta\varphi$ is the first variation for potential φ of self-consistent gravitating field.

If the first variations δf and $\delta\varphi$ are replaced by small perturbations f' and φ', then second variation $\delta^2 I$ (2.6) of the integral I will turn into a linear analogue E_1 of the full energy functional E

$$E_1 = 8\pi^2 \iint_0^\infty \frac{d^2\Phi}{df^2} \left(f^0 \right) f'^2 v^2 r^2 \, dr dv - \frac{1}{2} \int_0^\infty \left(r \frac{\partial \varphi'}{\partial r} \right)^2 dr$$

which will be preserved on the evolutionary solutions to system (2.5). It can be seen directly that the inequality

$$\frac{d^2\Phi}{df^2}\left(f^0\right) \le 0$$

or, equivalently,

$$\frac{1}{v}\frac{df^0}{dv} \ge 0 \tag{2.7}$$

is the sufficient condition for linear stability of exact stationary solutions (2.3), (2.4) to kinetic system (2.1) with regard to spherically symmetrical perturbations f', φ' (2.5).

However, any function f^0 is decreasing in accordance with the boundary conditions for system (2.1). So, inequality (2.7) cannot hold in principle. Hence, there is no stationary distribution functions f^0 (2.3), (2.4) which satisfy sufficient condition (2.7).

This means that the integral E_1 can be non-negative for decreasing functions f^0 if and only if the first term of functional E_1 is not less than its second term. In [6], V. Antonov obtained such condition for small spherically symmetrical perturbations in the form of normal modes – the well-known Antonov criterion [5]. However, for linear partial differential equations with variable coefficients, normal modes do not represent a complete closed system of functions. Thus, the Antonov criterion [5,6] for linear stability is formal: it is valid just for some incomplete unclosed subclass of small spherically symmetrical perturbations.

According to the Antonov criterion's conditional nature, we can provide a hypothesis about the absolute instability for exact stationary solutions (2.3), (2.4) to kinetic system (2.1) with respect to small spherically symmetrical perturbations f', φ' (2.5).

To verify this hypothesis, it is convenient to perform two non-degenerate changes of independent variables – the so-called hydrodynamic substitutions [8, 11]. So, we make two transitions from the Eulerian independent variables r, v, t to the mixed Eulerian-Lagrangian independent variables r, ν, t, where, according to the definition of Lagrangian coordinates, $d\nu/dt = 0$. Specifically,

$$v = u(r,\nu,t),\; v^2r^2f(r,v,t) = \rho_1(r,\nu,t)\left[\frac{\partial u}{\partial \nu}(r,\nu,t)\right]^{-1} \tag{2.8}$$

and

$$v = u(r,\nu,t),\; f(r,v,t) = \rho_2(r,\nu,t)\left[\frac{\partial u}{\partial \nu}(r,\nu,t)\right]^{-1}. \tag{2.9}$$

Hence, kinetic equations (2.1) are transformed, and we can write two systems of the gas-dynamic type equations for vortex shallow water in the Boussinesq approximation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} = -\frac{\partial \varphi}{\partial r},\quad \frac{\partial \rho_1}{\partial t} + \frac{\partial\left(u\rho_1\right)}{\partial r} = 0$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial \varphi}{\partial r}\right) = 16\pi^2\left(\int_0^\infty \rho_1(r,\nu,t)\,d\nu - r^2 n_g\right) \tag{2.10}$$

$$u(r,\nu,0) = u_0(r,\nu),\; \rho_1(r,\nu,0) = \rho_{10}(r,\nu)$$

and

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} = -\frac{\partial \varphi}{\partial r},\quad \frac{\partial \rho_2}{\partial t} + \frac{\partial\left(u\rho_2\right)}{\partial r} = 0$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial \varphi}{\partial r}\right) = 16\pi^2r^2\left(\int_0^\infty \rho_2(r,\nu,t)u^2\,d\nu - n_g\right) \tag{2.11}$$

$$u(r,\nu,0)=u_0(r,\nu),\ \rho(r,\nu,0)=\rho_{20}(r,\nu).$$

Here, u, ρ_1, and ρ_2 denote the velocity and two density fields of gas particles respectively with the initial data u_0, ρ_{10}, and ρ_{20}. We suppose that the functions u, ρ_1, and ρ_2 approach zero as $\nu\to\infty$ and, together with the function φ, are periodic on r or approach zero as $r\to\infty$.

At the exact evolutionary solutions to gas-dynamic system (2.10) for the first hydrodynamic substitution (2.8), the following functionals are preserved

$$E_2\equiv 8\pi^2\iint_0^\infty \rho_1 u^2\,d\nu dr-\frac{1}{2}\int_0^\infty\left(\frac{\partial\varphi}{\partial r}\right)^2 r^2\,dr=const \tag{2.12}$$

$$C_1\equiv 16\pi^2\iint_0^\infty \Phi_1(\kappa_1)\frac{\partial u}{\partial\nu}\,d\nu dr=const;\ \kappa_1\equiv\rho_1\left(\frac{\partial u}{\partial\nu}\right)^{-1}:\ \frac{\partial\kappa_1}{\partial t}+u\frac{\partial\kappa_1}{\partial r}=0$$

where E_2 is the integral of full energy, C_1 denotes the functional of motion, $\Phi_1=\Phi_1(\kappa_1)$ is an arbitrary function of its argument, and κ_1 denotes the reverse vorticity.

The last relation for function κ_1 from (2.12) guarantees the mutual uniqueness for replacement (2.8) of independent variables. Indeed, if $\partial u/\partial\nu\neq 0$ at $t=0$, then, in accordance with the equation for κ_1 in (2.12), it will remain nonzero at all $t>0$.

It is assumed that system (2.10) for hydrodynamic substitution (2.8) has the exact stationary solutions

$$u=u^0(r,\nu),\ \rho_1=\rho_1^0(r,\nu),\ \varphi=\varphi^0(r) \tag{2.13}$$

which satisfy the following stationary equations

$$u^0\frac{\partial u^0}{\partial r}=-\frac{d\varphi^0}{dr},\ \frac{\partial}{\partial r}\left(\rho_1^0u^0\right)=0 \tag{2.14}$$

$$\frac{d}{dr}\left(r^2\frac{d\varphi^0}{dr}\right)=16\pi^2\left(\int_0^\infty \rho_1^0(r,\nu)d\nu-r^2n_g\right).$$

It is supposed also that gas-dynamic system (2.11) for the second hydrodynamic substitution (2.9) has such exact stationary solutions:

$$u=u^0(\nu),\ \rho_2=\rho_2^0(\nu),\ \varphi=\varphi^0\equiv const. \tag{2.15}$$

Here, the functions u^0 and ρ_2^0 satisfy the following stationary equation

$$\int_0^\infty \rho_2^0(\nu)u^{02}\,d\nu=n_g. \tag{2.16}$$

Solutions (2.13), (2.14) and (2.15), (2.16) are equivalent to exact stationary solutions (2.3), (2.4) to kinetic system (2.1).

Next, these exact stationary solutions will be studied for linear stability with regard to spherically symmetrical perturbations.

To achieve such purpose, system (2.10) for hydrodynamic substitution (2.8) is linearized near exact stationary solutions (2.13), (2.14) and is written as

$$\frac{\partial u'}{\partial t}+u'\frac{\partial u^0}{\partial r}+u^0\frac{\partial u'}{\partial r}=-\frac{\partial\varphi'}{\partial r}$$

$$\frac{\partial\rho_1'}{\partial t}+\frac{\partial}{\partial r}\left(\rho_1^0u'+u^0\rho_1'\right)=0,\ \frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi'}{\partial r}\right)=16\pi^2\int_0^\infty\rho_1'\,d\nu \tag{2.17}$$

$$u'(r,\nu,0) = u'_0(r,\nu),\ \rho'_1(r,\nu,0) = \rho'_{10}(r,\nu).$$

In addition, gas-dynamic system (2.11) for hydrodynamic substitution (2.9) is linearized in the vicinity of exact stationary solutions (2.15), (2.16) and is written as

$$\frac{\partial u'}{\partial t} + u^0\frac{\partial u'}{\partial r} = -\frac{\partial \varphi'}{\partial r},\ \frac{\partial \rho'_2}{\partial t} + \rho_2^0\frac{\partial u'}{\partial r} + u^0\frac{\partial \rho'_2}{\partial r} = 0$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial \varphi'}{\partial r}\right) = 16\pi^2 r^2\int_0^\infty u^0\left(\rho'_2 u^0 + 2u'\rho_2^0\right)d\nu \tag{2.18}$$

$$u'(r,\nu,0) = u'_0(r,\nu),\ \rho'_2(r,\nu,0) = \rho'_{20}(r,\nu)$$

where $u' = u'(r,\nu,t)$, $\rho'_1 = \rho'_1(r,\nu,t)$, and $\rho'_2 = \rho'_2(r,\nu,t)$ are the small spherically symmetrical perturbations; $u'_0(r,\nu)$, $\rho'_{10}(r,\nu)$, and $\rho'_{20}(r,\nu)$ denote the initial data for perturbed velocity u', two density ρ'_1 and ρ'_2 fields of gas particles respectively.

Let $I_1 \equiv E_2 + C_1$ (see (2.12)). The first variation δI_1 of integral I_1 is calculated; the condition is written out

$$\frac{u^{02}}{2} + \varphi^0 = -\frac{d\Phi_1}{d\kappa_1}\left(\kappa_1^0\right)$$

under which the functional δI_1 is equal to zero (κ_1^0 is the stationary reverse vorticity). The second variation of integral I_1 is calculated as

$$\delta^2 I_1 = 8\pi^2\iint_0^\infty\left[2u^0\delta u\delta\rho_1 + \rho_1^0(\delta u)^2 + \frac{d^2\Phi_1}{d\kappa_1^2}(\kappa_1^0)(\delta\kappa_1)^2\frac{\partial u^0}{\partial \nu}\right]d\nu dr-$$

$$-\frac{1}{2}\int_0^\infty\left(r\frac{\partial \delta\varphi}{\partial r}\right)^2 dr. \tag{2.19}$$

Here, δu, $\delta\rho_1$, and $\delta\kappa_1$ denote the first variations of velocity, density, and reverse vorticity fields of gas particles respectively.

If the first variations δu, $\delta\rho_1$, $\delta\kappa_1$ and $\delta\varphi$ are replaced by small spherically symmetrical perturbations u', ρ'_1, κ'_1 and φ', then second variation $\delta^2 I_1$ (2.19) of functional I_1 will turn into a linear analogue E_3 of full energy integral E_2 (2.12)

$$E_3 = 8\pi^2\iint_0^\infty\left[2u^0 u'\rho'_1 + \rho_1^0 u'^2 + \frac{d^2\Phi_1}{d\kappa_1^2}(\kappa_1^0)\kappa_1'^2\frac{\partial u^0}{\partial \nu}\right]d\nu dr - \frac{1}{2}\int_0^\infty\left(r\frac{\partial \varphi'}{\partial r}\right)^2 dr \tag{2.20}$$

which will be preserved on the evolutionary solutions to linearized system (2.17) for hydrodynamic substitution (2.8).

Unfortunately, according to the Sylvester criterion [12], there is no condition for exact stationary solutions (2.13), (2.14) to gas-dynamic system (2.10) so that the functional E_3 is not positive or negative with respect to small spherically symmetrical perturbations u', ρ'_1, and φ' (2.17). Therefore, the Antonov criterion [5,6] for linear stability is formal: it holds only for some incomplete unclosed subclass of these small perturbations.

3 A priori exponential lower estimates

We will prove the absolute instability for exact stationary solutions (2.13), (2.14) and (2.15), (2.16) with regard to such subclasses of the corresponding small spherically symmetrical perturbations (2.17) and (2.18), which are determined by the relations

$$\frac{\partial \xi}{\partial t} = u' + \xi\frac{\partial u^0}{\partial r} - u^0\frac{\partial \xi}{\partial r} \tag{3.1}$$

and

$$\frac{\partial \xi}{\partial t} = u' - u^0 \frac{\partial \xi}{\partial r} \tag{3.2}$$

where $\xi = \xi(r, \nu, t)$ is the field of Lagrangian displacements [13].

The linearized systems of gas-dynamic equations (2.17) and (2.18) can be rewritten in terms of the fields of Lagrangian displacements ξ (3.1) and (3.2) as follows

$$\frac{\partial^2 \xi}{\partial t^2} + 2u^0 \frac{\partial^2 \xi}{\partial r \partial t} + u^0 \frac{\partial}{\partial r}\left(u^0 \frac{\partial \xi}{\partial r}\right) - \xi \frac{\partial}{\partial r}\left(u^0 \frac{\partial u^0}{\partial r}\right) = -\frac{\partial \varphi'}{\partial r}$$

$$\rho_1' = -\frac{\partial}{\partial r}\left(\xi \rho_1^0\right), \ \frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi'}{\partial r}\right) = -16\pi^2 \int_0^\infty \frac{\partial}{\partial r}\left(\xi \rho_1^0\right) d\nu \tag{3.3}$$

$$\xi(r, \nu, 0) = \xi_0(r, \nu), \ \frac{\partial \xi}{\partial t}(r, \nu, 0) = \left(\frac{\partial \xi}{\partial t}\right)_0 (r, \nu)$$

and

$$\frac{\partial^2 \xi}{\partial t^2} + 2u^0 \frac{\partial^2 \xi}{\partial r \partial t} + u^{02} \frac{\partial^2 \xi}{\partial r^2} = -\frac{\partial \varphi'}{\partial r}$$

$$\rho_2' = -\rho_2^0 \frac{\partial \xi}{\partial r}, \ \frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi'}{\partial r}\right) = 16\pi^2 r^2 \int_0^\infty u^0 \rho_2^0 \left(2\frac{\partial \xi}{\partial t} + u^0 \frac{\partial \xi}{\partial r}\right) d\nu \tag{3.4}$$

$$\xi(r, \nu, 0) = \xi_0(r, \nu), \ \frac{\partial \xi}{\partial t}(r, \nu, 0) = \left(\frac{\partial \xi}{\partial t}\right)_0 (r, \nu).$$

For small spherically symmetrical perturbations (3.1), (3.3) with the asymptotics

$$\int_0^\infty \left[\left(u^0 \kappa_1^0 u'^2\right)\Big|_{\nu\to\infty} - \left(u^0 \kappa_1^0 u'^2\right)\Big|_{\nu\to 0}\right] dr \to 0 \tag{3.5}$$

that imposes upper limit on allowable values of the kinetic energy for individual gas particles, linear analog E_3 (2.20) of full energy integral E_2 (2.12) takes the form

$$E_3 = -8\pi^2 \iint_0^\infty u^0 \frac{d\kappa_1^0}{d\nu}\left(\frac{\partial \xi}{\partial t} - \xi \frac{\partial u^0}{\partial r} + u^0 \frac{\partial \xi}{\partial r}\right)^2 d\nu dr - \frac{1}{2}\int_0^\infty \left(r \frac{\partial \varphi'}{\partial r}\right)^2 dr. \tag{3.6}$$

It follows that the inequality

$$u^0 \frac{d\kappa_1^0}{d\nu} \geq 0 \tag{3.7}$$

is the condition of definiteness in sign for functional E_3 (3.6) with respect to small spherically symmetrical perturbations (3.1), (3.3), and (3.5). This inequality is equivalent to condition (2.7). So, analogously, inequality (3.7) does not hold in principle: the functions κ_1^0 are decreasing in accordance with the boundary conditions for system of gas-dynamic type equations (2.10) in vortex shallow water and the Boussinesq approximations. Therefore, there is no stationary reverse vorticities κ_1^0 satisfying condition (3.7).

This means again that the integral E_3 can be non-negative for decreasing functions κ_1^0 only due to the Antonov criterion [5,6] when the first positive term of functional E_3 (3.6) is no less than the second negative one. However, the Antonov criterion for linear stability is formal: it is valid only for some incomplete unclosed subclass of small spherically symmetrical perturbations (3.1), (3.3).

Let us introduce the integrals M [8, 9, 11] – the Lyapunov functionals in our case:

$$M \equiv 16\pi^2 \iint_0^\infty \rho_i^0 \xi^2 d\nu dr;\ i = 1,\ 2. \tag{3.8}$$

The first and second derivatives of the integrals M with regard to time t are calculated along the corresponding evolutionary solutions to systems (3.1), (3.3) and (3.2), (3.4) ($i = 1, 2$):

$$\frac{dM}{dt} = 32\pi^2 \iint_0^\infty \rho_i^0 \xi \frac{\partial \xi}{\partial t} d\nu dr,\ \frac{d^2 M}{dt^2} = 32\pi^2 \iint_0^\infty \rho_i^0 \left[\left(\frac{\partial \xi}{\partial t} \right)^2 + \xi \frac{\partial^2 \xi}{\partial t^2} \right] d\nu dr. \tag{3.9}$$

By the relations (3.1)–(3.4), (3.8), (3.9), the following differential inequalities for the functionals M [9] hold:

$$\frac{d^2 M}{dt^2} - 2\lambda \frac{dM}{dt} + 2\left(\lambda^2 + \alpha_i\right) M \geq 0;\ i = 1,\ 2. \tag{3.10}$$

Here, λ denotes a constant, α_1 and α_2 are the known positive constant values.

Notice that the relation (3.10) is deduced for system of equations (3.2), (3.4) when the solutions ξ and φ' to this system are determined as $\xi = g(t)h(r,\nu)$ and $\varphi' = g(t)\varphi_0(r)$, where $g(t)$, $h(r,\nu)$, and $\varphi_0(r)$ denote some functions of their arguments. It is important that such definition of the solutions ξ and φ' to system of equations (3.2), (3.4) is not accompanied by a loss of generality in any way.

If $\lambda > 0$, then, according to the Chaplygin method [8, 9, 11], the inequalities (3.10) supplemented by the countable set of conditions [9]

$$\begin{gathered}
M\left(\frac{\pi n}{2\sqrt{\lambda^2 + 2\alpha_i}}\right) > 0; n = 0, 1, 2, \ldots;\ i = 1,\ 2 \\
\frac{dM}{dt}\left(\frac{\pi n}{2\sqrt{\lambda^2 + 2\alpha_i}}\right) \geq 2\left(\lambda + \frac{\alpha_i}{\lambda}\right) M \left(\frac{\pi n}{2\sqrt{\lambda^2 + 2\alpha_i}}\right) \\
M\left(\frac{\pi n}{2\sqrt{\lambda^2 + 2\alpha_i}}\right) \equiv M(0) \exp\left(\frac{\lambda \pi n}{2\sqrt{\lambda^2 + 2\alpha_i}}\right) \\
\frac{dM}{dt}\left(\frac{\pi n}{2\sqrt{\lambda^2 + 2\alpha_i}}\right) \equiv \frac{dM}{dt}(0) \exp\left(\frac{\lambda \pi n}{2\sqrt{\lambda^2 + 2\alpha_i}}\right) \\
M(0) > 0,\ \frac{dM}{dt}(0) \geq 2\left(\lambda + \frac{\alpha_i}{\lambda}\right) M(0)
\end{gathered} \tag{3.11}$$

imply the a priori exponential lower estimates [9]

$$M(t) \geq C_{0i} \exp(\lambda t);\ i = 1,\ 2. \tag{3.12}$$

Here, C_{0i} are the known positive constants.

Since we have obtained lower estimates (3.12) without any restrictions on corresponding exact stationary solutions (2.13), (2.14) and (2.15), (2.16) to systems (2.10) and (2.11), these solutions are absolutely unstable with respect to small spherically symmetrical perturbations (3.1), (3.3), (3.11) and (3.2), (3.4), (3.11) [8, 9, 11]. Then the Antonov criterion [5, 6] really plays the role of necessary and sufficient condition for linear stability of exact stationary solutions (2.3), (2.4) to kinetic system (2.1) with regard to small spherically symmetrical perturbations (2.17) and (2.18) from incomplete unclosed subclasses (3.1), (3.3), (3.11) and (3.2), (3.4), (3.11).

Note that the first two inequalities of (3.11) are the sufficient conditions for practical (at finite time intervals) instability of exact stationary solutions (2.13), (2.14) and (2.15), (2.16) to gas-dynamic systems (2.10) and (2.11) with respect to small spherically symmetrical perturbations (3.1), (3.3), (3.11) and (3.2), (3.4), (3.11). The same inequalities play the role of criterion for linear practical instability of exact stationary solutions (2.13), (2.14) and (2.15), (2.16) to systems (2.10) and (2.11) with regard to small spherically symmetrical perturbations (3.1), (3.3), (3.11) and (3.2), (3.4), (3.11) presented in the form of normal modes [8, 9].

4 Example for the gas-dynamic system (2.11)

For exact stationary solutions (2.15) that satisfy equation (2.16), according to the Euler-Poisson integral [14], we can find

$$u^0(\nu) = \nu,\ \rho_2^0(\nu) = \frac{4n_g}{\sqrt{\pi}} e^{-\nu^2},\ \varphi^0 \equiv const. \tag{4.1}$$

We need $\xi \to \infty$ when $t \to \infty$. So, let

$$\xi(r,\nu,t) = h(r,\nu)e^{\beta t}\ (\beta \equiv const > 0), \quad \varphi'(t) = \varphi_1 e^{\beta t}\ (\varphi_1 \equiv const) \tag{4.2}$$

where we take the function $g(t)$ (see comments after inequality (3.10)) as $\exp(\beta t)$. Substituting (4.1), (4.2) into system (3.4), we get

$$h(r,\nu) = \left(-\frac{4\nu}{\beta} + \frac{r}{\nu^2}\right) \exp\left(-\beta\frac{r}{\nu} - \nu^2\right).$$

Here, $\xi(r,\nu,t)$ is redefined by continuity as follows

$$\left(\xi|_{t=0,\,r=0}\right)\Big|_{\nu=0} = 0.$$

Thus, an analytical example of small spherically symmetrical perturbations (3.2), (3.4) in the form of normal modes (4.2) that are superimposed on decreasing stationary density field ρ_2^0 (2.15), (2.16), (4.1), but, meanwhile, are growing with time is constructed. The example describes a change in the distribution function due to the spread of gas particles from the vicinity of reference point (the sphere center) to infinity, so that the value of distribution function remains unchanged at the point of reference, decreases in the vicinity of reference point, and increases at infinity. Also, this example is a counterexample to the Antonov criterion [5,6]. The reason is that small spherically symmetrical perturbations (3.2), (3.4), (4.2) fall out of the applicability scope for this criterion.

5 Conclusion

The results of this work are consistent with the classic Earnshaw instability theorem. This theorem states that any equilibrium configuration of point electric charges is unstable if, besides its own Coulomb forces of attraction and repulsion, no other forces act on them.

The field of application for the Earnshaw theorem is extended from electrostatics to kinetics, that is, to the boundless collisionless self-gravitating Vlasov-Poisson gas of neutral particles.

Constructiveness is inherent in the sufficient conditions (see the first two inequalities of (3.11)) for linear practical instability established in this work, which enables them to be used as a testing and control mechanism for physical experiments and numerical calculations.

Since the differential inequality (3.10) is a very common relation, we expect it to be applied to other mathematical models of liquids, gases, and plasma.

Finally, the algorithm for constructing of the growing over time Lyapunov functional will be helpful in linear problems of both theoretical (at semi-infinite time intervals) and practical (at finite time intervals) stability of either gas-dynamic or kinetic type.

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