Keldysh's note

Theorem *If the boundary* Γ *of the domain* D *satisfies the condition*

$$|\theta(s_1) - \theta(s_2)| < K(s_1 - s_2)^\lambda,$$

then the Bieberbach polynomials satisfy the inequality

$$|f(z) - \Pi_n(z)| < \frac{C(\varepsilon)}{n^{\lambda-\varepsilon}}.$$

Proof The derivative $f'(z)$ satisfies the condition

$$|f'(z_1) - f'(z_2)| < M|z_1 - z_2|^\lambda,$$

therefore, one can construct $P_n(z)$ so that

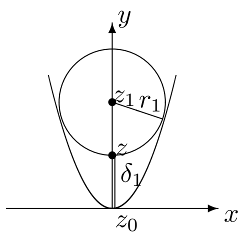
$$|f'(z) - P_n'(z)| < \frac{C_3}{n^{\lambda-\eta}}, \quad P_n(0) = 0, \quad P_n'(0) = 1,$$

$$|f(z) - P_n(z)| < \frac{C_3}{n^{\lambda-\eta}}.$$

Setting $\pi_n(z) = P_n(z) - \Pi_n(z)$, we get

$$J_n = \iint\limits_D |\pi_n'(z)|^2 d\sigma < \frac{2S\,C_3^2}{n^{2(\lambda-\eta)}}.$$

Let z_0 be a point of the boundary. Let us inscribe the parabola $y = k_1|x|^{1+\lambda}$.



r_1 and δ_1 are related by the relation

$$\delta_1 \sim (Cr_1)^{\frac{1+\lambda}{1-\lambda}},$$

besides,

$$r_2 \sim \delta_1,$$

$$r_{m+1} \sim \left[C^{\frac{(1+\lambda)^2}{2\lambda(1-\lambda)}} r_1\right]^{\left(\frac{1+\lambda}{1-\lambda}\right)^m} \cdot C^{\frac{1+\lambda}{2\lambda}}.$$

We take the point z_1 so that

$$q = C^{\frac{(1+\lambda)^2}{2\lambda(1-\lambda)}} r_1 < 1.$$

Choosing m points $z_1, z_2, \ldots, z_m$ we have

$$|\pi_n(z_k)| \le \pi_n(z_{k-1}) + \sqrt{\log(en)\frac{1}{\pi}\iint\limits_{|z-z_{k-1}|\le r_{k-1}} |\pi_n'(z)|^2\, d\sigma}$$

$$\begin{aligned} |\pi_n(z_{m+1})| &\le \pi_n(z_1) + \sqrt{\log(en)}\sum\left(\frac{1}{\pi}\iint\limits_{|z-z_{k-1}|\le r_{k-1}} |\pi_n'(z)|^2 d\sigma\right)^{\frac{1}{2}} \\ &\le \pi_n(z_1) + \sqrt{\log(en)\cdot m\cdot\frac{2}{\pi}J_n} \end{aligned} \qquad (1)$$

$$\begin{aligned} |\pi_n(z_0)| &\le |\pi_n(z_{m+1})| + a_1\rho + \ldots + a_n\rho^n \le \\ &\le |\pi_n(z_{m+1})| + \left[\sum\frac{1}{k}\left(\frac{\rho}{r_{m+1}}\right)^{2k}\cdot\frac{1}{\pi}\iint\limits_{|z-z_{m-1}|\le r_{m-1}} |\pi_n'(z)|^2\, d\sigma\right]^{\frac{1}{2}} \le \\ &\le |\pi_n(z_{m+1})| + \sqrt{\frac{1}{\pi}J_n\log(en)}\left(\frac{\rho}{r_{m+1}}\right)^n. \end{aligned} \qquad (2)$$

$$\rho \sim r_{m+1} + \delta_{m+1},$$
$$\frac{\rho}{r_{m+1}} \sim 1 + C^{\frac{1+\lambda}{1-\lambda}}\, r_{m+1}^{\frac{2\lambda}{1-\lambda}}.$$

Combining (1) and (2) we get

$$\begin{aligned} |\pi_n(z_0)| &\le |\pi_n(z_1)| + \sqrt{\frac{2}{\pi}J_n\ \log(en)}\left\{\sqrt{m} + \left(1 + C^{\frac{1+\lambda}{1-\lambda}} r_{m+1}^{\frac{2\lambda}{1-\lambda}}\right)^n\right\} \sim \\ &\sim |\pi_n(z_1)| + \sqrt{\frac{2}{\pi}J_n\ \log(en)}\left\{\sqrt{m} + \left(1 + C^{\frac{(1+\lambda)^2}{2\lambda(1-\lambda)}} q^{\left(\frac{1+\lambda}{1-\lambda}\right)^m}\right)^n\right\} \end{aligned}$$

Setting, for example,

$$m = 2\,\frac{\log\,\log n}{\log\frac{1+\lambda}{1-\lambda}},$$

we get

$$|\pi_n(z_0)| \le |\pi_n(z_1)| + \sqrt{\frac{4}{\pi\log\frac{1+\lambda}{1-\lambda}}J_n\log(en)\cdot\log\log n}.$$

Taking into account the estimate for J_n, we obtain

$$|\pi_m(z)| < \frac{C}{n^{\lambda-\eta}},$$

which proves the theorem.