International Workshop on Satellite Constellation & Formation Flying
16-19 July 2019, Glasgow, Scotland

Optimal design of spacecraft formations in Lissajous orbits

Sergey Shestakov

Maksim Shirobokov

Sergey Trofimov

Keldysh Institute of Applied Mathematics of RAS

Design of libration point formations

Two points of view on the design of a libration point FF:

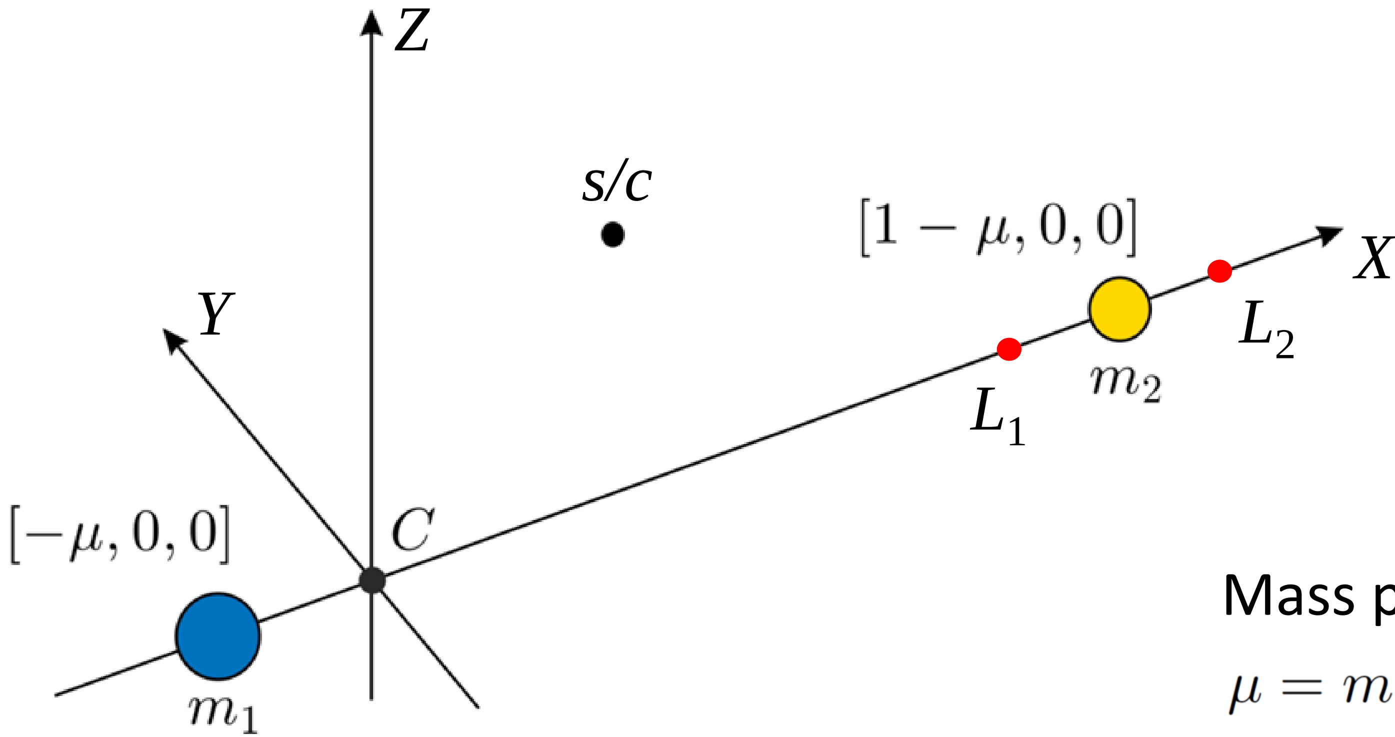
- <u>the optimal control problem</u>

  The reference relative motion is defined by hand; the control just ensures its tracking.

- <u>the natural motion search problem</u>

  Natural trajectories are sought that best fit mission requirements. The control ensures tracking and, if needed, refinement of the natural motion found.

Circular restricted three-body problem



Mass parameter

$$\mu = m_2/(m_1+m_2)$$

In the Sun-Earth system:

$$X_{L1} = 0.9899871, \quad X_{L2} = 1.0100740$$

Linearized dynamics in the vicinity of collinear libration points

New non-dimensional coordinates near the L1/L2 point:

$$x = \frac{X - X_L}{D}, \quad y = \frac{Y}{D}, \quad z = \frac{Z}{D}$$

$D = |X_L - 1 + \mu|$ is a distance from L1/L2 to the smaller primary

Solution to linearized equations:

$$\kappa = \frac{\omega_p^2 + 2\omega_v^2 + 1}{2\omega_p}$$

$$x = \alpha \cos(\omega_p t + \phi_1)$$

$$y = -\kappa\alpha \sin(\omega_p t + \phi_1)$$

$$z = \beta \cos(\omega_v t + \phi_2)$$

	Planar frequency	Vertical frequency
Sun-Earth L1	2.0864519	2.0152089
Sun-Earth L2	2.0570158	1.9850765

Differential and relative parameters for the description of relative motion

The relative position vector $\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ meets the same linearized equations

$$x = \alpha \cos\left(\omega_p t + \phi_1\right) \qquad \Delta x = A_x \cos\left(\omega_p t + \theta_1\right)$$

$$y = -\kappa\alpha \sin\left(\omega_p t + \phi_1\right) \qquad \Delta y = -\kappa A_x \sin\left(\omega_p t + \theta_1\right)$$

$$z = \beta \cos\left(\omega_v t + \phi_2\right) \qquad \Delta z = A_z \cos\left(\omega_v t + \theta_2\right)$$

Two sets of variables can be used for describing the relative motion in the linear approximation:

- differential amplitudes and phases $\Delta\alpha, \Delta\beta, \Delta\phi_1, \Delta\phi_2$
- relative amplitudes and phases $A_x, A_z, \theta_1, \theta_2$

Lindstedt-Poincaré series

- Lindstedt-Poincaré series approximate the central manifold
- For (quasi)periodic libration point orbits, two small parameters introduced are the in-plane and out-of-plane amplitudes
- Any invariant torus of (quasi)periodic trajectories is parameterized by two amplitudes and two phases

$$x = \sum x_{ijkm}\,\alpha^i \beta^j \gamma_1^k \gamma_2^m$$

$$y = \sqrt{-1}\,\sum y_{ijkm}\,\alpha^i \beta^j \gamma_1^k \gamma_2^m$$

$$z = \sum z_{ijkm}\,\alpha^i \beta^j \gamma_1^k \gamma_2^m$$

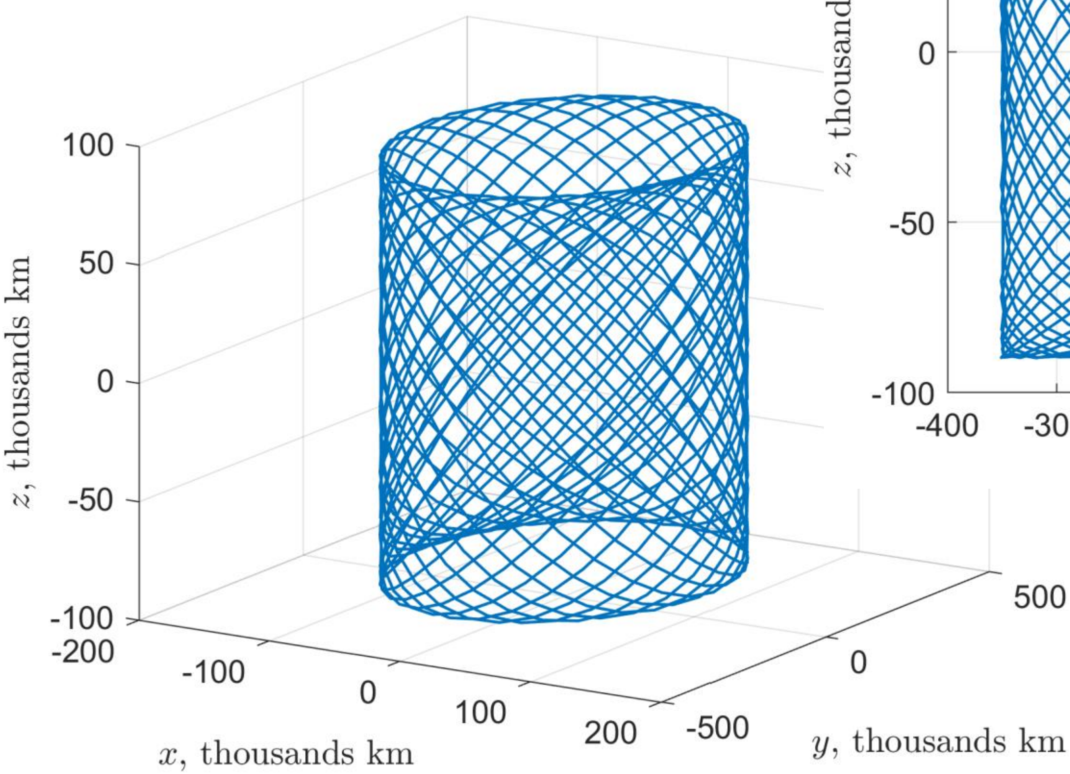
$$\omega_1 = \omega_p + \sum d_{ij}\,\alpha^i \beta^j$$

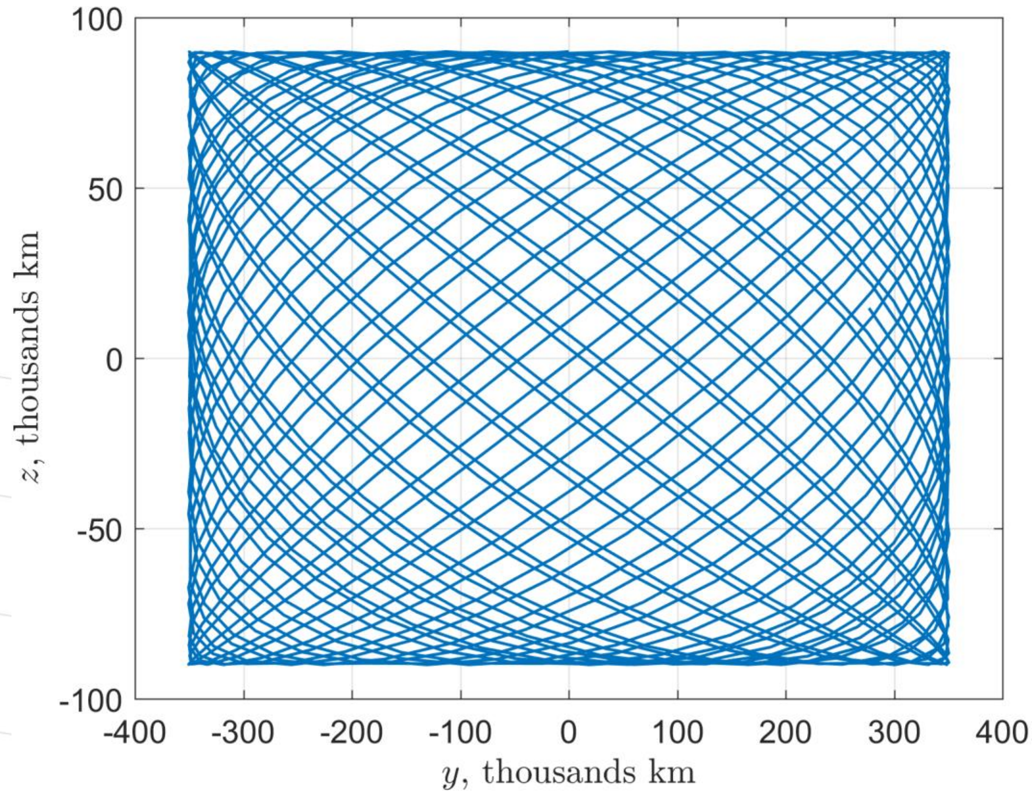
$$\omega_2 = \omega_v + \sum f_{ij}\,\alpha^i \beta^j$$

$$\gamma_i = \exp\left[\sqrt{-1}\,(\omega_i t + \phi_i)\right]$$

Reference orbit (linear approximation)

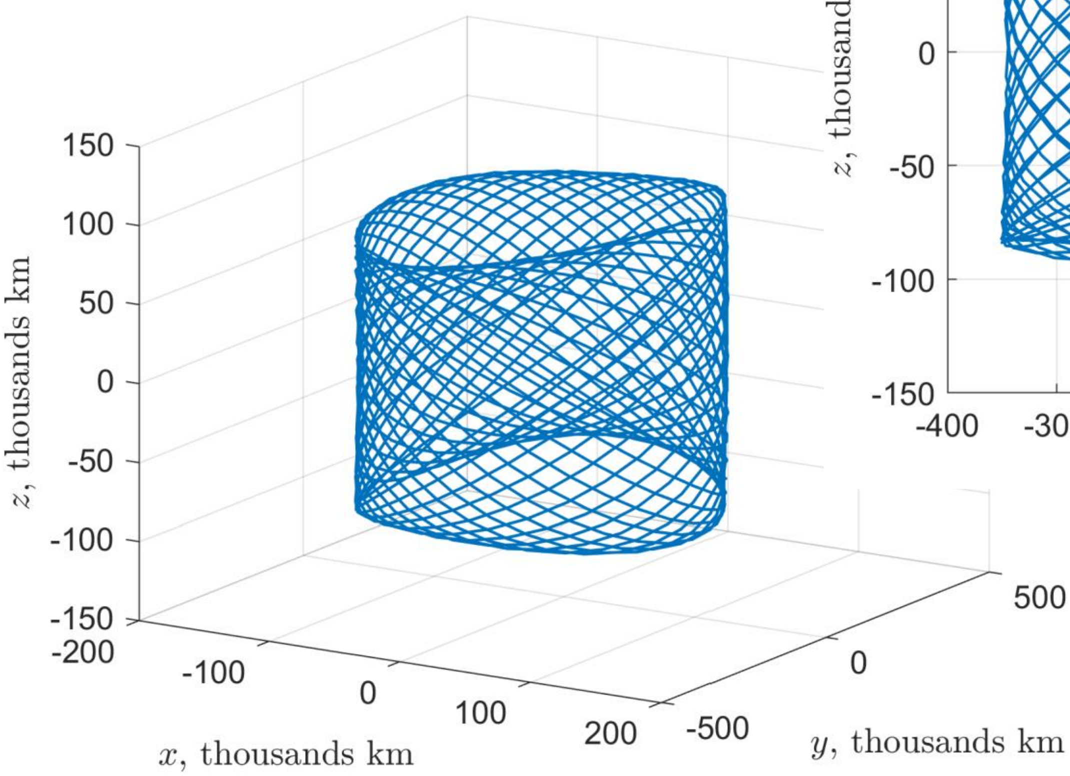
Reference orbit: Lissajous
110,000 km x 90,000 km

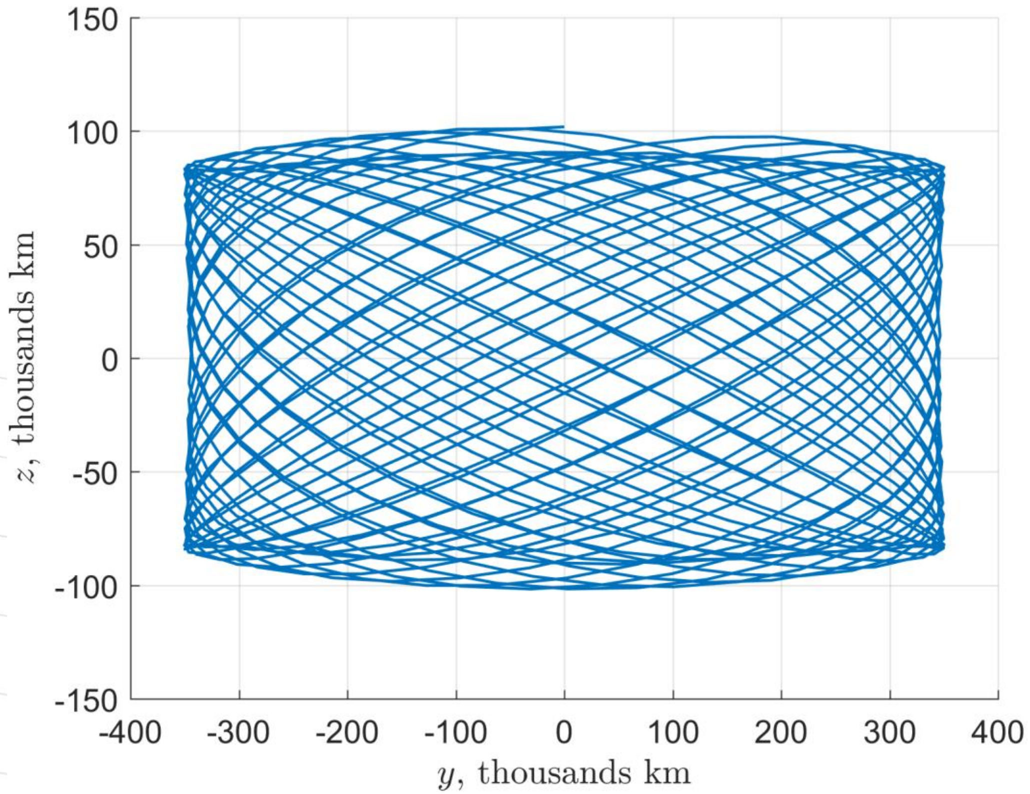




Reference orbit (15th-order LP series)

Reference orbit: Lissajous
110,000 km x 90,000 km





Some typical performance metrics

- Relative distance $\boxed{\Delta r^2 = \Delta x^2 + \Delta y^2 + \Delta z^2}$
  – Should keep the relative distance constant

- Projected relative distance $\boxed{\Delta r^2 - (\Delta \mathbf{r} \cdot \mathbf{n})^2}$
  – Should keep the projected relative distance constant (the relative trajectory is a projected circular orbit) – space interferometry missions

Design of libration point formations

- Search for a natural motion that produces good performance in linear model
- Optimize a motion in high-order LP series model
- Further adapt to ephemeris model

Advantages

- Linear model provides a lot of tools to possibly find the reference motion
- Only four parameters to optimize and Nelder-Mead simplex algorithm works well in a low-dimension search space
- Good first approximation so optimization methods converge fast
- No numerical integration in the highly unstable dynamical system

Performance metric #1: (squared) relative distance

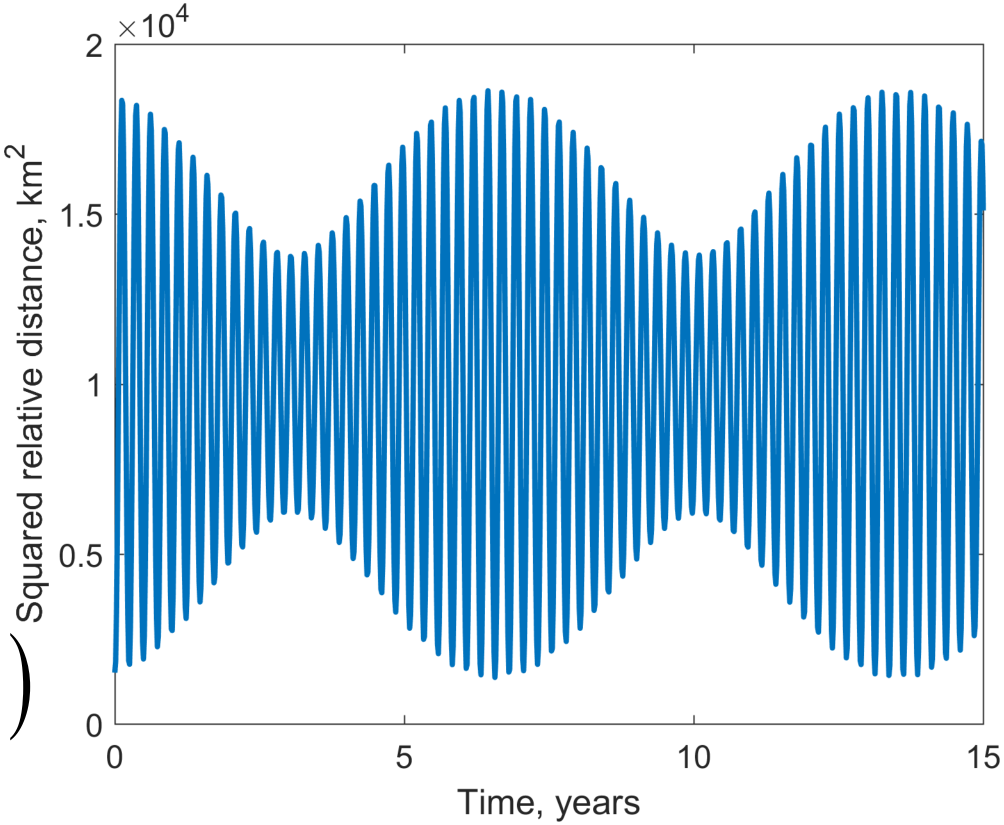
$$\Delta_1 = \Delta x^2 + \Delta y^2 + \Delta z^2$$

$$= \frac{A_x^2\left(\kappa^2+1\right)+A_z^2}{2}$$

$$+\frac{A_z^2}{2}\cos\left(2\omega_v t+2\theta_2\right)-\frac{A_x^2\left(\kappa^2-1\right)}{2}\cos\left(2\omega_p t+2\theta_1\right)$$

Performance metric #1: (squared) relative distance

$$\Delta_1 = \frac{A_x^2\left(\kappa^2+1\right)+A_z^2}{2} + \frac{A_z^2}{2}\cos\left(2\omega_v t+2\theta_2\right) - \frac{A_x^2\left(\kappa^2-1\right)}{2}\cos\left(2\omega_p t+2\theta_1\right)$$



Beating around mean value

Performance metric #1: (squared) relative distance

Constant part

$$\frac{A_x^2\left(\kappa^2+1\right)+A_z^2}{2}$$

Short-periodic part

$$\frac{A_z^2}{2}\cos\left(2\omega_v t+2\theta_2\right)-$$
$$-\frac{A_x^2\left(\kappa^2-1\right)}{2}\cos\left(2\omega_p t+2\theta_1\right)$$

Beating with the beat frequency $\delta=\omega_p-\omega_v$

Performance metric #1: (squared) relative distance

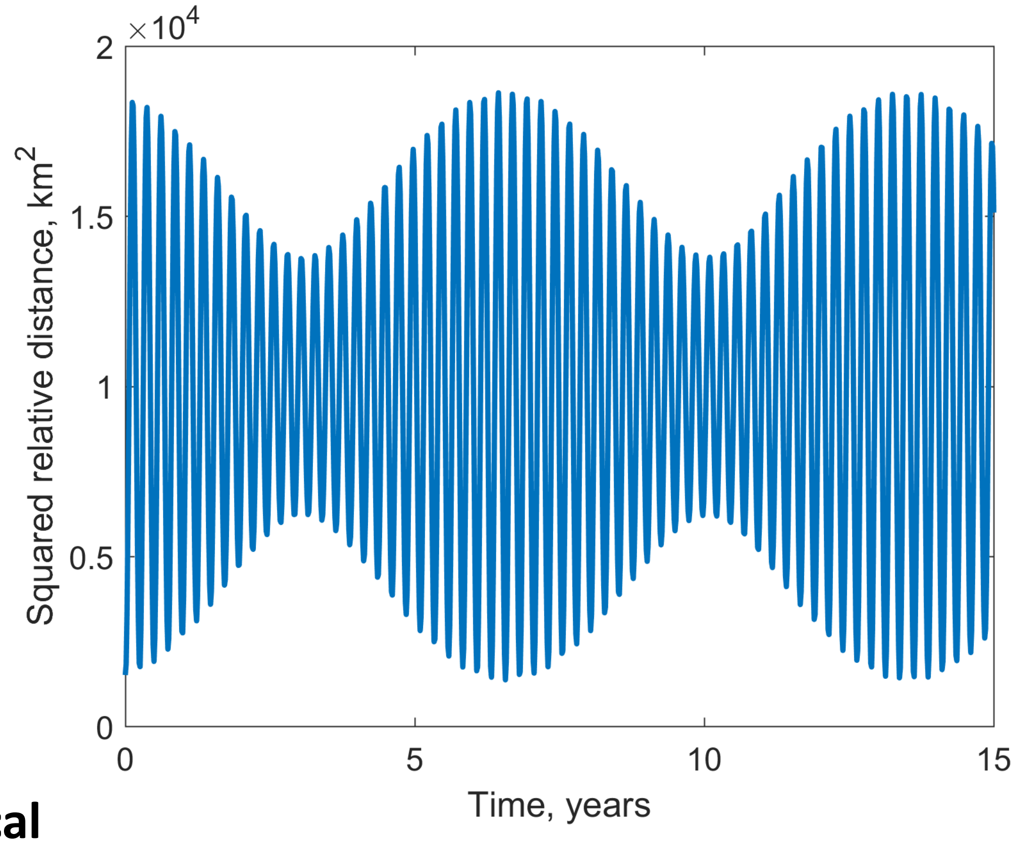
Minimum variation is more than 80%

⇩

It is unacceptable for most of real applications

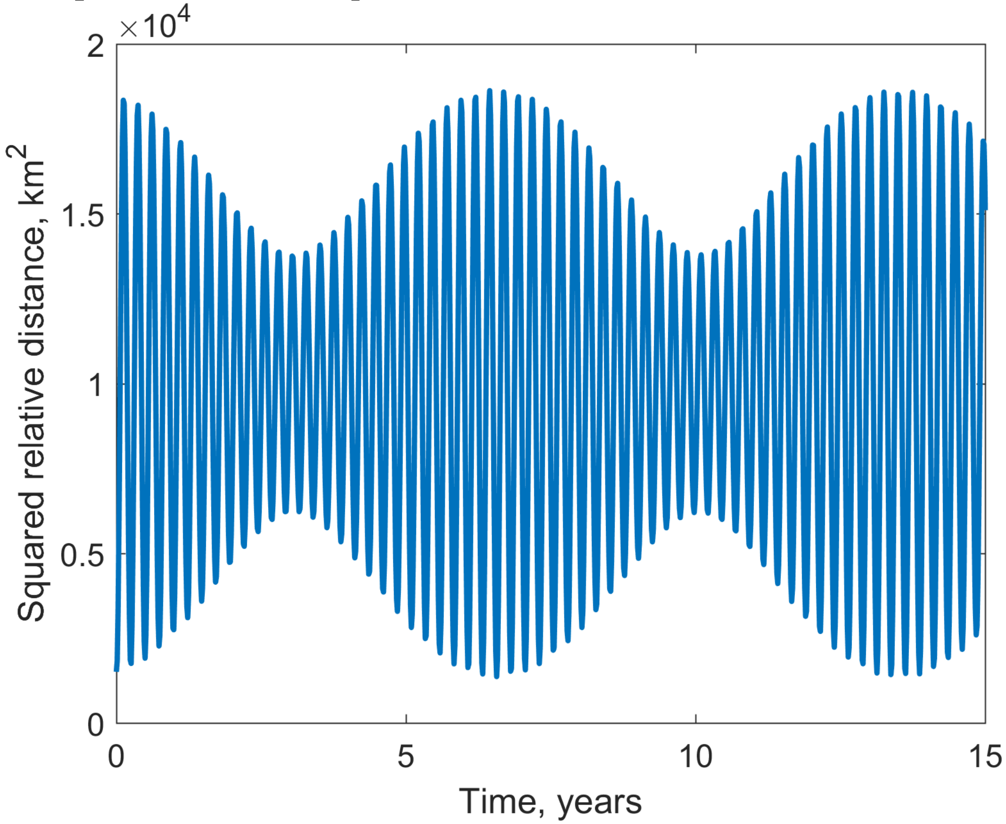
⇩

The performance can be good for a shorter interval



	Planar frequency	Vertical frequency
Sun-Earth L1	2.0864519	2.0152089
Sun-Earth L2	2.0570158	1.9850765

Performance metric #1: (squared) relative distance

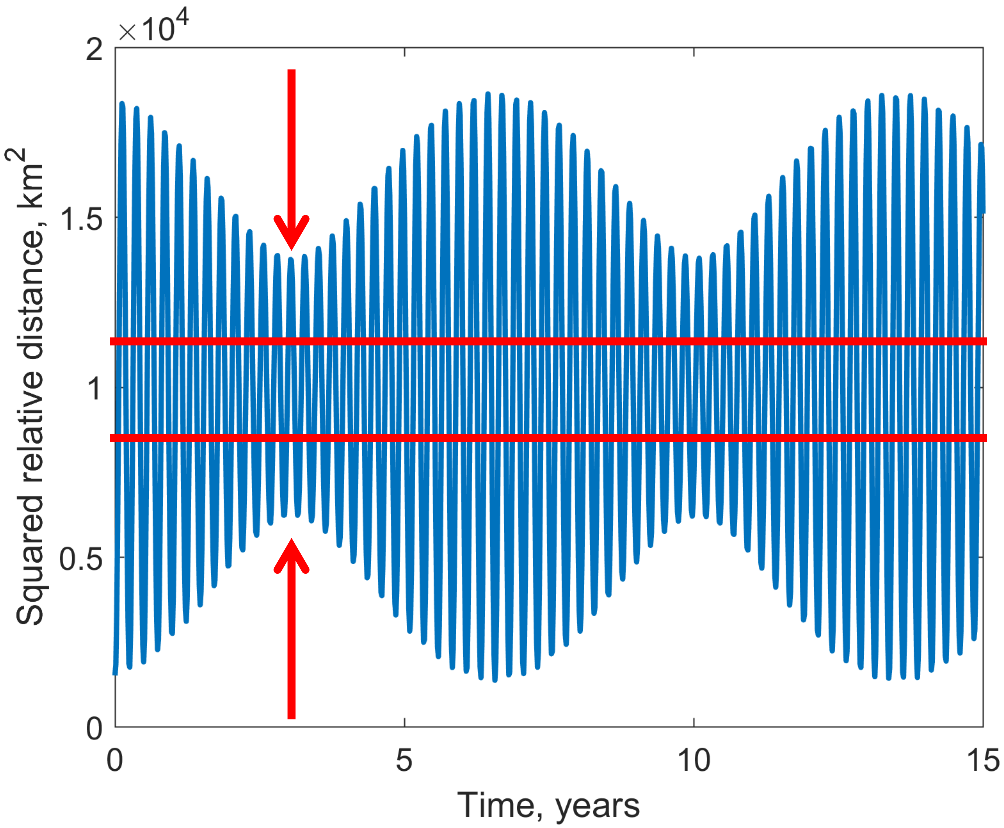


The upper and lower envelopes

$$\pm\sqrt{\frac{A_x^4\left(\kappa^2-1\right)^2}{4}+\frac{A_z^4}{4}-\frac{A_x^2A_z^2\left(\kappa^2-1\right)}{2}\cos\left(2\delta t-2\Delta\theta\right)}$$

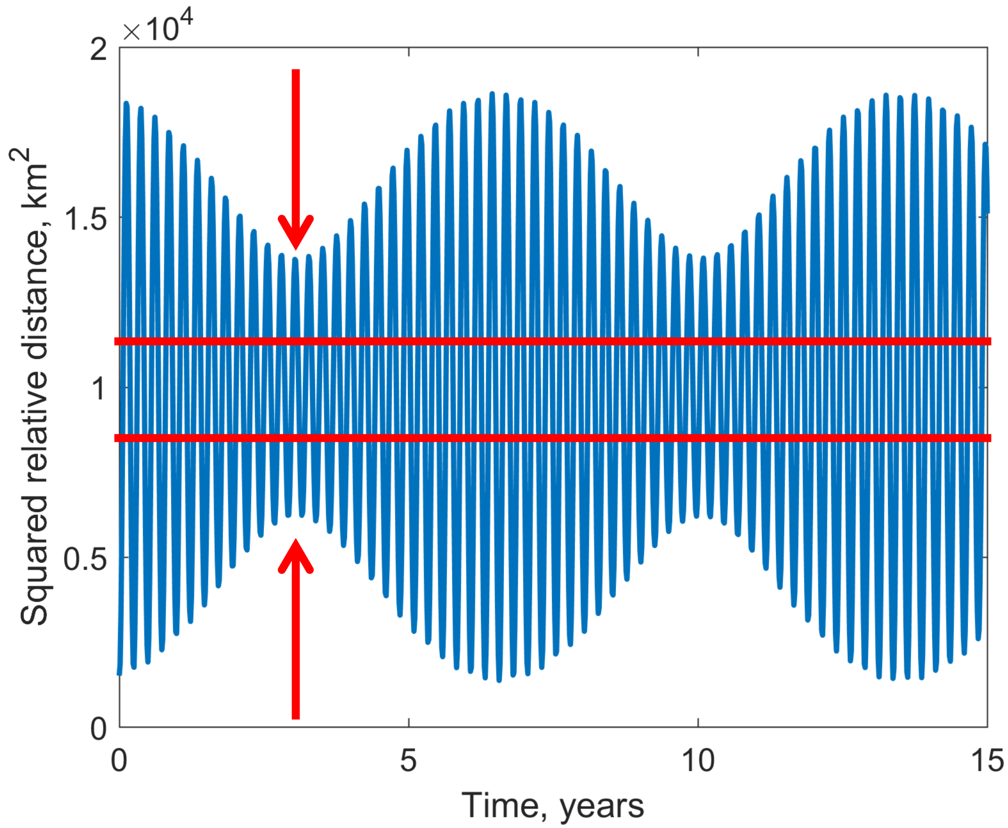
Performance metric #1: (squared) relative distance

$$\sqrt{\frac{A_x^4\left(\kappa^2-1\right)^2}{4}+\frac{A_z^4}{4}-\frac{A_x^2A_z^2\left(\kappa^2-1\right)}{2}}\cos\left(2\delta t-2\Delta\theta\right)=c\varepsilon$$



Adjust parameters to make beatings smaller for short period of time

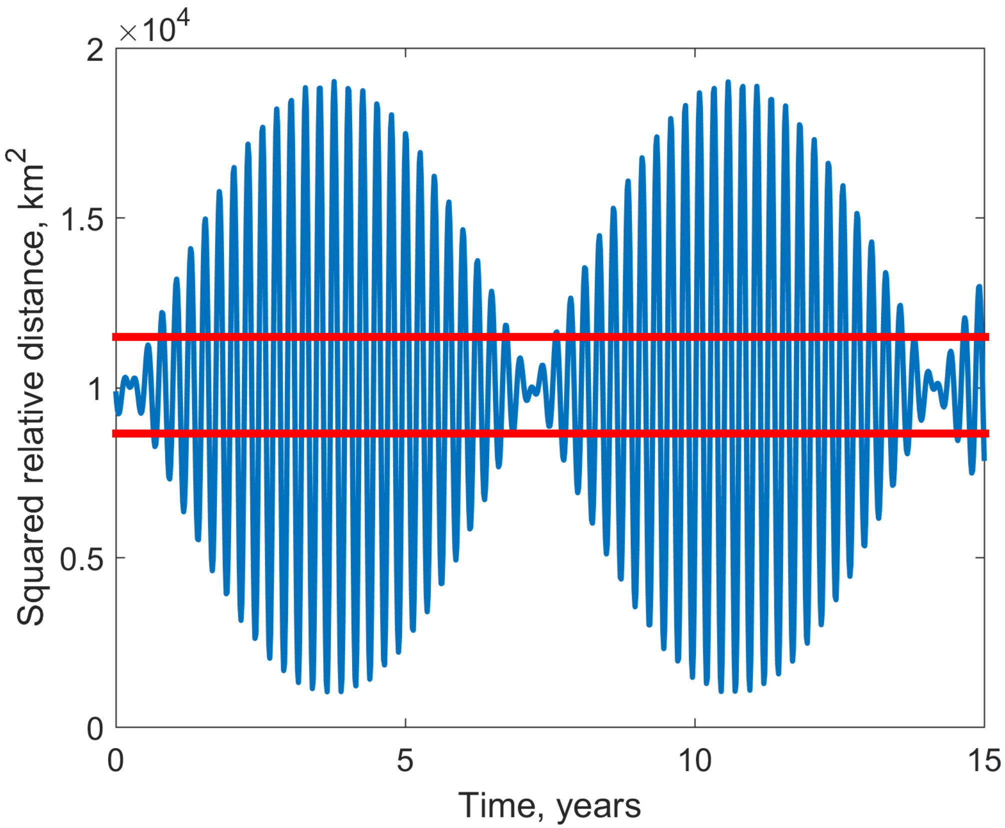
Performance metric #1: (squared) relative distance



Maximize the distance between adjacent roots

Performance metric #1: (squared) relative distance

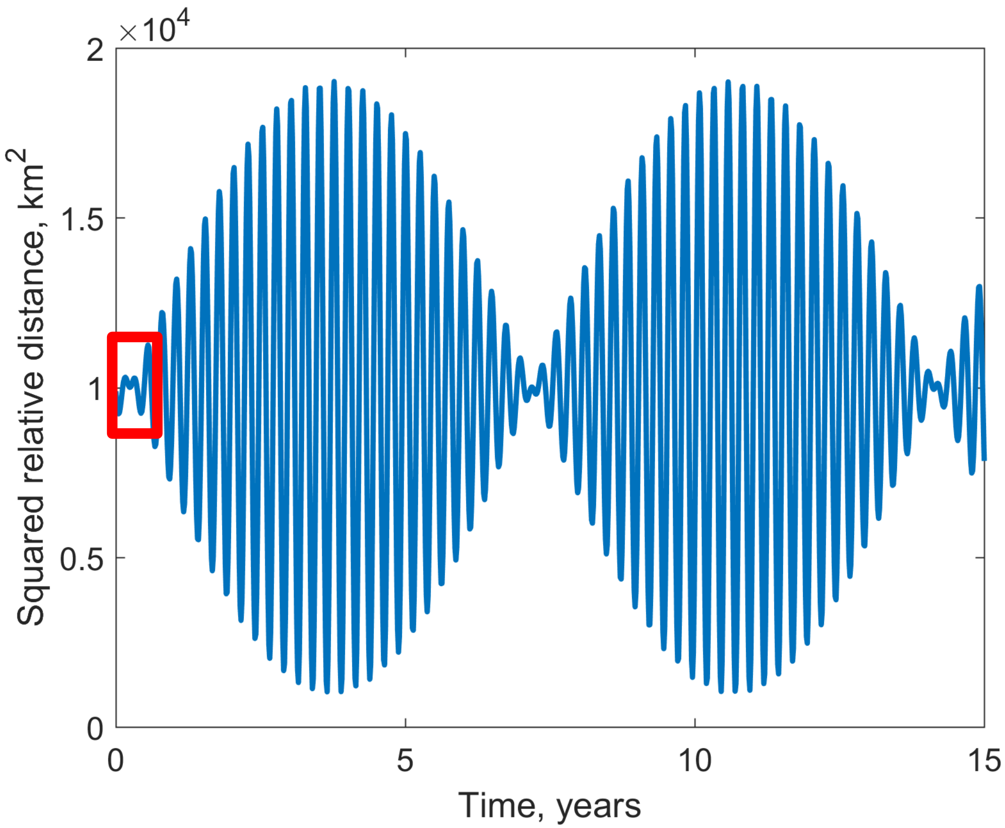
$$2\sqrt{a^2 + b^2 - 2ab\cos\left(2\delta t - 2\Delta\theta\right)} \leq 2\varepsilon$$



Adjust parameters to make beatings smaller for short period of time

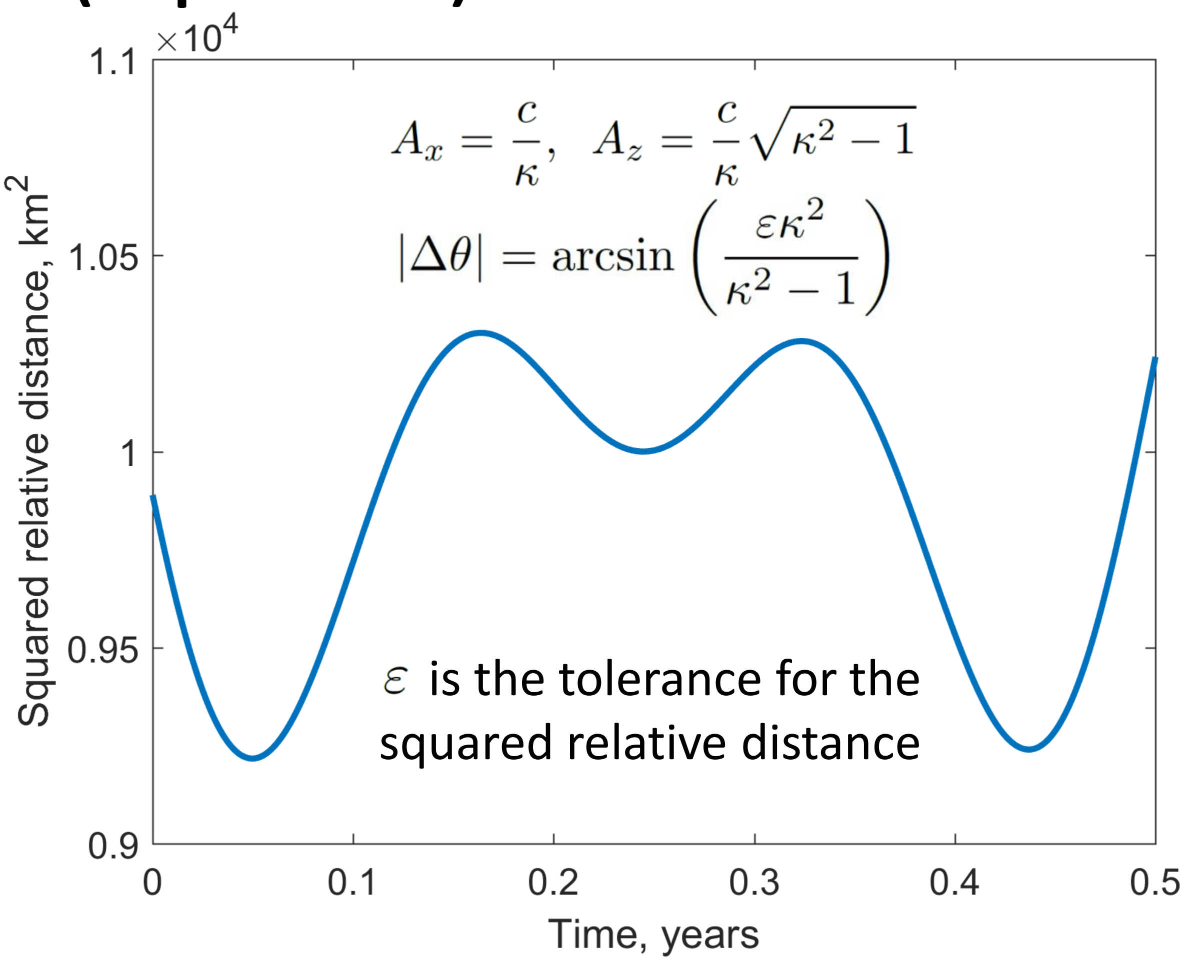
Performance metric #1: (squared) relative distance

$$2\sqrt{a^2+b^2-2ab\cos\left(2\delta t-2\Delta\theta\right)}\leq 2\varepsilon$$



Adjust parameters to make beatings smaller for short period of time

Performance metric #1: (squared) relative distance



The performance is good for half a year

Performance metric #2: projected relative distance

$$\Delta_2 = \Delta\mathbf{r}^2 - \left(\Delta\mathbf{r}\cdot\mathbf{n}\right)^2$$

If the projection plane is fixed in the rotating reference frame, $\mathbf{n}$ is a constant vector.

Performance metric #2: projected relative distance

$$\Delta_2 = \Delta\mathbf{r}^2 - \left(\Delta\mathbf{r}\cdot\mathbf{n}\right)^2$$

If the projection plane is fixed in the rotating reference frame, $\mathbf{n}$ is a constant vector.

Constant part

Short-periodic part

Long-periodic part

Performance metric #2: projected relative distance

$$\Delta_2 = \Delta\mathbf{r}^2 - \left(\Delta\mathbf{r}\cdot\mathbf{n}\right)^2$$

If the projection plane is fixed in the rotating reference frame, $\mathbf{n}$ is a constant vector.

Constant part

Short-periodic part

Long-periodic part

$$\Delta_2 = C + A\cos\left(\delta t + \Delta\theta\right) + B\sin\left(\delta t + \Delta\theta\right)$$
$$+ F_1(t)\cos\left(2\omega_p t + 2\theta_1\right) + F_2(t)\sin\left(2\omega_p t + 2\theta_1\right)$$

Performance metric #2: projected relative distance

Constant part

Short-periodic part

Long-periodic part

$$\Delta_2 = C + A\cos(\delta t + \Delta\theta) + B\sin(\delta t + \Delta\theta)$$
$$+ F_1(t)\cos(2\omega_p t + 2\theta_1) + F_2(t)\sin(2\omega_p t + 2\theta_1)$$

These oscillate with 2δ frequency

Performance metric #2: projected relative distance

The upper and lower envelopes

$$q = \delta t + \Delta\theta$$

$$C + A\cos q + B\sin q \pm \sqrt{A_0 + A_1\cos q + B_1\sin q + A_2\cos 2q + B_2\sin 2q}$$

$$\Downarrow$$

Need to have small distance between envelopes

$$\Downarrow$$

Maximize the distance between adjacent roots

$$2\sqrt{A_0 + A_1\cos q + A_2\cos 2q + B_1\sin q + B_2\sin 2q} \le \varepsilon \cdot c_{mean}$$

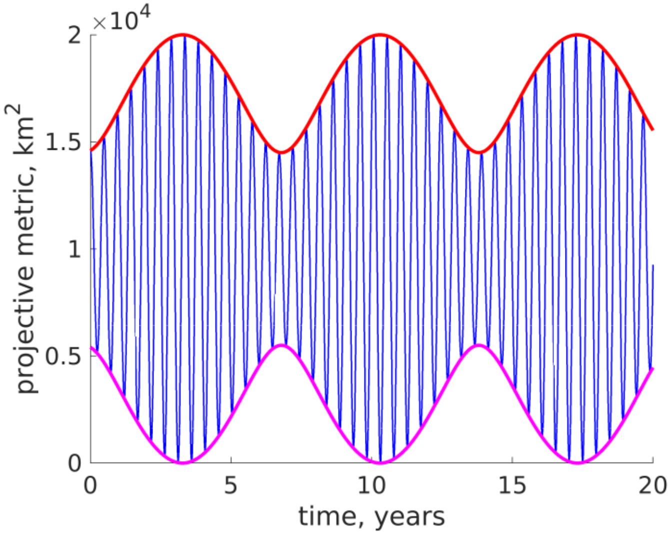
$$\Downarrow$$

Easier to do for polynomial

$$\sin q = \frac{2s}{1+s^2}, \cos q = \frac{1-s^2}{1+s^2}$$

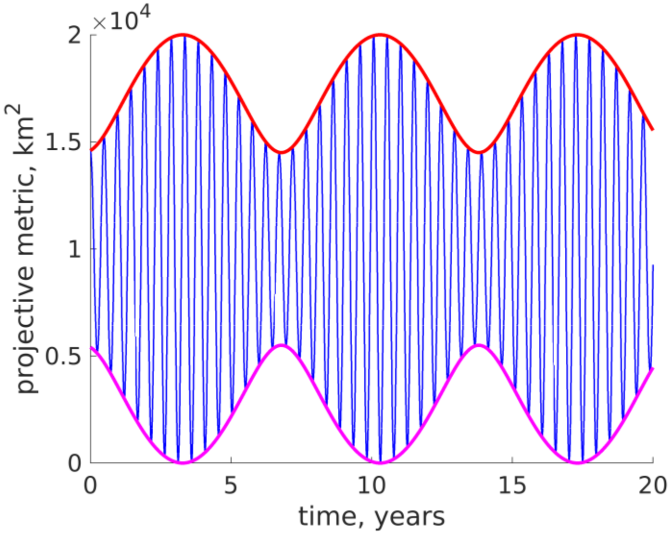
Performance metric #2: projected relative distance

$$\mathbf{n} = \begin{pmatrix} n_x \\ n_y \\ 0 \end{pmatrix}$$

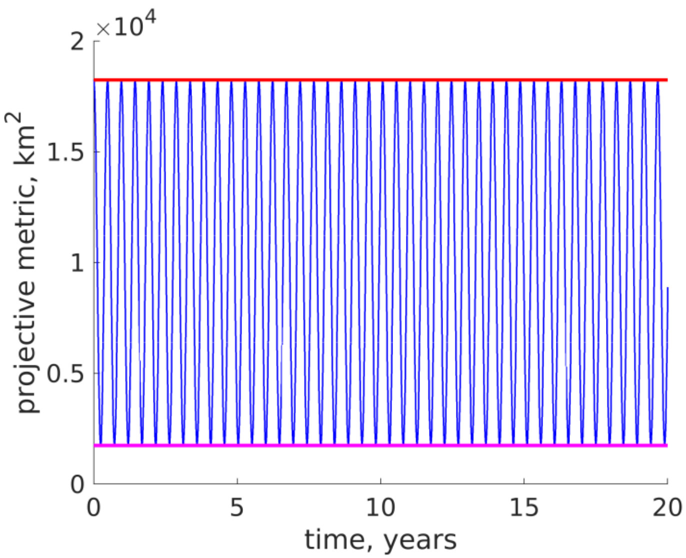


Performance metric #2: projected relative distance

$$\mathbf{n} = \begin{pmatrix} n_x \\ n_y \\ 0 \end{pmatrix}$$

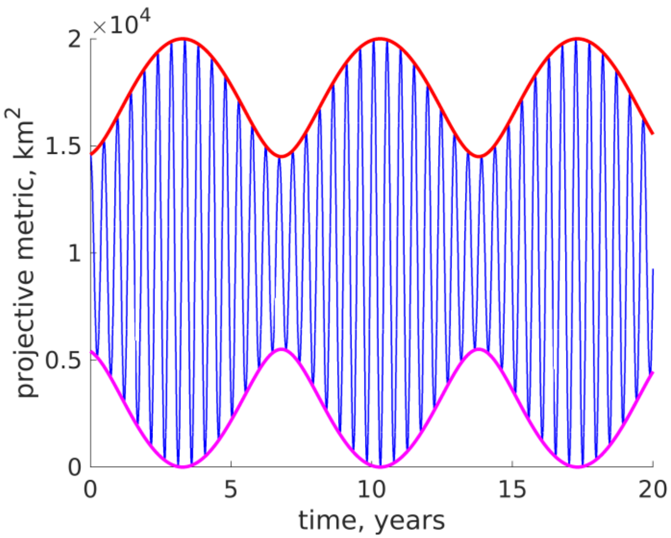


$$\mathbf{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

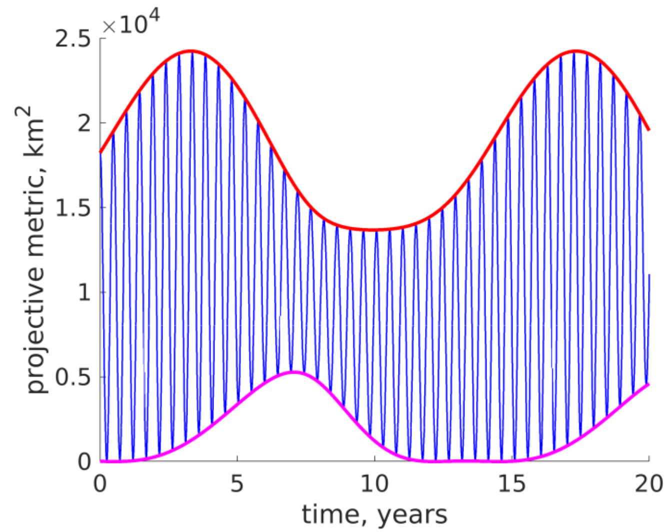


Performance metric #2: projected relative distance

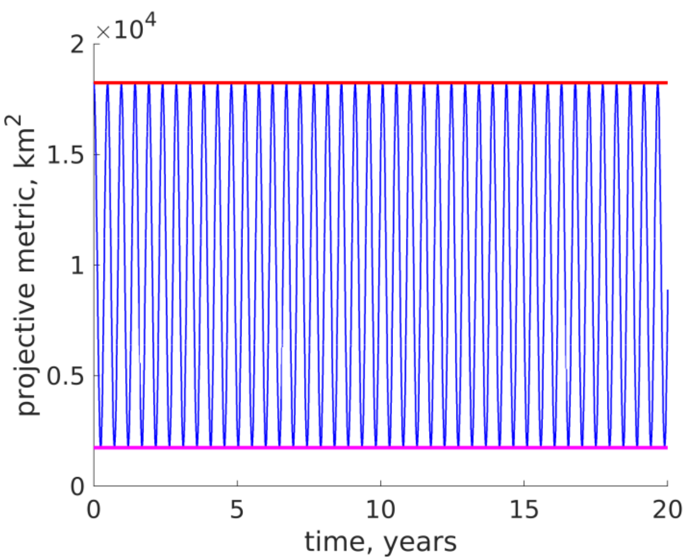
$$\mathbf{n} = \begin{pmatrix} n_x \\ n_y \\ 0 \end{pmatrix}$$



$$\mathbf{n} \sim \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

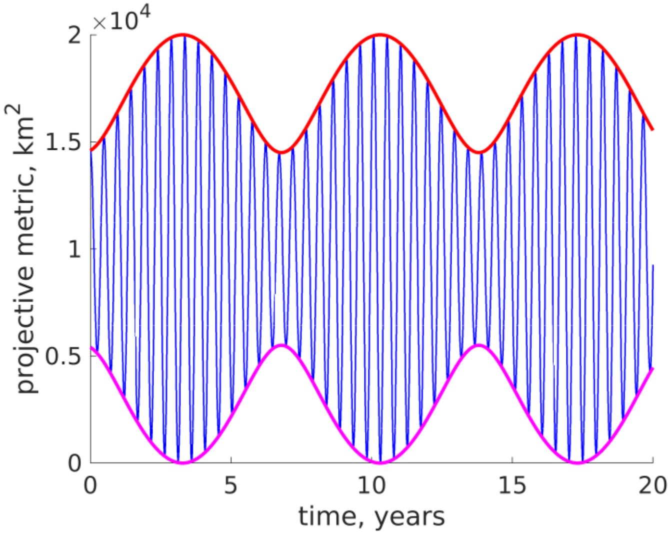


$$\mathbf{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

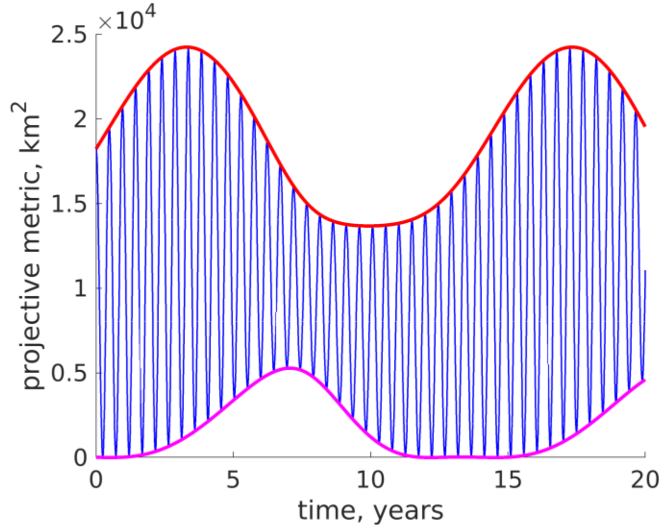


Performance metric #2: projected relative distance

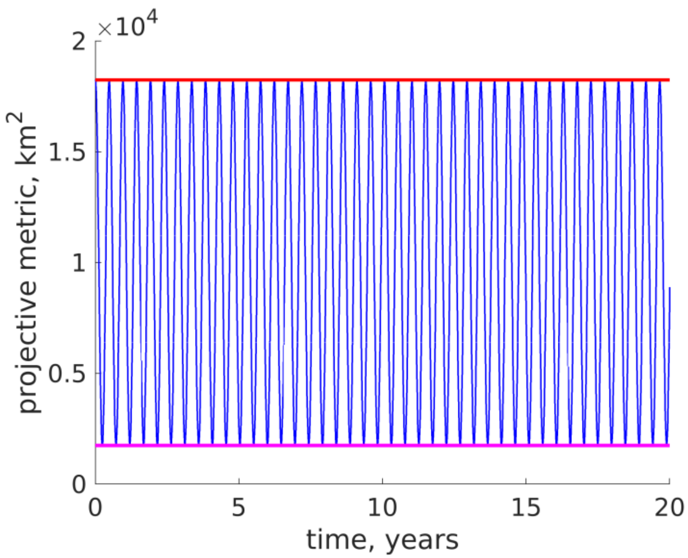
$$\mathbf{n} = \begin{pmatrix} n_x \\ n_y \\ 0 \end{pmatrix}$$



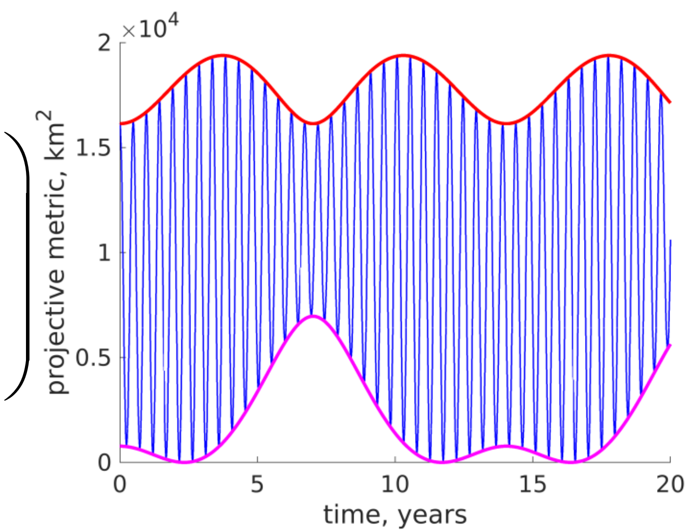
$$\mathbf{n} \sim \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$



$$\mathbf{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

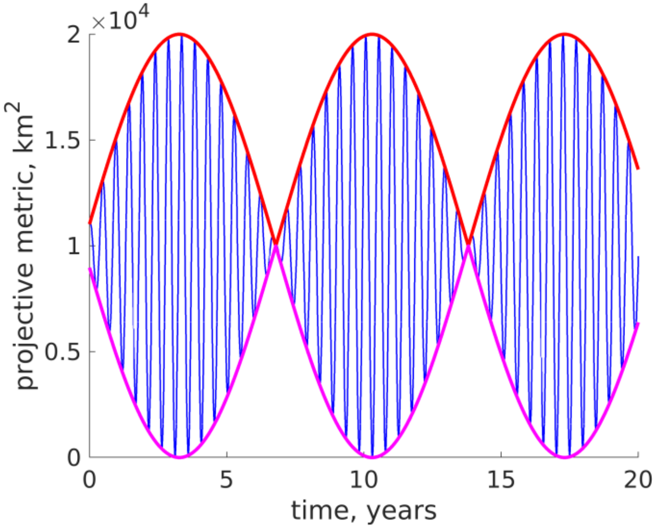


$$\mathbf{n} \sim \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

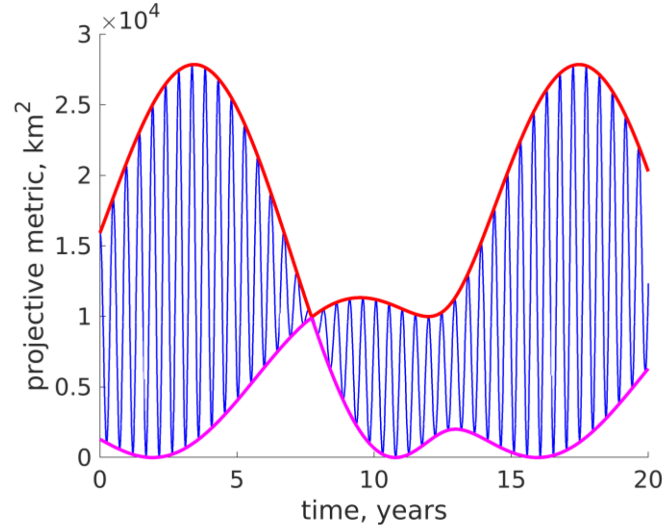


Performance metric #2: projected relative distance

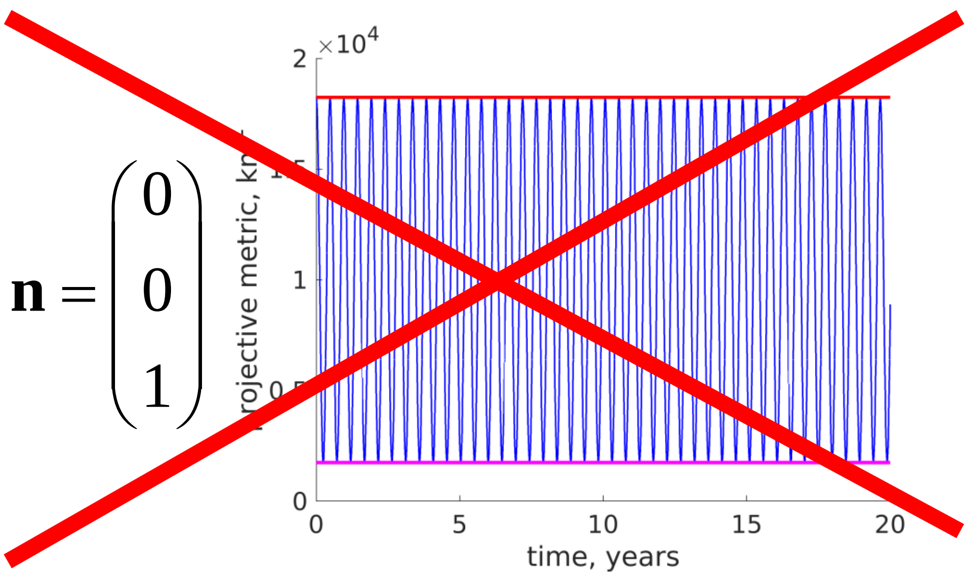
$$\mathbf{n} = \begin{pmatrix} n_x \\ n_y \\ 0 \end{pmatrix}$$



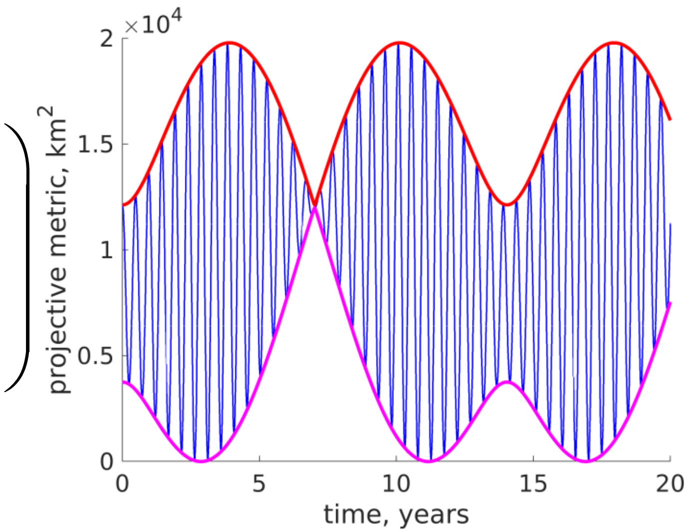
$$\mathbf{n} \sim \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$



$$\mathbf{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



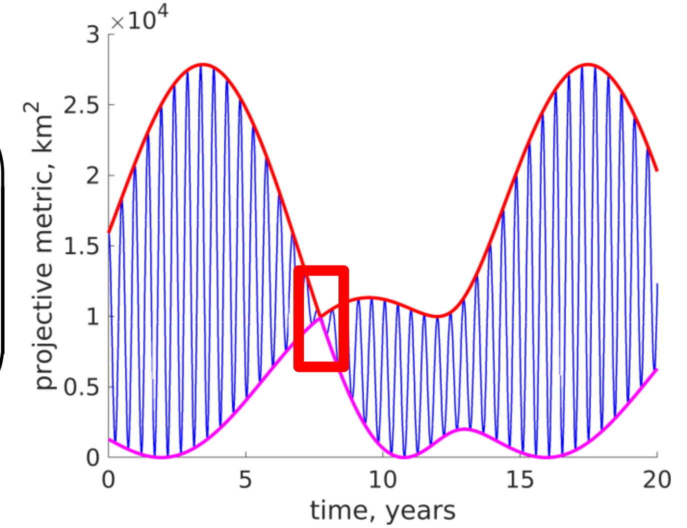
$$\mathbf{n} \sim \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

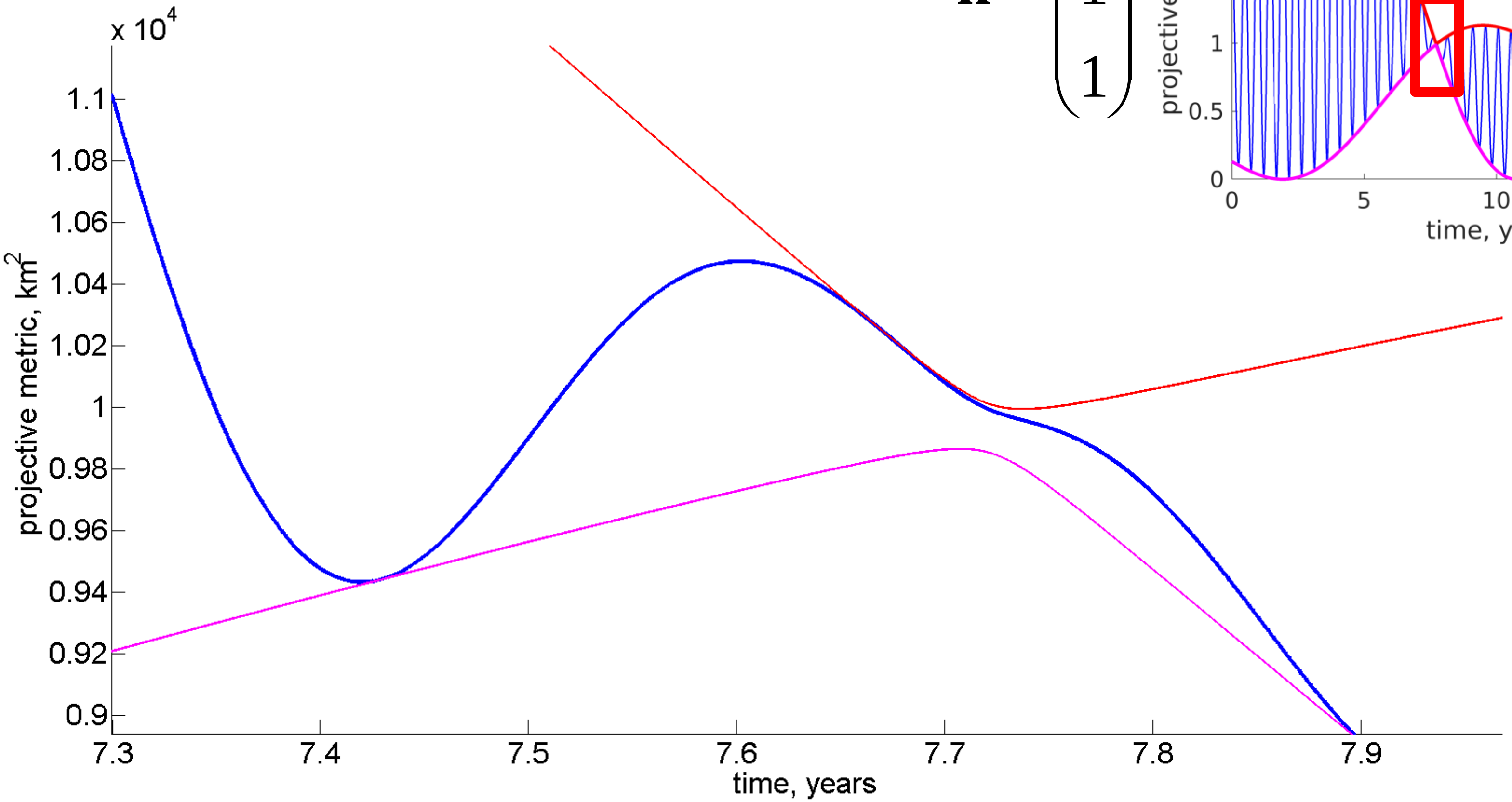


Performance metric #2: projected relative distance

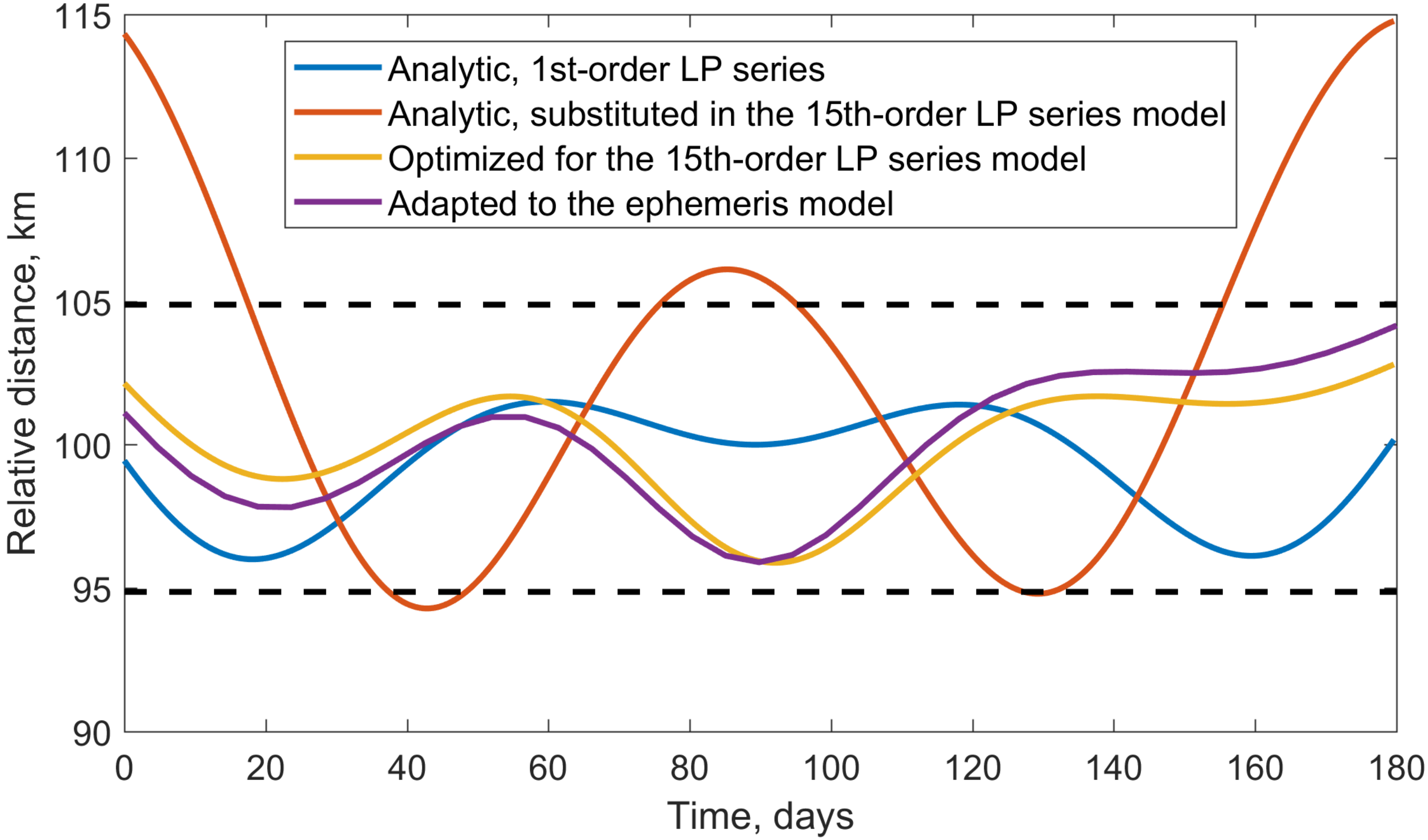
The performance is good for half a year

$$\mathbf{n} \sim \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

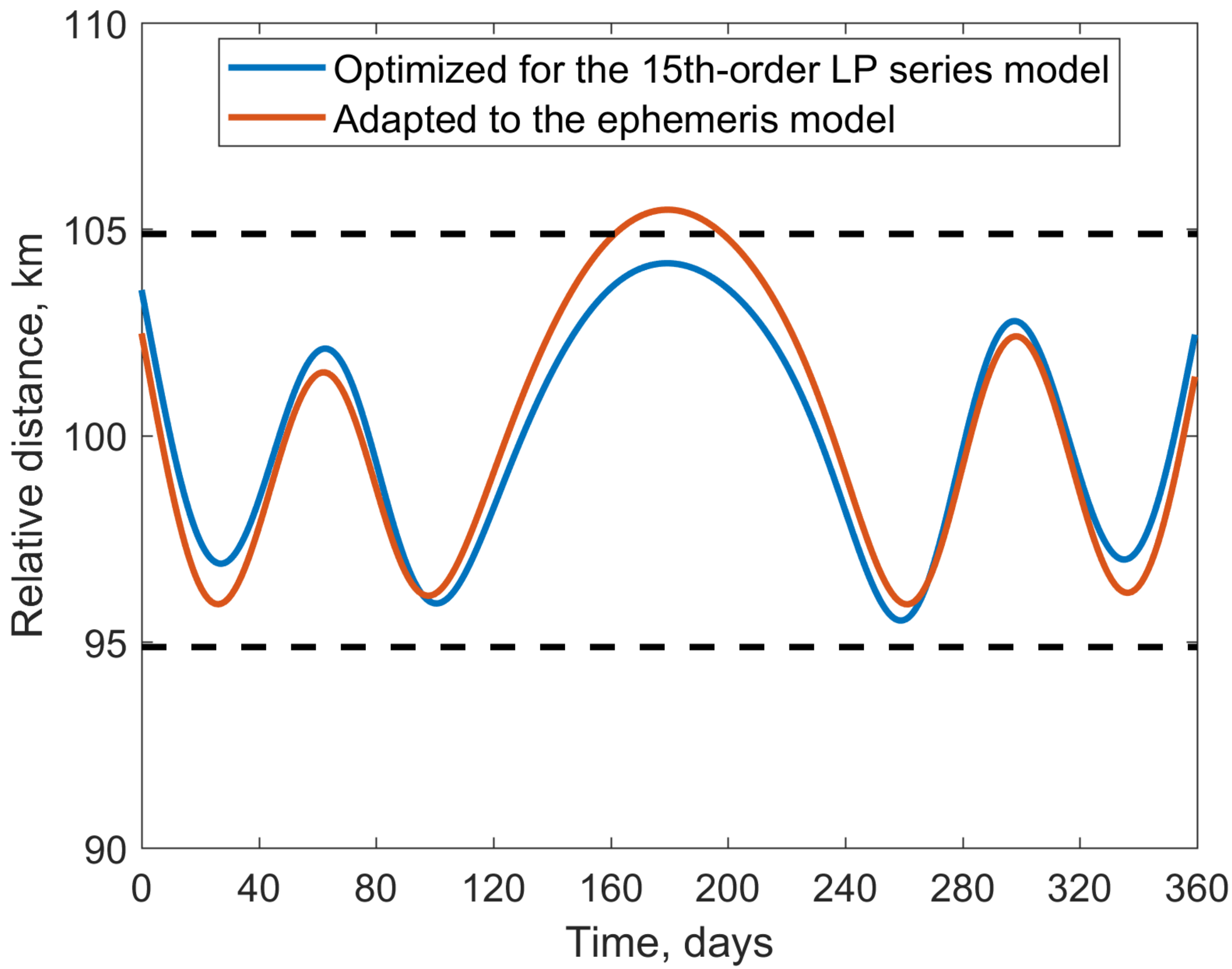




Performance metric #1: complex models adaptations



Performance optimized over the extended time interval (with the same initial guess)



Conclusions

- The analysis of linear expressions for the performance metric makes it often possible to obtain an initial guess for the numerical optimization procedure.
- Formation performance metrics are calculated w/o numerical integration of highly unstable trajectories
- The explicit analytical derivations have been presented for the relative distance-based performance metric and projective performance metric in the case of two-spacecraft formation
- The optimal design is found so that the performance metric variations for half a year are no greater than 10%
- The same stability level is achievable for the one-year ballistic flight

Questions?