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Optimal Design of Multi-Spacecraft Formations in Lissajous Orbits

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Design of libration point formations

Two points of view on the design of a libration point FF:

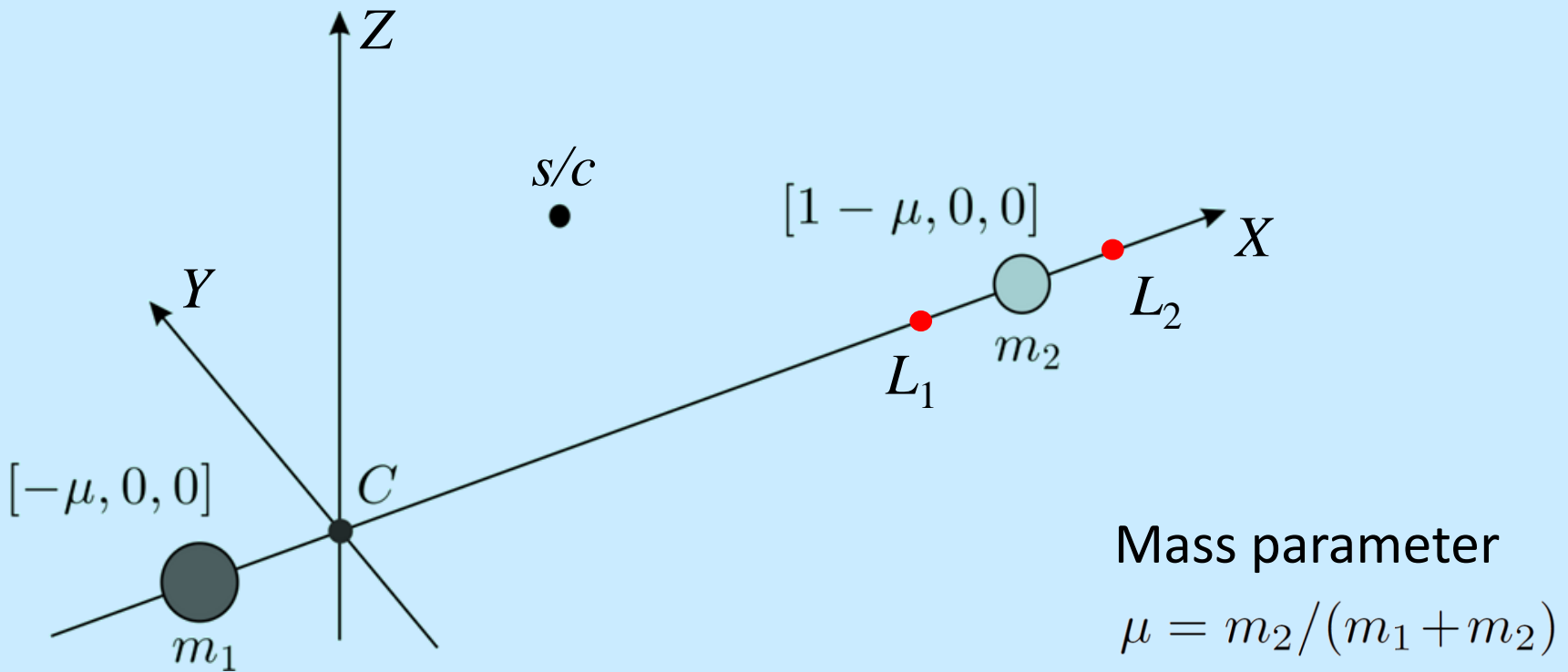
- the optimal control problem

The reference relative motion is defined by hand; the control just ensures its tracking.

- the natural motion search problem

Natural trajectories are sought that best fit mission requirements. The control ensures tracking and, if needed, refinement of the natural motion found.

Circular restricted three-body problem



In the Sun-Earth system:

$$X_{L_1} = 0.9899871, \quad X_{L_2} = 1.0100740$$

Linearized dynamics in the vicinity of collinear libration points

New non-dimensional coordinates near the L1/L2 point:

$$x = \frac{X - X_L}{D}, \quad y = \frac{Y}{D}, \quad z = \frac{Z}{D}$$

$D = |X_L - 1 + \mu|$ is a distance from L1/L2 to the smaller primary

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Solution to linearized equations:

$$x = \alpha \cos(\omega_p t + \phi_1)$$

$$y = -\kappa \alpha \sin(\omega_p t + \phi_1)$$

$$z = \beta \cos(\omega_v t + \phi_2)$$

$$\kappa = \frac{\omega_p^2 + 2\omega_v^2 + 1}{2\omega_p}$$

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	Planar frequency	Vertical frequency
Sun-Earth L1	2.0864519	2.0152089
Sun-Earth L2	2.0570158	1.9850765

Complex form of Lindstedt-Poincaré series for Lissajous orbits

$$x = \sum x_{ijklm} \alpha^i \beta^j \gamma_1^k \gamma_2^m$$

$$y = \sqrt{-1} \sum y_{ijklm} \alpha^i \beta^j \gamma_1^k \gamma_2^m$$

$$z = \sum z_{ijklm} \alpha^i \beta^j \gamma_1^k \gamma_2^m$$

$$\gamma_i = \exp \left[\sqrt{-1} (\omega_i t + \phi_i) \right]$$

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$$\omega_1 = \omega_p + \sum d_{ij} \alpha^i \beta^j$$

$$\omega_2 = \omega_v + \sum f_{ij} \alpha^i \beta^j$$

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The procedure of calculating the coefficients and the tables of coefficients for the Sun-Earth L1 and L2 points are presented in the Volume III of the famous monograph by Gómez et al.

Differential and relative parameters for the description of relative motion

The relative position vector $\Delta \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ meets the same linearized equations, so we can write

$$\Delta x = A_x \cos(\omega_p t + \theta_1)$$

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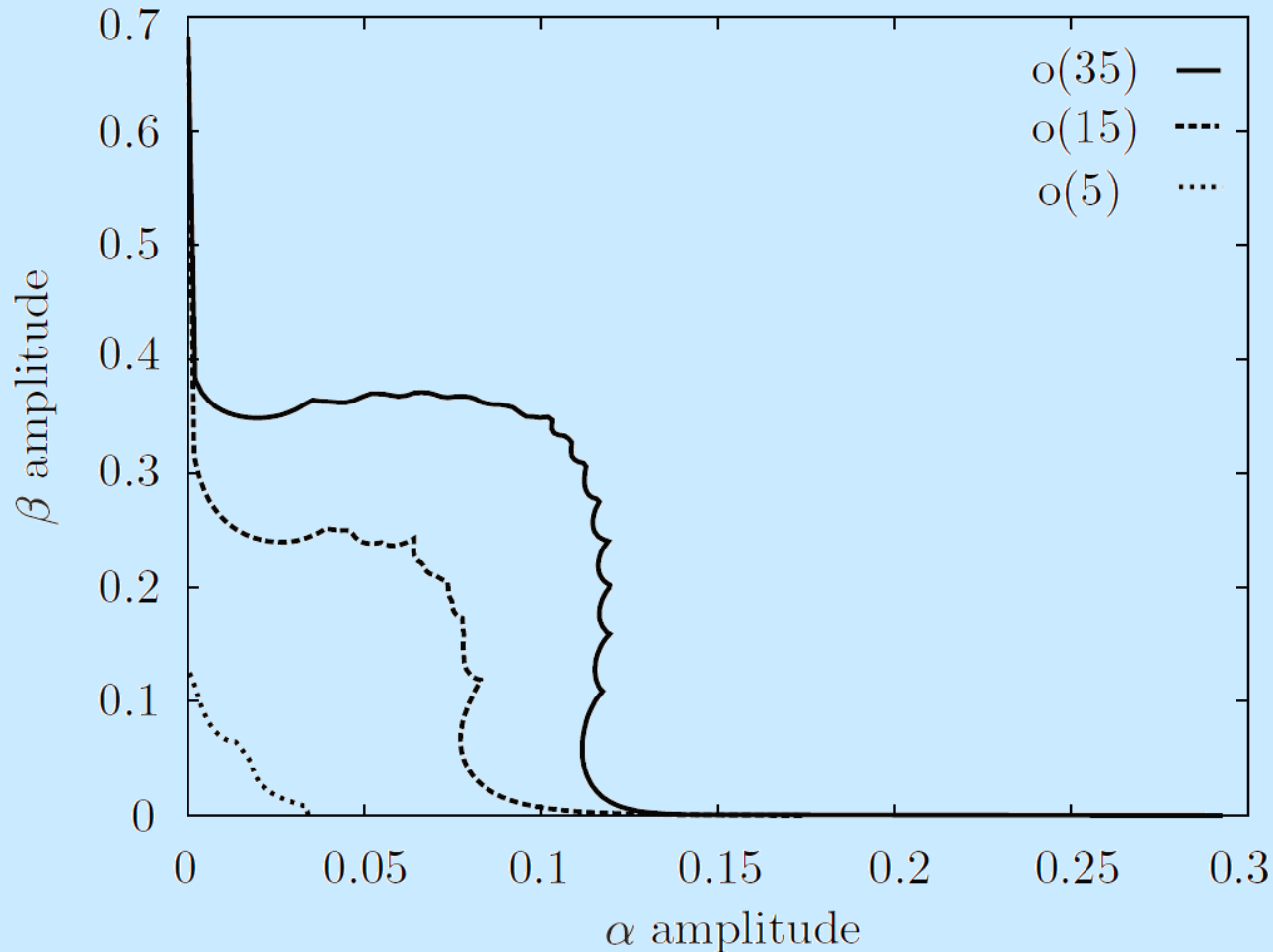
$$\Delta y = -\kappa A_x \sin(\omega_p t + \theta_1)$$

$$\Delta z = A_z \cos(\omega_v t + \theta_2)$$

Two sets of variables can be used for describing the relative motion in the linear approximation:

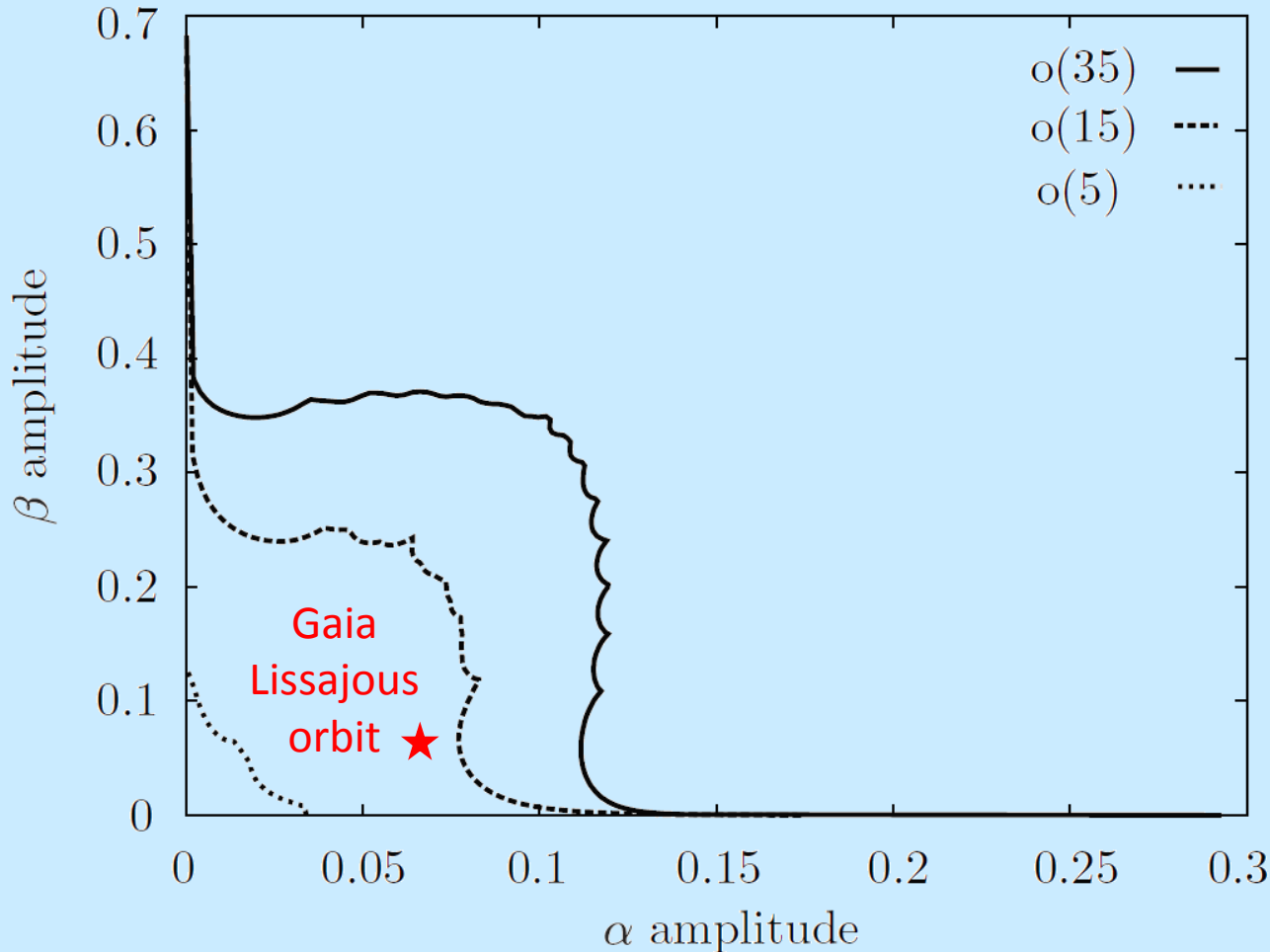
- differential amplitudes and phases $\Delta\alpha, \Delta\beta, \Delta\phi_1, \Delta\phi_2$
- relative amplitudes and phases $A_x, A_z, \theta_1, \theta_2$

Required order of approximation



Upon integrating the equations of motion at π time units, the error shall not exceed 10^{-6} distance units

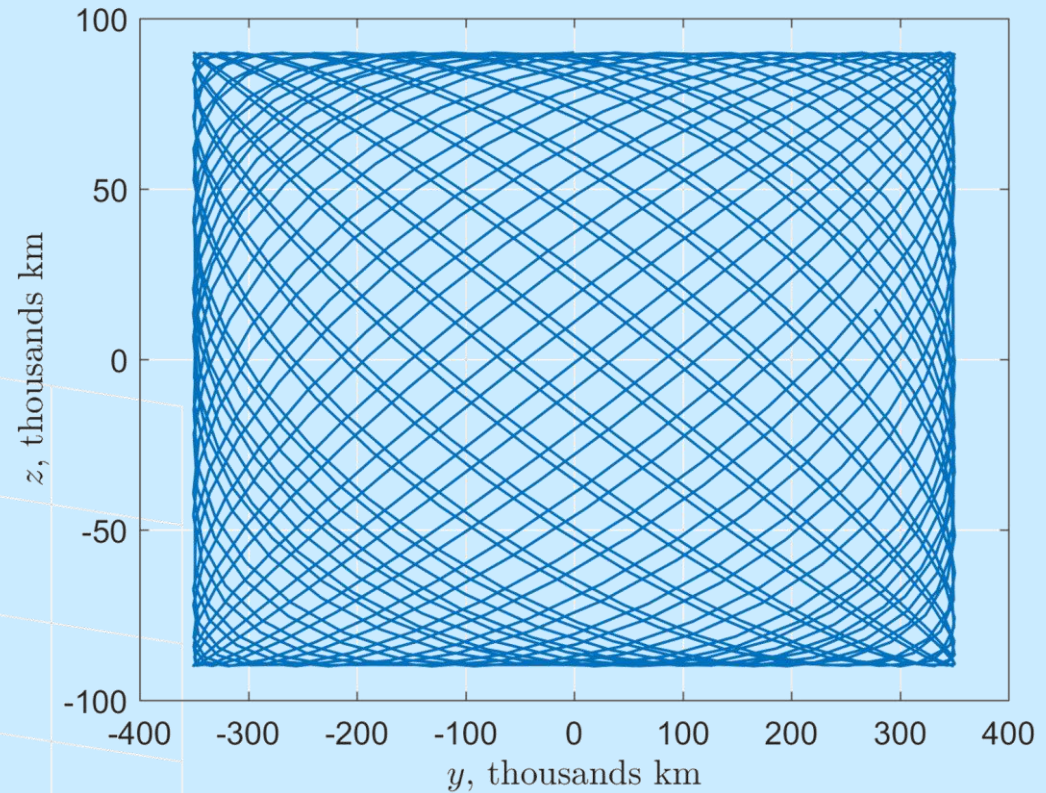
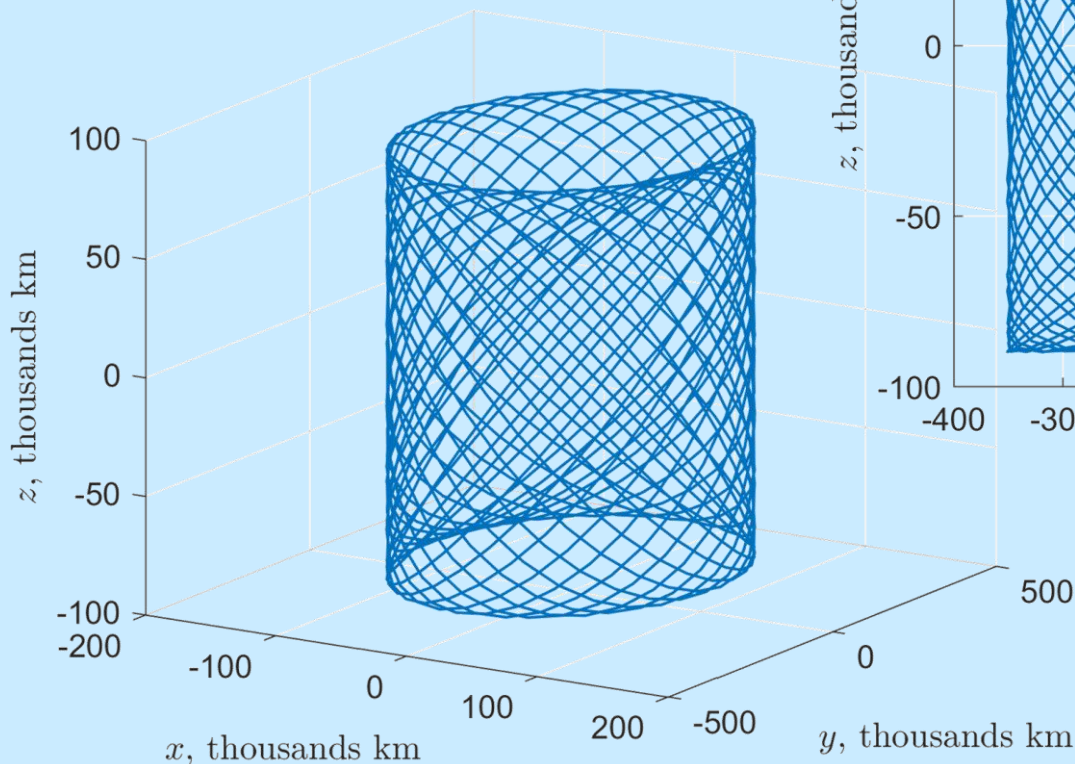
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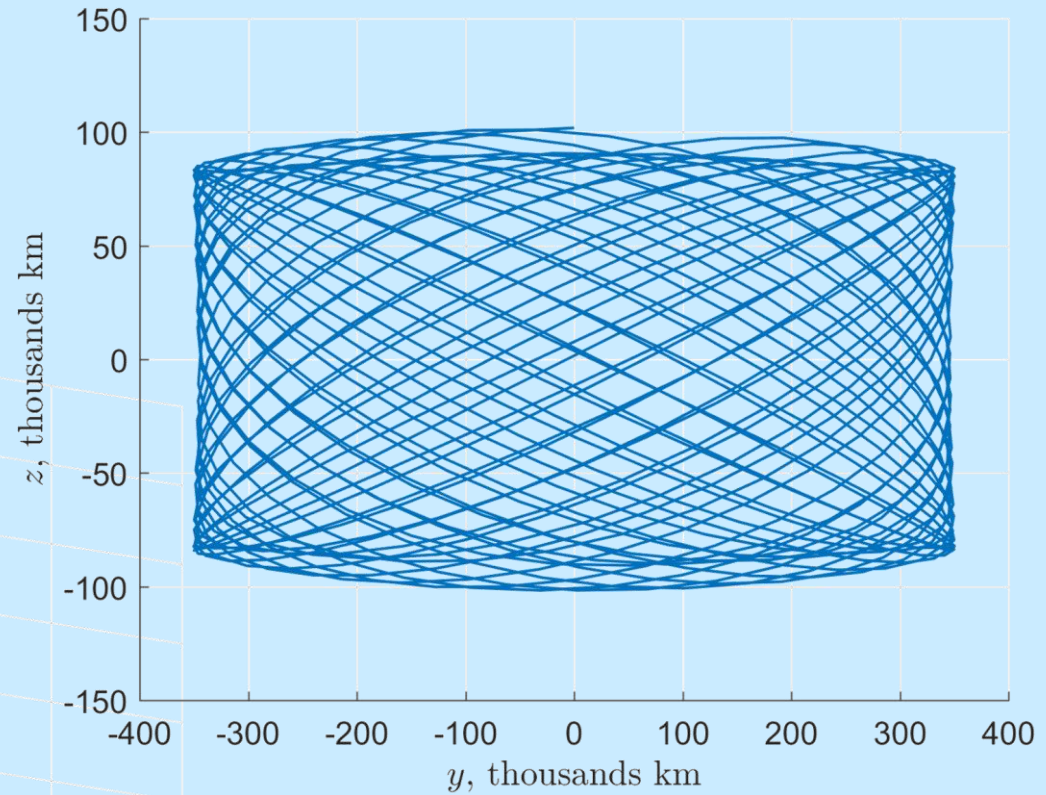
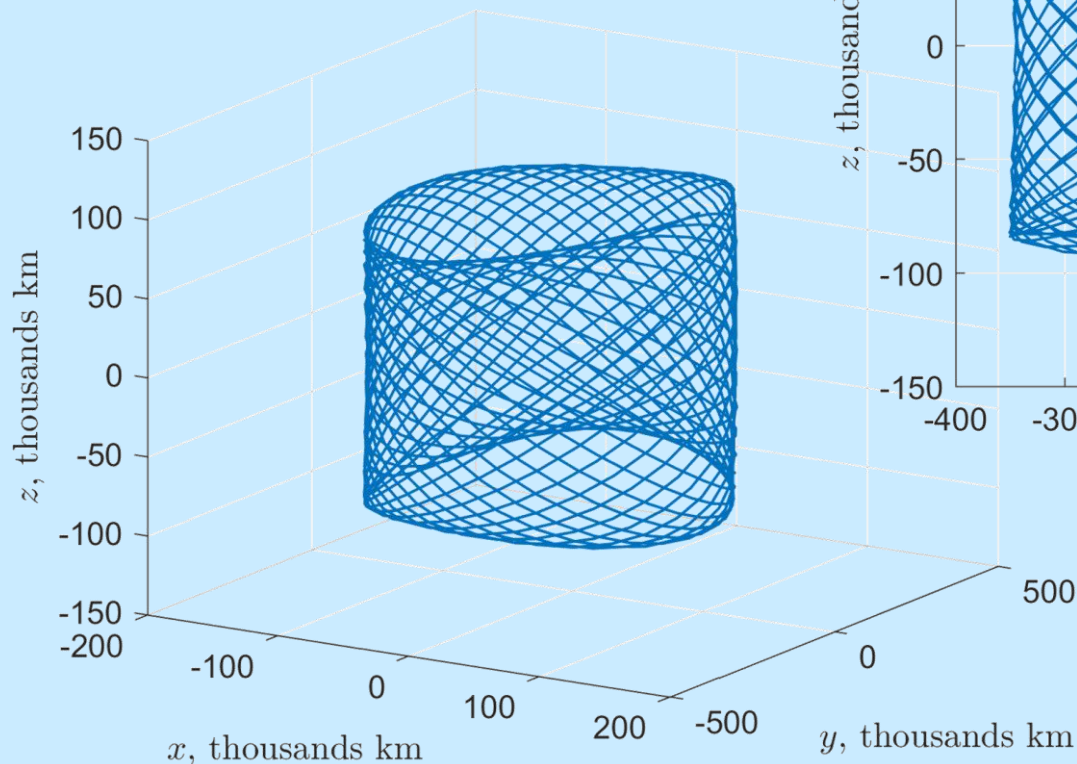
Reference orbit (linear approximation)

Reference orbit: Lissajous
110,000 km x 90,000 km



Reference orbit (15th-order LP series)

Reference orbit: Lissajous
110,000 km x 90,000 km



Some typical performance metrics

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- Projected relative distance $\Delta r^2 - (\Delta \mathbf{r} \cdot \mathbf{n})^2$
 - Should keep the projected relative distance constant (the relative trajectory is a projected circular orbit)
- Angle between the relative position vector and a given vector $\cos^2 \gamma = \frac{(\Delta \mathbf{r} \cdot \mathbf{n})^2}{\Delta r^2}$
 - Should be aligned along some direction

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The optimization problem

- Any formation performance metric can be expressed by symbolically manipulating the LP series
- Just four variables are to be optimized no matter the LP series of what order of approximation we exploit
- No numerical integration is required for calculating the objective function. It is very important in the highly unstable dynamical environment.
- An initial guess can often be obtained analytically from the linear approximation. The hierarchy of models with increasing approximation order can be leveraged.

What about constraints?

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$$+ k_1 \max(0, c(1 - \varepsilon_1) - m) + k_2 \max(0, M - c(1 + \varepsilon_2))$$

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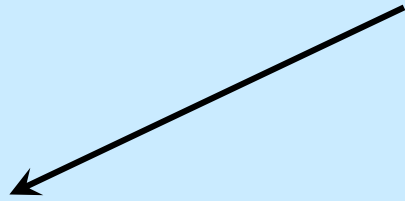
Here the angle brackets denote the average value over a time period of interest, k_1 and k_2 are some large penalty weight coefficients, $m = \min \sqrt{\Delta r^2}$, $M = \max \sqrt{\Delta r^2}$

Performance metric #1: (squared) relative distance

$$\Delta r^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$$

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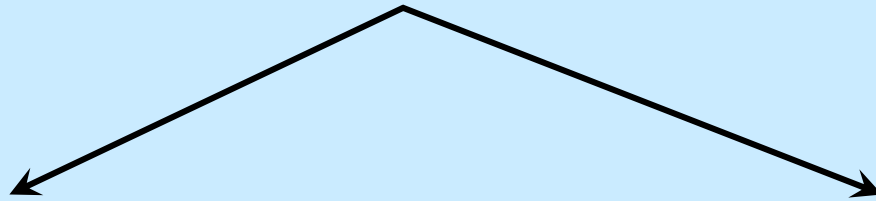


Constant part

$$\frac{A_x^2 (\kappa^2 + 1) + A_z^2}{2}$$

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Short-periodic part

$$\frac{A_z^2}{2} \cos(2\omega_v t + 2\theta_2) - \frac{A_x^2 (\kappa^2 - 1)}{2} \cos(2\omega_p t + 2\theta_1)$$

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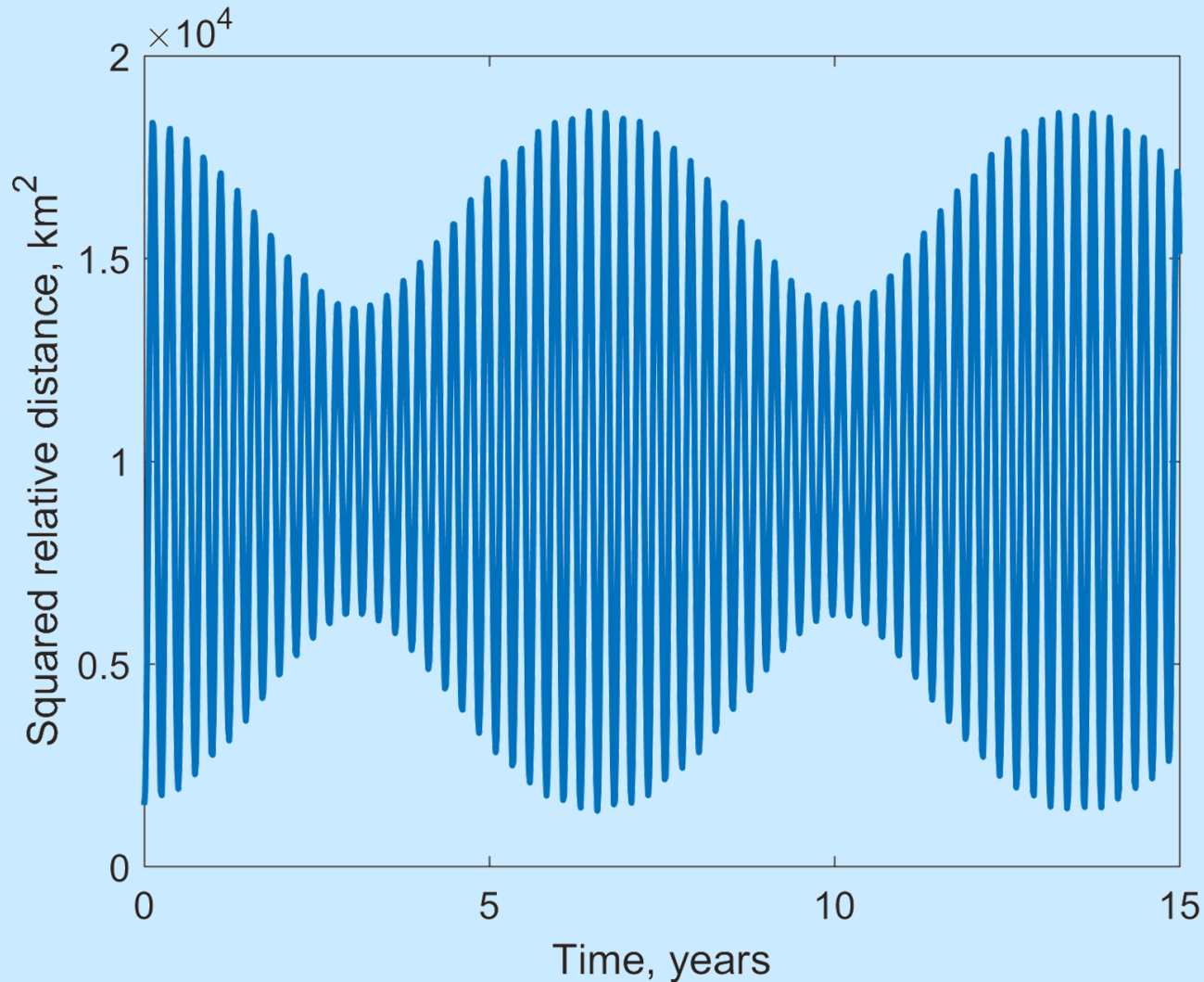
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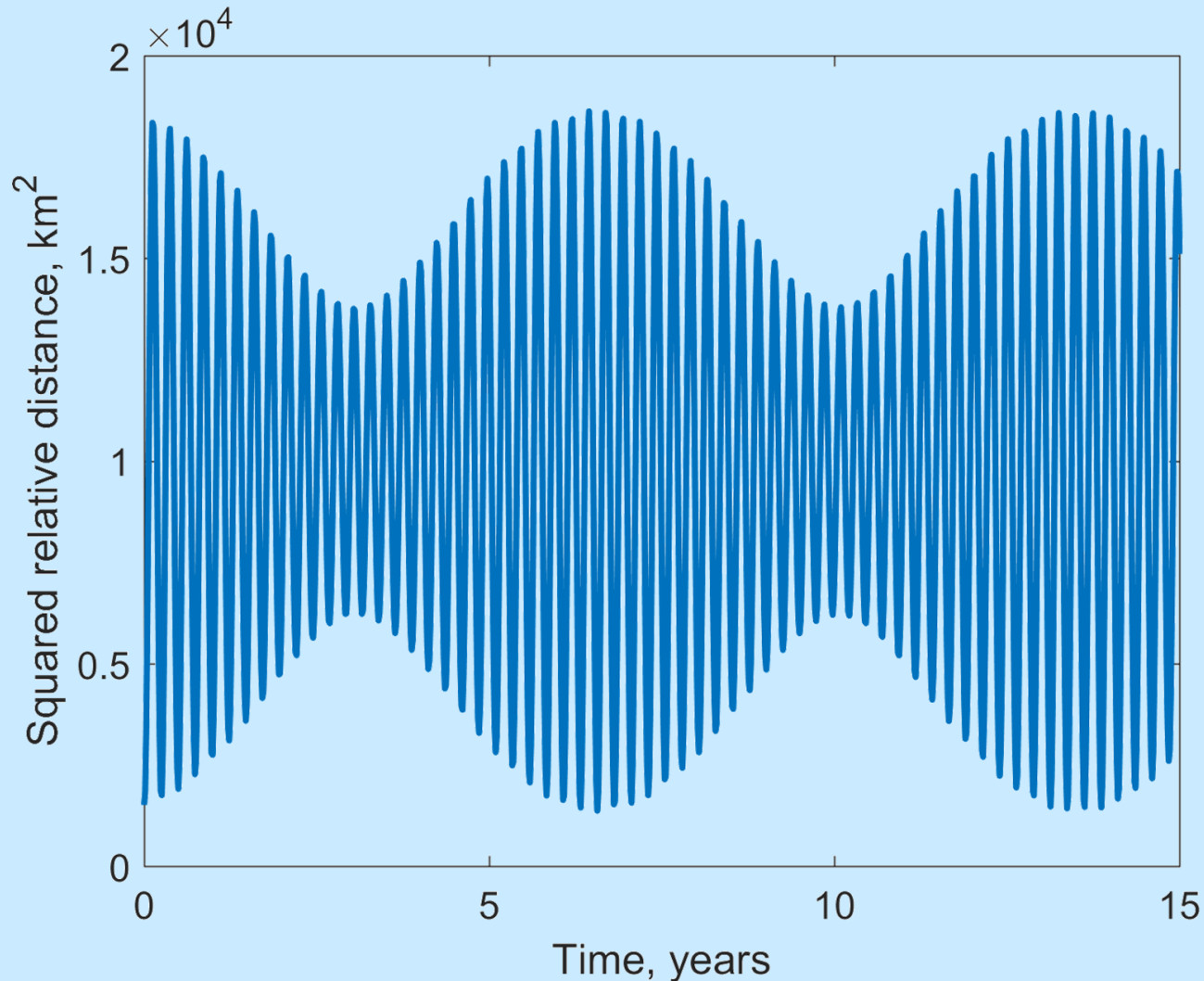
Beating with the beat frequency $\delta = \omega_p - \omega_v$

No solution for long time periods exist!



Minimum variation in the squared relative distance is more than 80%

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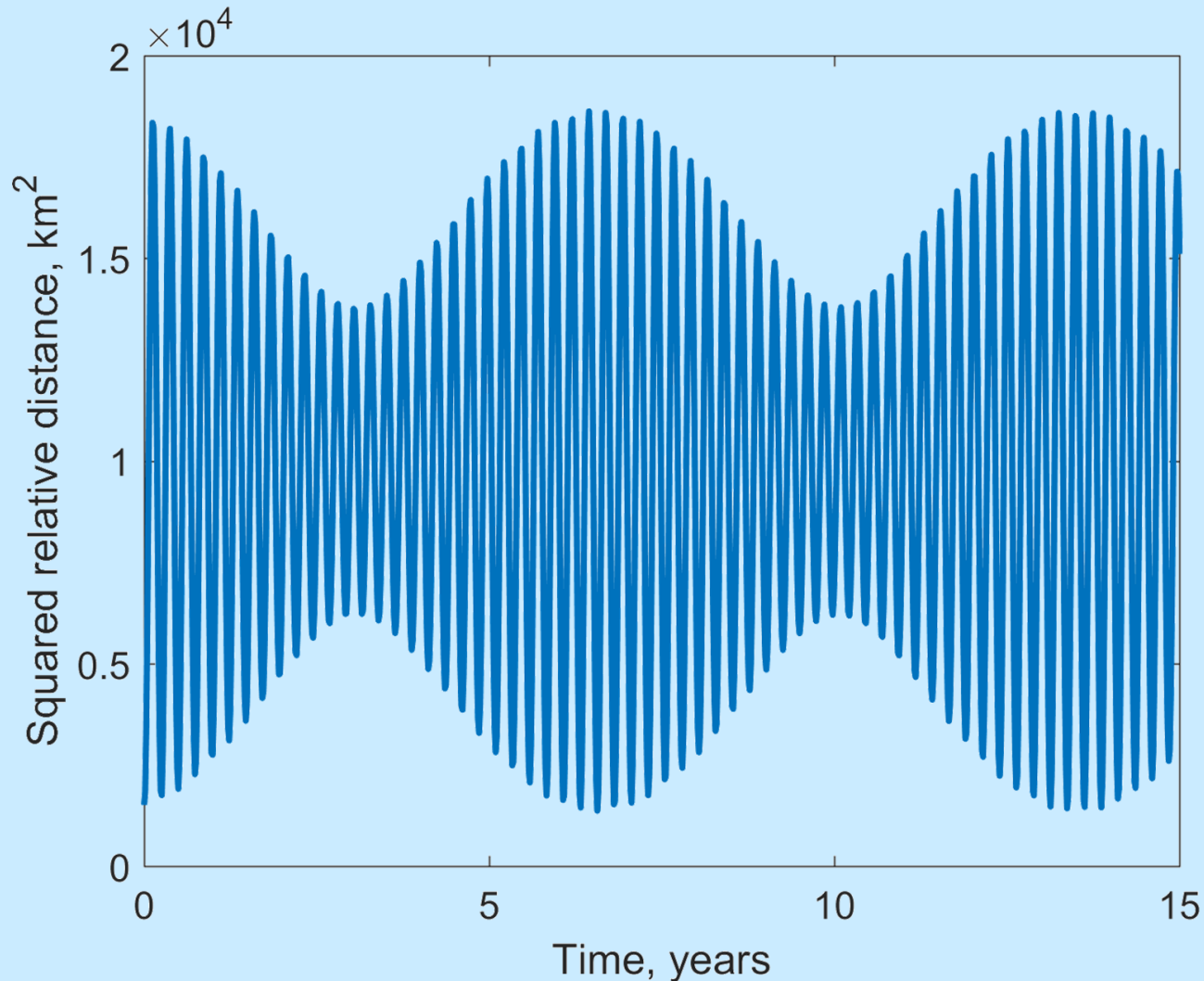


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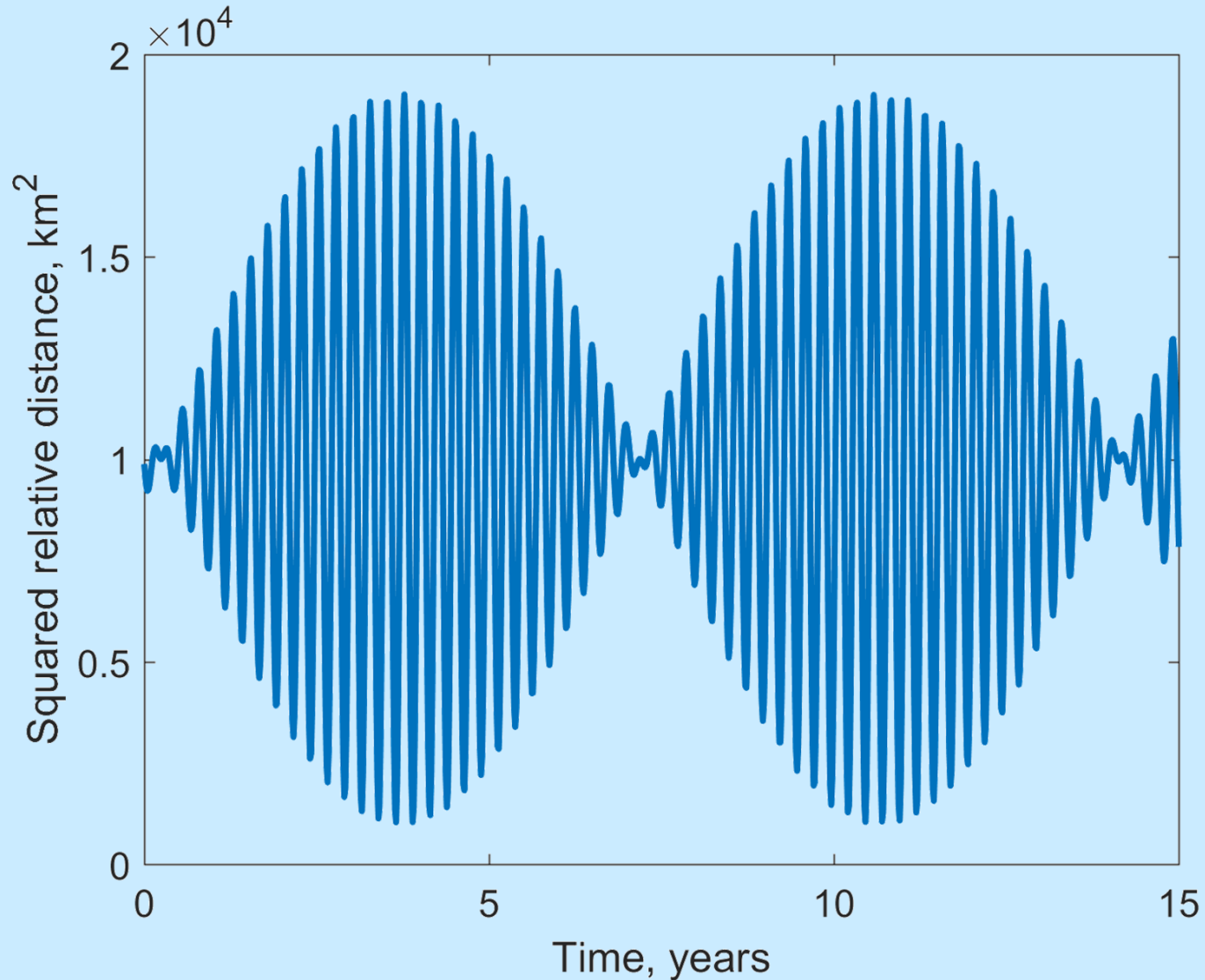


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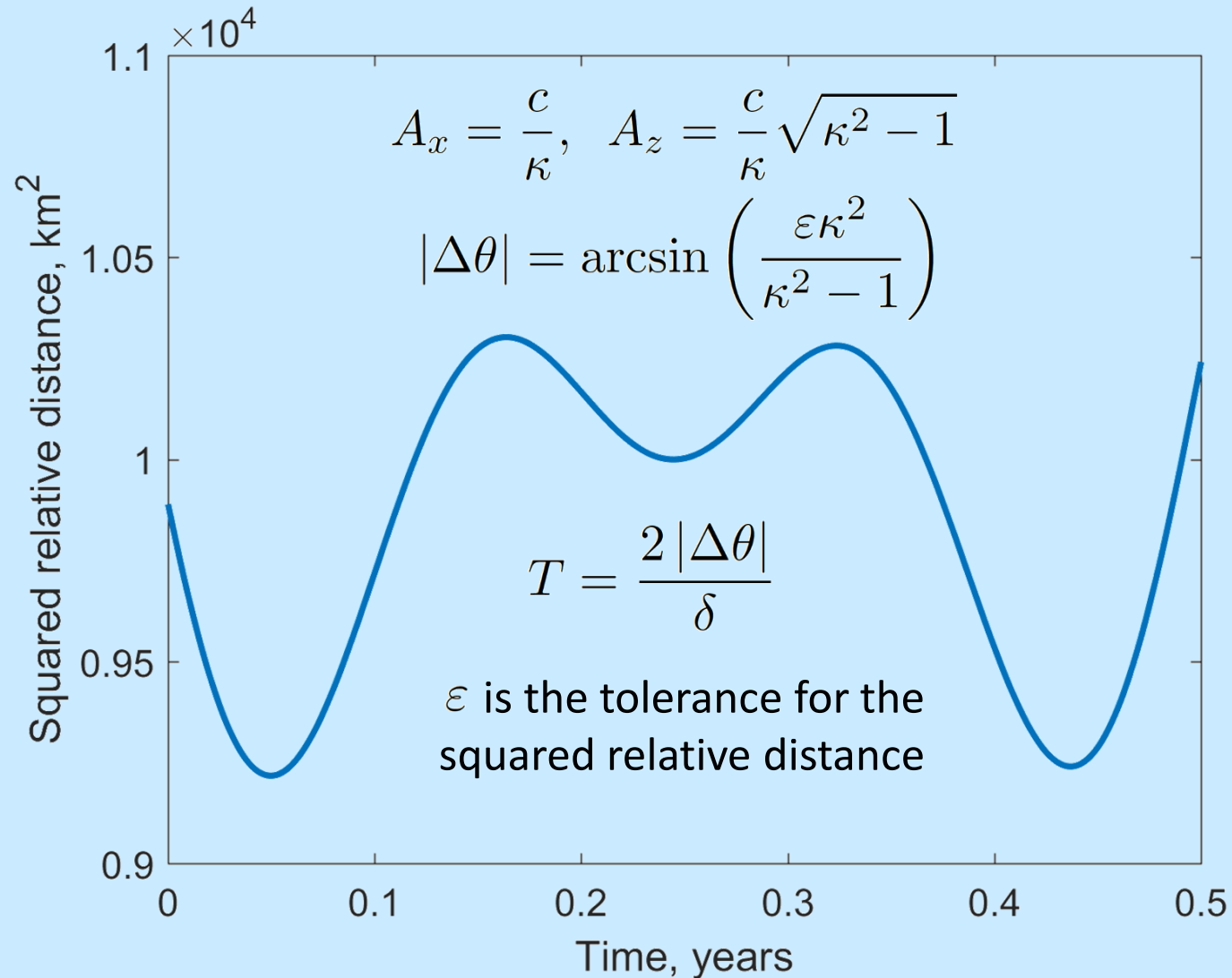


The performance
can be good for a
shorter interval if
phasing is correct

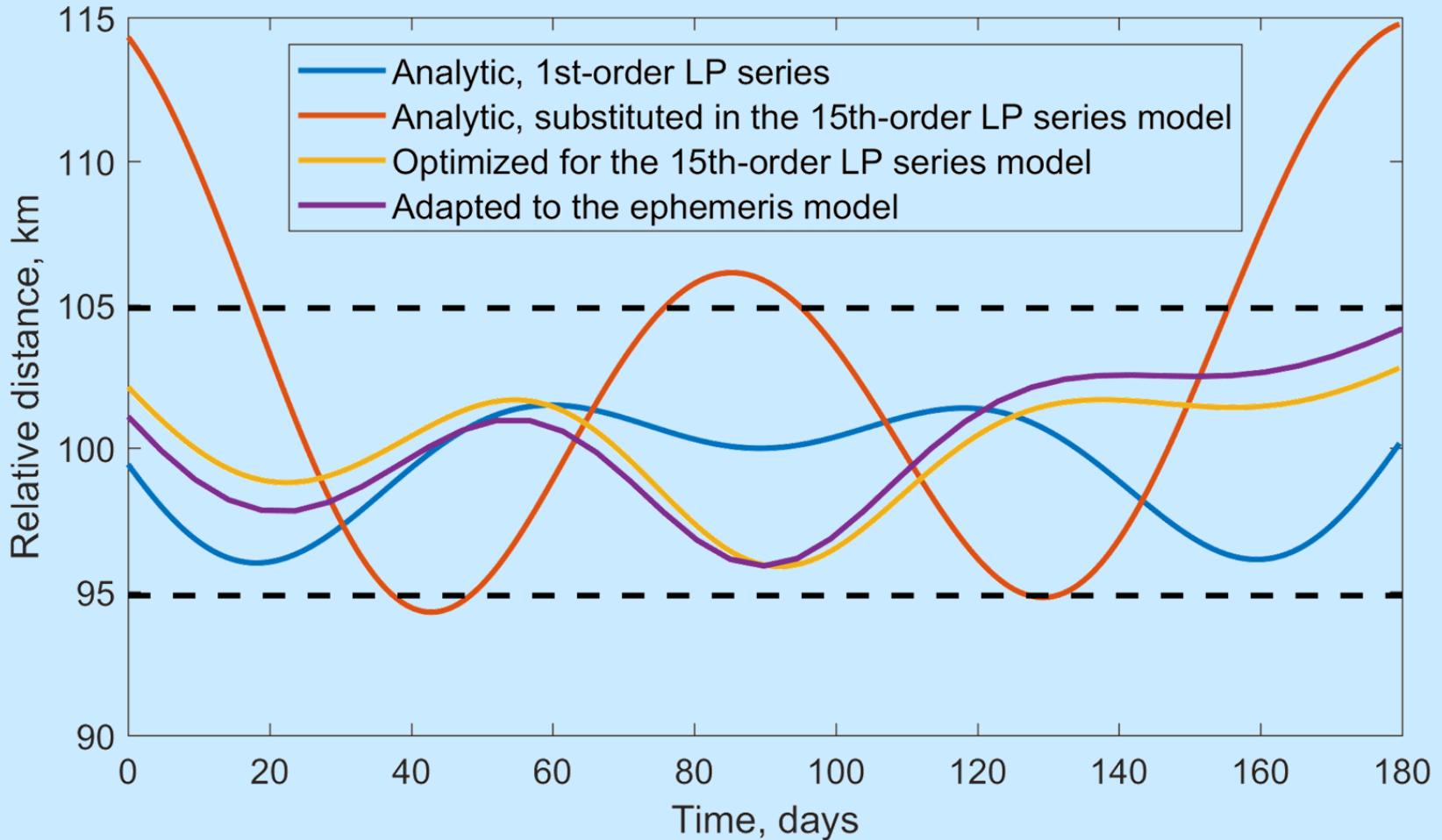
Analytically optimized solution



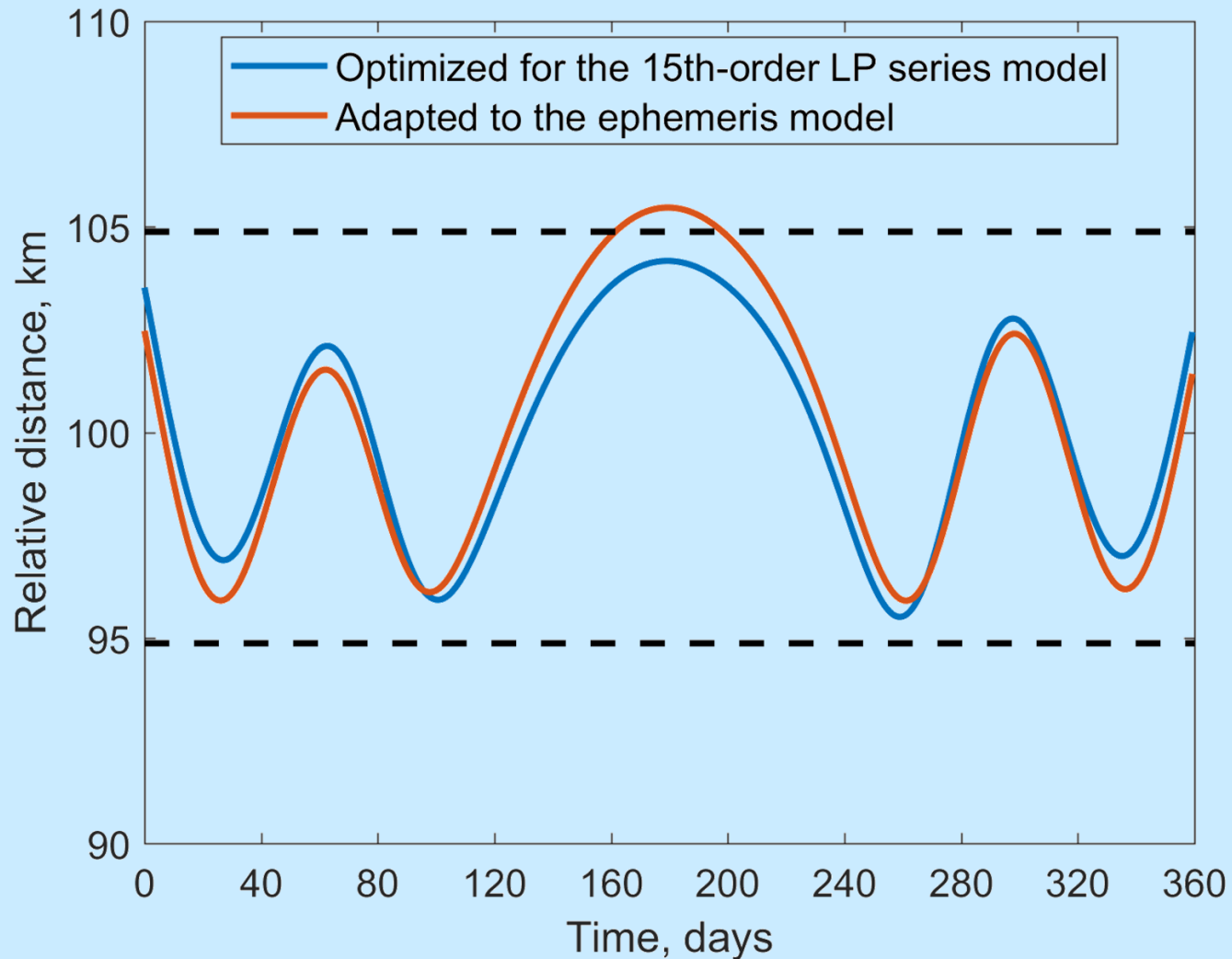
Analytically optimized solution



Performance: from the analytical guess to ephemeris trajectories



Performance optimized over the extended time interval (with the same initial guess)



Analytical initial guess is often critical!

Number of iterations for the Nelder-Mead algorithm to converge

	Smart initial guess	Zero initial guess
2 s/c, metric #1, predicted time	15	67
2 s/c, metric #1, extended time	23	74
2 s/c, metric #2, predicted time	5	39
3 s/c, metric #1, predicted time	92	Not converged

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Conclusions

- Lindstedt-Poincaré series allows parameterizing the relative motion by a quite small number of variables for any order of approximation
- Formation performance metrics are calculated w/o numerical integration of highly unstable trajectories
- Nelder-Mead simplex algorithm is appropriate, and it usually converges
- Analytical initial guess from the linear approximation is very helpful to ensure fast and regular convergence

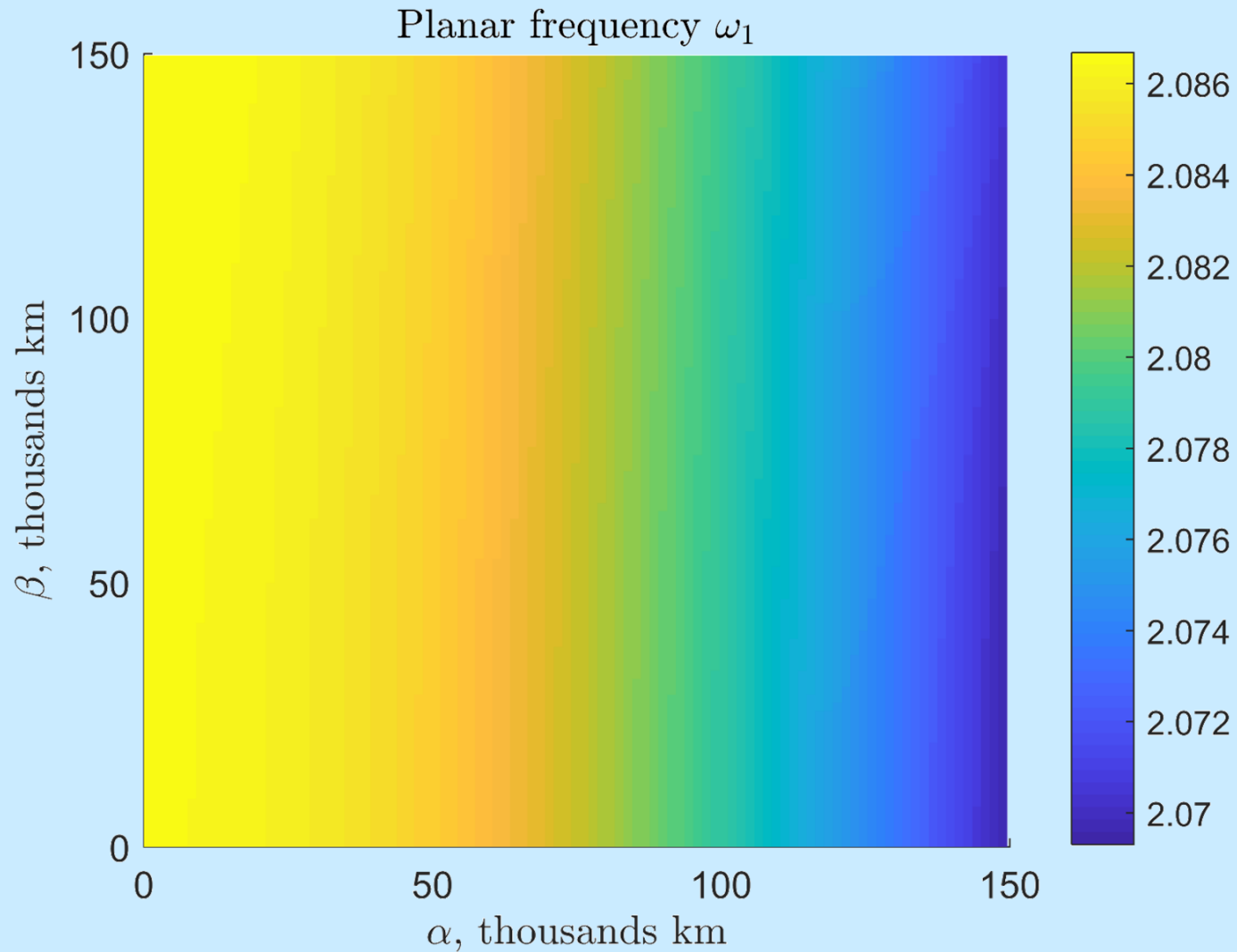
Thank you for your attention

The research is supported by the Russian Science Foundation (RSF) grant 19-11-00256.

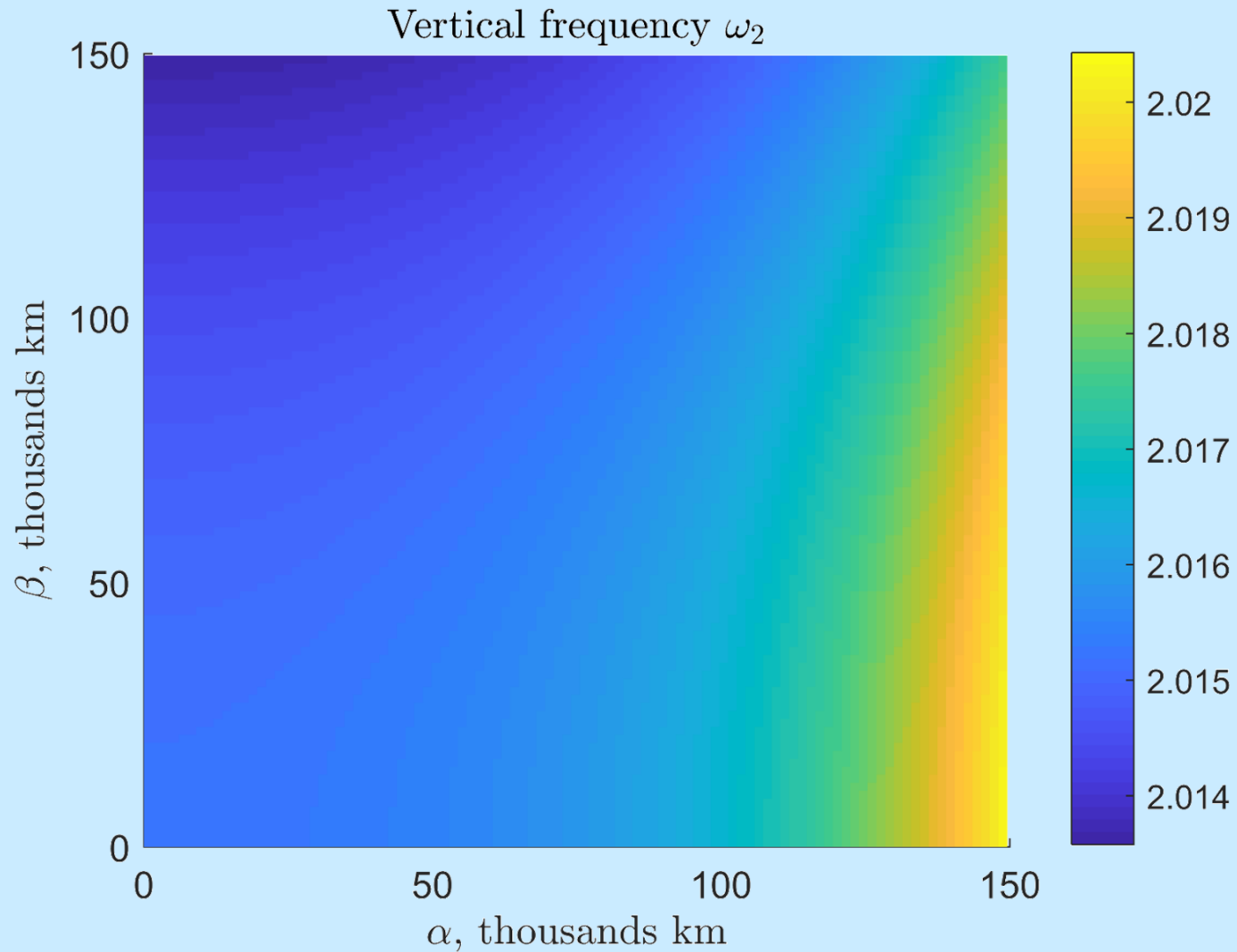
Lindstedt-Poincaré series

- Lindstedt-Poincaré series approximate the central manifold
- For (quasi-)periodic libration point orbits, two small parameters introduced are the in-plane and out-of-plane amplitudes
- Any invariant torus of (quasi-)periodic trajectories is parameterized by two amplitudes and two phases
- In this study, all the numerical examples are for Sun-Earth L2 Lissajous orbits

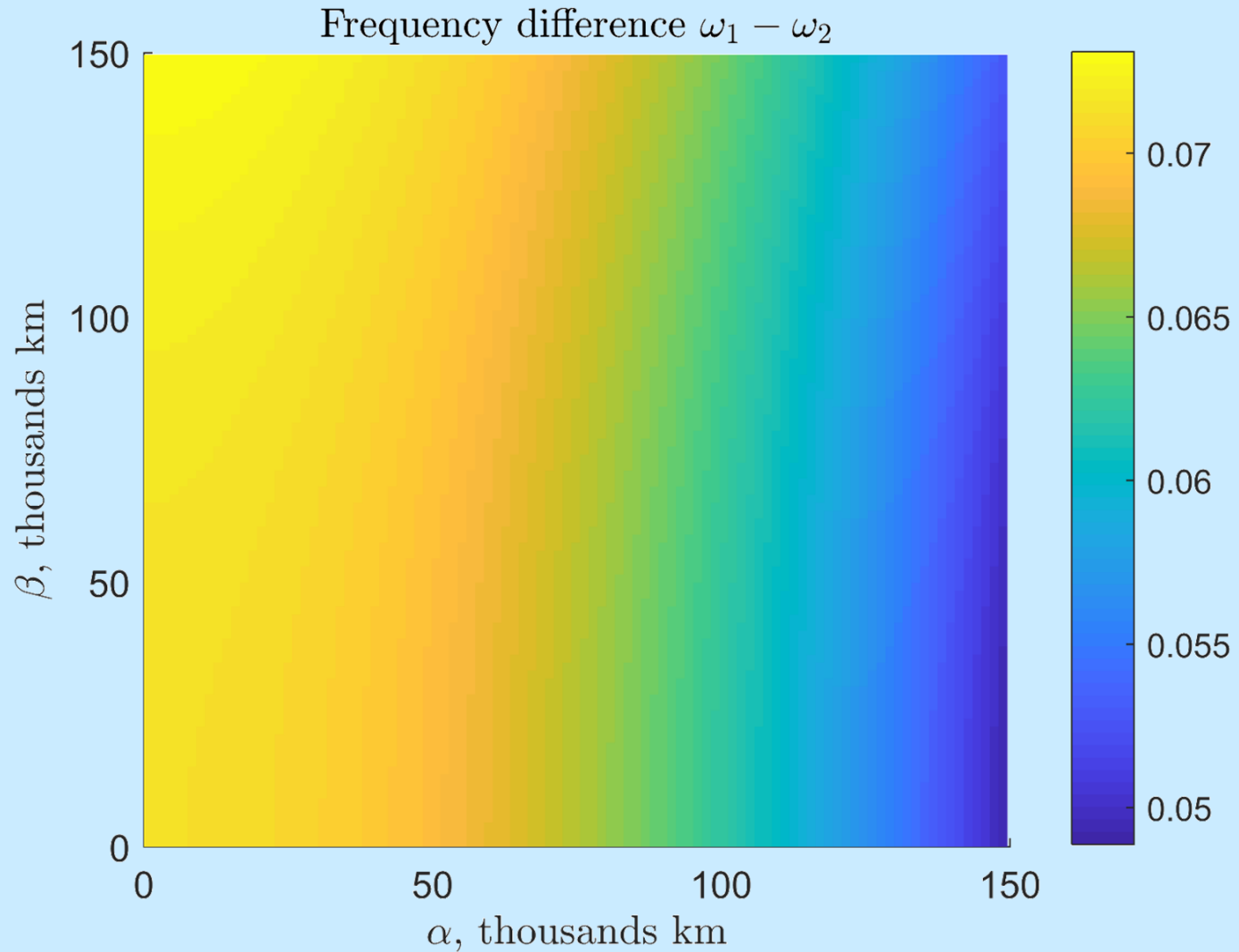
Planar frequency behavior



Vertical frequency behavior



Frequency difference behavior



Nelder-Mead simplex algorithm

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Nelder-Mead simplex algorithm

- One of the most popular derivative-free optimization methods
- The objective function is evaluated at the vertices of a simplex in the search space
- Based on the objective function values, this simplex is modified (reflected, expanded, contracted, shrunk)
- Implemented in Matlab (fminsearch)
- Works exceptionally well in a low-dimension search space

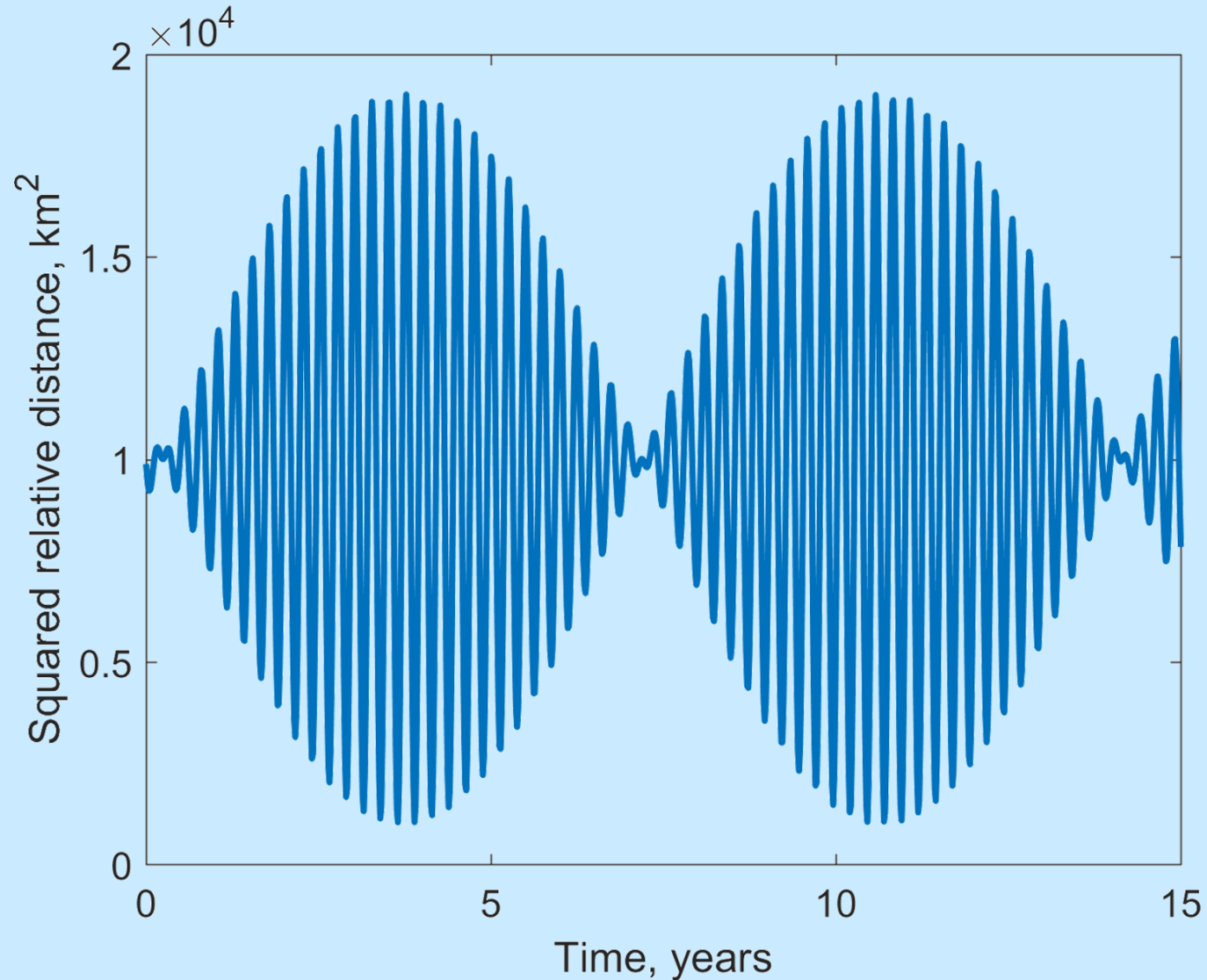
Performance metric #1: analytics

$$\Delta r^2 = \frac{A_x^2 (\kappa^2 + 1) + A_z^2}{2} + \frac{A_z^2}{2} \cos(2\omega_v t + 2\theta_2) - \frac{A_x^2 (\kappa^2 - 1)}{2} \cos(2\omega_p t + 2\theta_1)$$

The upper envelope of this beating curve is as follows:

$$\sqrt{\frac{A_x^4 (\kappa^2 - 1)^2}{4} + \frac{A_z^4}{4} - \frac{A_x^2 A_z^2 (\kappa^2 - 1)}{2} \cos(2\delta t - 2\Delta\theta)}$$

Performance metric #1: analytics



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We need to maximize the distance between the adjacent roots of the equation

$$\sqrt{\frac{A_x^4 (\kappa^2 - 1)^2}{4} + \frac{A_z^4}{4} - \frac{A_x^2 A_z^2 (\kappa^2 - 1)}{2} \cos(2\delta t - 2\Delta\theta)} = c^2 \varepsilon$$

s.t.

$$c^2 = \frac{A_x^2 (\kappa^2 + 1) + A_z^2}{2}$$

Performance metric #1: analytics

Rearranging yields

$$\cos(2\delta t - 2\Delta\theta) = \frac{c^4 \varepsilon^2 - a^2 - b^2}{2ab}$$

$$\text{s.t. } a - b/\chi = c^2$$

where

$$a = \frac{A_z^2}{2} \quad b = -\frac{A_x^2 (\kappa^2 - 1)}{2} \quad \chi = \frac{\kappa^2 - 1}{\kappa^2 + 1} \approx 0.82$$

Obviously, the right-hand side should be minimized.

Performance metric #1: analytics

Equivalently,

$$\eta(\xi) = \frac{(1 - \xi)^2 + \chi^2 \xi^2 - \varepsilon^2}{2\chi\xi(1 - \xi)} \longrightarrow \min$$

where

$$\xi = 1 - \frac{a}{c^2}, \quad \xi \in [0, 1]$$

Performance metric #1: analytics

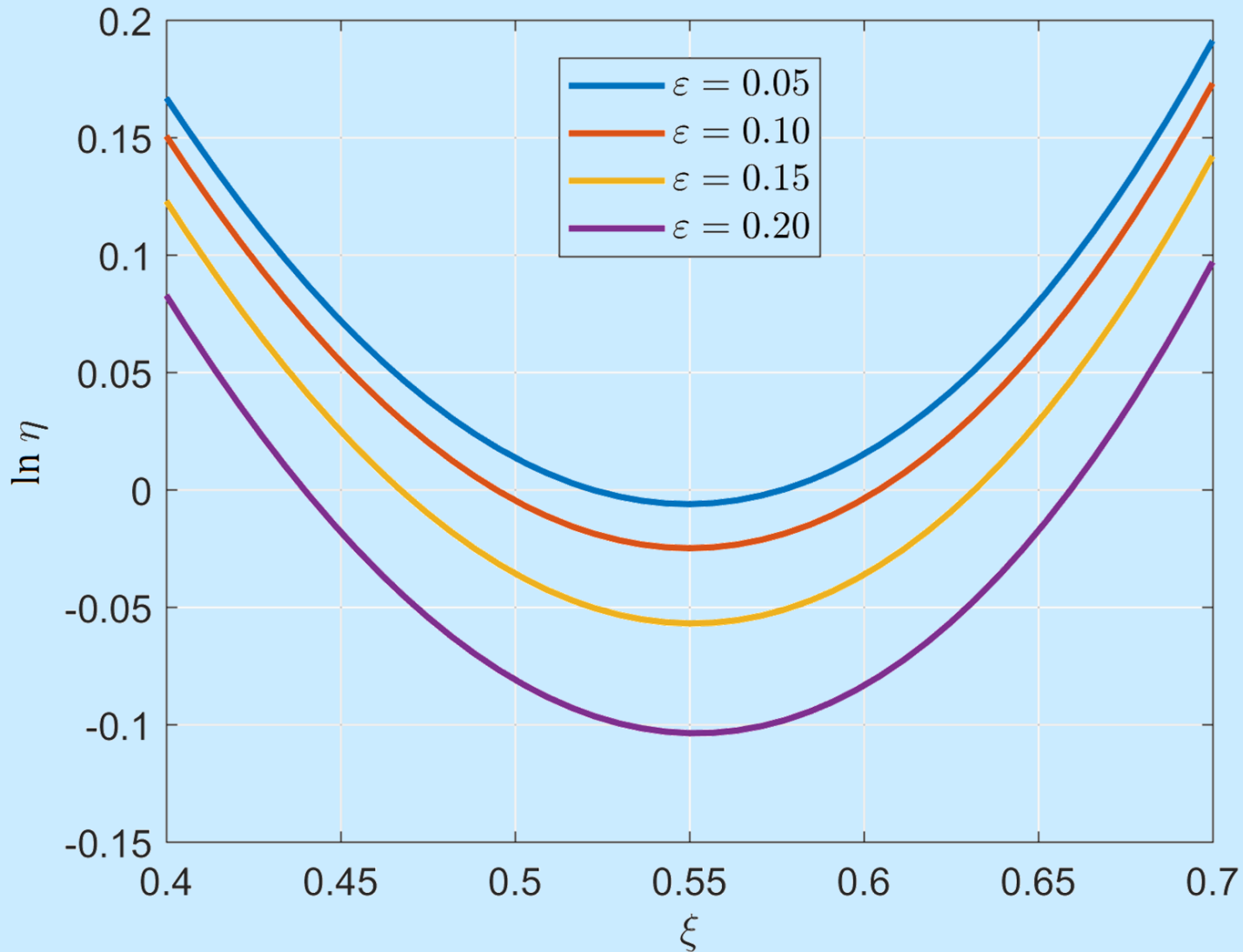
The minimum is attained at the point

$$\xi_{\min} = \frac{1 - \varepsilon^2 - \sqrt{(1 - \varepsilon^2)(\chi^2 - \varepsilon^2)}}{1 - \chi^2}$$

which weakly depends on ε

$$\xi_{\min} \approx \frac{1}{1 + \chi} \approx 0.55$$

Performance metric #1: analytics



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As a result, we have

$$A_x = \frac{c}{\kappa}, \quad A_z = \frac{c}{\kappa} \sqrt{\kappa^2 - 1}$$

$$|\Delta\theta| = \arcsin\left(\frac{\varepsilon\kappa^2}{\kappa^2 - 1}\right)$$

and the natural flight duration estimate

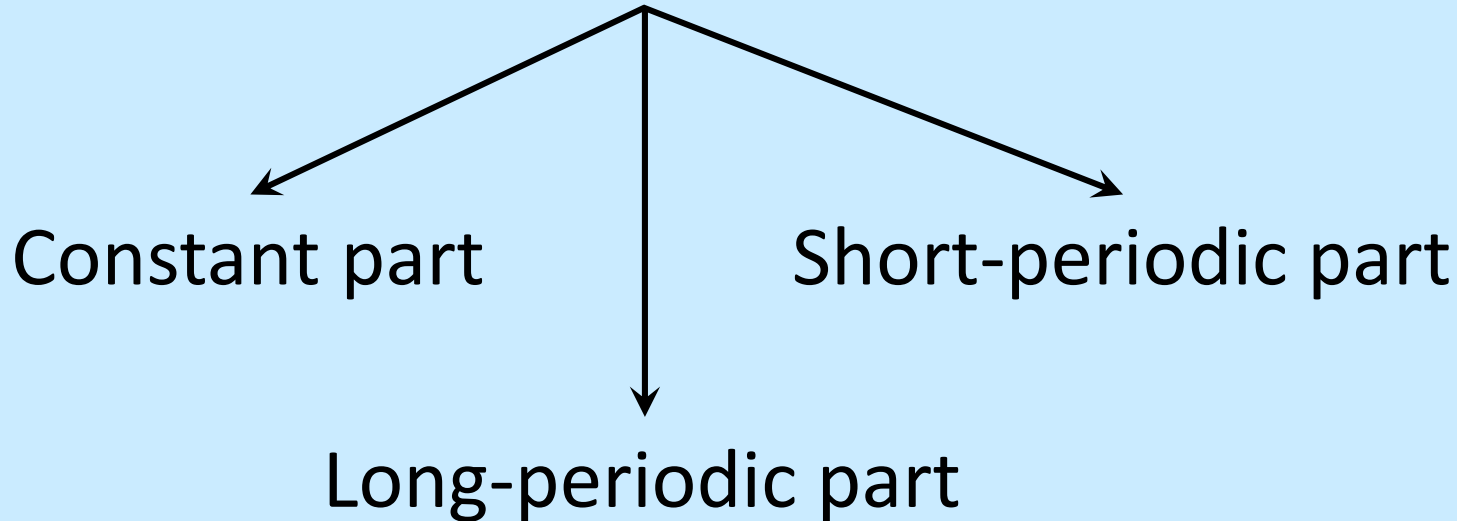
$$T = \frac{2|\Delta\theta|}{\delta}$$

Performance metric #2:
projected relative distance

$$\Delta r^2 - (\Delta \mathbf{r} \cdot \mathbf{n})^2$$

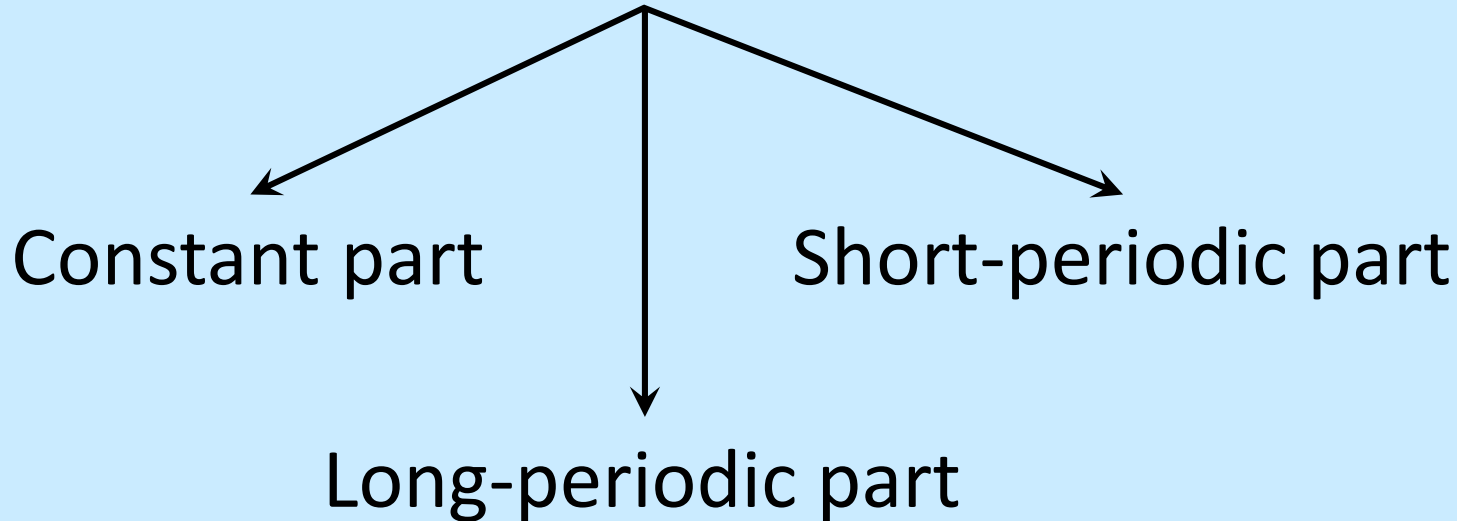
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Performance metric #2: projected relative distance

$$\Delta r^2 - (\Delta \mathbf{r} \cdot \mathbf{n})^2$$



Generally more difficult to analyze, but in some cases it is as simple as for the previous performance metric.

Performance metric #2: projected relative distance

$$\Delta r^2 - (\Delta \mathbf{r} \cdot \mathbf{n})^2$$

$$\mathbf{n} = [1, 0, 0]$$

sun vector

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Constant part

$$\frac{A_x^2 \kappa^2 + A_z^2}{2}$$

Short-periodic part

$$\frac{A_z^2}{2} \cos(2\omega_v t + 2\theta_2) -$$
$$- \frac{A_x^2 \kappa^2}{2} \cos(2\omega_p t + 2\theta_1)$$

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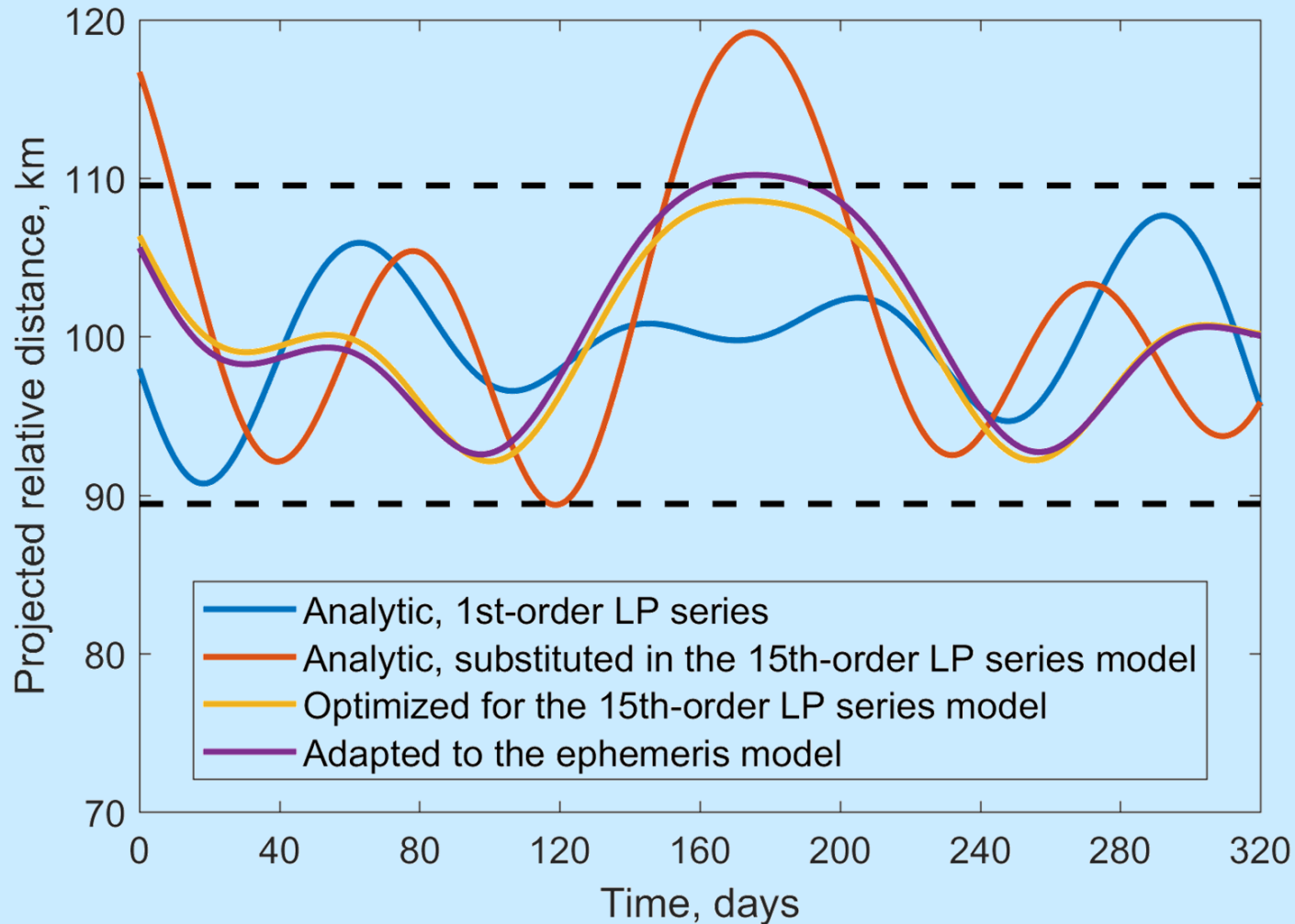
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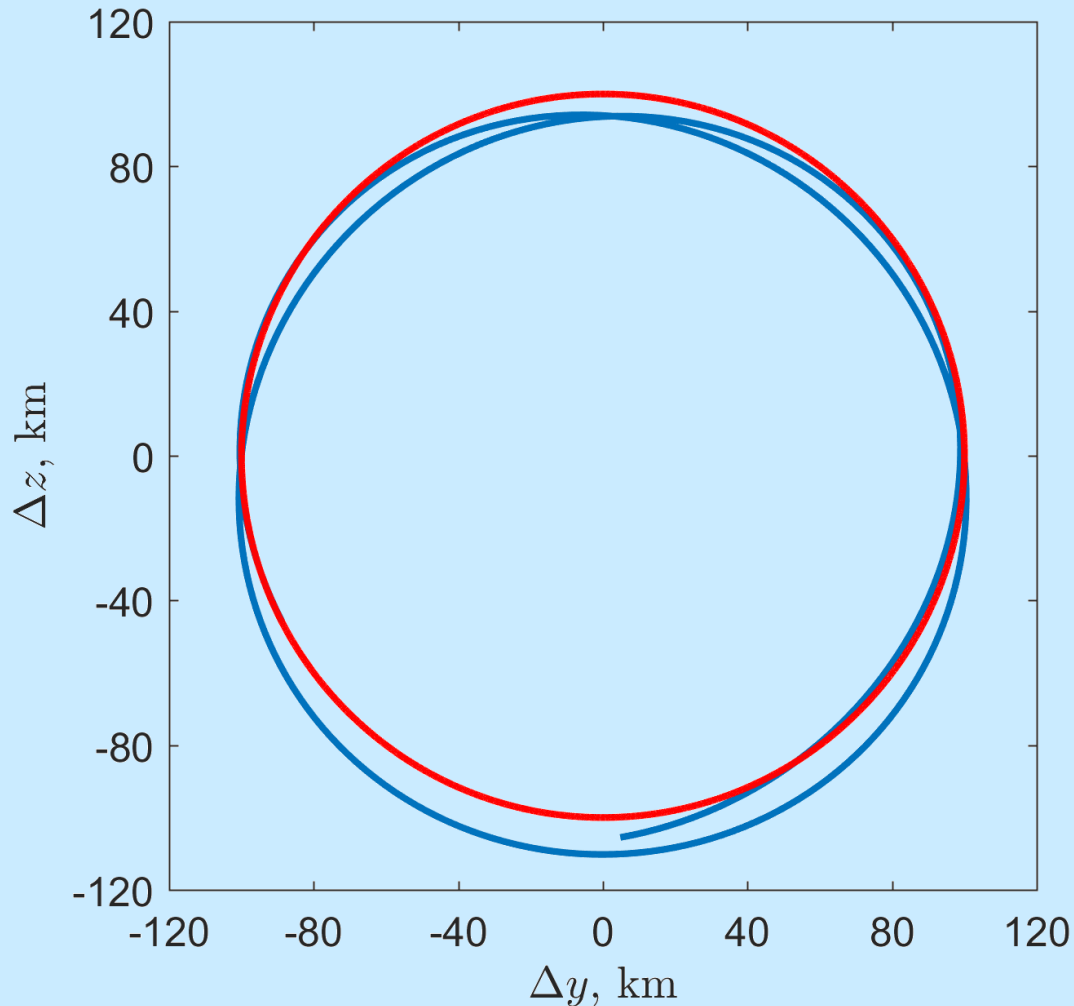
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Beating with the beat frequency $\delta = \omega_p - \omega_v$

Performance: from the analytical guess to ephemeris trajectories



Relative trajectory projected onto the plane orthogonal to the sun vector



Red: ideal 100-km circle

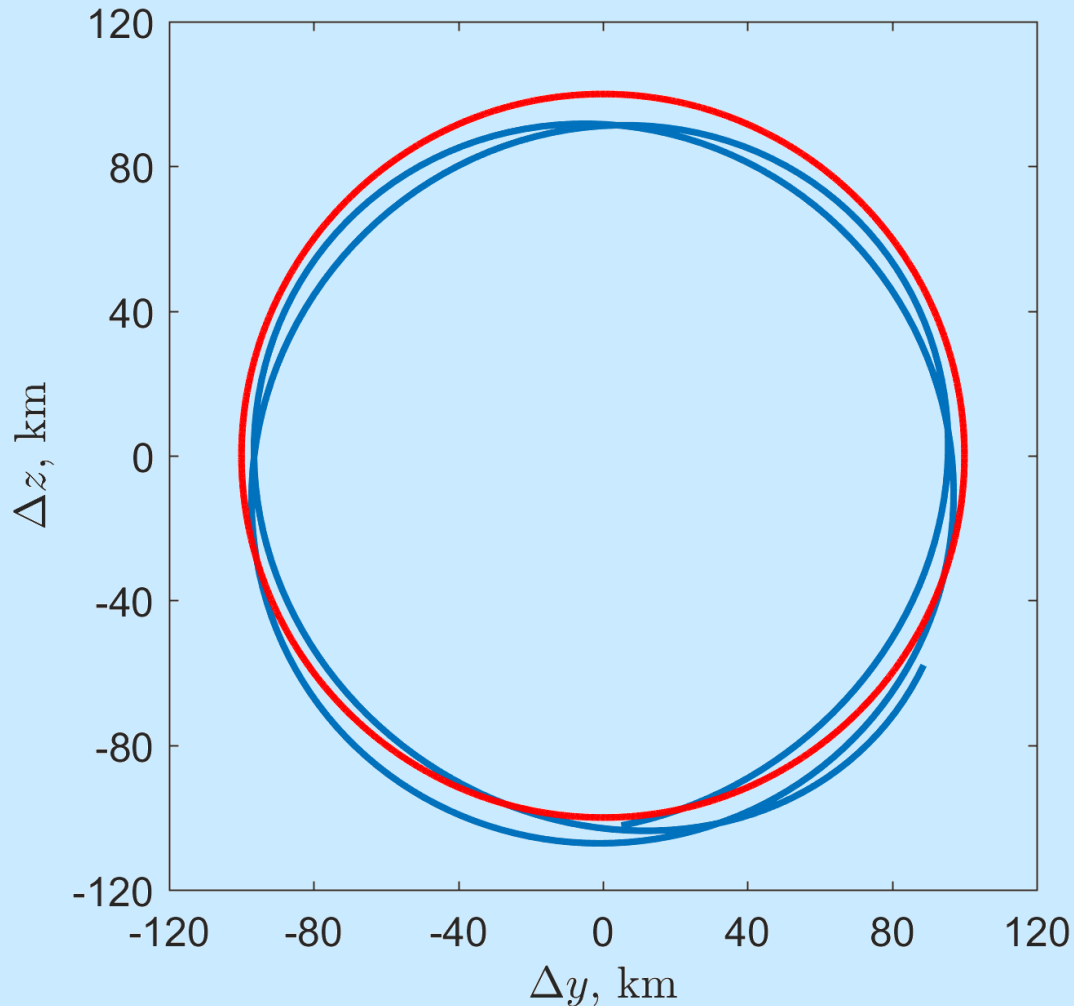
Blue: solution in the ephemeris model

Tolerance: 10%

Initial date: 01.01.2020

Flight duration: 320 days

The same initial guess works well for the 20% longer time interval



Red: ideal 100-km circle

Blue: solution in the ephemeris model

Tolerance: 10%

Initial date: 01.01.2020

Flight duration: 384 days

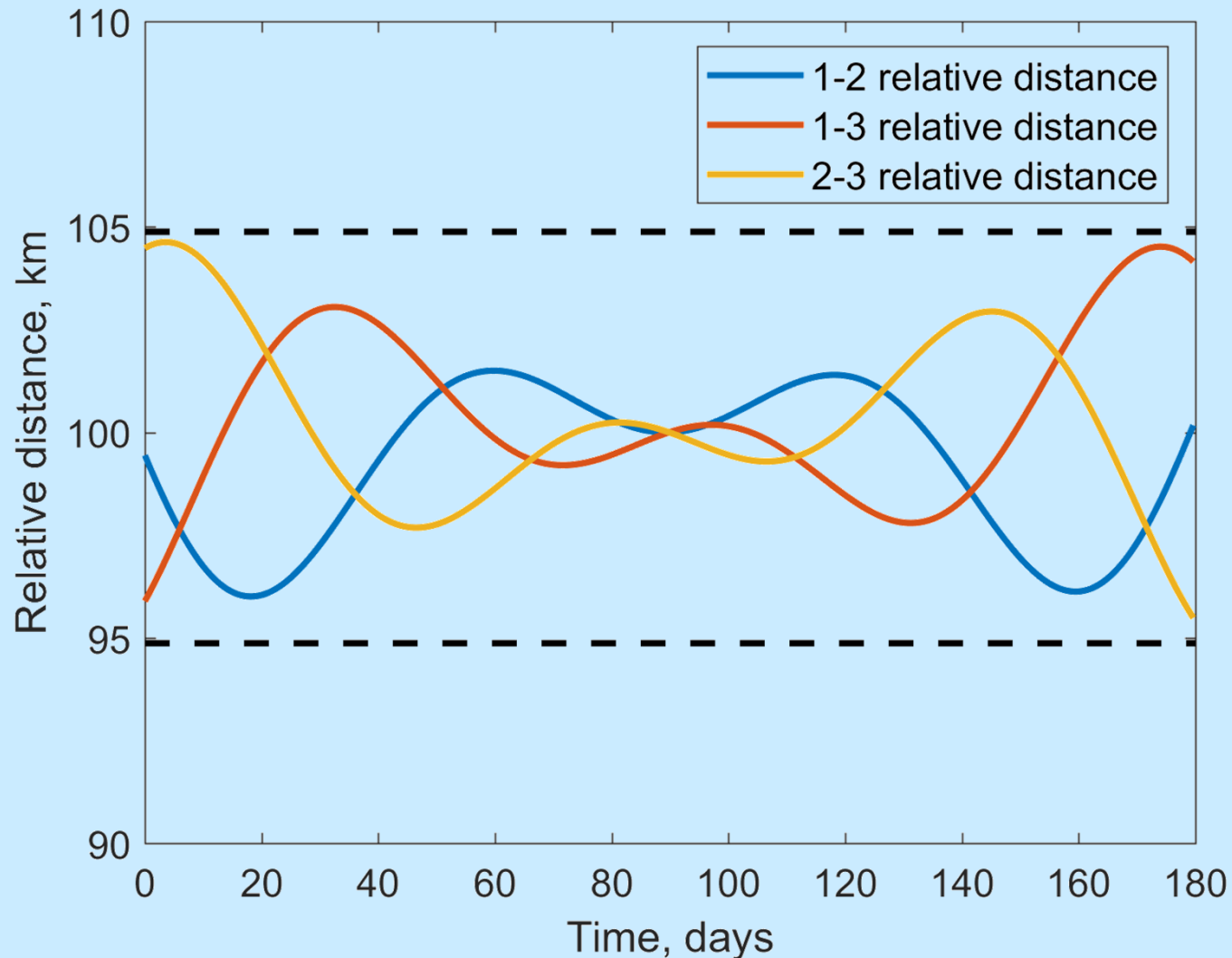
Equilateral triangle formation design

- In the case of an equilateral triangle formation, three intersatellite distances are to be maintained equal to a specified value
- From the linear approximation analysis: the relative amplitudes in each pair should be equal to

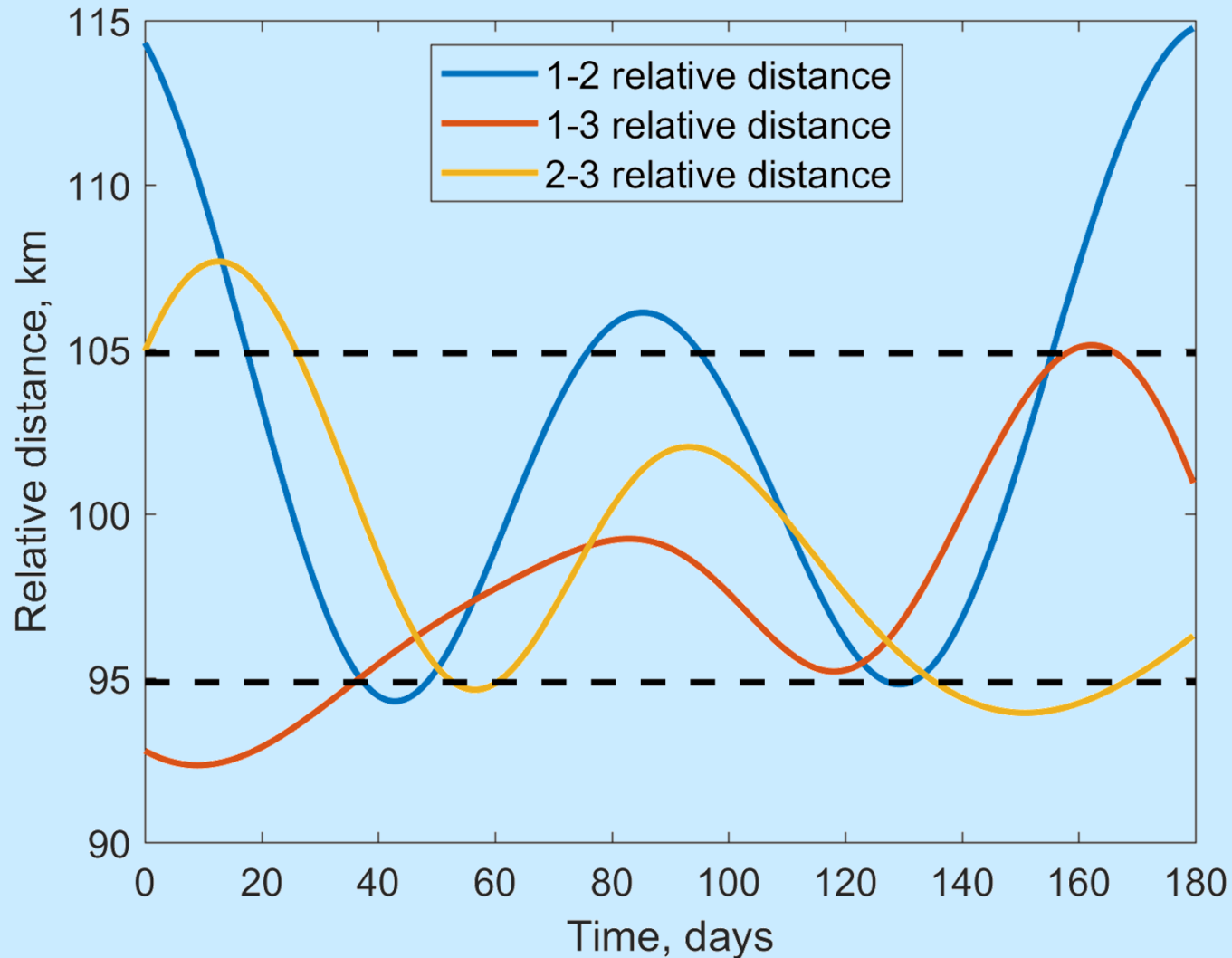
$$A_x = \frac{c}{\kappa}, \quad A_z = c$$

the corresponding phases should be shifted by $\pm\pi/3$

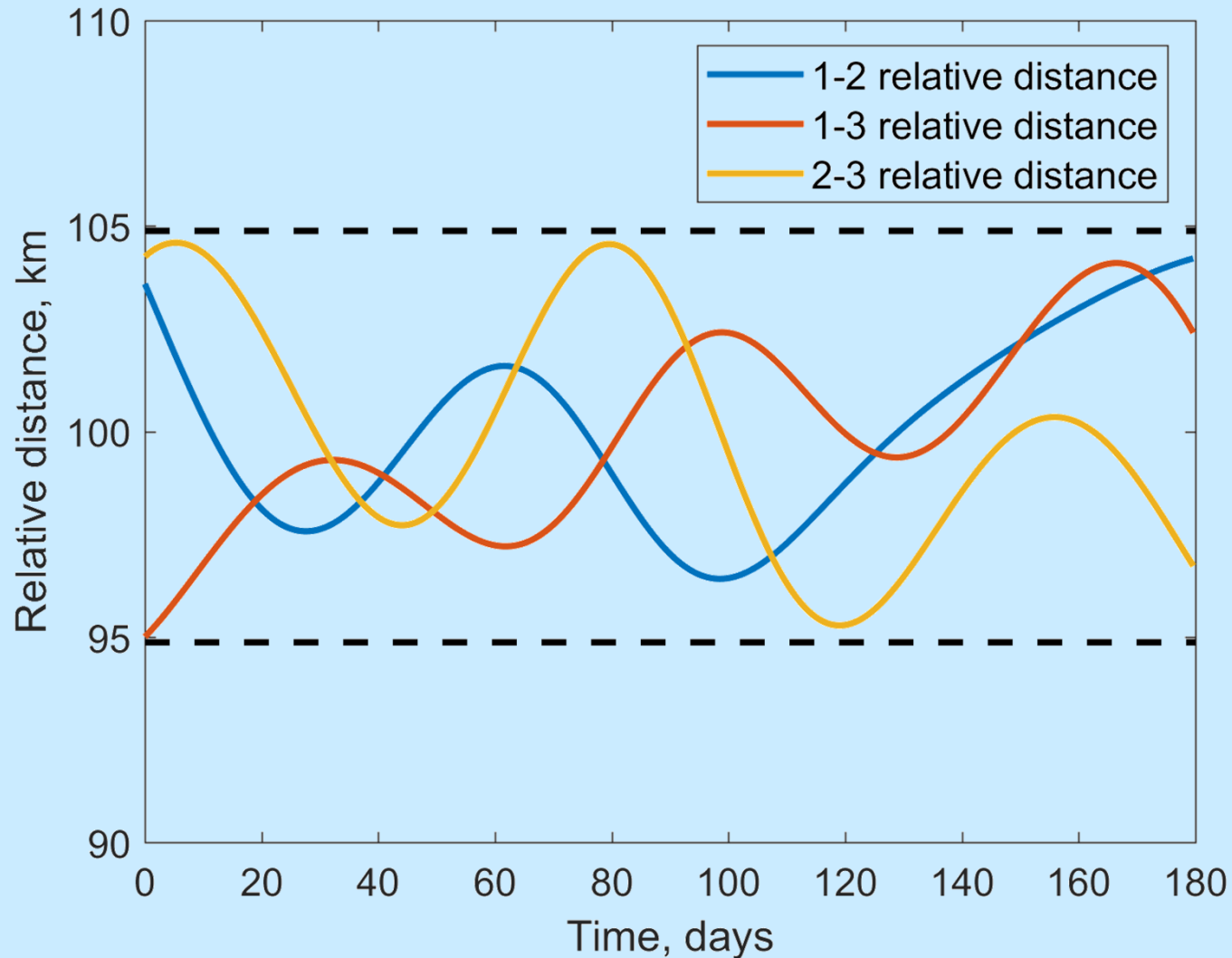
Analytical solution obtained in the linear approximation



Analytical solution substituted in the 15th-order approximation



Solution numerically optimized based on the analytical initial guess



Numerically optimized solution adapted to the ephemeris model

