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Algorithms of Solution of Parabolic Equations on Curvilinear Grids

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М.Ю. Заславский. Алгоритмы решения параболических уравнений на криволинейных сетках.

Аннотация. Рассмотрены различные численные методы для двумерных краевых задач для параболических уравнений. Предполагается, что коэффициенты уравнений разрывны и линии разрывов не совпадают с координатными. Это приводит к необходимости использования неортогональных сеток. Рассмотрены полностью неявная, регуляризованная и аддитивные схемы. Специальный метод регуляризации, учитывающий разрывы в коэффициентах, позволяет построить схемы более, чем первого порядка аппроксимации по времени, которые легко распараллеливаются.

M.Yu. Zaslavsky. Algorithms of solution of parabolic equations on curvilinear grids.

Abstract. The different computational methods for 2D parabolic boundary problems have been considered. It is assumed the coefficients of the equations are discontinuous and the lines of discontinuity don't coincide with coordinate ones. It results in the necessity of using the non-orthogonal grids. The fully implicit, additive and regularized schemes have been developed. The specific method of regularization taking into account discontinuity of coefficient allows constructing the schemes more than first order of time approximation, which are easily parallelized.

Contents

Introduction

Now there are many difference schemes for the heat conductivity parabolic equations with discontinuous coefficients, if the surfaces and lines of discontinuity being not coincident with coordinate ones. The discontinuity of coefficients in a heat conduction equation means that normal (to a discontinuity surface) derivatives are discontinuous. In this case, attempts of approximating the equations by means of the difference schemes without the taking into account the position of discontinuities may reduce the accuracy of approximation. For the best method of describing the phenomena it is expedient to use grids adapted to the media structure. Besides the schemes should satisfy the requirements of stability, monotonicity and conservativity. At the same time because of the usage of multiprocessing systems, it needs to provide the possibility of effective parallelezation of the difference schemes. But the explicit schemes, most simply parallelized, are not considered because of hard limitation of a time-step for schemes stability.

In the present work some methods of solution of the posed problem for the 2D equation are considered

$$\frac{\partial T}{\partial t} + div(\vec{Q}) = 0, \qquad (1.1)$$

where

$$\vec{Q} = -k(x,y) gradT, \qquad (1.2)$$

in domain Ω. It is supposed that the coefficient k is a piecewise continuous function. Boundary conditions may be general. The examined examples correspond to zero boundary conditions of the second type

$$\vec{Q}\Big|_{\Gamma} = 0, \qquad (1.3)$$

where Γ is the boundary of domain. Further, for simplicity we shall consider square domain $\Omega = [0;1] \times [0;1]$.

The cells of examined grids have a form of tetragons, though it should be noticed, that the proposed schemes might be generalized easy for grids containing triangles. The well-known support operator method has been used for a spatial approximation of the problem. The feature of this method is a definition of the divergence and gradient difference operators by using the difference analogy of the known identity:

$$\iint\limits_{\Omega} T div\vec{p} d\Omega + \iint\limits_{\Omega} \left(\vec{p}; gradT\right) d\Omega = \int\limits_{\Gamma} T\vec{p} d\vec{\Gamma}. \qquad (1.4)$$

If the operators of the divergence and gradient are in accordance with the difference approximation of (1.4), the approximation orders of both operators are matched.

Let $\left(\vec{p};\vec{n}\right)\Big|_{\Gamma}=0$. The equality (1.4) allows a definition of the divergence operator if the operator of gradient (or the flux operator $\vec{Q}$) is being determined. It is the reason of the method denomination. Really, the support operator method approximates the system (1.1), (1.2) instead of the approximation of the equation $\frac{\partial T}{\partial t}=div(kgradT)$ only. It is analogous to the ideas developed in the known article (Glowinski R., Wheeler M.F., 1988).

Five difference schemes were compared: fully implicit scheme on the basis of a of the support operators method (Samarskii A.A., Koldoba A.V., Poveshchenko Yu.A., Tishkin V.V., Favorskii A.P., 1996; Koldoba A.V., Poveshchenko Yu.A., Popov Yu.P., 1985), the scheme with regularizator (Samarskii A.A., 1989), two modifications for curvilinear grids of the scheme proposed by Yu.B. Radvogin, and also a variant of the locally one-dimensional scheme based on a variational-difference principle (Goloviznin V.M., Korshunov V.K., Samarskii A.A., Chudanov V.V., 1985). The results of comparison in midpoint of the process evolution and in a final are represented in the tables.

The main aim of the article is to create the effective methods of the 2D and 3D problems solution for the parabolic equations with piecewise continuous coefficients. There are many methods for rectangular grids but usually the surfaces and lines of the coefficients discontinuity don't coincide with coordinate ones. The parabolic equations have standard forms of the balance equations connecting the time derivatives of value with the flow divergence. The evident method is fully implicit but it results in the solution of the linear system of the large order. One possibility is determine the flow components at the intermediate moments of the time and following divergent closure of the system. This idea had been realized by Radvogin Yu.B. only for rectangular grids. The problem of the mixed derivatives for nonrectangular grid had not been discussed in the article. In this work it was shown that the contravariant component of flux (DFM) or covariant components (VDFM) are sufficient to be calculated. Then using the balance equation for each cell the divergence expression may be obtained. The closed equations for the flux components may be constructed if the mixed derivatives being omitted. The flux components equations are of the order O(1). However, it is sufficient for the final approximation of the second order for rectangular grid and of the first order for non-orthogonal grid. Similar principles are used for VLODM. It is a sample of the additive scheme, i.e. there is no approximation of each stage but only total.

The method considered is a regularization method based on Samarskii idea analogous to preconditioning the algebraic linear system. The difference approximation of the equations (1.1), (1.2) may be rewritten $B_1B_2\frac{\hat{T}-T}{\tau}=DIV(kGRADT)$, where $\hat{T}=T\left(t+\tau\right)$, $T=T\left(t\right)$, DIV and $GRAD$ are finite difference approximation of the operators by support operators method, B_1 and B_2 are positive one-dimensional operators, described below. It is essential the scheme is effective because it does not need to solve the complicated linear algebraic equations, as the elliptic operator is determined on the explicit layer. Besides, this scheme is easily parallelized.

The calculations for heat conductivity coefficients with different analytical properties have been realized for tasks on rectangular and curvilinear grids when the analytical solution being known.

It is evident the proposed schemes may be generalized on a 3D case.

Fully implicit scheme and support operator method

Now we describe the most obvious way of approximation: a fully implicit scheme and a method of operator $div(k(x,y)grad(\cdot))$ approximation. Let's mark, that hereinafter the temperature is considered in cell-centers and corresponding fluxes - in cell-edges. This method of discretisation allows the natural approximation of boundary conditions of Neumann type. Besides, it simplifies the construction of difference scheme near the surfaces or lines of coefficients discontinuities, since the fluxes normal to these lines are continuous (it is supposed that lines of discontinuities coincide with grid lines).

The fully implicit scheme has a form

$$\frac{\hat{T}-T}{\tau}=L_h(\hat{T}) \tag{2.1}$$

where $\hat{T}$ is a value T on an implicit time-layer, $L_h(\cdot)$ - difference approximation of the operator $div(k(x,y)grad(\cdot))$ by the support operator method.

Support operators method is a version of the finite volume method. The last one is based on the known identity $\int div\vec{Q}dS=\int Q_n dl$ for a vector $\vec{Q}=-kgradT$. Thus it needs to know components of vector $\vec{Q}$ in the same points of cell edges, but it is inconvenient, since the values of gradient different components are usually determined in different edges. It is offered to average the component values by usage of four neighbor bases (in a 2D case). Besides the integral identity (1.4) will be used instead of the standard formulation of the Gauss–Ostrogradskii theorem for calculating the divergence if the gradient vector being known. The last one allows constructing a difference scheme not on an initial grid, but on arbitrary one providing the best approximation.

The operator $grad(\cdot)$ was considered as the support operator. Let ξ and η be curvilinear coordinates, under level lines passing through cell-centers or their corresponding edges, and the distance between cell-centers for coordinates $(\xi;\eta)$ coincides with the distance in coordinates $(x;y)$. Besides it is supposed, that under transformation $\{f: x, y \to \xi, \eta\}$ the domain Ω is mapped into unit square (see fig. 2.1). The numbers of a grid points along coordinate lines are designated with N_1 and N_2 accordingly. Then the covariant components of a flux $\vec{Q}$ in local basis $\left(\vec{e}_\xi, \vec{e}_\eta\right)$ look like

$$Q_{\xi,i,j+1/2} = -k_{i,j+1/2} \frac{T_{i+1/2,j+1/2} - T_{i-1/2,j+1/2}}{h_{i,j+1/2}}, \tag{2.2}$$

$$Q_{\eta,i+1/2,j} = -k_{i+1/2,j} \frac{T_{i+1/2,j+1/2} - T_{i+1/2,j-1/2}}{h_{i+1/2,j}}, \tag{2.3}$$

$$k_{i,j+1/2} = \frac{k_{i+1/2,j+1/2} k_{i-1/2,j+1/2}}{(1-\gamma_1) k_{i+1/2,j+1/2} + \gamma_1 k_{i-1/2,j+1/2}}, \tag{2.4}$$

$$k_{i+1/2,j} = \frac{k_{i+1/2,j+1/2} k_{i+1/2,j-1/2}}{(1-\gamma_2) k_{i+1/2,j+1/2} + \gamma_2 k_{i+1/2,j-1/2}} \tag{2.5}$$

γ_1 and γ_2 are equal to the parts of segment $\left(i - 1/2 ; j + 1/2\right) - \left(i + 1/2 ; j + 1/2\right)$ and $\left(i + 1/2 ; j - 1/2\right) - \left(i + 1/2 ; j + 1/2\right)$, lying inside a cell $\left(i + 1/2 ; i + 1/2\right)$, $h_{i,j+1/2}$ and $h_{i+1/2,j}$ are the lengths of these segments.

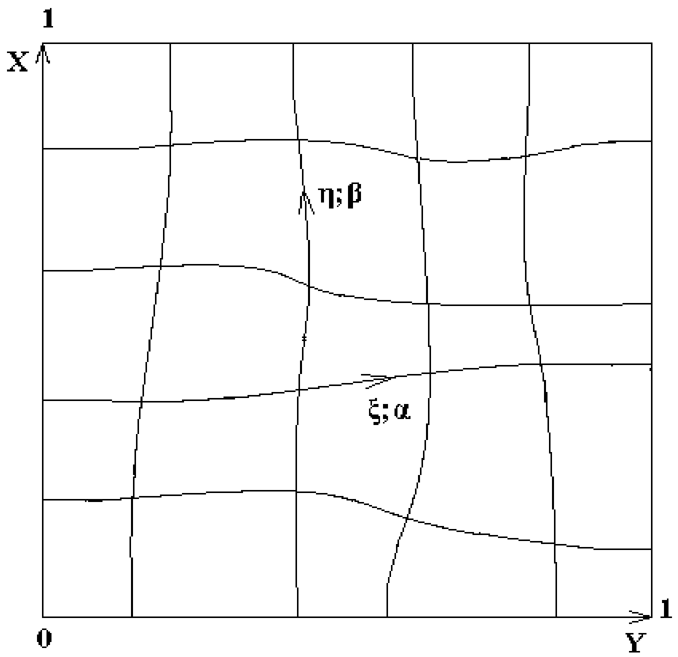

Fig. 2.1

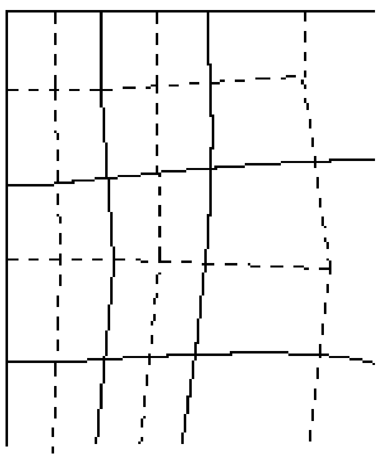
Fig. 2.2

Suppose, that the grid lines are orthogonal to the boundary of the domain (Fig. 2.1). Then we have according the boundary conditions (1.3)

$$Q_{\xi,0,j+1/2} = Q_{\xi,N_1,j+1/2} = Q_{\eta,i+1/2,0} = Q_{\eta,i+1/2,N_2} \qquad (2.6)$$

For approximating the second term of the left hand part of (1.4) the additional grid M' has been constructed. The edges of the grid M' connect the cell centers of the grid M. For covering of whole area Ω we shall add some cells formed by segments of boundary and normals to it (see a fig. 2.2, the continuous lines bound cells of a grid M, dashed - M').

Let's consider an internal cell $(i;j)$ of grid M'. Its vertices are denoted with F, G, H, I. The notation of angles is represented in the fig. 2.3. Then the inner product of vectors $\vec{p}$ and $\vec{Q}$ in the vicinity of vertex F has a form

$$\left(\vec{p};\vec{Q}\right)_F = \frac{1}{\sin^2 F}\left(p_{FG}Q_{FG} + p_{FI}Q_{FI} - \left(p_{FG}Q_{FI} + p_{FI}Q_{FG}\right)\cos F\right) \qquad (2.7)$$

Similarly in vertex G:

$$\left(\vec{p};\vec{Q}\right)_G = \frac{1}{\sin^2 G}\left(p_{FG}Q_{FG} + p_{GH}Q_{GH} - \left(p_{FG}Q_{GH} + p_{GH}Q_{FG}\right)\cos G\right) \quad (2.8)$$

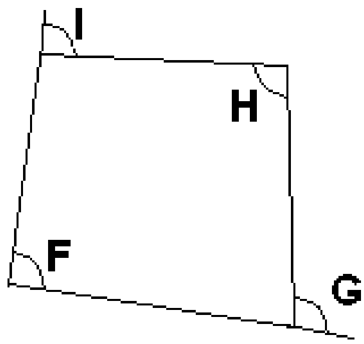

Fig. 2.3

Using (2.6), (2.7) and analogous expressions for vertices H and I the quantity of integral over the cell may be represented

$$\int_{FGHI} \left(\vec{p};\vec{Q}\right) d\Omega = S_F\left(\vec{p};\vec{Q}\right)_F + S_G\left(\vec{p};\vec{Q}\right)_G + S_H\left(\vec{p};\vec{Q}\right)_H + S_I\left(\vec{p};\vec{Q}\right)_I . \quad (2.9)$$

Here S_F, S_G, S_H, S_I are some non-negative values, the sum of which equals the square of FGHI. Let the vertices F, G, H, I have indexes $\left(i-1/2; j-1/2\right)$, $\left(i+1/2; j-1/2\right)$, $\left(i+1/2; j+1/2\right)$, $\left(i-1/2; j+1/2\right)$ respectively. Then the expression (2.9) may be rewritten

$$\int\limits_{FGHI}\left(\vec{p};\vec{Q}\right)d\Omega = W_{i,j} = \left(\frac{S_F}{\sin^2 F}\right)_{i,j}\left[p_{\xi,i,j-1/2}Q_{\xi,i,j-1/2} + p_{\eta,i-1/2,j}Q_{\eta,i-1/2,j} - \right.$$
$$\left.\left(p_{\xi,i,j-1/2}Q_{\eta,i-1/2,j} + p_{\eta,i-1/2,j}Q_{\xi,i,j-1/2}\right)\left(\cos F\right)_{i,j}\right] +$$
$$+\left(\frac{S_G}{\sin^2 G}\right)_{i,j}\left[p_{\xi,i,j-1/2}Q_{\xi,i,j-1/2} + p_{\eta,i+1/2,j}Q_{\eta,i+1/2,j} - \right.$$
$$\left.\left(p_{\xi,i,j-1/2}Q_{\eta,i+1/2,j} + p_{\eta,i+1/2,j}Q_{\xi,i,j-1/2}\right)\left(\cos G\right)_{i,j}\right] +$$
$$+\left(\frac{S_H}{\sin^2 H}\right)_{i,j}\left[p_{\xi,i,j+1/2}Q_{\xi,i,j+1/2} + p_{\eta,i+1/2,j}Q_{\eta,i+1/2,j} - \right.$$
$$\left.\left(p_{\xi,i,j+1/2}Q_{\eta,i+1/2,j} + p_{\eta,i+1/2,j}Q_{\xi,i,j+1/2}\right)\left(\cos H\right)_{i,j}\right] +$$
$$+\left(\frac{S_I}{\sin^2 I}\right)_{i,j}\left[p_{\xi,i,j+1/2}Q_{\xi,i,j+1/2} + p_{\eta,i-1/2,j}Q_{\eta,i-1/2,j} - \right.$$
$$\left.\left(p_{\xi,i,j+1/2}Q_{\eta,i-1/2,j} + p_{\eta,i-1/2,j}Q_{\xi,i,j+1/2}\right)\cos I_{i,j}\right] \qquad (2.10)$$

For boundary cell, but not for an angular one, the sum (2.10) will include only two terms. For example, for a cell $(i;0)$ we obtain the following according to (2.6):

$$\int\limits_{FGHI}\left(\vec{p};\vec{Q}\right)d\Omega = W_{i,0} = \left(\frac{S_H}{\sin^2 H}\right)_{i,0}\left[p_{\xi,i,1/2}Q_{\xi,i,1/2} - \right.$$
$$\left. - p_{\eta,i+1/2,0}Q_{\xi,i,1/2}\left(\cos H\right)_{i,0}\right] +$$
$$+\left(\frac{S_I}{\sin^2 I}\right)_{i,0}\left[p_{\xi,i,1/2}Q_{\xi,i,1/2} - p_{\eta,i-1/2,0}Q_{\xi,i,1/2}\left(\cos I\right)_{i,0}\right] \qquad (2.11)$$

For angular cells there will be only one term, which equals to 0 owing to (2.6).

The integral over domain $\int\limits_{\Omega}\left(\vec{p};\vec{Q}\right)d\Omega = \sum W_{i,j}$, where summation is performed over all cells of M'.

Let grid vector $\vec{p}$ be a difference gradient:

$p_{\xi,i,j+1/2}=\dfrac{v_{i+1/2,j+1/2}-v_{i-1/2,j+1/2}}{h_{i,j+1/2}}$, $p_{\eta,i+1/2,j}=\dfrac{v_{i+1/2,j+1/2}-v_{i+1/2,j-1/2}}{h_{i+1/2,j}}$. It is possible to consider, that vector $\vec{p}$ component normal to boundary is equal to 0, i.e. $p_{\xi,0,j+1/2}=p_{\xi,N_1,j+1/2}=p_{\eta,i+1/2,0}=p_{\eta,i+1/2,N_2}$.

Then

$$\sum W_{i,j}=\sum v_{i-1/2,j-1/2}d_{i-1/2,j-1/2}, \tag{2.12}$$

Where

$d_{i-1/2,j-1/2}=l_{i,j-1/2}Q^{\xi}_{i,j-1/2}-l_{i-1,j-1/2}Q^{\xi}_{i-1,j-1/2}+l_{i-1/2,j}Q^{\eta}_{i-1/2,j}-l_{i-1/2,j-1}Q^{\eta}_{i-1/2,j-1}$.

Here for $i>0,\ j>0$

$$\begin{aligned}Q^{\xi}_{i,j-1/2}=\frac{1}{l_{i,j-1/2}h_{i,j-1/2}}\Bigg[&\left(\frac{S_F}{\sin^2 F}\right)_{i,j}\left(Q_{\xi,i,j-1/2}-Q_{\eta,i-1/2,j}\left(\cos F\right)_{i,j}\right)+\\&+\left(\frac{S_G}{\sin^2 G}\right)_{i,j}\left(Q_{\xi,i,j-1/2}-Q_{\eta,i+1/2,j}\left(\cos G\right)_{i,j}\right)+\\&+\left(\frac{S_H}{\sin^2 H}\right)_{i,j-1}\left(Q_{\xi,i,j-1/2}-Q_{\eta,i+1/2,j-1}\left(\cos H\right)_{i,j-1}\right)+\\&+\left(\frac{S_I}{\sin^2 I}\right)_{i,j-1}\left(Q_{\xi,i,j-1/2}-Q_{\eta,i-1/2,j-1}\cos I_{i,j-1}\right)\Bigg]\end{aligned} \tag{2.13}$$

$$\begin{aligned}Q^{\eta}_{i-1/2,j}=\frac{1}{l_{i-1/2,j}h_{i-1/2,j}}\Bigg[&\left(\frac{S_F}{\sin^2 F}\right)_{i,j}\left(Q_{\eta,i-1/2,j}-Q_{\xi,i,j-1/2}\left(\cos F\right)_{i,j}\right)+\\&+\left(\frac{S_I}{\sin^2 I}\right)_{i,j}\left(Q_{\eta,i-1/2,j}-Q_{\xi,i,j+1/2}\left(\cos I\right)_{i,j}\right)+\\&+\left(\frac{S_H}{\sin^2 H}\right)_{i-1,j}\left(Q_{\eta,i-1/2,j}-Q_{\xi,i-1,j+1/2}\left(\cos H\right)_{i-1,j}\right)+\\&+\left(\frac{S_G}{\sin^2 G}\right)_{i-1,j}\left(Q_{\eta,i-1/2,j}-Q_{\xi,i-1,j-1/2}\left(\cos G\right)_{i-1,j}\right)\Bigg]\end{aligned} \tag{2.14}$$

Remark, the sum (2.12) does not include expressions with $Q^{\xi}_{0,j+1/2}, Q^{\xi}_{N_1,j+1/2}, Q^{\eta}_{i+1/2,0}, Q^{\eta}_{i+1/2,N_2}$, that corresponds to boundary conditions (1.3).

Here $l_{i,j+1/2}$ and $l_{i+1/2,j}$ are length of some segments. For example, they may be the segments connecting centers of gravity of cells $(i; j+1)$, $(i; j)$ and $(i; j)$, $(i+1; j)$ of grid M' accordingly. Then it's natural to consider the expression

$$(div\vec{Q})_{i+1/2,j+1/2} = \frac{d_{i+1/2,j+1/2}}{V_{i+1/2,j+1/2}}, \tag{2.15}$$

as a difference analogue of the operator $div(\cdot)$, where $V_{i+1/2,j+1/2}$ are squares of some domains, associated with nodes $\left(i+1/2; j+1/2\right)$. They may be squares of a tetragon with vertices in centers of gravity of cells $(i; j+1)$, $(i; j)$, $(i+1; j)$ and $(i+1; j+1)$ of grid M' (it is assumed that for boundary cells these vertices coincide with centers of boundary segments, and for angular - with its angle vertex). It is possible to lie $V_{i+1/2,j+1/2} = \dfrac{1}{N_1 N_2}$ for simpler variant. In fact it means the theorem of Gauss-Ostrogradskii for this domain, and the grid functions $Q^{\xi}_{i,j-1/2}$ and $Q^{\eta}_{i-1/2,j}$ are contravariant components of a flux on cell edges of a grid M'.

The finite volume method was used in works of Edwards M.G. and Aavatsmark I. for investigation of single-phase and multiphase problems of underground hydrodynamics (M.G. Edwards, C.F. Rogers, 1998; Aavatsmark I., Barkve T., O. Boe, T. Mannseth, 1996). They define the divergence using the Gauss therem for one cell. The feature of the support operator method is a definition of flux $\vec{Q}$ from an equality (1.4). It results to the following: the order of the divergence and gradient operators approximations is the same. This method of difference schemes construction based on reviewing of the system (1.1), (1.2) instead of the equation $\dfrac{\partial T}{\partial t} = div(kgradT)$, that is similar to work Glowinski R., Wheeler M.F., 1988, where the same representation is used for a mixed finite element method.

The regularized scheme

The following type of scheme is developed as the regularized scheme:

$$B_h \frac{\hat{T} - T}{\tau} = L_h(T), \qquad (3.1)$$

where B_h operator is a regularizator (Samarskii A.A., 1989).

As known, for an absolute stability of the scheme it is sufficient to examine

$$B_h = \left(\left(E - 0.5\tau k_{\max} \frac{\partial^2}{\partial x^2} \right) \left(E - 0.5\tau k_{\max} \frac{\partial^2}{\partial y^2} \right) \right)_h \qquad (3.2)$$

with boundary conditions

$$\left. \frac{\partial}{\partial x} \right|_{x=0;1} = \left. \frac{\partial}{\partial y} \right|_{y=0;1} = 0 \qquad (3.2\,')$$

for the heat conductivity equation with a variable coefficient on a rectangular grid (Samarskii A.A., 1989). Here E is a unit operator, $k_{\max}$ is a max value of a heat conductivity. The obtained scheme is a scheme of the first order of approximation by time. Under drawing an analogy with rectangular grids it is possible to use the regularizator of the following form for curvilinear grids

$$B_h = \left(\left(E - 0.5\tau \bar{k}_\alpha k_{\max} \frac{\partial^2}{\partial \alpha^2} \right) \left(E - 0.5\tau \bar{k}_\beta k_{\max} \frac{\partial^2}{\partial \beta^2} \right) \right)_h \qquad (3.3),$$

with boundary conditions

$$\left. \frac{\partial}{\partial \alpha} \right|_{\alpha=0;1} = \left. \frac{\partial}{\partial \beta} \right|_{\beta=0;1} = 0 \qquad (3.3\,')$$

where $(\alpha;\beta)$ are curvilinear coordinates. Coordinate lines of the system $(\alpha;\beta)$ coincide with lines of coordinates $(\xi;\eta)$, but the initial grid maps into uniform rectangular one in a plane $(\alpha;\beta)$ with steps h_α and h_β (see fig. 2.1), $\bar{k}_\alpha$ and $\bar{k}_\beta$ are factors depending on "irregularity" of a grid and for a rectangular grid equaling 1.

However, the regularizator of the type mentioned above gives unsatisfactory results of calculations for the discontinuous coefficient of the heat conductivity, as it does not provide the necessary order of the approximation in the vicinity of the discontinuity line.

Because of this fact the following form of the regularizator B_h is offered:

$$B_h = \left(\left(E - 0.5\tau \bar{k}_\alpha \frac{\partial}{\partial \alpha} \left(k \frac{\partial}{\partial \alpha} \right) \right) \left(E - 0.5\tau \bar{k}_\beta \frac{\partial}{\partial \beta} \left(k \frac{\partial}{\partial \beta} \right) \right) \right)_h \qquad (3.4)$$

with boundary conditions (3.3 ').

Usually regularized schemes are schemes of the first order of approximation by time. The exception are the schemes on rectangular grids being the schemes of the second order of approximation, if $L_h = L_{h,1} + L_{h,2}$ and operators $B_i = -L_{h,1}$. Therefore it is possible to expect that the schemes of the type mentioned above with regularized operators similar to main one of the problem near coefficients discontinuity lines will be by more precise, than additive schemes described below.

It is easy to find expressions for factors $\overline{k}_\alpha$ and $\overline{k}_\beta$, corresponding to absolute stability of the scheme (3.1) with regularizator (3.4), for a concrete grid. Let $T^n = T(n\tau)$ be a grid function, obtained on n-th time step ($n = 0..N$). Under stability the following will be understood

$$\left(-L_h T^N; T^N\right)_{xy} + \left(\frac{\hat{T}-T}{\tau}; \frac{\hat{T}-T}{\tau}\right)_{xyt} + \left(T^0; T^0\right)_{xy} \le$$

$$\le \left(-L_h T^0; T^0\right)_{xy} + \left(T^0; T^0\right)_{xy} \qquad (*)$$

Here is an inner product in the space of grid functions, defined in cells of a grid M (the summation is performed over all cells of a grid M):

$$\left(f; g\right)_{xy} = \sum_{i,j} f_{i+1/2, j+1/2}\, g_{i+1/2, j+1/2}\, V_{i+1/2, j+1/2} \qquad (3.5)$$

as well as

$$\left(f; g\right)_{xyt} = \sum_{i,j,n} \tau\, f^{n+1/2}_{i+1/2, j+1/2}\, g^{n+1/2}_{i+1/2, j+1/2}\, V_{i+1/2, j+1/2} \qquad (3.5')$$

It should be noted that it needs to introduce last terms in both parts of (*). It is concerned with the fact the expressions in the (*) without terms are not a norms, but just as semi-norms. So the left part is a norm in solution space and right part is a norm in space of initial data.

Let's prove, that the satisfaction of the inequality

$$B_h \ge -0.5\tau L_h + 0.5E. \qquad (3.6)$$

is sufficient for stability of the scheme (3.1) (in sense of (*)).

Really, having multiplied both parts of (3.1) by $2\tau \dfrac{\hat{T}-T}{\tau}$ (in inner product $\left(\cdot;\cdot\right)_{xy}$), we have:

$$0 = 2\tau\left(B_h \frac{\hat{T}-T}{\tau}; \frac{\hat{T}-T}{\tau}\right)_{xy} + 2\tau\left(-L_h T; \frac{\hat{T}-T}{\tau}\right)_{xy} =$$

$$= 2\tau\left(B_h \frac{\hat{T}-T}{\tau}; \frac{\hat{T}-T}{\tau}\right)_{xy} + 2\tau\left(-L_h\left(\frac{\hat{T}+T}{2} - \frac{\tau}{2}\frac{\hat{T}-T}{2}\right); \frac{\hat{T}-T}{\tau}\right)_{xy} =$$

$$= 2\tau\left(\left(B_h + 0.5\tau L_h\right)\frac{\hat{T}-T}{\tau};\frac{\hat{T}-T}{\tau}\right)_{xy} + \left(-L_h\left(\hat{T}+T\right);\hat{T}-T\right)_{xy} =$$

$$= 2\tau\left(\left(B_h + 0.5\tau L_h\right)\frac{\hat{T}-T}{\tau};\frac{\hat{T}-T}{\tau}\right)_{xy} + \left(-L_h\hat{T};\hat{T}\right)_{xy} - \left(-L_h T;T\right)_{xy}$$

Thus, if (3.6) is fair, then $\left(-L_h\hat{T};\hat{T}\right)_{xy} + \tau\left(\frac{\hat{T}-T}{\tau};\frac{\hat{T}-T}{\tau}\right) \le \left(-L_h T;T\right)_{xy}$,

from which (*) immediately follows.

Besides the inner product $\left(\cdot;\cdot\right)_{xy}$ we shall introduce

$$\left(f;g\right)_{\alpha\beta} = \sum_{i,j} f_{i+1/2,j+1/2}\, g_{i+1/2,j+1/2}\, h_\alpha h_\beta ,$$

where the summation is performed over all cells of a grid M. It should be marked, that norms induced by inner products $\left(\cdot;\cdot\right)_{xy}$ and $\left(\cdot;\cdot\right)_{\alpha\beta}$ are equivalent:

$$\sqrt{\frac{\min_{i,j} V_{i+1/2,j+1/2}}{h_\alpha h_\beta}\left(f;f\right)_{\alpha\beta}} \le \sqrt{\left(f;f\right)_{xy}} \le \sqrt{\frac{\max_{i,j} V_{i+1/2,j+1/2}}{h_\alpha h_\beta}\left(f;f\right)_{\alpha\beta}}$$

Then,

$$B = B_1 + B_2, \tag{3.8}$$

$$B_1 = E + 0.25\tau^2 \bar{k}_\alpha \bar{k}_\beta \frac{\partial}{\partial\alpha}\left(k\frac{\partial}{\partial\alpha}\left(\frac{\partial}{\partial\beta}\left(k\frac{\partial}{\partial\beta}\right)\right)\right)_h , \tag{3.9}$$

$$B_2 = -0.5\tau\left(\bar{k}_\alpha \frac{\partial}{\partial\alpha}\left(k\frac{\partial}{\partial\alpha}\right) + \bar{k}_\beta \frac{\partial}{\partial\beta}\left(k\frac{\partial}{\partial\beta}\right)\right)_h . \tag{3.8}$$

and

$$\left(B_2 T;T\right)_{\alpha\beta} = 0.5\tau h_\alpha h_\beta\left[\sum_{i,j}\left(\bar{k}_\alpha k'_{i,j+1/2}\left(\frac{T_{i+1/2,j+1/2} - T_{i-1/2,j+1/2}}{h_\alpha}\right)^2 + \bar{k}_\beta k'_{i+1/2,j}\left(\frac{T_{i+1/2,j+1/2} - T_{i+1/2,j-1/2}}{h_\beta}\right)^2\right)\right] \tag{3.10}$$

Here $k'_{i,j-1/2} = \frac{2k_{i+1/2,j+1/2}k_{i-1/2,j+1/2}}{k_{i+1/2,j+1/2}+k_{i-1/2,j+1/2}},\; k'_{i-1/2,j} = \frac{2k_{i+1/2,j+1/2}k_{i+1/2,j-1/2}}{k_{i+1/2,j+1/2}+k_{i+1/2,j-1/2}}$

For approximation of $\left(-0.5\tau L_h T;T\right)_{xy}$ it is assigned in (1.4) $\vec{p}=gradT$ just as it was done in support operator method. Then the right part of (1.4) becomes 0 and according to the formulas (2.13), (2.14)

$$\left(-0.5\tau L_h T;T\right)_{xy}=0.5\tau\left(kgradT;gradT\right)_{xy}=$$
$$0.5\tau\sum_{i,j}\Bigg[\left(\frac{S_A}{\sin^2 A}\right)_{i,j}\Bigg(k_{i,j-1/2}\left(T'_{i,j-1/2}\right)^2+k_{i-1/2,j}\left(T'_{i-1/2,j}\right)^2-$$
$$-T'_{i,j-1/2}T'_{i-1/2,j}\left(k_{i,j-1/2}+k_{i-1/2,j}\right)\cos A_{i,j}\Bigg)+$$
$$+\left(\frac{S_B}{\sin^2 B}\right)_{i,j}\Bigg(k_{i,j-1/2}\left(T'_{i,j-1/2}\right)^2+k_{i+1/2,j}\left(T'_{i+1/2,j}\right)^2-$$
$$-T'_{i,j-1/2}T'_{i+1/2,j}\left(k_{i,j-1/2}+k_{i+1/2,j}\right)\cos B_{i,j}+$$
$$+\left(\frac{S_C}{\sin^2 C}\right)_{i,j}\Bigg(k_{i,j+1/2}\left(T'_{i,j+1/2}\right)^2+k_{i+1/2,j}\left(T'_{i+1/2,j}\right)^2-$$
$$-T'_{i,j+1/2}T'_{i+1/2,j}\left(k_{i,j+1/2}+k_{i+1/2,j}\right)\cos C_{i,j}\Bigg)+$$
$$+\left(\frac{S_D}{\sin^2 D}\right)_{i,j}\Bigg(k_{i,j+1/2}\left(T'_{i,j+1/2}\right)^2+k_{i-1/2,j}\left(T'_{i-1/2,j}\right)^2-$$
$$-T'_{i,j+1/2}T'_{i-1/2,j}\left(k_{i,j+1/2}+k_{i-1/2,j}\right)\cos D_{i,j}\Bigg)\Bigg], \tag{3.11}$$

where $T'_{i,j+1/2}=\dfrac{T_{i+1/2,j+1/2}-T_{i-1/2,j+1/2}}{h_{i,j+1/2}}, T'_{i+1/2,j}=\dfrac{T_{i+1/2,j+1/2}-T_{i+1/2,j-1/2}}{h_{i+1/2,j}}$, and the summation is performed over all cells of a grid M'.

Thus,

$$\left(-0.5\tau L_h T;T\right)_{xy}\le 0.25\tau C_1 C_2\sum_{i,j}\Bigg(k_{i,j-1/2}\left(T'_{i,j-1/2}\right)^2+k_{i+1/2,j}\left(T'_{i+1/2,j}\right)^2+$$
$$k_{i,j+1/2}\left(T'_{i,j+1/2}\right)^2+k_{i-1/2,j}\left(T'_{i-1/2,j}\right)^2\Bigg), \tag{3.12}$$

where

$$C_1 = 4\max_{i,j}\left(\max\left(\left(\frac{S_A}{\sin^2 A}\right)_{i,j},\left(\frac{S_B}{\sin^2 B}\right)_{i,j},\left(\frac{S_C}{\sin^2 C}\right)_{i,j},\left(\frac{S_D}{\sin^2 D}\right)_{i,j}\right)\right)$$

$$C_2 = 1+\max_{i,j}\left(\frac{\left(k_{i,j-1/2}+k_{i-1/2,j}\right)\left|\cos A_{i,j}\right|}{\sqrt{k_{i,j-1/2}k_{i-1/2,j}}},\frac{\left(k_{i,j-1/2}+k_{i+1/2,j}\right)\left|\cos B_{i,j}\right|}{\sqrt{k_{i,j-1/2}k_{i+1/2,j}}},\right.$$

$$\left.\frac{\left(k_{i,j+1/2}+k_{i+1/2,j}\right)\left|\cos C_{i,j}\right|}{\sqrt{k_{i,j+1/2}k_{i+1/2,j}}},\frac{\left(k_{i,j+1/2}+k_{i-1/2,j}\right)\left|\cos D_{i,j}\right|}{\sqrt{k_{i,j+1/2}k_{i-1/2,j}}}\right)$$

Since each edge appears twice in the sum (3.12) except the boundary ones, it is sufficient to suppose

$$\bar{k}_\alpha = \frac{C_1}{h_\alpha h_\beta}C_2\max_{i,j}\frac{k_{i,j-1/2}}{k'_{i,j-1/2}}\frac{h_\alpha^2}{h_{i,j-1/2}^2} \tag{3.13}$$

$$\bar{k}_\beta = \frac{C_1}{h_\alpha h_\beta}C_2\max_{i,j}\frac{k_{i-1/2,j}}{k'_{i-1/2,j}}\frac{h_\beta^2}{h_{i-1/2,j}^2} \tag{3.14}$$

for satisfaction of $\left(B_2T;T\right)_{\alpha\beta} \ge \left(-0.5\tau L_hT;T\right)_{xy}$

Let's assume $V_{i+1/2,j+1/2} = \dfrac{1}{N_1N_2} = h_\alpha h_\beta$. Then both introduced inner products coincide and it is necessary to prove $B_1 \ge 0.5E$

$B_1 = E + 0.25\tau^2\bar{k}_\alpha\bar{k}_\beta L_1L_2$, where $L_1 = -\dfrac{\partial}{\partial\alpha}\left(k\dfrac{\partial}{\partial\alpha}\right)_h$,

$L_2 = -\dfrac{\partial}{\partial\beta}\left(k\dfrac{\partial}{\partial\beta}\right)_h$ with boundary conditions (1.3). Since both operators L_i are non-negative determined (with respect to inner product $(\cdot;\cdot)_{\alpha\beta}$, which coincides with $(\cdot;\cdot)_{xy}$ in a considered case), all their eigenvalues λ_i^j, corresponding to eigenfunctions φ_i^j, are non-negative. Consider the case $k = const$. Since all grid functions are decomposed by basis $\psi^{kl} = \varphi_1^k \varphi_2^l$: $T = \sum_{k,l} T_{kl}\psi^{kl}$, then

$$\left(B_1 T;T\right)_{\alpha\beta} = \left(T;T\right)_{\alpha\beta} + \sum_{k,l}\left(\lambda_1^k \lambda_2^l T_{kl}\psi^{kl};T\right)_{\alpha\beta} \ge \left(T;T\right)_{\alpha\beta} \ge 0.5\left(T;T\right)_{\alpha\beta}.$$

Thus, we have proved, if having selected regularizing factors according to the formulas (3.13) and (3.14) the scheme (3.1) will be absolutely stable. As several tests without analytical solution have shown it will be also stable for variable k. In other cases the regularizator (3.3) should be used.

It should be noted, that for the considered class of problems the scheme (3.1) with $V_{i+1/2,j+1/2}$ equal to squares of tetragons with vertices in center of gravity of cells $(i;j+1)$, $(i;j)$, $(i+1;j)$ and $(i+1;j+1)$ of grids M', was also absolutely stable (with some $\bar{k}_\alpha$ and $\bar{k}_\beta$). Moreover, the scheme (3.1) with regularizator of the form

$$B_h = \left(\left(E - 0.5\tau\bar{k}_\xi \frac{\partial}{\partial\xi}\left(k\frac{\partial}{\partial\xi}\right)\right)\left(E - 0.5\tau\bar{k}_\eta \frac{\partial}{\partial\eta}\left(k\frac{\partial}{\partial\eta}\right)\right)\right)_h \quad (3.15)$$

with boundary conditions

$$\left.\frac{\partial}{\partial\xi}\right|_{\xi=0;1} = \left.\frac{\partial}{\partial\eta}\right|_{\eta=0;1} = 0 \quad (3.15')$$

was absolutely stable for some $\bar{k}_\xi$ and $\bar{k}_\eta$.

Let's remark, that the scheme (3.1) with regularizator (3.4) or (3.15) is economic, since the corresponding system of linear equations may be easily inverted by two sweep methods.

Difference flux method

Flux method is understood as the scheme of following sort: at the first step (predictor) the fluxes on edges of cells are calculated on "a intermediate time layer using the equations for each component of a vector $\vec{Q}$. Usually they are difference approximations of the equation for a flux $\frac{1}{k}\frac{\partial\vec{Q}}{\partial t} = grad(div\vec{Q})$ by some methods. The second step (corrector) is a divergent closure due to initial equation.

Let's remind the scheme proposed by Yu.B. Radvogin for a heat conductivity equation with variable coefficient on a rectangular uniform grid.

Predictor

$$\bar{Q}^x_{i+1,j+1/2} = Q^x_{i+1,j+1/2} + \frac{\sigma k_{i+1,j+1/2}}{2}\frac{\tau}{h_x^{\,2}}\left(\bar{Q}^x_{i+2,j+1/2} - 2\bar{Q}^x_{i+1,j+1/2} + \bar{Q}^x_{i,j+1/2}\right) \quad (4.1)$$

$$\bar{Q}^y_{i+1/2,j+1} = Q^y_{i+1/2,j+1} + \frac{\sigma k_{i+1/2,j+1}}{2}\frac{\tau}{h_y^{\,2}}\left(\bar{Q}^y_{i+1/2,j+2} - 2\bar{Q}^y_{i+1/2,j+1} + \bar{Q}^y_{i+1/2,j}\right) \quad (4.2)$$

Corrector

$$\hat{T}_{i+1/2,j+1/2} = T_{i+1/2,j+1/2} + \frac{\tau}{h_x}\left(\bar{Q}^x_{i+1,j+1/2} - \bar{Q}^x_{i,j+1/2}\right) + \frac{\tau}{h_y}\left(\bar{Q}^y_{i+1/2,j+1} - \bar{Q}^y_{i+1/2,j}\right)$$

(4.3)

Here h_x and h_y are grid steps, $\bar{Q}^x_{i,j+1/2}$ and $\bar{Q}^y_{i+1/2,j}$ are fluxes normal to the cell edges. If $\sigma \ge 2$ the scheme is absolutely stable, i.e. the scheme with surpassing definition of fluxes is stable.

On a step of predictor we effectively calculate components of fluxes normal to the cell edges referred to intermediate time layer. It should be noted, that each equation of (4.1) and (4.2) approximates equation for a flux with zero order, i.e. it is supposed that for each layer of cells the flux through its lateral boundaries is equal to zero. On a step of corrector using these values we calculate values of temperature for an implicit time layer.

The natural generalizing for curvilinear grids of the scheme proposed by Yu.B. Radvogin looks like the following:

predictor

$$\bar{Q}_{\xi,i+1,j+1/2} = Q_{\xi,i+1,j+1/2} + \frac{\sigma}{2}\frac{\tau k_{i+1,j+1/2}}{h_{i+1,j+1/2}} \cdot$$
$$\cdot\left(\left(div\vec{\bar{Q}}\right)_{i+3/2,j+1/2} - \left(div\vec{\bar{Q}}\right)_{i+1/2,j+1/2}\right) \quad (4.4)$$

$$\bar{Q}_{\eta,i+1/2,j+1} = Q_{\eta,i+1/2,j+1} + \frac{\sigma}{2}\frac{\tau k_{i+1/2,j+1}}{h_{i+1/2,j+1}} \cdot$$
$$\cdot\left(\left(div\vec{\bar{Q}}\right)_{i+1/2,j+3/2} - \left(div\vec{\bar{Q}}\right)_{i+1/2,j+1/2}\right) \quad (4.5)$$

Here $div\vec{\bar{Q}}$ is defined by support operators method for each layer along directions ξ and η under the following assumption. It is supposed that flux along the same direction in other layers and flux along other direction are equal to 0.

Corrector

$$\frac{\hat{T}_{i+1/2,j+1/2} - T_{i+1/2,j+1/2}}{\tau} = L_h\left(\bar{Q}^{\xi}, \bar{Q}^{\eta}\right) \tag{4.6}$$

The operator L_h is defined by a support operator method: using contravariant components of a flux we calculate the covariant ones by a method depicted above. The equations (4.4) and (4.5) also may be solved effectively.

Variational difference flux method

Let the operator $grad(\cdot)$ be employed to both parts of (1.1) and then outcome be multiplied by $k(x,y)$. Then due to (1.2) the equation for flux is obtained

$$\frac{1}{k}\frac{\partial \vec{Q}}{\partial t} - graddiv(\vec{Q}) = 0 \tag{5.1}$$

Discretising in time with the help of the implicit scheme ($\bar{\tau} = \sigma\tau$), we have

$$\frac{1}{k}\frac{\bar{\vec{Q}} - \vec{Q}}{\bar{\tau}} - graddiv(\bar{\vec{Q}}) = 0 \tag{5.2}$$

Multiplying (5.2) by an arbitrary variation $\delta\bar{\vec{Q}}$ and integrating over arbitrary domain G we obtain due to boundary conditions (1.3), if boundary fluxes being equal to zero:

$$\iint_G \left\{ \frac{1}{k}\frac{\bar{\vec{Q}} - \vec{Q}}{\bar{\tau}} \delta\bar{\vec{Q}} + \delta\left(\frac{1}{2}(div\bar{\vec{Q}})^2\right) \right\} d\Omega = 0. \tag{5.3}$$

Thus, functional

$$\Phi(\bar{\vec{Q}}) = \iint_G \left\{ \frac{1}{k}\frac{\left(\bar{\vec{Q}} - \vec{Q}\right)^2}{2\bar{\tau}} + \frac{1}{2}(div\bar{\vec{Q}})^2 \right\} d\Omega \tag{5.4}$$

reaches its stationary value on a solution of the equations (1.1) and (1.2) with boundary conditions (1.3).

Let the curvilinear coordinates $(\alpha;\beta)$ in domain Ω be introduced by the same way as in the third item. For a definiteness we suppose that a Jacobian $J = \dfrac{\partial(x,y)}{\partial(\alpha,\beta)} > 0$. Designating N_α and N_β number of cells along directions α and β respectively, the expressions in (5.4) may be rewritten in local contravariant base of a curvilinear coordinate system $(\alpha;\beta)$.

The orts of contravariant base look like

$$\vec{n}_\alpha = \frac{1}{l_\alpha}\left(\frac{\partial y}{\partial \alpha}; -\frac{\partial x}{\partial \alpha}\right),\ \vec{n}_\beta = -\frac{1}{l_\beta}\left(\frac{\partial y}{\partial \beta}; -\frac{\partial x}{\partial \beta}\right), \tag{5.5}$$

where $l_\alpha = \sqrt{\left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\right)^2}$, $l_\beta = \sqrt{\left(\frac{\partial x}{\partial \beta}\right)^2 + \left(\frac{\partial y}{\partial \beta}\right)^2}$ are Lame coefficients of a curvilinear coordinate system.

Since

$$Q^x = -\frac{1}{J}\left(\frac{\partial x}{\partial \beta} l_\alpha Q^\alpha + \frac{\partial x}{\partial \alpha} l_\beta Q^\beta\right),\ Q^y = -\frac{1}{J}\left(\frac{\partial y}{\partial \beta} l_\alpha Q^\alpha + \frac{\partial y}{\partial \alpha} l_\beta Q^\beta\right),$$

where Q^α and Q^β are contravariant components of flux then having made substitution $Q^\alpha = -Q^\alpha$, $Q^\beta = -Q^\beta$ we obtain

$$\vec{Q}^2 = \left(Q^x\right)^2 + \left(Q^y\right)^2 =$$

$$= \frac{l_\alpha^2 l_\beta^2}{J^2}\left(\left(Q^\alpha\right)^2 + \left(Q^\beta\right)^2 + 2\left(\frac{\partial x}{\partial \alpha}\frac{\partial x}{\partial \beta} + \frac{\partial y}{\partial \alpha}\frac{\partial y}{\partial \beta}\right) Q^\alpha Q^\beta / l_\alpha l_\beta\right), \tag{5.6}$$

$$div\vec{Q} = \frac{\partial Q^x}{\partial x} + \frac{\partial Q^y}{\partial y} = \frac{1}{J}\left(\frac{\partial\left(l_\beta Q^\beta\right)}{\partial \alpha} + \frac{\partial\left(l_\alpha Q^\alpha\right)}{\partial \beta}\right). \tag{5.7}$$

As well as in a difference flux method, we suppose that for each layer of cells the flux through its boundary is equal to zero. Let's consider the approximation of a functional (5.4), having taken into account of any layer of cells along α or β as G (see fig. 5.1).

For a layer $j + \frac{1}{2}$ along α we have:

$$\Phi_{j+\frac{1}{2}}(\bar{\vec{Q}}) = \sum_{i=0}^{N_\alpha - 1}\left\{\frac{\left\langle \bar{\vec{Q}} - \vec{Q}\right\rangle^2_{i+\frac{1}{2},j+\frac{1}{2}}}{2\bar{\tau}k_{i+\frac{1}{2},j+\frac{1}{2}}} + \frac{1}{2}\left\langle div\bar{\vec{Q}}\right\rangle^2_{i+\frac{1}{2},j+\frac{1}{2}}\right\} S_{i+\frac{1}{2},j+\frac{1}{2}}, \tag{5.8}$$

where

$$\left\langle \bar{\vec{Q}} - \vec{Q}\right\rangle^2_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{1}{S_{i+\frac{1}{2},j+\frac{1}{2}}} h_\alpha^2 l^2_{\alpha,i+\frac{1}{2},j+\frac{1}{2}} h_\beta^2 l^2_{\beta,i+\frac{1}{2},j+\frac{1}{2}} \times$$

$$\times\frac{1}{2}\left(\left(\bar{Q}^{\beta}_{i+1,j+1/2}-Q^{\beta}_{i+1,j+1/2}\right)^2+\left(\bar{Q}^{\beta}_{i,j+1/2}-Q^{\beta}_{i,j+1/2}\right)^2\right) \quad (5.9)$$

$$\left(div\vec{\bar{Q}}\right)_{i+1/2,j+1/2}=\frac{1}{S_{i+1/2,j+1/2}}\left(h_\beta l_{\beta,i+1,j+1/2}\bar{Q}^{\beta}_{i+1,j+1/2}-h_\beta l_{\beta,i,j+1/2}\bar{Q}^{\beta}_{i,j+1/2}\right) \quad (5.10)$$

$$h_\alpha l_{\alpha,i+1/2,j+1/2}=0.5\left(h_\alpha l_{\alpha,i+1/2,j+1}+h_\alpha l_{\alpha,i+1/2,j}\right) \quad (5.11)$$

$$h_\beta l_{\beta,i,j+1/2}=\sqrt{\left(x_{i,j+1}-x_{i,j}\right)^2+\left(y_{i,j+1}-y_{i,j}\right)^2}\equiv L_{i,j+1/2} \quad (5.12)$$

$$h_\alpha l_{\alpha,i+1/2,j}=\sqrt{\left(x_{i+1,j}-x_{i,j}\right)^2+\left(y_{i+1,j}-y_{i,j}\right)^2}\equiv L_{i+1/2,j} \quad (5.13)$$

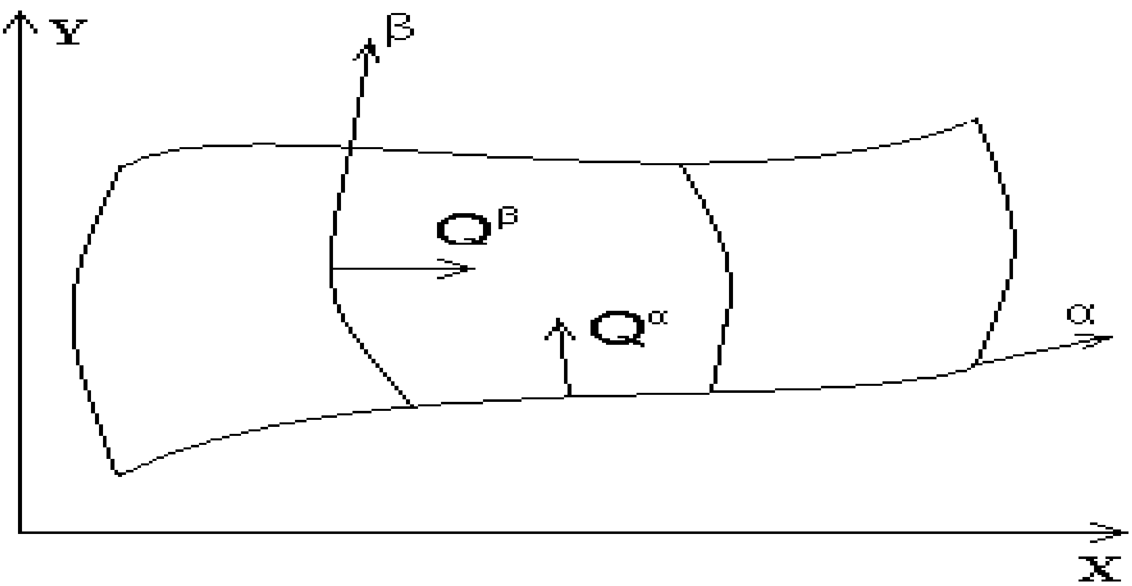

Fig. 5.1

Then functional $\Phi_{j+1/2}(\vec{\bar{Q}})$ may be represented as

$$\Phi_{j+1/2}(\vec{\bar{Q}})=\sum_{i=0}^{N_\alpha-1}\left\{G_{i+1/2,j+1/2}\frac{1}{2}\left(\left(\bar{Q}^{\beta}_{i+1,j+1/2}-Q^{\beta}_{i+1,j+1/2}\right)^2+\right.\right.$$

$$\left.+\left(\bar{Q}^{\beta}_{i,j+1/2}-Q^{\beta}_{i,j+1/2}\right)^2\right)$$

$$\left.+P_{i+1/2,j+1/2}\left(L_{i+1,j+1/2}\bar{Q}^{\beta}_{i+1,j+1/2}-L_{i,j+1/2}\bar{Q}^{\beta}_{i,j+1/2}\right)\right\}, \quad (5.14)$$

where

$$P_{i+1/2,j+1/2}=\frac{1}{2S_{i+1/2,j+1/2}}, \tag{5.15}$$

$$G_{i+1/2,j+1/2}=\frac{1}{2\bar{\tau}k_{i+1/2,j+1/2}}\frac{\left(L^2_{i+1,j+1/2}+L^2_{i,j+1/2}\right)\left(L^2_{i+1/2,j+1}+L^2_{i+1/2,j}\right)}{4S_{i+1/2,j+1/2}}. \tag{5.16}$$

The difference analogue of the equation (5.1) follows from a condition of a minimum of a functional (5.8): $\frac{\partial\Phi(\vec{\bar{Q}})}{\partial\bar{Q}^{\beta}_{i,j+1/2}}=0$ or

$$\begin{gathered}\left(G_{i+1/2,j+1/2}-G_{i-1/2,j+1/2}\right)\left(\bar{Q}^{\beta}_{i,j+1/2}-Q^{\beta}_{i,j+1/2}\right)-\\ -2P_{i+1/2,j+1/2}L_{i,j+1/2}\left(L_{i+1,j+1/2}\bar{Q}^{\beta}_{i+1,j+1/2}-L_{i,j+1/2}\bar{Q}^{\beta}_{i,j+1/2}\right)+\\ +2P_{i-1/2,j+1/2}L_{i,j+1/2}\left(L_{i,j+1/2}\bar{Q}^{\beta}_{i,j+1/2}-L_{i-1,j+1/2}\bar{Q}^{\beta}_{i-1,j+1/2}\right)\end{gathered} \tag{5.17}$$

Using boundary conditions $\bar{Q}^{\beta}_{0,j+1/2}=\bar{Q}^{\beta}_{N_\alpha,j+1/2}=0$ the equation is solved by sweep method.

For a layer $i+1/2$ along β the reasonings are similar, therefore we list only finite outcome - difference equation:

$$\begin{gathered}\left(G_{i+1/2,j+1/2}-G_{i+1/2,j-1/2}\right)\left(\bar{Q}^{\alpha}_{i+1/2,j}-Q^{\alpha}_{i+1/2,j}\right)-\\ -2P_{i+1/2,j+1/2}L_{i+1/2,j}\left(L_{i+1/2,j+1}\bar{Q}^{\alpha}_{i+1/2,j+1}-L_{i+1/2,j}\bar{Q}^{\alpha}_{i+1/2,j}\right)+\\ +2P_{i+1/2,j-1/2}L_{i+1/2,j}\left(L_{i+1/2,j}\bar{Q}^{\alpha}_{i+1/2,j}-L_{i+1/2,j-1}\bar{Q}^{\alpha}_{i+1/2,j-1}\right)\end{gathered} \tag{5.18}$$

and boundary conditions $\bar{Q}^{\alpha}_{i+1/2,0}=\bar{Q}^{\alpha}_{i+1/2,N_\beta}=0$.

For divergent closure we use the initial equation (1.1) and formula (5.7):

$$\begin{gathered}\frac{\hat{T}_{i+1/2,j+1/2}-T_{i+1/2,j+1/2}}{\tau}=\frac{1}{S_{i+1/2,j+1/2}}\Big(L_{i+1,j+1/2}\bar{Q}^{\beta}_{i+1,j+1/2}-\\ -L_{i,j+1/2}\bar{Q}^{\beta}_{i,j+1/2}+L_{i+1/2,j+1}\bar{Q}^{\beta}_{i+1/2,j+1}-L_{i+1/2,j}\bar{Q}^{\beta}_{i+1/2,j}\Big).\end{gathered} \tag{5.19}$$

The variational locally one-dimensional method

The developed scheme is very similar with previous one. Let's specify their differences. In a flux method using known distribution of temperature the fluxes on an intermediate time layer are calculated, and then divergent closure is performed. In the given scheme using known temperature covariant component of a flux only along one direction (for example, along β) is calculated, then the temperature on an intermediate time layer is obtained under supposition that the fluxes along α are equal to zero. Then using the new distribution of temperature on an intermediate time layer covariant component of a flux along α is calculated. Finally, the temperature on an implicit time layer is obtained under assumption that of a component of the flux along β is zero. Since the calculations coincide with one described above we represent only final formulas – the difference equations:

$$\begin{aligned}&\left(G_{i+1/2,j+1/2}-G_{i-1/2,j+1/2}\right)\left(\breve{Q}^{\beta}_{i,j+1/2}-Q^{\beta}_{i,j+1/2}\right)-\\&-2P_{i+1/2,j+1/2}L_{i,j+1/2}\left(L_{i+1,j+1/2}\breve{Q}^{\beta}_{i+1,j+1/2}-L_{i,j+1/2}\breve{Q}^{\beta}_{i,j+1/2}\right)+\\&+2P_{i-1/2,j+1/2}L_{i,j+1/2}\left(L_{i,j+1/2}\breve{Q}^{\beta}_{i,j+1/2}-L_{i-1,j+1/2}\breve{Q}^{\beta}_{i-1,j+1/2}\right)\end{aligned} \qquad (6.1)$$

$$\frac{\bar{T}_{i+1/2,j+1/2}-T_{i+1/2,j+1/2}}{\tau}=\frac{1}{S_{i+1/2,j+1/2}}\left(L_{i+1,j+1/2}\breve{Q}^{\beta}_{i+1,j+1/2}-L_{i,j+1/2}\breve{Q}^{\beta}_{i,j+1/2}\right) \qquad (6.2)$$

$$\begin{aligned}&\left(G_{i+1/2,j+1/2}-G_{i+1/2,j-1/2}\right)\left(\tilde{Q}^{\alpha}_{i+1/2,j}-\bar{Q}^{\alpha}_{i+1/2,j}\right)-\\&-2P_{i+1/2,j+1/2}L_{i+1/2,j}\left(L_{i+1/2,j+1}\tilde{Q}^{\alpha}_{i+1/2,j+1}-L_{i+1/2,j}\tilde{Q}^{\alpha}_{i+1/2,j}\right)+\\&+2P_{i+1/2,j-1/2}L_{i+1/2,j}\left(L_{i+1/2,j}\tilde{Q}^{\alpha}_{i+1/2,j}-L_{i+1/2,j-1}\tilde{Q}^{\alpha}_{i+1/2,j-1}\right)\end{aligned} \qquad (6.3)$$

$$\frac{\hat{T}_{i+1/2,j+1/2}-\bar{T}_{i+1/2,j+1/2}}{\tau}=\frac{1}{S_{i+1/2,j+1/2}}\left(L_{i+1/2,j+1}\tilde{Q}^{\beta}_{i+1/2,j+1}-L_{i+1/2,j}\tilde{Q}^{\beta}_{i+1/2,j}\right) \qquad (6.4)$$

Calculation results

All five algorithms were tested on a rectangular grid, scalene grid (see fig. 7.1) and sine grid (see fig. 7.2). The square Ω was divided into 10x10, 20x20 and 40x40 of cells. The exact solutions were the following

1. $T = e^{-\pi^2 t}\cos(\pi y)$, $k = 1$
2. $T = e^{-9\pi^2 t}\cos(3\pi y)$, $k = 1$
3. $T = e^{-10\pi^2 t}\cos(3\pi y)\cos(\pi x)$, $k = 1$
4. $T = \begin{cases} e^{-\lambda t}\cos\sqrt{\dfrac{\lambda}{k_0}}y, & y < 0.5 \\ e^{-\lambda t}\dfrac{\cos\frac{1}{2}\sqrt{\frac{\lambda}{k_0}}}{\cos\frac{1}{2}\sqrt{\frac{\lambda}{k_1}}}\cos\sqrt{\dfrac{\lambda}{k_0}}y, & y \ge 0.5 \end{cases}$, $k = \begin{cases} 1, & y < 0.5 \\ 0.1, & y \ge 0.5 \end{cases}$,

where λ is obtained from the equation

$$\left(\sqrt{k_0} + \sqrt{k_1}\right)\sin\frac{\sqrt{\lambda}}{2}\left(\frac{1}{\sqrt{k_0}} + \frac{1}{\sqrt{k_1}}\right) - \\ -\left(\sqrt{k_1} - \sqrt{k_0}\right)\sin\frac{\sqrt{\lambda}}{2}\left(\frac{1}{\sqrt{k_0}} - \frac{1}{\sqrt{k_1}}\right) = 0 \qquad (7.1)$$

Besides, the problem with sources have been tested too. Instead of (1.1) the following equation have been considered $\dfrac{\partial T}{\partial t} + div\vec{Q} = f$. At the same time the regularized and fully implicit schemes have been developed. These schemes were also tested on solution

5. $T = \sin(t)x^2(1-x)^2,\ k = 1,\ f = \cos(t)(2 - 12x + 12x^2)$

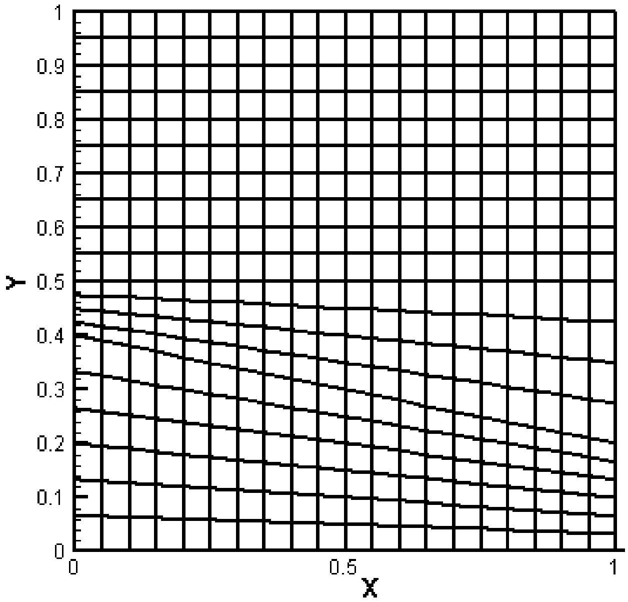

Fig. 7.1

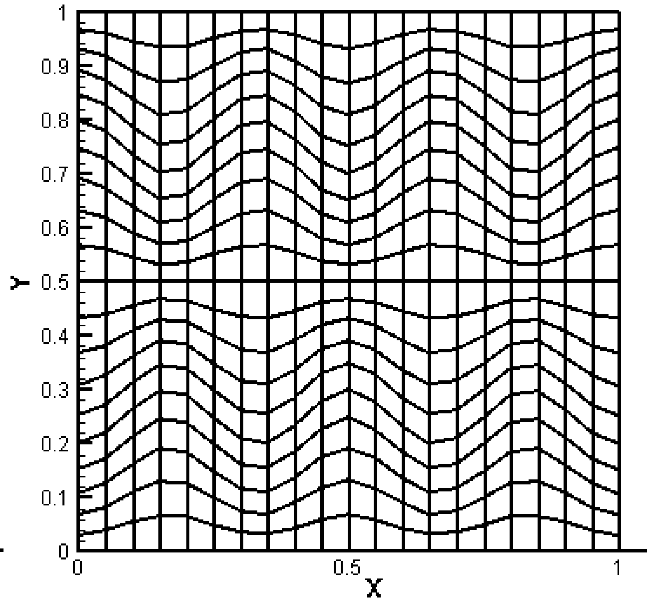

Fig. 7.2

		The middle of calculation			Final moment		
		Rectang.	Scalene	Sine	Rectang.	Scalene	Sine
10	Implicit	0.002543	0.005791	0.017498	0.004267	0.004786	0.005831
	Regul.	0.001028	0.005610	0.018306	0.001701	0.002891	0.004672
	DFM	0.003288	0.009470	0.044702	0.005555	0.006121	0.007731
	VDFM	0.003288	0.006604	0.015847	0.005555	0.005988	0.006746
	VLODM	0.003288	0.006745	0.017656	0.005555	0.006006	0.006412
20	Implicit	0.001802	0.002757	0.003224	0.003002	0.003170	0.002974
	Regul.	0.000255	0.002479	0.002564	0.000419	0.001269	0.001628
	DFM	0.002562	0.014380	0.068315	0.004299	0.004519	0.008980
	VDFM	0.002562	0.003858	0.009687	0.004299	0.004412	0.004330
	VLODM	0.002562	0.003945	0.010538	0.004299	0.004419	0.004205
40	Implicit	0.001614	0.001661	0.001674	0.002684	0.002732	0.002675
	Regul.	5.910e-5	0.001393	0.001048	9.692e-5	0.001078	0.001036
	DFM	0.002378	0.037927	0.043497	0.003983	0.005954	0.006813
	VDFM	0.002378	0.003501	0.003585	0.003983	0.003988	0.003924
	VLODM	0.002378	0.003427	0.003743	0.003983	0.003990	0.003912

Table 7.1

		The middle of calculation			Final moment		
		Rectang.	Scalene	Sine	Rectang.	Scalene	Sine
10	Implicit	0.060256	0.084513	0.052579	6.954e-9	0.003246	7.405e-4
	Regul.	0.035828	0.066245	0.034633	1.694e-9	0.004266	0.001353
	DFM	0.076859	0.099951	0.093450	1.642e-8	0.003478	7.715e-4
	VDFM	0.076859	0.100001	0.064092	1.642e-8	0.003300	0.001186
	VLODM	0.076859	0.100644	0.062796	1.642e-8	0.003302	0.001289
20	Implicit	0.042323	0.049574	0.039917	2.493e-9	7.776e-4	9.130e-5
	Regul.	0.016587	0.024491	0.021972	4.006e-10	0.001244	4.014e-4
	DFM	0.059831	0.069960	0.087101	6.660e-9	9.922e-4	1.112e-4
	VDFM	0.059831	0.066459	0.058264	6.660e-9	8.126e-4	6.367e-4
	VLODM	0.059831	0.066603	0.058629	6.660e-9	8.135e-4	6.438e-4
40	Implicit	0.037857	0.039757	0.037575	1.895e-9	1.930e-4	4.612e-6
	Regul.	0.011800	0.015915	0.017438	2.436e-10	6.094e-4	1.101e-4
	DFM	0.055590	0.062290	0.072428	5.273e-9	4.033e-4	5.492e-6
	VDFM	0.055590	0.057125	0.055641	5.273e-9	2.220e-4	1.999e-4
	VLODM	0.055590	0.057186	0.055720	5.273e-9	2.223e-4	2.004e-4

Table 7.2

		The middle of calculation			Final moment		
		Rectang.	Scalene	Sine	Rectang.	Scalene	Sine
10	Implicit	0.058951	0.080214	0.038918	9.32e-10	0.002992	2.058e-4
	Regul.	0.033464	0.060647	0.042225	1.762e-10	0.004628	2.669e-5
	DFM	0.058421	0.077163	0.055781	9.03e-10	0.003042	1.950e-4
	VDFM	0.058421	0.079921	0.045288	9.03e-10	0.003015	2.455e-4
	VLODM	0.066577	0.087797	0.048211	1.468e-9	0.003015	2.640e-4
20	Implicit	0.043829	0.050063	0.040376	3.47e-10	7.128e-4	1.048e-5
	Regul.	0.017012	0.024025	0.020200	4.493e-11	0.001601	3.104e-5
	DFM	0.043922	0.051174	0.065077	3.49e-10	7.568e-4	2.731e-5
	VDFM	0.043922	0.050462	0.045016	3.49e-10	7.251e-4	3.991e-5
	VLODM	0.052266	0.058096	0.050214	5.93e-10	7.254e-4	4.490e-5
40	Implicit	0.040019	0.041671	0.039675	2.68e-10	1.760e-4	4.102e-7
	Regul.	0.012873	0.016171	0.015776	2.893e-11	8.474e-4	7.318e-6
	DFM	0.040273	0.048669	0.055690	2.72e-10	2.191e-4	6.596e-6
	VDFM	0.040273	0.042710	0.042085	2.72e-10	1.856e-4	9.758e-6
	VLODM	0.048662	0.050290	0.050019	4.69e-10	1.864e-4	1.131e-5

Table 7.3

		The middle of calculation			Final moment		
		Rectang.	Scalene	Sine	Rectang.	Scalene	Sine
10	Implicit	0.029916	0.139758	0.111295	0.068440	0.132269	0.068473
	Regul.	0.027471	0.143325	0.117483	0.062569	0.126339	0.071540
	DFM	0.031644	0.139261	0.118225	0.072636	0.136601	0.066644
	VDFM	0.031644	0.143370	0.119056	0.072636	0.137619	0.070704
	VLODM	0.031644	0.144451	0.117742	0.072636	0.137898	0.071240
20	Implicit	0.010815	0.152431	0.024321	0.022945	0.056930	0.016478
	Regul.	0.008049	0.153958	0.021091	0.017029	0.051391	0.012137
	DFM	0.012768	0.147452	0.068678	0.027176	0.060852	0.021627
	VDFM	0.012768	0.155467	0.025705	0.027176	0.061600	0.021214
	VLODM	0.012768	0.157788	0.025298	0.027176	0.061783	0.021195
40	Implicit	0.005784	0.097919	0.006320	0.012025	0.028871	0.011572
	Regul.	0.002941	0.096378	0.003688	0.006122	0.023593	0.006318
	DFM	0.007790	0.092611	0.042937	0.016247	0.032853	0.017442
	VDFM	0.007790	0.098255	0.008891	0.016247	0.033321	0.015941
	VLODM	0.007790	0.100245	0.008740	0.016247	0.033407	0.015942

Table 7.4

		The middle of calculation			Final moment		
		Rectang.	Scalene	Sine	Rectang.	Scalene	Sine
10	Implicit	8.242e-5	8.480e-5	6.975e-4	0.010368	0.010438	0.014873
	Regul.	2.162e-5	2.398e-5	4.494e-5	3.519e-4	3.905e-4	0.001466
20	Implicit	2.250e-5	2.959e-5	8.245e-5	0.002562	0.002585	0.002981
	Regul.	2.452e-5	2.536e-5	2.656e-5	9.644e-5	1.422e-4	2.441e-4
40	Implicit	4.942e-6	9.133e-6	6.943e-6	6.071e-4	6.142e-4	6.127e-4
	Regul.	2.552e-5	2.574e-5	2.532e-5	6.182e-5	7.687e-5	7.351e-5

Table 7.5

The results of calculations are represented in tables 7.1…7.5 according to number of the test (1…5). Final time for tests 1…4 was 0.25, for the test 5 was $\frac{\pi}{2}$. Numbers of steps by time are 100 and 600 respectively. Factors $\overline{k}_\xi$ and $\overline{k}_\eta$ are supposed to be equal to 1.5 in the scheme with regularizator (3.15).

Conclusions

The computer simulation of the different boundary and initial problem demonstrates some features of the methods developed. All additive and almost additive schemes DFM, VDFM, VLODM have the same accuracy and CPU times.

The fully implicit scheme accuracy is better than for additive scheme but CPU time is essentially more and there is a serious problem of parallelelizing this scheme. The best scheme is a regularized one. The error of the approximation and CPU time are less than for other methods except the case of the large values of function decrements. Besides the order of the time approximation is more than the first as it has been discussed above.

If the spatial part of the function is $\cos 3\pi x \cos \pi y$ and the decrement is $10\pi^2$ the asymptotic stability of the scheme is not satisfied. Really at the end of evolution the absolute error of the solution is 1.694e-9 for rectangular grid, 0.00426 for scalene grid and 0.00135 for sinusoidal one, if the step is about of 1/10. For N=20, 40 correspondingly the values of errors are 0.00124, 6.094e-4 for scalene grid. As for other methods the error were less. It may be explained by the fact that regularization operators are not strictly agreed with the approximation of the elliptic operator by the support operator method.

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