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ON THE METHOD OF DIFFERENCE POTENTIALS

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Abstract

The concise look at the Difference Potentials Method and at the causes of new possibilities which are provided by this method; three examples of solved applied problems and also the notes about connections between Calderon-Seely and new potentials.

В.С. Рябенький

О методе разностных потенциалов

Аннотация

Общий взгляд на метод разностных потенциалов и на причины новых возможностей, которые дает этот метод; три примера из числа решенных прикладных задач; замечания о связях между новыми потенциалами и потенциалами Кальдерона-Сили.

The method of difference potentials (DPM) is intended for digital simulation and numerical solution of some problems of mathematical physics DPM was proposed by the author in his doctoraite (D. Sc.) thesis in 1969 and was significantly developed at Keldysh Institute Russian Academy of Sciences, at the Department of Computational Mathematics of the Moscow Institute of Physics and Technology, at the Institute for Mathematics Modeling of the Russian Academy of Sciences, at ICASE (NASA Lengley Reserch Center) as well as some other Institutes.

This report will be necessarily fragmentary. The almost modern state-of-the art of the DPM is reflected in [1].

There (in [1]) are also named others people who take part in developing of DPM. Here I would like to mention only Professor Semeon Tsynkov, whose participation was very bright and significant.

The new possibilities provided by the DPM originate from the fact, that DPM combines several advantages of the classical potentials and Calderon's-type potentials with the universality and constructively of difference schemes.

The main advantage of classical potential method for discretisation and numerical solution the different problems of mathematical physics (Laplas and Helmholtz equations, the Lame, Stocks, Maxwell, Cauhy-Riemann systems) is the possibility to digitize the potentials and boundary integral equations by means of quadratures. But the kernels of integral operators of this kind are constructed by means of fundamental solutions, which must be sufficiently simple. Therefore the field of applicability of classical potentials method is bounded, to say the least, by equations with constant coefficients, and can't be used for equations with variable coefficients.

A.P.Calderon [2] and R.T.Seeley [3] have constructed and studied the pseudodifferential potentials, boundary pseudodifferential equations with projectors for elliptic equations with variable coefficients. But Calderon-Seeley equations can’t be digitized by means of quadratures, because they do not contain any integrals.

We have constructed some simple auxiliary boundary problems which are used instead of using symbols of differential operators. These modified potentials are defined not only for elliptic equations, they have similar properties, but they also can’t be digitized by means of quadratures because they also do not contain the integrals. However, we have constructed earlier the difference potentials. Those can be used for discrete approximation of new potentials and boundary pseudodifferential equations with projectors, connected with them. It is possible to say that the difference potentials play the role of non-classic quadratures for modified Calderon-Seeley potentials.

Thus the modified Calderon-Seeley potentials can be used for discrete simulation and numerical solution of different problems of mathematical physics, not only elliptic.

Note, however, that really the difference potentials were proposed much earlier then modification of Calderon-Seeley potentials was made (1983). This modification was made firstly to close the constructions of difference potentials when grid step tends to zero.

The difference potentials can be used also for discrete simulation and numerical solution of different problems directly, without using the modified Calderon-Seeley potentials.

Note also, that theory of DPM is mainly algebraic and algorithmic. The general metric properties of new potentials are studied only partially (see [1], part I-III). We will speak here below only about one sort of difference potentials, namely, about the so-called difference potentials of Cauchy-type and about three examples of applications of DPM to some numerical problems.

I. The constructions and properties of Cauchy-type difference potentials.

The difference potentials of Cauchy-type and more general potentials are constructed for solutions of linear system of difference equations of general form. But we will speak here only about difference potentials of Cauchy-type, using the Poisson equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(x,y)$ and its simplest difference analogue

$$\sum_{n\in N_m} a_{mn}u_n = f_m,$$

where N_m is five-point stencil of difference scheme. Namely, we use grid $m=(m_1h,\ m_2h)$, with step h; $m_1, m_2 = 0, \pm 1, \ldots,$ and stencil N_m, which contain following five points:

$$N_m = \{(m_1h,\ m_2h),\ ((m_1 \pm 1)h,\ m_2h)),\ (m_1h, (\ m_2 \pm 1)h)\}.$$

I. 1. The auxiliary difference problem.

To construct the difference potentials we use the following auxiliary difference problems.

Let D^0 be a bounded domain in the *xy*-plane. We will assume that D^0 is some square, whose sides are lying on grid line $x=kh$, or $y=lh$, where k and l are any integers (Fig.1).

Let M^0 be the set of points $m=(m_1h,\ m_2h)$, which belong to interior part of square D^0 (the black points on Fig.1). We consider the following difference equation

$$\sum_{n\in N_m} a_{mn}u_n = f_m, \qquad m\in M_0. \tag{1}$$

Obviously, the left-hand side of this equation is meaningful for the functions $u_{N^0}=\{u_n\},\ \ n\in N^0$, whose grid domain N^0 is $N^0=\bigcup N_m,\ \ m\in M^0$. We add some linear homogeneous condition to the difference equation (1). For example we assume

$$u_n = 0, \qquad n \in N^0 \bigcap \partial \bar{D}^0.$$

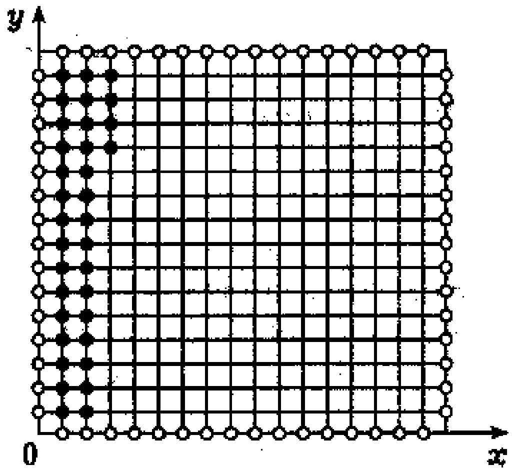

Fig.1.

We can rewrite these boundary conditions as inclusion

$$u_{N^0} \in U_{N^0} \tag{2}$$

where U_{N^0} is the space of all functions u_{N^0}, which equal to zero on boundary $N^0 \bigcap \partial D^0$.

I. 2. The grid boundary and Cauchy-type potential with jump.

Let $D^+ \subset D^0$ be given subdomain of D^0 (Fig.2). By M^+ we denote the set of points *m* lying in the interior of D^+ or on its boundary $\Gamma = \partial \bar{D}^+$, and consider the equation

$$\sum a_{mn} u_n = f_m, \qquad m \in M^+. \tag{3}$$

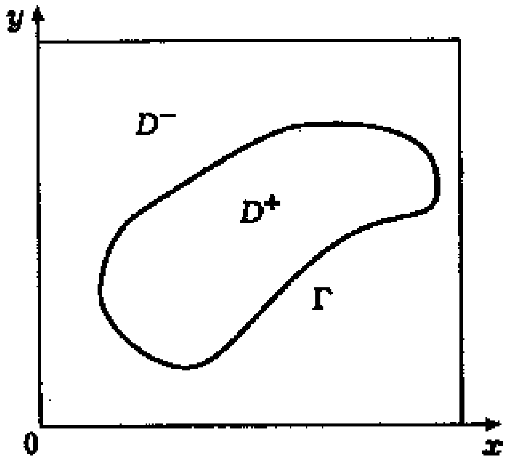

Fig.2.

The left-hand side of this equation is meaningful only for the functions $\{u_n\}$, defined on the set

$$N^+ = \bigcup N_m, \quad m \in M^+.$$

By D^- we denote the domain $D^- = D^0 \setminus \bar{D}^+$ and by M^- the set of grid points lying in D^-:

$$M^- = \{m \mid \ m \in D^-\} = M^0 \setminus M^+.$$

We consider system

$$\sum a_{mn} u_n = f_m, \quad m \in M^-, \tag{4}$$

on the set M^-. The left-hand side of system (4) is meaningful for the function $\{u_n\}$ that are defined on the set

$$N^- = \bigcup N_m, \quad m \in M^-.$$

Thus system (1) splits into two subsystems (3) and (4) whose solutions are defined on N^+ and N^- respectively.

Let us define the boundary γ between grid domains N^+ and N^-, by setting

$\gamma = N^+ \bigcap N^-$ (see Fig.3).

We shall speak that functions $u_{N^0} \in U_{N^0}$ from space U_{N^0} are regular.

Let $u^+_{N^0} \in U_{N^0}$ and $u^-_{N^0} \in U_{N^0}$ be two arbitrary regular functions. We define the piecewise-regular function $u^\pm_n$, $n \in N^0$, by setting

$$u^\pm_n = \begin{cases} u^+_n & \text{if} \quad n \in N^+, \\ u^-_n & \text{if} \quad n \in N^-. \end{cases} \tag{5}$$

Let us introduce the linear space $U^\pm_{N^0}$ all functions of the form (5). The function (5) takes two values u^+_n and u^-_n at each point $n \in \gamma$ of the grid boundary γ.

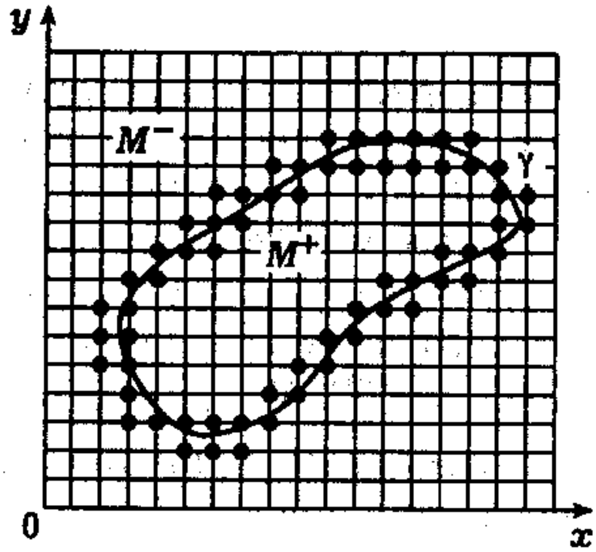

Fig.3.

A single-value function v_γ, defined at the points $n \in \gamma$ by the formula

$$v_\gamma\big|_n = [u^\pm]_n = u^+_n - u^-_n, \quad n \in \gamma,$$

will be called a jump of the piecewise-regular function $u^\pm$ on the grid boundary γ. The piecewise-regular function (5) will be called a piecewise-regular solution of the problem

$$\sum a_{mn} u_n^{\pm} = 0, \quad u^{\pm} \in U^{\pm} \tag{6}$$

if the functions u_n^+, $n \in N^+$, and u_n^-, $n \in N^-$, satisfy the homogeneous equations

$$\sum a_{mn} u_n^+ = 0, \quad m \in M^+,$$
$$\sum a_{mn} u_n^- = 0, \quad m \in M^-.$$

Theorem. Let v_γ be an arbitrary function which is defined on γ. There exists one and only one piecewise-regular solution $u^{\pm}$ of the problem (6) with jump v_γ.

Definition. The piecewise-regular solution $u^{\pm}$ of problem (6) with given jump v_γ will be called a difference potential $u^{\pm} = P^{\pm} v_\gamma$ with density v_γ.

I. 3. An analogy between the Cauchy-type difference potential and the classical Cauchy-type integral.

Suppose that Γ is a non-self-intersecting closed contour that divides the complex plane $z=x+iy$ into the bounded part D^+ and the complimentary unbounded part D^-. The classical Cauchy-type integral

$$u^{\pm}(z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{v_\Gamma(\xi)}{\xi - z} d\xi, \quad \Gamma = \partial D^+ \tag{7}$$

can be determined as a piecewise analytic function tending to zero at infinity and exhibiting the jump $v_\Gamma(\xi) = [u^{\pm}]_\Gamma$ on the contour Γ. Here $u^+(z)$ and $u^-(z)$ are the values of the Cauchy-type integral (7) for $z \in D^+$ and $z \in D^-$ accordingly.

A Cauchy-type integral can be interpreted as a potential for the solutions of the Cauchy-Riemann system

$$\frac{\partial a}{\partial x} = \frac{\partial b}{\partial y}, \quad \frac{\partial b}{\partial x} = -\frac{\partial a}{\partial y},$$

connecting the real and imaginary parts of an analytic function.

Thus the Cauchy-type difference potentials $\boldsymbol{u}^{\pm} = \boldsymbol{P}^{\pm} \boldsymbol{v}_{\gamma}$ plays the same role for solutions of general linear difference equations as the Cauchy-type integral plays for solutions of the Cauchy-Riemann system.

II. The examples of applications of DPM.

The difference Cauchy-type potentials play the same role for general difference schemes as Cauchy-type integral play for Cauchy-Riemann system. The Cauchy-type integral has many applications to different problems, which are connected with Cauchy-Riemann system. Therefore it is naturally to expect, that the difference Cauchy-type potentials must have many applications to the different and distant one to other problems. Really, it is so.

We give here several examples of applications of DPM.

These and many others ones are reflected in the book [1] and many papers.

II. 1. Artificial boundary conditions for stationary problems.

A typical example of problem requiring the construction of artificial boundary conditions is the problem of calculating the velocity and pressure of the air flow around a body, usually in the close vicinity of the body. However, for computation we have to take considerably larger neighborhood.

The DPM gives the possibilities to construct the nonlocal artificial boundary conditions on the external boundary of the finite computational subdomain, which demonstrate some advantages in comparison with ordinary classical artificial boundary conditions. It was demonstrated by means of some NASA tests ([1], part V).

II. 2. Nonreflecting artificial boundary conditions for long-time calculations of acoustic and electromagnetic waves propagation.

Let we need to calculate a table of values some function $u(t,x,y,z)$ on grid of points $(t_{m_0}, x_{m_1}, y_{m_2}, z_{m_3}) = (m_0\tau,\ m_1 h,\ m_2 h,\ m_3 h,),\quad \tau = \text{const}\cdot h,\ \ h>0,\ \ m_0 = 0,1,...;$ $m_1 = 0,\pm1,...;\ \ m_2 = 0,\pm1,...;\ \ m_3 = 0,\pm1,...$ which are lying inside of the unique sphere

$$(m_1 h)^2 + (m_2 h)^2 + (m_3 h)^2 < 1$$

on the time-levels $t = m_0\tau,\ \ m_0 = 0,1,...,\left[\frac{T}{\tau}\right]$ and T can be arbitrary great number. Obviously, the amount of grid points, where we have to find function $u(t,x,y,z)$ has the order of $O(Th^{-4})$. Therefore there is no method, which requires less then $O(Th^{-4})$ arithmetic operations to calculate this table.

We consider now this problem in case when function $u(t,x,y,z)$ is the solution of the following problem

$$\frac{\partial^2 u}{\partial t^2} - c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = f(t,x,y,z)$$
$$u(t,x,y,z) = 0, \quad \text{if} \quad t \le 0,$$
$$f(t,x,y,z) = 0, \quad \text{if} \quad t \le 0 \quad \text{or} \quad x^2 + y^2 + z^2 > 1.$$

The calculation by means of ordinary difference scheme with grid mech h for x,y,z and $\tau \le \frac{\sqrt{3}}{3}h$ for t requires $O(T^4 h^{-4})$ arithmetic operations and becomes impossible even for not very large values of T.

But using the lacunas of wave equation in 3 dimensional space x,y,z together whit DPM we constructed algorithm which takes only $O(Th^{-4})$ and which therefore can't be improved.

Recently this result was obtained also for solution $u = \begin{pmatrix} E \\ H \end{pmatrix}$ Maxwell system, namely, were constructed non-reflecting artificial boundary conditions for change the Maxwell system in vacuum out computational subdomain $x^2 + y^2 + z^2 \le 1$. These results valid also, if inside sphere $x^2 + y^2 + z^2 \le 1$ there are some scattering, cavities or any non linearity.

These results can be used for long-time calculations of some problems of acoustic and electrodynamics (see [1], part VII).

II. 3. Difference simulating of active shielding problem.

We will consider following problem, which describes single-frequency harmonic sound

$$\begin{aligned} &\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \mu^2(x,y)u = f(x,y), \quad (x,y) \in D^0, \\ &u\big|_{\partial D^0} = 0 \end{aligned} \tag{8}$$

on domain D^0 (see Fig.2). We will assume that $f(x,y) = f^+(x,y) + f^-(x,y)$, where

$$f^+(x,y) = \begin{cases} f(x,y), & (x,y) \in D^+, \\ 0, & (x,y) \in D^- \end{cases}$$

is a density of useful sound of sources, and function

$$f^-(x,y) = \begin{cases} 0, & (x,y) \in D^+, \\ f(x,y), & (x,y) \in D^- \end{cases}$$

is a density of noise sources accordingly. We will assume also, that neither $f(x,y)$ nor $u(x,y)$ is given. We know only two functions $u\big|_{\partial D^+}$ and $\left.\frac{\partial u}{\partial n}\right|_{\partial D^+}$ on boundary $\Gamma = \partial D^+$, which reflect the sum either the useful sound and noise.

The problem of active shielding the subdomain D^+ against sources, which are located outside D^+, we are setting in the following way: to construct function $g(x,y)$, $(x,y) \in D^0$, in order that the solution $v(x,y)$ of the problem

$$\begin{aligned} &\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \mu^2 v = f(x,y) + g(x,y), \quad (x,y) \in D^0, \\ &v\big|_{\partial D^0} = 0 \end{aligned} \tag{9}$$

and the solution $w(x,y)$ of the problem

$$\begin{aligned} &\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \mu^2 w = f^+(x,y), \quad (x,y) \in D^0, \\ &w\big|_{\partial D^0} = 0 \end{aligned} \tag{10}$$

were coinciding on domain D^+:

$$v(x,y) \equiv w(x,y), \quad \text{if} \quad (x,y) \in D^+.$$

The function $g(x,y)$ in this case will be called the active suppress of noise.

We will construct now digital simulation of this problem and give its general solution.

Instead of problem (8) we consider its five-points difference analogue (Fig.2)

$$\begin{aligned} &\sum a_{mn} u_n = f_m, \quad m \in M^0, \\ &u_{N^0} \in U_{N^0}. \end{aligned} \tag{11}$$

We consider the difference analogue of problem (9)

$$\sum a_{mn} v_n = f_m + g_m, \qquad m \in M^0, \\ v_{N^0} \in U_{N^0}. \tag{12}$$

where g_m, $m \in M^0$, is an arbitrary grid function.

We also consider the problem

$$\sum a_{mn} w_n = f_m^+, \qquad m \in M^0, \\ w_{N^0} \in U_{N^0}$$

where

$$f_m^+ = \begin{cases} f_m, & \text{if} \quad m \in M^+, \\ 0, & \text{if} \quad m \in M^-. \end{cases}$$

The grid function will be called active digital suppress function, if the following equality is valid:

$$v_n \equiv w_n, \quad \text{if} \quad n \in N^+.$$

Theorem. All active suppress digital functions $g_{M^0} = \{g_m\}$, $m \in M^0$, have the form

$$g_m = \begin{cases} 0, & \text{if} \quad m \in M^+, \\ -\sum a_{mn} z_n, & \text{if} \quad m \in M^-, \end{cases} \tag{13}$$

where $z_{N^0} \in U_{N^0}$ is an arbitrary function from U_{N^0} which coincides with given u_n, $n \in \gamma$:

$$z_n = u_n, \quad n \in \gamma \tag{14}$$

on the grid boundary γ.

Note that the function u_{N^0}, v_{N^0}, w_{N^0}, f_{M^0} are unknown. We know only u_n, $n \in \gamma$ which can be measured. And we know, that influence of active suppress function g_{M^0} has the same effect as if we switch off all noise sources, i.e. change f_m, $m \in M^-$, supposing

$$f_m \equiv 0, \quad \text{if} \quad m \in M^-.$$

About the more general setting and solution of active shielding problem see [1], part VIII.

III. On some properties of modified Calderon potentials and approximation of these potentials by means of difference potentials.

The general scheme of modified Calderon-Seeley potentials is based on using of auxiliary boundary-value problems instead of the symbols of differential operators. The modified in accordance with this general scheme potentials save the main properties of Calderon-Seeley potentials and have some additional useful characteristics. Namely, modified potentials contains some unrestraint of choice of the auxiliary problems, which can be taken in attention of the particularity of problem we have to consider. The modified Calderon-Seeley potential can be approximated by difference potential and then can be used for numerical solution of the original problem.

The examples of application of DPM given above and many other ones, which book [1] contains or reflects, allow to say that DPM is one among other numerical methods.

At the same time the book [1] contains mainly the algebraic and algorithmic parts of theory DPM, but metric part is developed much less.

Maybe it would be interesting to obtain the theorems about metric characteristics of modified Calderon-Seeley potentials and accordance pseudo-differential boundary projectors for different classes of differential equations like that, as it is made for Calderon potentials and boundary pseudodifferential projectors of elliptic equations by Seeley [3].

Maybe it would be interestingly also to obtain some theorems about metric characteristics of difference potentials and about approximation of modified Calderon's potential by means of difference ones, when mesh step tends to zero.

The example of similar investigation contains in the part I of book [1], where the problems, which are connected with Poisson equation and corresponding fife point difference scheme, were considered .

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