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NEW GENERALIZATIONS OF THE CONTINUED FRACTION

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In Introductions we discuss the history of the continued fraction and of its generalizations. In Part I authors propose a new generalization of the continued fraction that gives periodicity for cubic irrationalities with positive discriminant. In Part II we propose a new generalization giving periodicity for cubic irrationalities with negative discriminant. We consider the simultaneous rational approximations of a number and its square. At first we describe the structure of the best integer approximations in homogeneous coordinates when three or two real forms are given. After that we propose an algorithm to compute the approximations. Examples of computations are given as well.

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Introduction

The algorithm of computation of the regular continued fraction [1] of a number has several fine properties. In particular,

(i) it is simple;

(ii) it gives the best rational approximations to the number;

(iii) it is periodic for square irrational numbers.

In 18–20 centuries a lot of mathematicians attempted to generalize the algorithm for vectors (Euler [2], Hermit [3], Jacobi [4], Dirichlet, Poincaré [5], Hurwitz, Brun [6], Minkowski [7], Klein [8], Voronoi [9], Perron [10], Skubenko [11], Arnold [12] etc.). But their algorithms had not properties (i) and (ii) together with the property

(iii′) periodicity for cubic irrational numbers.

Only the Voronoi algorithm [9] has properties (ii) and (iii′), but it is too complicated. Polyhedra of Klein [8]–Skubenko [11]–Arnold [12] do not give a basis for a good algorithm that was clarified in papers [13–20]. The interest of one of the authors to generalizations of the continued fraction arose in connection with his article [21], repeated by Lang [22].

Here we propose a new generalization of the continued fraction which has properties (i), (ii), (iii′) for cubic irrationalities with positive discriminant.

In a three-dimensional space we consider three homogeneous linear forms. In another three-dimensional space, where coordinates are absolute values of these forms, we consider the convex hull of points corresponding to all integer points of the first space, except the origin [23]. The proposed generalization of the continued fraction is a motion along the surface of the convex hull [25]. In [24] there are results of computation of the surfaces for eleven cubic forms being products of three linear forms. The result show periodic structures of surfaces and confirm the correctness of the proposed generalized algorithm. Below we consider the generic case only. All vectors are lines and asterisk means transposition.

1. The polyhedral surface

Let in the space $\mathbb{R}^3$ with coordinates $X = (x_1, x_2, x_3)$ be given three real homogeneous linear forms

$$l_i(X) = \langle L_i, X\rangle, \quad i = 1, 2, 3, \quad \det(L_1 L_2 L_3) \neq 0, \tag{1.1}$$

where $L_i = (l_{i1}, l_{i2}, l_{i3})$ belong to the space $\mathbb{R}^3_*$, which is dual to $\mathbb{R}^3$, and $\langle\cdot,\cdot\rangle$ denotes the scalar product. We put

$$m_i(X) = |l_i(X)|, \ i = 1, 2, 3, \ \text{and} \ M(X) = (m_1(X), m_2(X), m_3(X)).$$

The transformation $M(X)$ maps the space $\mathbf{R}^3$ into the first octant $M \geq 0$ of the space $\mathbf{R}^3$ and the set $\mathbb{Z}^3 \setminus 0$ of all integer points except the origin into the set $\mathbf{Z}^3$. The convex hull of the set $\mathbf{Z}^3$ is a polyhedral set $\mathbf{M}$. Its boundary $\partial\mathbf{M}$ is a polyhedral surface containing vertices V_i, edges R_i and faces Γ_i.

Let $V_1, V_2, V_3 \in \mathbf{Z}^3$ and

$$V_j = M(B_j),\ B_j \in \mathbb{Z}^3,\ j = 1, 2, 3. \tag{1.2}$$

Define $\omega(V_1, V_2, V_3) = |\det(B_1^* B_2^* B_3^*)|$.

Obviously ω takes nonnegative integer values. For a face Γ_i of the surface $\partial\mathbf{M}$, we define $\omega(\Gamma_i)$ as the minimum of $\omega(V_1, V_2, V_3)$ over all the triples of different $V_1, V_2, V_3 \in \mathbf{Z}^3$ that lie on the face Γ_i.

A face Γ_i of $\partial\mathbf{M}$ is said to be *simple* if it is a triangle with vertices (1.2). A face Γ_i is said to be *semi-simple* if it is a triangle that contains exactly one inner point from $\mathbf{Z}^3$ and has $\omega(\Gamma_i) = 1$. We have $\omega(\Gamma_i) = |\det(B_1^* B_2^* B_3^*)|$ for a simple face Γ_i with vertices (1.2).

Theorem 1. *If $\omega(\Gamma_i) = 0$ for a simple face Γ_i with a vertices* (1.2), *then one of the vectors B_1, B_2, B_3 is the sum of the other two.*

Theorem 2. *In the generic case, all faces Γ_i of the surface $\partial\mathbf{M}$ are simple ore semi-simple ones and $\omega(\Gamma_i) \leq 2$.*

A semi-simple face Γ_i is naturally partitioned into three triangles, in each of which two vertices are vertices of Γ_i and the third vertex is the interior point of Γ_i belonging to $\mathbf{Z}^3$ (see Fig. 1). Therefore, in the generic case, the surface $\partial\mathbf{M}$ has a natural triangulation.

Theorem 3. *If the forms* (1.1) *are such that the matrix $(L_1^* L_2^* L_3^*) = SW$, where S is a nonsingular matrix with rational elements and W is the Wandermond matrix for a cubic polynomial $P_3(\lambda)$ with positive discriminant, then the surface $\partial\mathbf{M}$ has two independent periods.*

Example 1. The equation

$$F_3(\lambda) \stackrel{\text{def}}{=} \lambda^3 - 2\lambda^2 - \lambda + 1 = 0$$

has three real roots

$$\lambda_1 \approx -.80193773580,\ \ \lambda_2 \approx .55495813209,\ \ \lambda_3 \approx 2.24697960371.$$

We put

$$L_i = (1, \lambda_i^2, \lambda_i^2 - 2\lambda_i),\ \ i = 1, 2, 3 \tag{1.3}$$

and consider corresponding forms (1.1) (see [24]). The logarithmic projection

$$n_1 = \log m_1(X),\ \ n_2 = \log m_2(X) \tag{1.4}$$

product of the linear forms

$$h(X) = l_1(X)l_2(X)l_3(X) \tag{1.5}$$

is the seventh extremal cubic form found in [26] (see also [20]).

For each point $Y = M(X)$ shown in Fig. 2, there are written the value $|h(X)|$ and the vector X. The boldface lines show the boundary of the fundamental domain. Values $\omega(\Gamma_i)$ are written for each face Γ_i lying in it. $|h(X)| = 1$ for all vertices $V = M(X)$ of the surface $\partial\mathbf{M}$. The fundamental domain consists of 6 faces. Two faces have $\omega = 1$ and four faces have $\omega = 0$.

The surfaces $\partial\mathbf{M}$ for ten other sets of forms (1.1) are shown in [24, 27].

2. Algorithm of motion along the surface ∂M

Let in $\mathbb{R}^3$ be given three linear forms (1.1) and such a vector $A = (\alpha_1, \alpha_2, \alpha_3)$ that $l_1(A) = l_2(A) = 0$. The surface $\partial\mathbf{M}$ corresponds to the forms (1.1). **Our aim** is to construct the integer approximations $B_k \in \mathbb{Z}^3$ to the straight line $l_1(X) = l_2(X) = 0$ or $X = \mu A, \mu \in \mathbb{R}$.

We denote by the bar the orthogonal projection of a point from $\mathbf{R}^3$ into the plane (m_1, m_2). For instance, if $M = (m_1, m_2, m_3)$ then $\bar{M} = (m_1, m_2)$.

Lemma 2.1. *Let three points $U, V, W \in \mathbf{R}^3$. The plane passing through them intersects the third axis in the point*

$$y_3(U, V, W) = \frac{\det(U^*V^*W^*)}{|\bar{U}^*\bar{V}^*| + |\bar{V}^*\bar{W}^*| + |\bar{W}^*\bar{U}^*|}, \tag{2.1}$$

where $|\bar{U}^*\bar{V}^*| \stackrel{\text{def}}{=} \det\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$.

Let integer vectors $B_1, B_2, B_3 \in \mathbb{Z}^3$ form a lattice base, i.e. $\det(B_1^* B_2^* B_3^*) = \pm 1$. Then we have $l_{ij} \stackrel{\text{def}}{=} l_i(B_j)$, $i, j = 1, 2, 3$ and the vector $\Lambda = (\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_1 B_1 + \lambda_2 B_2 + \lambda_3 B_3 = \mu_0 A$. Here $\sum\limits_{j=1}^{3} l_{ij}\lambda_j = 0,\ i = 1, 2$. These initial data can be written as the table

$$\begin{matrix} B_1 & l_{11} & l_{21} & l_{31} & \lambda_1 \\ B_2 & l_{12} & l_{22} & l_{32} & \lambda_2 \\ B_3 & l_{13} & l_{23} & l_{33} & \lambda_3. \end{matrix} \tag{2.2}$$

Denote $M_i = M(B_i)$, i.e. $m_{ij} = |l_{ij}|,\ i, j = 1, 2, 3$. Below $X = (x_1, x_2, x_3)$ are coordinates of a point X in the base B_1, B_2, B_3, i.e. $X = x_1 B_1 + x_2 B_2 + x_3 B_3$.

A transition to another lattice base B_1', B_2', B_3' consists of the following 5 steps.

the plane (m_1, m_2). They are vertices of a triangle. Each its side has an external normal vector. We take such a side for which the external normal vector has both components negative. *Points* $\bar{M}_i$ lying in that side *are distinguished.* Distinguishing can be made by a picture or by computations described in [23, 25]. Let for definiteness the distinguished pair be $\bar{M}_1, \bar{M}_2$. The straight line passing through them is denoted as $\mathcal{L}$ (see Fig. 3).

In $\mathbb{R}^3$ the line μA intersects the plane $x_3 = 0$ in the point $x_1 = x_2 = 0$ and the plane $x_3 = \operatorname{sign} \lambda_3 \stackrel{\text{def}}{=} a_3 = \pm 1$ in the point $X = (\lambda_1/|\lambda_3|, \lambda_2/|\lambda_3|, \lambda_3/|\lambda_3|)$.

Step 2. We compute $a_i = [\lambda_i/|\lambda_3|],\ i = 1, 2, 3$, where square brackets $[\alpha]$ mean the integral part of α.

Step 3. In the planes $x_3 = 0$ and $x_3 = a_3$ we take integer points nearest to the line μA. In the plane $x_3 = 0$ there are two such points $(B_1 + B_2)$ and $(B_1 - B_2)$ (see Fig. 4). In the plane $x_3 = a_3$ there are four such points

$$\begin{array}{rrrr} W_1 = & a_1 B_1 & +a_2 B_2 & +a_3 B_3, \\ W_2 = & (a_1+1) B_1 & +a_2 B_2 & +a_3 B_3, \\ W_3 = & a_1 B_1 & +(a_2+1) B_2 & +a_3 B_3, \\ W_4 = & (a_1+1) B_1 & +(a_2+1) B_2 & +a_3 B_3 \end{array} \tag{2.3}$$

(see Fig. 5). We denote $U_1 = M(B_1 + B_2), U_2 = M(B_1 - B_2), V_j = M(W_j),\ j = 1, 2, 3, 4$. The points U_i, V_j, which projections $\bar{U}_i, \bar{V}_j$ lie on the right side of the line $\mathcal{L}$, are rejected, and the points U_i, V_j on its left side (i.e. more close to the origin) are kept.

Step 4. For each kept point U_i and V_j, by Lemma 1 we compute the point of intersection of the plane, passing through points M_1, M_2 and the point U_i or V_j, with the third coordinate axis. We select the smallest computed value (2.1) and the corresponding point U_i or V_j.

Step 5. If the point selected in the step 4 is U_i, we do the step 5a. If it is V_j, we do the step 5b.

Step 5*a*. Let the point U_1 be selected. In the triangle with vertices $\bar{M}_1, \bar{M}_2, \bar{U}_1$ we take such a side, containing the point $\bar{U}_1$, which has the negative external normal vector. Let the point $\bar{M}_2$ belong to that side. Then we keep the vector B_2 and replace the vector B_1 by the vector $B_1 + B_2$, i.e. we pass from the base B_1, B_2, B_3 to the base $B_4 = B_1 + B_2, B_2, B_3$:

$$\begin{pmatrix} B_1' \\ B_2' \\ B_3' \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} B_4 \\ B_2 \\ B_3 \end{pmatrix} = N \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The matrix N^{*-1} gives the transformation

$$\Lambda^{*\prime} = N^{*-1}\Lambda^*, \quad N^{*-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Step 5*b*. Let the point $V_j = M(W_j)$ be selected, where $W_j = \tilde{a}_1 B_1 + \tilde{a}_2 B_2 + a_3 B_3$ according to (2.3). Then we pass from base B_1, B_2, B_3 to the base B_1, B_2, W_j:

$$\begin{pmatrix} B_1' \\ B_2' \\ B_3' \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} B_1 \\ B_2 \\ W_j \end{pmatrix} = N \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \tilde{a}_1 & \tilde{a}_2 & a_3 \end{pmatrix}.$$

Here

$$\Lambda^{*\prime} = N^{*-1}\Lambda^*, \quad N^{*-1} = \begin{pmatrix} 1 & 0 & -a_3\tilde{a}_1 \\ 0 & 1 & -a_3\tilde{a}_2 \\ 0 & 0 & a_3 \end{pmatrix}.$$

Vectors B_1', B_2', B_3' form the new base, and the vector Λ' is the vector $\Lambda = \mu_0 A$ in that new base. Thus, the transition from the base B_1, B_2, B_3 to the new base B_1', B_2', B_3' is finished and we can write new table (2.2) of initial values for the next transition.

First, we assume that on the surface $\partial\mathbf{M}$:

$$\text{all the faces } \Gamma_i \text{ are simple or semi–simple} \tag{2.4}$$

and have

$$\omega(\Gamma_i) \leq 1. \tag{2.5}$$

According to Theorem 2 property (2.4) holds in the generic case.

Theorem 4. *Under assumptions* (2.4) *and* (2.5), *if the points M_1 and M_2 are associated with the edge of a natural triangulation of $\partial\mathbf{M}$, then the indicated transition to another base yields the natural triangle on $\partial\mathbf{M}$ with vertices M_1, M_2 and U_i or V_j.*

Theorem 5. *Let B_1, B_2, B_3 be the initial base and a pair of distinguished points from $M(B_i)$, $i = 1, 2, 3$ belong to an edge of the natural triangulation of $\partial\mathbf{M}$. If the final base $\tilde{B}_1, \tilde{B}_2, \tilde{B}_3$ is obtained after several indicated transitions to intermediate bases and, in the last transition, a point of V_j is chosen, then under assumption* (2.4) *and* (2.5) *the points $M(\tilde{B}_i)$, $i = 1, 2, 3$ are vertices of natural triangle on $\partial\mathbf{M}$.*

If the assumption (2.5) is violated then the described algorithm passes through a face Γ_i with $\omega(\Gamma_i) = 2$ using a point P_i. A pyramid Δ_i with the base Γ_i and the vertex P_i corresponds to the face Γ_i with $\omega(\Gamma_i) = 2$. The subtraction of all such pyramids Δ_i from the set $\mathbf{M}$ gives a concave-convex set $\mathbf{N}$ with the boundary $\partial\mathbf{N}$.

Thus, the proposed algorithm is the directed motion along the surface $\partial\mathbf{N}$, and one stage of this motion gives the transition from a triangle with $\omega = 1$

in [18, 29].

Example 2 (continuation of Example 1). Let for forms (1.1),(1.3), we start from the base $B_1 = (1,0,0)$, $B_2 = (0,1,0)$, $B_3 = (1,0,1)$, then the distinguished points are $M(B_2)$ and $M(B_3)$. In the first transition, we replace B_3 by the vector $B_4 = B_2 - B_3 = (-1,1,-1)$. Now the distinguished points are $M(B_2)$ and $M(B_4)$. In the second transition, we replace B_1 by the vector $B_5 = -B_1 + 3B_2 = (-1,3,0)$. Vectors B_2, B_4, B_5 form a base and points $M(B_2), M(B_4), M(B_5)$ are vertices of the surface $\partial\mathbf{M}$. Now points $M(B_2), M(B_5)$ are distinguished and the new vector $B_6 = B_5 - B_2 = (-1,2,0)$ should be taken instead of B_2. Now points $M(B_5), M(B_6)$ are distinguished and the new vector $B_7 = B_4 + 3B_5 = (-4,10,-1)$. The final base is B_6, B_5, B_7. According to Fig. 2 in logarithmic coordinates (1.4), the triangle $\{\bar{M}(B_6), \bar{M}(B_7), \bar{M}(B_5)\}$ can be obtained from the initial triangle $\{\bar{M}(B_1), \bar{M}(B_2), \bar{M}(B_3)\}$ by a parallel translation. Thus, the linear transformation $X = YT$ with

$$\begin{pmatrix} B_6 \\ B_7 \\ B_5 \end{pmatrix} = T \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}$$

induces the linear automorphism of the surface $\partial\mathbf{M}$, i.e.

$$T = \begin{pmatrix} B_6 \\ B_7 \\ B_5 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 2 & 0 \\ -3 & 10 & -1 \\ -1 & 3 & 0 \end{pmatrix}.$$

Introduction

Hear we propose a generalization of the continued fraction giving periodic expansion for cubic irrationality with negative discriminant. In 1850, Hermit [3] proposed his generalizations of the continued fraction, which were used by Sharve [30] for finding unities of simplest number fields. In 1896, Voronoi [9, part II] considered linear and quadratic forms, and suggested his algorithm for finding their successive relative minima (or best approximations). Other algorithms were proposed in [31-34] (see [35, Ch. 5]). All these algorithms consist in computation of the sequence of integer bases with imbedded first octants containing the ray to be approximated. Two latter properties of the sequence of bases are superfluous restrictions on the algorithm leading to its unjustifiable complication.

In [25, §§ 1,2], we considered various types of continuous fractions and their plain interpretations. The most appropriate for generalization is the diagonal continued fraction, which was first introduced by Minkowski in 1896 [8, part I, case $\Omega = 1$] (without the name). In 1902, he introduced it once again with the name [36] (see also [37, 38]). A detailed expounding see in [39]. Below all vectors are lines and asterisk means transposition.

1. Statement of the problem

Let two forms be given in the space $\mathbb{R}^3$ with coordinates $X = (x_1, x_2, x_3)$: the linear

$$l_1(X) = \langle J, X \rangle \stackrel{\text{def}}{=} j_1 x_1 + j_2 x_2 + j_3 x_3 \tag{1.1}$$

and the quadratic

$$l_2(X) = \langle K, X \rangle \langle \overline{K}, X \rangle, \tag{1.2}$$

where $J = (j_1, j_2, j_3)$ is a real vector, $K = (k_1, k_2, k_3)$ is a complex vector, $\overline{K}$ is complex conjugate vector, and the brackets $\langle K, X \rangle \stackrel{\text{def}}{=} k_1 x_1 + k_2 x_2 + k_3 x_3$ mean scalar product. We denote $L(X)$ the product of forms $l_i(X)$:

$$L(X) = l_1(X) l_2(X). \tag{1.3}$$

It is obvious that for $X \in \mathbb{R}^3$, both forms $l_1(X)$ and $l_2(X)$ are real. Let the vector A be such that $l_2(A) = 0$. Then the form $l_2(X)$ vanishes at the straight line $X = \mu A, \mu \in \mathbb{R}$. We assign

$$m_i(X) = |l_i(X)|, \ i = 1, 2, \tag{1.4}$$

$$M(X) = (m_1(X), m_2(X)). \tag{1.5}$$

straight line $X = \mu A$ or to the plane $l_1(X) = 0$ if there is no such an integer point $Y \in \mathbb{Z}^3, Y \neq 0$ that

$$M(Y) \leq M(X), \quad m_1(Y) + m_2(Y) < m_1(X) + m_2(X). \tag{1.6}$$

Problem: Find the best integer approximations $\tilde{X}$ (not necessarily all of them) to the straight line $X = \mu A$ with arbitrary small $m_2(\tilde{X})$.

2. Principal construction

Introduced by (1.4), (1.5) the vector-function $M(X)$ maps $\mathbb{R}^3$ into the first quadrant $\mathbf{R}^2_+$ of the plane with coordinates $M = (m_1, m_2)$. Two circles of the space $\mathbb{R}^3$ are mapped in one point $M \in \mathbf{R}^2_+$, i.e. one circle for $l_1 < 0$, and another for $l_1 > 0$. Hence we restrict ourselves to the semi-space $\pi\mathbb{R}^3 = \mathbb{R}^3 \cap \{l_1 > 0\}$. Let the set of integer points $X \in \mathbb{Z}^3$, excluding $X = 0$, be mapped into the set $\mathbf{Z}^2$ under this map, i.e. $\mathbf{Z}^2 = M(\mathbb{Z}^3 \setminus 0)$. Generically, no more than one integer point $X \in \pi\mathbb{Z}^3 \stackrel{\text{def}}{=} \pi\mathbb{R}^3 \cap \mathbb{Z}^3$ is mapped in a point M, i.e. the integer pre-image in $\pi\mathbb{Z}^3$ is unique.

Let $\mathbf{M}$ be the convex hull of the set $\mathbf{Z}^2$, and $\partial\mathbf{M}$ be its boundary. Obviously, $\partial\mathbf{M}$ is the convex polygonal line. It consists of vertices and edges. All vertices are images $M(X)$ of integer points $X \in \mathbb{Z}^3$. But on $\partial\mathbf{M}$, there may be other images of integer points placed inside the edges. Let all images of integer points be numbered sequentially by integer indices in the direction of growth of m_1. We denote these points as $U_k = (u_{1k}, u_{2k}), u_{1k} < u_{1,k+1}$, and their integer pre-images with $l_1 > 0$ as F_k, i.e.

$$U_k = M(F_k), \quad k \in \mathbb{Z}. \tag{2.1}$$

Lemma 1. *The minors of the matrix* $(F^*_{k-1} F^*_k)$ *have no common divisors.*

For every k, we define

$$\omega(k) = |\det(F^*_{k-2} F^*_{k-1} F^*_k)|. \tag{2.2}$$

Obviously, $\omega(k)$ takes integer non-negative values.

Lemma 2. *For every integer* $l \geq 0$*, there exist forms* (1.1) *and* (1.2) *such that* $\omega(k) \geq l$ *for some* k*.*

Similar proposition for consecutive minima was proved in [34].

The algorithm of the regular continued fraction gives a sequence of bases, and the transition to the following base is given by a unimodular matrix. In our three-dimensional case, we can form the sequence of bases consisting only of vectors F_k and linked by unimodular transformations only if $\omega(k)$ takes the values 0 and 1. If, on the other hand, $\omega(k)$ has values grater formed.

We assign to each pair of neighboring points U_{k-1}, U_k a point $V \in \mathbf{R^2}$ by the following rule. Among all points $G \in \mathbb{Z}^3$, we choose those for which

$$\det(F_{k-1}^* F_k^* G^*) = -1, 0, +1. \tag{2.3}$$

According to Lemma 1, for each among these tree values, there exists a two-dimensional lattice of points G with the property (2.3). Among these points G, we keep only those for which $m_1(G) > m_1(F_k)$, and we denote as $\mathbf{G_k}$ the set of these points G. Now among the points $M(G)$ with $G \in \mathbf{G_k}$, we choose the point (denote it as $\tilde{M}$) for which the segment $[U, M(G)]$ is inclined the most to the axis m_1, i.e. all points $M(G)$ with $G \in \mathbf{G_k}$ lie no lower than the straight line going through points U_k and $\tilde{M}$. Technically, we can do this marking the point of intersection of the straight line going through points U_k and $M(G)$ with the axis m_1. It is easy to note that for $M = M(G)$, this value

$$m_1 = \eta_1(U_k, M) \stackrel{\text{def}}{=} \frac{\det|U^* M^*|}{m_2 - u_{2k}}. \tag{2.4}$$

If for all $G \in \mathbf{G_k}$, the points $M(G)$ have $m_2(G) > u_{2k}$, then $\eta_1(U_k, M(G)) < u_{1k}$. In this case, we take the $M(G)$ as $\tilde{M}$, where $\min \eta_1(U_k, M)$ is attained over $G \in \mathbf{G_k}$.

If there are $G \in \mathbf{G_k}$ with $m_2(G) < u_{2k}$, then for them $\eta_1(U_k, M(G)) > u_{1k}$. In this case, we take such $M(G)$ where $\min \eta_1(U_k, M)$ is attained for $G : m_2(G) < u_{2k}$. This selection can be simplified if we put

$$\zeta_1(U_k, M(G)) \stackrel{\text{def}}{=} \frac{1}{\eta_1 - u_{1k}} = \frac{m_2 - u_{2k}}{u_{2k}(u_{1k} - m_1)}. \tag{2.5}$$

As $\tilde{M}$, we take the point $M(G)$ with $G \in \mathbf{G_k}$ where $\zeta_1(U_k, M(G))$ attains maximum. If there are several such points, then we take the closest to the point U_k.

Thus, the point $V_k = \tilde{M}$ is assigned to each pair of neighboring points U_{k-1}, U_k. If $\omega(k+1) = 0$ or $\omega(k+1) = 1$, then the point V_k coincides with the point U_{k+1}. If $\omega(k+1) > 1$, then the point V_k is different from the point U_{k+1} and lies inside the convex hull $\mathbf{M}$ (Fig. 6).

Theorem 1. $u_{1,k} < v_{1,k} < u_{1,k+1}$.

Analogous inequality for second coordinates, $v_{2,k} < u_{2,k}$, generally speaking, is not correct: often $v_{2,k} > u_{2,k}$.

Let the pre-image of the point V_k in $\pi\mathbb{Z}^3$ be G_k, i.e. $V_k = M(G_k)$.

Theorem 2. *If* $G_k \neq F_{k+1}$, *then* $|\det(F_k^* G_k^* F_{k+1}^*)| \leq 1$, $|\det(G_k^* F_{k+1}^* G_{k+1}^*)| \leq 1$.

Let $P(\lambda) = \lambda^3 + a\lambda^2 + b\lambda + c$ be a polynomial with integer coefficients and negative discriminant. It has tree roots $\lambda_1, \lambda_2, \lambda_3$, $\lambda_1 \in \mathbb{R}$, $\lambda_2 = \overline{\lambda}_3 \in \mathbb{C}$, where the bar at the top means complex conjugate. We will assume that the number λ_1 is irrational.

Theorem 3. *For the forms* (1.1), (1.2) *with vectors* (2.6), *the sequence* $\{F_k\}$ *is periodic, i.e. there exist natural* t *and unimodular matrix* T *such that*

$$F_{k+t} = TF_k, \;\; k \in \mathbb{Z}. \tag{2.7}$$

Using period, one can easily find the fundamental unity of the correspon-ding number field.

3. Algorithm of generalized continued fraction

Let there be a base

$$B_1, B_2, B_3, \tag{3.1}$$

which is ordered in such a way that $m_{11} < m_{12} < m_{13}$, where

$$M_i = M(B_i) = (m_{1i}, m_{2i}), \;\; i = 1, 2, 3. \tag{3.2}$$

We describe one transition to another ordered base B_1', B_2', B_3'. Consider all points of the type

$$G = a_1 B_1 + a_2 B_2 + a_3 B_3, \tag{3.3}$$

where

$$a_1 = -1, 0, +1, \;\; a_2, a_3 \in \mathbb{Z}, \tag{3.4}$$

and we choose among them the point for which

(i) $m_1(G) > m_{13}$,

(ii) $\zeta_1(M_3, M(G))$ has the greatest value over all points G.

If $a_1 \neq 0$ for the chosen point (3.3), then instead of B_1, we include into the base the vector G, and we obtain the new base $B_2, B_3, B_4 = G$. If $a_1 = 0$ in (3.3), then G is taken instead of B_2, and the new base now is $B_1, B_3, B_4 = G$.

Technically, it can be done as follows. Let the vector A in the base B_1, B_2, B_3 be $\Lambda = (\lambda_1, \lambda_2, \lambda_3)$, i.e.

$$A \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \Lambda, \tag{3.6}$$

where all vectors are lines. We compute integers

$$\begin{array}{lll} a_i = [\lambda_i/|\lambda_1|], & i = 1, 2, 3, & a_1 = \pm 1, \\ b_i = [\lambda_i/|\lambda_2|], & i = 2, 3, & b_2 = \pm 1, \\ c = [|b_3|/10] + 1, & & \end{array} \tag{3.5}$$

$$
\begin{array}{lllll}
X_0 = & a_1 B_1 & + a_2 B_2 & + a_3 B_3, & \\
X_k = & X_0 & & + k B_3, & -c \le k \le c, \\
Y_0 = & X_0 & + B_2, & & \\
Y_k = & Y_0 & & + k B_3, & -c \le k \le c, \\
Z_l = & & b_2 B_2 & + l B_3, & -|b_3| - 1 \le l \le |b_3| + 1.
\end{array}
$$

As points G, we take all points X_k, Y_k, Z_l. We select those from them for which the property (i) holds, and among them we find $\max \zeta_1(M_3, M(G))$.

If we choose one of the points X_k or Y_k, then instead of B_1, we take the selected point G and obtain the new base $B_2, B_3, B_4 = G = \tilde{a}_1 B_1 + \tilde{a}_2 B_2 + \tilde{a}_3 B_3$, i.e.

$$
\begin{pmatrix} B_1' \\ B_2' \\ B_3' \end{pmatrix} = N \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \tilde{a}_1 & \tilde{a}_2 & \tilde{a}_3 \end{pmatrix}.
$$

The matrix N^{*-1} gives the transformation

$$
\Lambda^{*\prime} = N^{*-1} \Lambda^*, \quad N^{*-1} = \begin{pmatrix} -\tilde{a}_1 \tilde{a}_2 & 1 & 0 \\ -\tilde{a}_1 \tilde{a}_3 & 0 & 1 \\ \tilde{a}_1 & 0 & 0 \end{pmatrix}.
$$

If one of the points Z_l is chosen, then instead of B_2, we take the chosen point $G = Z_l = \tilde{b}_2 B_2 + \tilde{b}_3 B_3$ and obtain the new base $B_1, B_3, B_4 = G$, i.e.

$$
\begin{pmatrix} B_1' \\ B_2' \\ B_3' \end{pmatrix} = N \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \tilde{b}_2 & \tilde{b}_3 \end{pmatrix}.
$$

Here

$$
\Lambda' = N^{*-1} \Lambda, \quad N^{*-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\tilde{b}_2 \tilde{b}_3 & 1 \\ 0 & \tilde{b}_2 & 0 \end{pmatrix}.
$$

This algorithm is programmed.

Example 1. Voronoi [9, part II] computed the period of expansion of the vector $(1, \rho, \rho^2)$, where $\rho^3 = 23$. The period consists of 21 transitions of his algorithm. Our algorithm has period of 16 transitions. The corresponding fundamental unity is inverse to the unity found by Voronoi, and it coincides with the unity found by Markov.

Example 2. We consider the equation $\lambda^3 + 22\lambda^2 + 11\lambda + 25 = 0$, $J \approx (1, 0.393582, 0.861731)$.

k	B_k			d	L	$l_1(B_k)$	$l_2(B_k)$	a_1	a_2	a_3
1	1	0	0		1	.1000e+1	.1000e+1			
2	0	1	0		25	.2154e+2	.1160e+1			
3	0	0	1	1	625	.4641e+3	.1347e+1	1	0	1
4	1	0	1	1	109	.4651e+3	.2344e+0	1	0	2
5	2	1	2	1	113	.9087e+3	.1244e+0	-1	2	3
6	8	3	7	-1	157	.3192e+4	.4918e-1	-1	0	2
7	15	6	13	1	85	.5919e+4	.1436e-1	-1	1	3
8	51	20	44	-1	235	.2004e+5	.1173e-1	-1	0	2
9	94	37	81	1	1	.3689e+5	.2711e-4	0	1	21
10	2025	797	1745	0	25	.7947e+6	.3146e-4	1	-9	22
11	43719	17207	37674	-1	109	.1716e+8	.6353e-5	0	-1	2
12	85413	33617	73603	0	113	.3352e+8	.3371e-5	1	1	3
13	300052	118095	258564	-1	157	.1178e+9	.1333e-5	-1	0	2
14	556385	218983	479454	1	85	.2184e+9	.3893e-6	-1	1	3
15	1883794	741427	1623323	-1	235	.7393e+9	.3178e-6			

Here

$$d = d_k = \det(B^*_{k-2} B^*_{k-1} B^*_k), \;\; k \geq 3.$$

The period of the algorithm is $t = 7$. Here the convex polygonal line $\partial \mathbf{M}$ contains the points $M(B_k)$ with numbers $k = 1, 4, 5, 6, 7, 9, 11, 12, 13, 14, 15$. Correspondingly, the points with numbers $k = 2, 3, 8, 10$ lie higher than the convex polygonal line. Absolute values of determinants $\det(B^*_i B^*_j B^*_k)$, where $M(B_i), M(B_j), M(B_k)$ are neighboring points of the convex polygonal line, are equal to $1, 1, 1, 2, 22, 1, 1, 1, 1$ respectively.

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