Asymptotics of entropy of orthogonal polynomials

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Program of the talk

- Definitions

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- Statement of problem. The first heuristic asymptotical result

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- L_p norm method. L_p Steklov problem.
- Logarithmic potential theory approach. Mutual energy.
- "Equilibrium" and "Non-equilibrium" parts of the asymptotics of orthogonal polynomials

Orthogonal polynomials and related probability measures

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- some discrete measures - zero counting measure, quadrature formulae measure, ...

Definition of entropies. Shannon discrete entropy

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- Examples :

$$\{q_j\}_{j=1}^n := \{\tfrac{1}{n}, \ldots, \tfrac{1}{n}\} \quad \Rightarrow S_n = \log(n) \,,$$

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- In general : $\qquad \log(n) \geq S_n \geq 0$.

- Boltzmann entropy for continues measure $d\mu = w(x)dx$:

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- More general. Boltzmann-Gibbs entropy for two measures :

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- Relative, mutual (Kullback) entropy for two measures :

$$K(d\mu, d\nu) := -\int \log(\frac{d\mu}{d\nu})\,d\mu\,.$$

Continues entropy of Orthogonal polynomials

- Given $\{p_n(x)\}_{n=0}^{\infty} \quad \perp \quad d\mu$. Consider :

$$dr_n(x) \; = \; p_n^2(x)\, d\mu(x)\,.$$

Motivation - probability density of quantum mechanical systems.

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- Remark on discrete entropy of orthogonal polynomials.

Statement of problem

- Consider Entropies of Orthogonal Polynomials :

$$B_n := B(dr_n(x), d\mu(x)) = -\int p_n^2(x)\log(p_n^2(x))\, d\mu(x),$$

or

$$\widetilde{B_n} := B(dr_n(x), dx) = B_n + \int \log(\frac{d\mu}{dx})p_n^2(x)\, d\mu(x).$$

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- **Problem :** Find limit

$$B_n \to ?\,, \quad n \to \infty\,.$$

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- Heuristic answer for *supp* $\mu = [-1, 1]$:

$$B_n \to B := \log 2 - 1 + K(d\lambda, d\mu),$$

where

$$d\lambda(x) := \frac{dx}{\pi\sqrt{1-x^2}}, \quad K(d\lambda, d\mu) := -\int \log(\frac{d\lambda}{d\mu})\, d\lambda.$$

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- Still conjecture ...

heuristic derivation of the asymptotics

$$B_n := -\int_{-1}^{1} \log(p_n^2(x))\, p_n^2(x) w(x) dx \qquad \rightarrow$$

$$B := \log 2 - 1 - \int_{-1}^{1} \log\left(\frac{1}{\pi\sqrt{1-x^2}\,w(x)}\right) \frac{dx}{\pi\sqrt{1-x^2}} \;?$$

- Plug in B_n asymptotics of p_n:

$$B_n \simeq -\int_{-1}^{1} \log\left(\frac{2\cos^2(n\theta+\gamma(\theta))}{\pi\sqrt{1-x^2}\,w(x)}\right) \frac{2\cos^2(n\theta+\gamma(\theta))dx}{\pi\sqrt{1-x^2}}\,.$$

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$$\int_{-1}^{1} \left[\frac{1}{\pi\sqrt{1-x^2}} \log\Big(\frac{1}{\pi\sqrt{1-x^2}\, w(x)}\Big)\right] 2\cos^2(n\theta) dx .$$

Useful Lemma

Lemma. Given

$$g \in C(\mathbb{R}), \quad g(\theta+\pi) = g(\theta),$$
$$f \in L_1([0,\pi]),$$

and $\gamma < \infty$ a.e. on $[0,\pi]$.

Then when $n \to \infty$

$$\int_0^\pi g(n\theta + \gamma(\theta))\, f(\theta)\, d\theta \quad \to \quad \frac{1}{\pi} \int_0^\pi g(\theta)\, d\theta \int_0^\pi f(\theta)\, d\theta\,.$$

L_p norm method

Connection L_{2q}-norm and entropy

- Given $\{p_n(x)\}_{n=0}^{\infty} \quad \perp \quad d\mu$. Consider :

$$N_n(q) \; = \; \int p_n^{2q}(x)\, d\mu(x)\,.$$

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- Thus, problem $N_n(q) \to ?$ as $n \to \infty$.

- Remark (L_p Steklov problem) :

$$N_n(1) = 1, \qquad\qquad N_n(\infty) - \text{Steklov Problem.}$$

L_p-norm method (cont.)

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- Asymptotics for Entropy Jacobi polynomials.

- Case of unbounded support of μ (L_p-norms):

L_p-norm method (cont.)

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- Freud weights : $w(x) := \exp(-|x|^\rho)$ on $\mathbb{R}$

$$B_n = -\frac{2n+1}{\rho} + \frac{1}{\rho}\ln 2n - \frac{1}{\rho}\ln\left(\frac{\sqrt{\pi}}{2}\right)\frac{\Gamma(\rho/2)}{\Gamma(\rho/2+1/2)} - 1 + \ln\pi + o(1).$$

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- Laguerre weights : $w(x) := x^\alpha \exp(-x)$ on $(0, \infty)$

$$B_n = -2n + (\alpha+1)\ln n - \alpha - 2 + \ln(2\pi) + o(1).$$

Logarithmic potential theory approach

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Logarithmic potential theory approach

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- A.Aptekarev, V.Buyarov, J.Dehesa and W.Van Assche *Asymptotics for entropy integrals of orthogonal polynomials*, Doklady Russian Ac.Sci.,1996, **346**(4),439-441

Logarithmic potential theory approach

Some notations:

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- where $\nu_n := \nu[Q_n], \quad r_n(x) := p_n^2(x)\, d\mu(x)$

Logarithmic potential theory approach (cont.)

$$B_n = nW(\nu_n, r_n) - \ln k_n \quad !$$

Weak asymptotics of B_n

- $$\frac{B_n}{n} \quad \longrightarrow \quad ?$$

Weak asymptotics of B_n

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Mutual Energy $W(\nu_n, r_n)$ and Potential V_{r_n}.

- $$W(\nu_n, r_n) = -\frac{1}{2}\int \ln\left(\prod_{j=1}^{n}(x - x_{j,n})\right)^2 p_n^2(x)dx = \frac{1}{n}\sum_{j=1}^{n} V_{r_n}(x_{j,n}).$$

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- (A. Martinez-Finkelshtein and J. Sanches-Lara - 2005): $(*)$ is not true for non-Szegö weights !

Asymptotic entropy in the Szegö class

- $$B_n \to B := \log 2 - 1 + K(d\lambda, d\mu),$$

  where

  $$d\lambda(x) := \frac{dx}{\pi\sqrt{1-x^2}}, \quad K(d\lambda, d\mu) := -\int \log(\frac{d\lambda}{d\mu})\, d\lambda.$$

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- B. Beckermann, A. Martinez-Finkelshtein, E.A. Rakhmanov and F. Wielonsky, *Asymptotic upper bounds for the entropy of orthogonal polynomials in the Szegö class*, Journal of Mathematical Physics, 2004, **45**(11), 4239 -4254.

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- **Theorem.** Measure of orthogonality μ is from the Szegö class. Then, for all $M > \sqrt{2}$

$$B_n \;=\; \log 2 - 1 \;+\; K(d\lambda, d\mu) \quad -$$

$$\int_{\Delta_n(M)} p_n^2(x)\log^+(p_n^2(x))\,d\mu(x) \;+\; o(1), \quad n \to \infty.$$

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