Varying weights of orthogonality for family of polynomials with varying recurrence coefficients

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Paper in JAT

A. I. Aptekarev, J. S. Geronimo, and W. Van Assche :

Varying weights for orthogonal polynomials with monotonically varying recurrence coefficients

Program of the talk

- Statement of problem (varying weight and rec. coef. and their limits)

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- Monotonic bounded rec.coef. - direct problem
- Some open directions

Statement of problem - varying weight

- We consider families of polynomials $N \in \mathbb{N}$

$$\{P_{n,N}(\lambda)\}_{n=0}^{\infty} \quad : \quad P_{n,N}(\lambda) := \prod_{j=1}^{n} (\lambda - \lambda_{j,n,N}),$$

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- Example : $W_N(\lambda) := e^{-2NQ(\lambda)}$

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- Three-term recurrence relation

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- Example : $a(x),\ b(x) \quad x \in \mathbb{R}^+ \quad h := \frac{1}{N}$

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- Relation between limits

$$\left\{\begin{matrix} a(x) \\ b(x) \end{matrix}\right\} \leftrightarrow \{Q(\lambda)\}.$$

Spectral theory point of view

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- EXAMPLE :

  P. Deift and K. T-R McLaughlin, *A Continuum Limit of the Toda Lattice*, Memoirs Amer. Math. Soc. **624**, Providence, RI, 1998.

Continuum Limit of the Toda Lattice

- Cauchy problem : $a(x,t)$, $b(x,t)$, where $(x,t) \in \mathbb{R}^+ \times \mathbb{R}^+$

$$\begin{cases} \dfrac{\partial a}{\partial t} = 2b\,\dfrac{\partial b}{\partial x}\,, \\ \dfrac{\partial b}{\partial t} = \dfrac{b}{2}\,\dfrac{\partial a}{\partial x}\,. \end{cases}$$

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- The Toda lattice $(T := Nt)$:

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- $\lim_{N\to\infty, k/N\to x}\{a_{k,N}(Nt),\, b_{k,N}(Nt)\} = \{a(x,t),\, b(x,t)\}$

Solution of the Toda equations

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- $Q(\lambda, t) = \lambda t + Q(\lambda, 0)$

Solution of C.P. of Cont. Toda PDE system

- Direct spectral problem

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Potentials with presence of external field

- External field (semi-continues from below) :

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- Energy, potential

$$J(\mu):=I(\mu)+2\int Q(\lambda)d\mu(\lambda),$$

where

$$I(\mu):=\int V^\mu(\lambda)d\mu(\lambda), \quad V^\mu(\lambda):=-\int\log|\lambda-z|\,d\mu(z).$$

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$$\nu_{n,N}(\lambda) = \frac{1}{N}\sum_{j=1}^{n}\delta(\lambda - \lambda_{j,n,N}), \qquad \nu_{n,N}(\lambda) \xrightarrow[\substack{n/N\to x\\ N\to\infty}]{*} \frac{1}{x}\tau_x(\lambda)$$

Ratio asymptotics

- Class of varying weights (external fields) + extra condition :

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where Φ_x- mapping funct. (inverse Zhukovskii) for $\bar{\mathbb{C}} \backslash \Delta_x$.

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- Limit of Recurrence coefficients $\exists a(x), b(x)$:

$$(*) \ \Rightarrow \ \exists\, a(x) = \frac{\alpha(x) + \beta(x)}{2}, \qquad b(x) = \frac{\beta(x) - \alpha(x)}{4}\,.$$

Existence of the rec.coef. limit

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- Strong asymptotics of Orthogonal Polynomials with Varying weight guaranties .... (Starting from end of 80-s)

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- $(\alpha = -1, \beta = +1 \Rightarrow a = 0, b = \frac{1}{2})$

Direct problem

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- *the local condition*

$$\int_0^{\epsilon} |\log b(x)|\, dx < \infty, \quad \epsilon > 0.$$

Statement of the result

Theorem

If the varying recurrence coefficients $a_{n,N}$ and $b_{n,N}$ are in $\mathcal{M}$, then uniformly on compact subsets of $(-1,1) \setminus \{a(0)\}$

$$\lim_{N\to\infty} \frac{1}{2N} \log W_N(\lambda) = -\int_0^{\infty} g_{\Delta_x}(\lambda)\,dx =: -Q(\lambda),$$

where

$$g_{\Delta_x}(\lambda) = \log \left| \frac{\lambda - a(x) + \sqrt{(\lambda - a(x))^2 - 4b(x)^2}}{2b(x)} \right|$$

is the Green's function for the interval $\Delta_x = [\alpha(x), \beta(x)]$.

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- Measure of orthogonality $d\mu_N(\lambda) = W_N(\lambda)\, d\lambda$ and for each fixed N:

$$\frac{2}{\pi}\frac{\sqrt{1-\lambda^2}}{W_N(\lambda)} = |\xi_N(\lambda)|^2 \prod_{j=1}^{\infty} |\Phi_{(\frac{j}{N})}(\lambda)|^2, \quad \lambda \in (-1,1)$$

here Φ_x is the mapping function of $\overline{\mathbb{C}}\setminus[\alpha(x),\beta(x)]$, and

$$\xi_N(\lambda) = 1 + \sum_{k=1}^{\infty}\left\{\frac{1}{\Phi_{(\frac{k}{N})}(\lambda)} - \frac{b_{k,N}/b_{k+1,N}}{\Phi_{(\frac{k+1}{N})}(\lambda)}\right\}\frac{p_{k-1,N}(\lambda)}{\prod_{j=1}^{k}\Phi_{(\frac{j}{N})}(\lambda)},$$

About proof - Limit $N \longrightarrow \infty$

- Our goal is $\lim_{N\to\infty} W_N(\lambda)^{1/N} = W(\lambda) = e^{-2Q(\lambda)}$, where

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- Difficult part :

$$|\log|\xi_N|| \le C_K \sqrt{N}.$$

Example

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$$p_n^{(\alpha_N,\beta_N)} \quad \alpha_N := AN, \beta_N := BN.$$

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$$\exp[-2Q(\lambda)] = e^{-2Q(0)}(1-\lambda)^A(1+\lambda)^B,$$

Classical Orthogonality with Varying weight

Direct problems. Three terms rec. coef. come from :

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Find a model which provides spectral data for $\beta(x), \alpha(x)$ with several local extremum.

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