Ovchinnikov M.Yu., Roldugin D.S.

Active magnetic attitude control system of a satellite equipped with a flywheel

Recommended form of bibliographic references: Ovchinnikov M.Yu., Roldugin D.S. Active magnetic attitude control system of a satellite equipped with a flywheel. Keldysh Institute preprints, 2011, No. 21, 28 p. URL: **http://library.keldysh.ru/preprint.asp?id=2011-21&lg=e**

KELDYSH INSTITUTE OF APPLIED MATHEMATICS
RUSSIAN ACADEMY OF SCIENCES

M.Yu. Ovchinnikov, D.S. Roldugin

Active magnetic attitude control system of a satellite equipped with a flywheel

Moscow, 2011

Магнитная система ориентации спутника, оснащенного тангажным маховиком. М.Ю. Овчинников, Д.С. Ролдугин. ИПМ им. М.В.Келдыша РАН, Москва, 2011г., 28 с., библиография: 13 наименований, 10 рисунков, 1 таблица

Рассматривается спутник, оснащенный магнитной системой ориентации и тангажным маховиком. Исследуется быстродействие системы в переходном режиме в зависимости от наклонения орбиты спутника. Проводится сравнение с системой, включающей только магнитные катушки. В установившемся режиме гравитационной ориентации рассматриваются малые движения в окрестности положения равновесия. Исследуется точность ориентации и быстродействие алгоритма демпфирования. Исследуется алгоритм произвольной, но заданной ориентации спутника в плоскости орбиты. Проводится численное моделирование.

Ключевые слова: магнитная система ориентации, тангажный маховик, осредненная модель магнитного поля Земли, быстродействие системы ориентации

Active magnetic attitude control system of a satellite equipped with a flywheel. M.Yu. Ovchinnikov, D.S. Roldugin. Keldysh Institute of Applied Mathematics of Russian Academy of Sciences, 2011, 28 p., 13 items of bibliography, 10 figures, 1 table

Attitude motion of a satellite equipped with a single flywheel and active magnetic attitude control system is considered. Time-response of the system in a transient mode with respect to the orbit inclination is studied. Comparison with the system consisting of magnetic coils only is conducted. Small steady-state motion near the gravitational orientation is considered. Accuracy and time-response is studied. An algorithm of arbitrary but given orientation in the orbital plane is proposed and studied. Numerical analysis is carried out.

Key words: active magnetic control, flywheel, averaged geomagnetic field model, time-response

Introduction

Active magnetic attitude control systems (MACS) are widely used for satellite orientation and stabilization. MACS are especially attractive for small satellites. There is no consumption of any limited resources, the system has low mass, dimensions, cost and energy requirements. Other common actuators like reaction wheels or propulsion system cannot match these requirements. MACS however have less accuracy. Another fundamental problem of MACS is impossibility to provide arbitrary control torque. The torque lies in a plane perpendicular to the vector of geomagnetic induction. So MACS are often used for the initial angular velocity damping or with other actuators.

Spin stabilization was one of the first methods used in satellite orientation since the beginning of the space era. A satellite spinned fast about its axis of symmetry acquires the properties of a gyroscope and maintains its axis attitude in the inertial space for a considerable amount of time. MACS are often used on spin-stabilized satellites. A satellite equipped with a flywheel is considered in this paper. Such a wheel can have big angular velocity and angular momentum which are considered constant during the functioning. This allows for the same principle to be used without spinning the whole satellite. In this case the axis of a flywheel maintains its attitude in the inertial space. If the satellite is subjected to the gravity-gradient torque the attitude when the axis of a flywheel is perpendicular to the orbital plane is stable. Stability problem of a satellite equipped with a flywheel is studied, for example, in [1] and [2]. If the whole system is dissipative the satellite tends to orientate so the axis of a flywheel is perpendicular to the orbital plane. Passive damping devices may be used [3]. But the problem of satellite attitude in the orbital plane arises. Gravity-gradient torque may be used again. Principal axes of inertia of a satellite coincide with the radius-vector of a satellite and normal to it in the orbital plane. However, gravity-gradient torque may be of the same or even smaller order as disturbing torques if principal moments of inertia are commensurable. Besides, arbitrary orientation of some axis in the orbital plane may be required. For example, if one conducts an orbital maneuvering, it is necessary to have particular orientation of the axis of thruster for the optimal delta vee. To provide arbitrary orientation in the orbital plane, initial angular velocity damping and flywheel axis attitude MACS can be used.

Attitude control system with magnetorquers and a flywheel is thoroughly covered in the literature. The main focus in most papers is placed on the numerical analysis and flight results. Small satellite GURWIN [4] launched in 1998 reached the accuracy of 2°. In the paper [5] MACS algorithm and flight results of REIMEI small satellite, launched in 2004, are presented. Using MACS and a flywheel this satellite was stabilized with accuracy of 0.2°. In [6] equations of motion are analytically solved using asymptotical methods. This work, however, only deals with the small oscillations in the vicinity of the gravitational orientation. It is shown that the time-response of chosen algorithm increases when the orbit inclination rises[1].

In present paper transient motion is studied along with the steady-state motion. In the transient mode MACS provide damping of initial angular velocity. Flywheel is considered spinning already. Note that initial angular velocity may be the result of a flywheel spinup as well as the result of the separation from the launch vehicle. A well-known "*-Bdot*" algorithm is used for the damping. The time-responses of systems with and without a flywheel are compared. In the steady-state motion the accuracy of the gravitational orientation is studied. An algorithm of arbitrary orientation in the orbital plane is proposed and studied.

1. Problem statement

Let's describe the geomagnetic field model, reference frames and equations of motion used in this work.

1.1. Reference frames

Here the frames used in this paper are presented.

$O_aZ_1Z_2Z_3$ is the inertial reference frame, O_a is the Earth center, O_aZ_3 axis is normal to the orbital plane, O_aZ_1 lies in the equatorial plane and is directed to the ascending node, O_aZ_2 is directed such that the reference frame is right-handed.

$OL_1L_2L_3$ is the frame associated with the angular momentum of the satellite. O is the satellite's center of mass, OL_3 axis is directed along the angular momentum, OL_2 axis is perpendicular to OL_3 and lies in the plane parallel to the $O_aZ_1Z_2$ plane and containing O, OL_1 is directed such that the reference frame is right-handed.

$OX_1X_2X_3$ is the orbital reference frame, OX_1 lies in the orbital plane, is perpendicular to the radius vector and directed as the orbital velocity does, OX_3 axis is directed along the radius-vector of a satellite, OX_2 is directed such that the reference frame is right-handed.

$Ox_1x_2x_3$ is the bound frame, its axes are directed along the principal axes of inertia of the satellite.

---

[1] By inclination rise we mean the change in inclination from zero to 90°.

Reference frames mutual orientation is described using direct cosine matrices $\mathbf{Q},\mathbf{A},\mathbf{D}$ expressed in tables

$$\begin{array}{cccc} & L_1 & L_2 & L_3 \\ Z_1 & q_{11} & q_{12} & q_{13} \\ Z_2 & q_{21} & q_{22} & q_{23} \\ Z_3 & q_{31} & q_{32} & q_{33} \end{array} \qquad \begin{array}{cccc} & x_1 & x_2 & x_3 \\ L_1 & a_{11} & a_{12} & a_{13} \\ L_2 & a_{21} & a_{22} & a_{23} \\ L_3 & a_{31} & a_{32} & a_{33} \end{array} \qquad \begin{array}{cccc} & x_1 & x_2 & x_3 \\ X_1 & d_{11} & d_{12} & d_{13} \\ X_2 & d_{21} & d_{22} & d_{23} \\ X_3 & d_{31} & d_{32} & d_{33} \end{array}.$$

We introduce indices Z,L,X,x to denote any vector components in frames $O_aZ_1Z_2Z_3$, $OL_1L_2L_3$, $OX_1X_2X_3$ and $Ox_1x_2x_3$ respectively. For example, for the first component of a torque in these frames we write $M_{1Z},M_{1L},M_{1X},M_{1x}$.

1.2. Equations of motion

We will use Beletsky-Chernousko variables and Euler angles to describe satellite dynamics. Beletsky-Chernousko variables is a set $L,\rho,\sigma,\varphi,\psi,\theta$ [7], where L is the angular momentum magnitude, angles ρ,σ introduce its orientation with respect to the $O_aZ_1Z_2Z_3$ frame (Fig.1). Orientation of the $Ox_1x_2x_3$ frame with respect to the $OL_1L_2L_3$ is described using Euler angles φ,ψ,θ. Direct cosine matrix $\mathbf{Q}$ takes form

$$\mathbf{Q}=\begin{pmatrix} \cos\rho\cos\sigma & -\sin\sigma & \sin\rho\cos\sigma \\ \cos\rho\sin\sigma & \cos\sigma & \sin\rho\sin\sigma \\ -\sin\rho & 0 & \cos\rho \end{pmatrix}. \qquad (0.1)$$

Direct cosine matrix $\mathbf{A}$ takes form

$$\mathbf{A}=\begin{pmatrix} \cos\varphi\cos\psi-\cos\theta\sin\varphi\sin\psi & -\sin\varphi\cos\psi-\cos\theta\cos\varphi\sin\psi & \sin\theta\sin\psi \\ \cos\varphi\sin\psi+\cos\theta\sin\varphi\cos\psi & -\sin\varphi\sin\psi+\cos\theta\cos\varphi\cos\psi & -\sin\theta\cos\psi \\ \sin\theta\sin\varphi & \sin\theta\cos\varphi & \cos\theta \end{pmatrix}. \qquad (0.2)$$

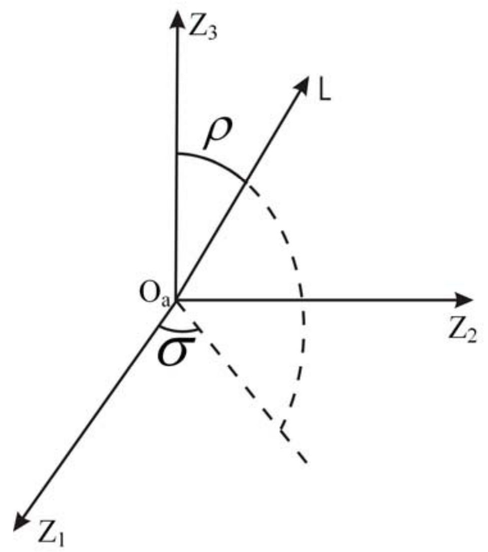

Fig. 1. Angular momentum orientation in the inertial space

Let $\mathbf{J}_x = diag(A,B,C)$ be the tensor of inertia of the satellite. Its center of mass moves on the circular keplerian orbit. The motion of a satellite not equipped with a flywheel is described with equations [7]

$$\frac{dL}{dt} = M_{3L},\ \frac{d\rho}{dt} = \frac{1}{L}M_{1L},\ \frac{d\sigma}{dt} = \frac{1}{L\sin\rho}M_{2L},$$

$$\begin{aligned}
\frac{d\theta}{dt} &= L\sin\theta\sin\varphi\cos\varphi\left(\frac{1}{A}-\frac{1}{B}\right)+\frac{1}{L}\left(M_{2L}\cos\psi - M_{1L}\sin\psi\right),\\
\frac{d\varphi}{dt} &= L\cos\theta\left(\frac{1}{C}-\frac{\sin^2\varphi}{A}-\frac{\cos^2\varphi}{B}\right)+\frac{1}{L\sin\theta}\left(M_{1L}\cos\psi + M_{2L}\sin\psi\right),\\
\frac{d\psi}{dt} &= L\left(\frac{\sin^2\varphi}{A}+\frac{\cos^2\varphi}{B}\right)-\frac{1}{L}M_{1L}\cos\psi\,\mathrm{ctg}\,\theta-\frac{1}{L}M_{2L}\left(\mathrm{ctg}\,\rho+\sin\psi\,\mathrm{ctg}\,\theta\right)
\end{aligned} \tag{0.3}$$

where M_{1L}, M_{2L}, M_{3L} are torque components in the $OL_1L_2L_3$ frame.

If a satellite is equipped with a flywheel (let it be directed along the Ox_2 axis), equations (0.3) are slightly different. We now find the analog of (0.3) for a satellite with a flywheel. Three first equations in (0.3) don't change. But the angular momentum is now the sum of the angular momentum of a satellite and of a flywheel. Consider equations for φ, ψ, θ. Satellite angular velocity components may be expressed as

$$\omega_1 = \frac{La_{31}}{A},\ \omega_2 = \frac{La_{32}-h}{B},\ \omega_3 = \frac{La_{33}}{C} \tag{0.4}$$

where h is the angular momentum magnitude of a flywheel directed along the Ox_2 axis. These components may be expressed [7] as

$$\begin{aligned}
\omega_1 &= \frac{d\theta}{dt}\cos\varphi + \frac{d\psi}{dt}a_{31} + \frac{d\rho}{dt}a_{21} + \frac{d\sigma}{dt}\left(a_{11}q_{21}+a_{31}q_{23}\right),\\
\omega_2 &= -\frac{d\theta}{dt}\sin\varphi + \frac{d\psi}{dt}a_{32} + \frac{d\rho}{dt}a_{22} + \frac{d\sigma}{dt}\left(a_{12}q_{21}+a_{32}q_{23}\right),\\
\omega_3 &= \frac{d\varphi}{dt} + \frac{d\psi}{dt}a_{33} + \frac{d\rho}{dt}a_{23} + \frac{d\sigma}{dt}\left(a_{13}q_{21}+a_{33}q_{23}\right).
\end{aligned} \tag{0.5}$$

These equalities can be easily derived from kinematics. We now multiply the first equation in (0.5) by $\cos\varphi$, the second by $-\sin\varphi$ and adding taking into account (0.4) . All transformations in the right sides of (0.5) are omitted since they reproduce those in [7]. The result is

$$\frac{d\theta}{dt} = L\sin\theta\sin\varphi\cos\varphi\left(\frac{1}{A}-\frac{1}{B}\right)+\frac{1}{L}\left(M_{2L}\cos\psi - M_{1L}\sin\psi\right)+\frac{h}{B}\sin\varphi.$$

We multiply the first equation in (0.5) by $\sin\varphi$, second by $\cos\varphi$ and adding. This leads to

$$\frac{d\psi}{dt}=L\left(\frac{\sin^2\varphi}{A}+\frac{\cos^2\varphi}{B}\right)-\frac{1}{L}M_{1L}\cos\psi\,\mathrm{ctg}\,\theta-\frac{1}{L}M_{2L}\left(\mathrm{ctg}\,\rho+\sin\psi\,\mathrm{ctg}\,\theta\right)-\frac{h}{B}\frac{\cos\varphi}{\sin\theta}.$$

Substituting these two relations to (0.5) we get

$$\frac{d\varphi}{dt}=L\cos\theta\left(\frac{1}{C}-\frac{\sin^2\varphi}{A}-\frac{\cos^2\varphi}{B}\right)+\frac{1}{L\sin\theta}\left(M_{1L}\cos\psi+M_{2L}\sin\psi\right)+\frac{h}{B}\frac{\cos\varphi\cos\theta}{\sin\theta}.$$

This allows us to write the equations of motion of a satellite equipped with a flywheel directed along the Ox_2 axis

$$\frac{dL}{dt}=M_{3L},\ \frac{d\rho}{dt}=\frac{1}{L}M_{1L},\ \frac{d\sigma}{dt}=\frac{1}{L\sin\rho}M_{2L},$$

$$\frac{d\theta}{dt}=L\sin\theta\sin\varphi\cos\varphi\left(\frac{1}{A}-\frac{1}{B}\right)+\frac{1}{L}\left(M_{2L}\cos\psi-M_{1L}\sin\psi\right)+\frac{h}{B}\sin\varphi, \tag{0.6}$$

$$\frac{d\varphi}{dt}=L\cos\theta\left(\frac{1}{C}-\frac{\sin^2\varphi}{A}-\frac{\cos^2\varphi}{B}\right)+\frac{1}{L\sin\theta}\left(M_{1L}\cos\psi+M_{2L}\sin\psi\right)+\frac{h}{B}\frac{\cos\varphi\cos\theta}{\sin\theta},$$

$$\frac{d\psi}{dt}=L\left(\frac{\sin^2\varphi}{A}+\frac{\cos^2\varphi}{B}\right)-\frac{1}{L}M_{1L}\cos\psi\,\mathrm{ctg}\,\theta-\frac{1}{L}M_{2L}\left(\mathrm{ctg}\,\rho+\sin\psi\,\mathrm{ctg}\,\theta\right)-\frac{h}{B}\frac{\cos\varphi}{\sin\theta}$$

where the angular momentum is the sum of those of a satellite and of a flywheel.

Beletsky-Chernousko variables are convenient for the analysis of transient motion. The angular velocity magnitude is described with the magnitude of angular momentum, which is only one variable. Steady-state motion is better studied using Euler angles. In this case variables $\omega_1,\omega_2,\omega_3,\alpha,\beta,\gamma$ are used. Here ω_i are the components of satellite angular velocity in the $Ox_1x_2x_3$ frame (i=1,2,3), Euler angles α,β,γ describe the orientation of the $Ox_1x_2x_3$ frame with respect to the $OX_1X_2X_3$. Direct cosine matrix $\mathbf{D}$ takes form

$$\mathbf{D}=\begin{pmatrix}\cos\alpha\cos\beta+\sin\alpha\sin\beta\sin\gamma & -\cos\alpha\sin\beta+\sin\alpha\cos\beta\sin\gamma & \sin\alpha\cos\gamma\\ \sin\beta\cos\gamma & \cos\beta\cos\gamma & -\sin\gamma\\ -\sin\alpha\cos\beta+\cos\alpha\sin\beta\sin\gamma & \sin\alpha\sin\beta+\cos\alpha\cos\beta\sin\gamma & \cos\alpha\cos\gamma\end{pmatrix}. \tag{0.7}$$

Equations of motions of the satellite subjected to the gravity-gradient torque and torque produced by the interaction between MACS and geomagnetic field are

$$A\frac{d\omega_1}{dt}=h\omega_3+(B-C)\left(\omega_2\omega_3-3\omega_0^2 d_{32}d_{33}\right)+M_{1x},$$

$$B\frac{d\omega_2}{dt}=-(A-C)\left(\omega_1\omega_3-3\omega_0^2 d_{31}d_{33}\right)+M_{2x},$$

$$
\begin{aligned}
&C\frac{d\omega_3}{dt} = -h\omega_1 - (B-A)\left(\omega_1\omega_2 - 3\omega_0^2 d_{31}d_{32}\right) + M_{3x},\\
&\frac{d\alpha}{dt} = \frac{1}{\cos\gamma}(\omega_1\sin\beta + \omega_2\cos\beta) - \omega_0,\\
&\frac{d\beta}{dt} = \omega_3 + \mathrm{tg}\,\gamma(\omega_1\sin\beta + \omega_2\cos\beta),\\
&\frac{d\gamma}{dt} = \omega_1\cos\beta - \omega_2\sin\beta
\end{aligned} \tag{0.8}
$$

where M_{1x}, M_{2x}, M_{3x} are torque components in the $Ox_1x_2x_3$ frame, ω_0 is the orbital velocity.

1.3. Averaged geomagnetic field model

Let's describe the geomagnetic field model used in this work. Geomagnetic induction vector is often modeled using Gauss decomposition [8]. It is impossible to use this model for an analytical study, so a number of simplifications are introduced. We describe three most common models. If only three first terms in the decomposition are taken into account one has the inclined dipole model. The geomagnetic field is one of the dipole inclined at the angle 168°26' to the Earth axis. This model allows rather compact analytical expression but still too complicated to be used in an analytical investigation. Further simplification, the direct dipole model, is widely used. In this case the field is one of the dipole directed along the Earth axis and antiparallel to it. Geomagnetic induction vector moves almost uniformly on the almost circular cone as a satellite moves along the orbit. This model however doesn't allow to obtain the solution of equations of motion in the explicit form in the cases considered in this work. We introduce further simplification, modeling the geomagnetic induction vector moving uniformly on the circular cone. To do so, we need inertial reference frame $O_aY_1Y_2Y_3$. Here O_a is the Earth center, O_aY_3 axis is directed along the Erath axis, O_aY_1 axis lies in the equatorial plane and is directed to the ascending node, O_aY_2 is directed so the whole frame is right-handed. If we now translate the geomagnetic induction vector to the O_a point, the cone is tangent to the O_aY_3, its axis lies in the plane $O_aY_2Y_3$ (Fig. 2). Cone half-opening angle satisfies [9] the relation

$$
\mathrm{tg}\,\Theta = \frac{3\sin 2i}{2\left(1 - 3\sin^2 i + \sqrt{1+3\sin^2 i}\right)} \tag{0.9}
$$

where i is the orbit inclination. Geomagnetic induction vector moves uniformly on the cone with the doubled orbital velocity, $\chi = 2u + \chi_0$ where $u = \omega_0 t$ is argument of latitude. Without loss of generality we assume $\chi_0 = 0$.

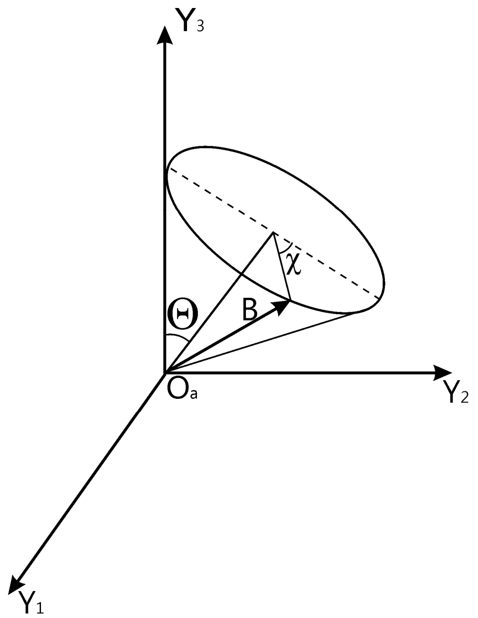

Fig. 2. Averaged geomagnetic field model

This model, sometimes called averaged, is used in the present work. This model doesn't allow to take into account non-uniformity of the induction vector movement (as right dipole model does) and its diurnal change (as inclined dipole model does). It however may be considered as a good trade-off between authenticity and simplicity of geomagnetic field modeling. Comprehensive comparison of models may be found in [8].

We need to know the geomagnetic induction vector in the frames $O_aZ_1Z_2Z_3$ and $OX_1X_2X_3$. In the frame $OX_1X_2X_3$ it has form

$$\mathbf{B}_X = B_0 \begin{pmatrix} \sin i \cos u \\ \cos i \\ -\sin i \sin u \end{pmatrix} \qquad (0.10)$$

where B_0 is its magnitude. To find it in the $O_aZ_1Z_2Z_3$ frame we introduce auxiliary frame $O_aS_1S_2S_3$. It is the frame $O_aY_1Y_2Y_3$ turned by angle Θ about O_aY_1 axis. Geomagnetic induction vector in this frame is

$$\mathbf{B} = B_0 \begin{pmatrix} \sin\Theta \sin 2u \\ \sin\Theta \cos 2u \\ \cos\Theta \end{pmatrix}.$$

Note that the $O_aZ_1Z_2Z_3$ frame is the frame $O_aY_1Y_2Y_3$ turned by angle i about O_aY_1 axis or the frame $O_aS_1S_2S_3$ turned by angle $\Theta - i$. So the magnetic induction vector in the $O_aZ_1Z_2Z_3$ frame may be expressed as

$$\mathbf{B}_Z = B_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\delta & \sin\delta \\ 0 & -\sin\delta & \cos\delta \end{pmatrix} \begin{pmatrix} \sin\Theta \sin 2u \\ \sin\Theta \cos 2u \\ \cos\Theta \end{pmatrix} = \begin{pmatrix} \sin\Theta \sin 2u \\ \sin\Theta \cos\delta \cos 2u + \sin\delta \cos\Theta \\ -\sin\Theta \sin\delta \cos 2u + \cos\delta \cos\Theta \end{pmatrix} \qquad (0.11)$$

where $\delta = \Theta - i$. Lets exploit the expression (0.9). Figures 3 and 4 introduce the relationships between Θ and $\Theta - i$ and the orbit inclination.

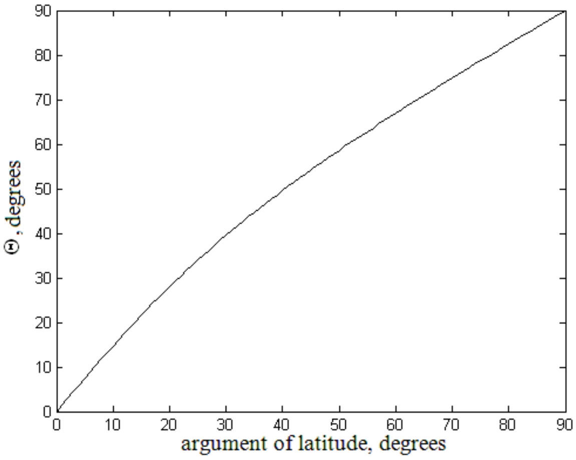

Fig. 3. Angle Θ

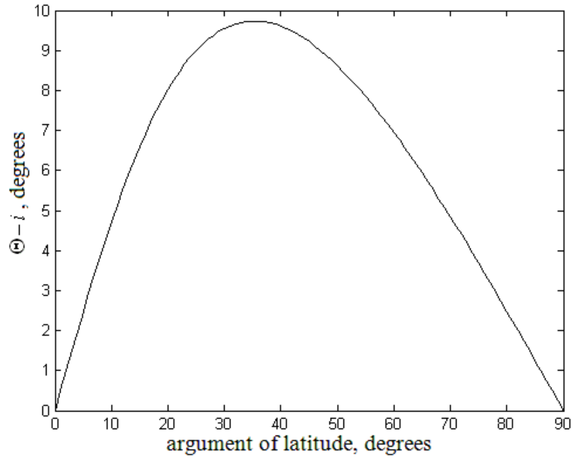

Fig. 4. The difference $\Theta - i$

It is seen that $\delta = \Theta - i$ doesn't exceed 10°. Then (0.11) may be expressed as

$$\mathbf{B}_Z \approx B_0 \begin{pmatrix} \sin\Theta \sin 2u \\ \sin\Theta \cos 2u \\ \cos\Theta \end{pmatrix} + B_0 \delta \begin{pmatrix} 0 \\ \cos\Theta \\ -\sin\Theta \cos 2u \end{pmatrix}. \quad (0.12)$$

Note that the second term may be neglected if the orbit is not close to the equatorial or polar because $\cos\Theta$ and $\sin\Theta$ are commensurable and δ is small ($\delta = o(1)$). Consider near-equatorial orbit. In this case $\sin\Theta \approx \Theta \approx i$. Again δ is less than i by magnitude and the second term is less than the first by magnitude too. Near-polar orbit case is considered in the same way. So we can neglect the second term in (0.12) in comparison with the first for any orbit inclination.

2. Analytical study

Let's consider transient and steady-state motions of a satellite.

2.1. Asymptotical methods. Averaging technique

We now describe the general technique for analyzing systems with slow and fast variables. Consider equations

$$\frac{dx}{dt} = \varepsilon X(x, y), \qquad \frac{dy}{dt} = y_0(x, y) + \varepsilon Y(x, y). \quad (0.13)$$

Parameter ε is supposed to be small. All variables are divided into fast y and slow x. Function X is 2π-periodic in y. We try to find the change of variables

$$x = \overline{x} + \varepsilon u_1(\overline{x}, \overline{y}), y = \overline{y} + \varepsilon v_1(\overline{x}, \overline{y}) \quad (0.14)$$

leading to the equations

$$\frac{d\bar{x}}{dt} = \varepsilon A_1\left(\bar{x}\right), \frac{d\bar{y}}{dt} = y_0\left(\bar{x}, \bar{y}\right) + \varepsilon B_1\left(\bar{x}\right). \qquad (0.15)$$

This is the common way of asymptotical methods for systems of the form (0.13) [10]. Change (0.14) contains variables of higher order of smallness in general case. But the reasoning became rather complicated if y is indeed a vector. Functions u_1, v_1 are finite. Substituting (0.14) to (0.13) and taking into account (0.15) we get the equation for u_1

$$\sum_{i=1}^{k} \frac{\partial u_1}{\partial y_i} y_{0i}\left(\bar{x}\right) = X\left(\bar{x}, \bar{y}\right) - A_1\left(\bar{x}\right) \qquad (0.16)$$

where k is a number of slow variables. Since X is periodic in y it can be decomposed to the Fourier series

$$X\left(\bar{x}, \bar{y}\right) = \sum a_{n_1,\ldots,n_k}\left(\bar{x}\right) \exp i\left(\sum n_j \bar{y}_j\right).$$

We try to find u_1 in a form

$$u_1\left(\bar{x}, \bar{y}\right) = \sum b_{n_1,\ldots,n_k}\left(\bar{x}\right) \exp i\left(\sum n_j \bar{y}_j\right) + \sum c_j\left(\bar{x}\right) \bar{y}_j.$$

Taking into account (0.16) we get

$$b_{n_1,\ldots,n_k}\left(\bar{x}\right) = \frac{a_{n_1,\ldots,n_k}\left(\bar{x}\right)}{i\sum n_j \bar{y}_j},$$

$$\sum c_j\left(\bar{x}\right) y_{0j}\left(\bar{x}\right) = a_{0,\ldots,0}\left(\bar{x}\right) - A_1\left(\bar{x}\right).$$

Function u_1 is finite so all c_j are equal to zero. In opposite case u_1 may rise to infinite value as $\bar{y}$ rise. This leads to

$$a_{0,\ldots,0}\left(\bar{x}\right) = A_1\left(\bar{x}\right),$$

but $a_{0,\ldots,0}\left(\bar{x}\right)$ is the mean value of X. So

$$A_1\left(\bar{x}\right) = \frac{1}{(2\pi)^k} \underbrace{\int \ldots \int}_{k} X\left(\bar{x}, \bar{y}\right) dy_1 \ldots dy_k = \overline{X}\left(\bar{x}\right),$$

That means that A_1 is the X averaged by all fast variables y. Taking into account only the first order of smallness we obtain equations for slow variables evolution

$$x = \bar{x}, \frac{d\bar{x}}{dt} = \varepsilon \overline{X}\left(\bar{x}\right).$$

For the time interval $t \sim {}^{1}/_{\varepsilon}$ the accuracy of slow variables determination is $\left|x-\bar{x}\right| \sim \varepsilon$. Equation for slow variables evolution are obtained by averaging initial equations by all fast variables.

2.2. Transient motion

Let's consider transient motion of a satellite. Control torque generated by the MACS is

$$\mathbf{M} = \mathbf{m} \times \mathbf{B}$$

where $\mathbf{m}$ is the magnetic dipole moment of a satellite. Control utilizes the "-*Bdot*" algorithm. The magnetic dipole moment is [11]

$$\mathbf{m}_x = -k \frac{d\mathbf{B}_x}{dt}$$

where k is the positive constant. The geomagnetic induction vector derivative in the $Ox_1x_2x_3$ frame is determined by its derivative in the $O_aZ_1Z_2Z_3$ frame by the expression

$$\frac{d\mathbf{B}_x}{dt} = \mathbf{A}^T\mathbf{Q}^T \frac{d\mathbf{B}_Z}{dt} - \boldsymbol{\omega}_x \times \mathbf{B}_x . \qquad (0.17)$$

We consider fast rotations of a satellite. In this case the first term in (0.17) describing the change of $\mathbf{B}$ in the inertial space may be neglected in comparison with the second. Such a case may occur on the first stage of satellite functioning. Satellites often have big angular velocity after the separation from the launch vehicle. Note that the angular momentum of a flywheel may be still higher than angular momentum of a satellite itself by at least a magnitude. We consider the control torque

$$\mathbf{M}_x = k\left(\boldsymbol{\omega}_x \times \mathbf{B}_x\right) \times \mathbf{B}_x . \qquad (0.18)$$

We rewrite equations (0.6) introducing argument of latitude u instead of time, dimensionless angular momentum magnitude l by the relation $L = L_0 l$ where L_0 is the initial magnitude and dimensionless torque $\overline{\mathbf{M}}$,

$$\frac{dl}{du} = \varepsilon l \overline{M}_{3L}, \quad \frac{d\rho}{du} = \varepsilon \overline{M}_{1L}, \quad \frac{d\sigma}{du} = \frac{\varepsilon}{\sin\rho} \overline{M}_{2L},$$

$$\begin{aligned}
\frac{d\varphi}{du} &= \eta_1 l \cos\theta + \frac{\varepsilon}{\sin\theta}\left(\overline{M}_{1L}\cos\psi + \overline{M}_{2L}\sin\psi\right) + \mu\frac{\cos\varphi\cos\theta}{\sin\theta}, \\
\frac{d\psi}{du} &= \eta_2 l - \varepsilon\overline{M}_{1L}\cos\psi\,\mathrm{ctg}\,\theta - \varepsilon\overline{M}_{2L}\left(\mathrm{ctg}\,\rho + \sin\psi\,\mathrm{ctg}\,\theta\right) - \mu\frac{\cos\varphi}{\sin\theta}, \qquad (0.19) \\
\frac{d\theta}{du} &= l\eta_3 \sin\theta\sin\varphi\cos\varphi + \varepsilon\left(\overline{M}_{2L}\cos\psi - \overline{M}_{1L}\sin\psi\right) + \mu\sin\varphi
\end{aligned}$$

where $\varepsilon = \frac{kB_0^2}{L_0}$, $\eta_1 = \frac{L_0}{\omega_0}\left(\frac{1}{C} - \frac{\sin^2\varphi}{A} - \frac{\cos^2\varphi}{B}\right)$, $\eta_2 = \frac{L_0}{\omega_0}\left(\frac{\sin^2\varphi}{A} + \frac{\cos^2\varphi}{B}\right)$, $\eta_3 = \frac{L_0}{\omega_0}\left(\frac{1}{A} - \frac{1}{B}\right)$, $\mu = \frac{h}{B\omega_0}$.

We assume that the control torque is small in a sense of small change of angular momentum for one revolution on the orbit in comparison with its value. It means that ε is small. Variables l,ρ,σ are obviously slow, variable u is fast. We need to know which of the variables φ,ψ,θ are slow and which are fast. Consider the angular momentum of a flywheel to be prevailing in comparison with the angular momentum of a satellite itself, without a flywheel. This assumption is not valid if a flywheel is used on a satellite with a mass by a magnitude greater than the design of a flywheel suggests or the angular velocity of a satellite exceeds several tens degrees per second (for a small satellite). If the angular momentum of a flywheel is big, the angular momentum of a system satellite-flywheel is directed almost along the Ox_2 axis. In this case angle φ us close to zero, angle θ is close to 90°. Variables φ,θ are slow, variable ψ is fast. We now can obtain the equations for slow variables. We need first to find the torque in the $OL_1L_2L_3$ frame. Angular momentum in the frame $OL_1L_2L_3$ is $\mathbf{L}_L = (0,0,L)^T$. Angular momentum in the frame $Ox_1x_2x_3$ can be found as $\mathbf{L}_x = \mathbf{A}^T\mathbf{L}_L$. That leads to $\mathbf{L}_x = L(a_{31},a_{32},a_{33})^T$ and the angular velocity in the $Ox_1x_2x_3$ frame is $\boldsymbol{\omega}_x = L\left(\frac{1}{A}a_{31},\frac{1}{B}\left(a_{32}-\frac{h}{L}\right),\frac{1}{C}a_{33}\right)^T$. Next,

$$\boldsymbol{\omega}_L = \mathbf{A}\boldsymbol{\omega}_x = L\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} \frac{1}{A}a_{31} \\ \frac{1}{B}\left(a_{32}-\frac{h}{L}\right) \\ \frac{1}{C}a_{33} \end{pmatrix} = L\begin{pmatrix} \frac{1}{A}a_{11}a_{31} + \frac{1}{B}a_{12}\left(a_{32}-\frac{h}{L}\right) + \frac{1}{C}a_{13}a_{33} \\ \frac{1}{A}a_{21}a_{31} + \frac{1}{B}a_{22}\left(a_{32}-\frac{h}{L}\right) + \frac{1}{C}a_{23}a_{33} \\ \frac{1}{A}a_{31}^2 + \frac{1}{B}a_{32}\left(a_{32}-\frac{h}{L}\right) + \frac{1}{C}a_{33}^2 \end{pmatrix}. \qquad (0.20)$$

Taking into account (0.12) the geomagnetic induction vector in the frame $OL_1L_2L_3$ is

$$\mathbf{B}_L = \mathbf{Q}^T\mathbf{B}_Z = B_0\begin{pmatrix} q_{11}B_1 + q_{21}B_2 + q_{31}B_3 \\ q_{12}B_1 + q_{22}B_2 + q_{32}B_3 \\ q_{13}B_1 + q_{23}B_2 + q_{33}B_3 \end{pmatrix} \qquad (0.21)$$

where

$$\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} \sin\Theta\sin 2u \\ \sin\Theta\cos 2u \\ \cos\Theta \end{pmatrix} + \delta \begin{pmatrix} 0 \\ \cos\Theta \\ -\sin\Theta\cos 2u \end{pmatrix}.$$

According to (0.20) and (0.21) fast variable ψ is present only in (0.20) while fast variable u is present only in (0.21). Taking into account the form of the control torque (0.18) we can average (0.20) by ψ first and then find the torque averaging (0.18) by u. To average (0.20) by ψ we write matrix (0.2) taking into account $\varphi \approx 0, \theta \approx \frac{\pi}{2}$,

$$\mathbf{A} = \begin{pmatrix} \cos\psi & -\varphi\cos\psi - \theta\sin\psi & \sin\psi \\ \sin\psi & -\varphi\sin\psi + \theta\cos\psi & -\cos\psi \\ \varphi & 1 & \theta \end{pmatrix}.$$

That leads to

$$\langle \boldsymbol{\omega}_L \rangle_\psi = \left(0, 0, \frac{1}{B}(L-h) \right)^T .$$

Control torque averaged by ψ is

$$\langle \mathbf{M}_L \rangle_\psi = k\left(\langle \boldsymbol{\omega}_L \rangle_\psi \times \mathbf{B}_L \right) \times \mathbf{B}_L = \frac{kL_0 B_0^2}{B}(l-h_0)\left(B_{1L}B_{3L}, B_{2L}B_{3L}, -B_{1L}^2 - B_{2L}^2 \right)^T \quad (0.22)$$

where $h_0 = h / L_0$.

In order to average (0.22) by u we need expressions

$$B_{ij} = \frac{1}{2\pi}\int_0^{2\pi} B_i B_j du, \ (i, j = 1,2,3).$$

Some calculation leads to $B_{11} = B_{22} = \frac{1}{2}\sin^2\Theta$, $B_{33} = \cos^2\Theta$, $B_{23} = \delta\left(\cos^2\Theta - \frac{1}{2}\sin^2\Theta \right)$, $B_{12} = B_{13} = 0$. Taking into account (0.22) we can now write equations for slow variables evolution. Equations for l, ρ, σ are separated from the equations for φ, θ. We consider only the first group of equations. For the transient motion the main variable of interest is the angular momentum magnitude l. Equations for the evolution of l, ρ, σ are

$$\frac{dl}{du} = -\varepsilon(l-h_0)l\left[\left(1-q_{13}^2\right)B_{11} + \left(1-q_{23}^2\right)B_{22} + \left(1-q_{33}^2\right)B_{33} - B_{23}\left(q_{21}q_{31} + q_{22}q_{32}\right) \right],$$

$$\frac{d\rho}{du} = \varepsilon(l-h_0)\left[q_{11}q_{13}B_{11} + q_{21}q_{23}B_{22} + q_{31}q_{33}B_{33} + B_{23}\left(q_{21}q_{33} + q_{31}q_{23}\right) \right], \qquad (0.23)$$

$$\frac{d\sigma}{du} = \varepsilon(l-h_0)\frac{1}{\sin\rho}\left[q_{12}q_{13}B_{11} + q_{22}q_{23}B_{22} + B_{23}(q_{22}q_{33} + q_{32}q_{23})\right].$$

Here new parameter $\varepsilon = \dfrac{kB_0^2}{\omega_0 B}$ is introduced. But if the torque is small in a sense described above this new parameter is small again. All the reasoning about slow and fast variables is valid. Taking into account (0.1) and expressions for B_{ij} equations (0.23) take form

$$\frac{dl}{du} = -\varepsilon l(l-h_0)\left(\sin^2\Theta + \eta\sin^2\rho - \delta\eta\cos\rho\sin\rho\sin\sigma\right),$$

$$\frac{d\rho}{du} = -\varepsilon(l-h_0)\left(\eta\sin\rho\cos\rho + \delta\eta\sin\sigma\left(\cos^2\rho - \sin^2\rho\right)\right),$$

$$\frac{d\sigma}{du} = \varepsilon\delta(l-h_0)\eta\cos\sigma\,\mathrm{ctg}\,\rho$$

where $\eta = \cos^2\Theta - \frac{1}{2}\sin^2\Theta$. We neglect terms of the order $\varepsilon\delta$ since $\delta = o(1)$. The system takes form

$$\frac{dl}{du} = -\varepsilon l(l-h_0)\left(\sin^2\Theta + \eta\sin^2\rho\right),$$

$$\frac{d\rho}{du} = -\varepsilon(l-h_0)\eta\sin\rho\cos\rho, \qquad (0.24)$$

$$\frac{d\sigma}{du} = 0.$$

These equations are similar to those obtained in [11] except for the multiplier $(l-h_0)$. We divide this first equation in (0.24) by the second

$$\frac{dl}{d\rho} = l\frac{\sin^2\Theta + \eta\sin^2\rho}{\eta\sin\rho\cos\rho}.$$

Integrating this relation we get

$$\ln l = \sin^2\Theta\ln(\mathrm{tg}\,\rho) + \frac{\eta}{2}\ln\left(\mathrm{tg}^2\rho + 1\right) - C_0$$

where $C_0 = \sin^2\Theta\ln(\mathrm{tg}\,\rho_0) + \frac{\eta}{2}\ln\left(\mathrm{tg}^2\rho_0 + 1\right)$ is integration constant. This leads to

$$l(\rho) = \exp\left(\sin^2\Theta\ln(\mathrm{tg}\,\rho) + \frac{\eta}{2}\ln\left(\mathrm{tg}^2\rho + 1\right) - C_0\right) \qquad (0.25)$$

and

$$\int_{\rho_0}^{\rho} \frac{d\rho}{\left(l(\rho)-h_0\right)\eta \sin\rho \cos\rho} = -\varepsilon\left(u-u_0\right). \tag{0.26}$$

Expressions (0.26) and (0.25) deliver the solution of (0.24) in quadratures.

Figure 5 introduces the result of decreasing of the angular momentum magnitude for a set of inclinations and $\varepsilon = 0.1$, $h_0 = 0.9$.

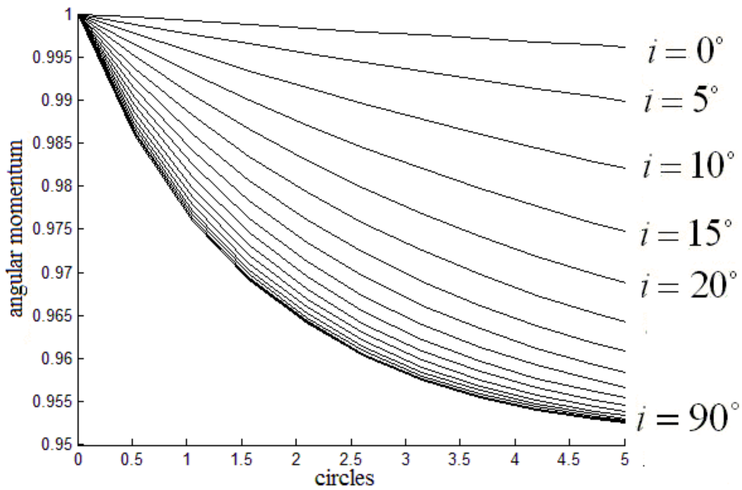

Fig. 5. Angular momentum damping

This figure shows that the time-response of the algorithm, that is the rate of angular momentum damping, rises when orbit inclination increases for a given h_0. Similar result was obtained for the "*-Bdot*" algorithm in case of a satellite without a flywheel [12] and for a spin-stabilized satellite [13]. The cone half-opening angle in the averaged geomagnetic field model increases with the inclination. This results in the increased amplitude of the geomagnetic induction vector change. This vector is constant for the equatorial orbit and angular momentum is almost not damped. In case of polar orbit geomagnetic induction vector lies in the orbital plane. It has the most change in the direction and damping rate is maximal.

Let's consider the impact of a flywheel's angular momentum magnitude h_0 on the time-response of the damping algorithm. Figures 6 and 7 introduce the value of the angular momentum of a satellite itself with respect to its initial value in percents. Figure 6 corresponds to the value of angular momentum after 10 circles of a satellite on the orbit, Figure 7 corresponds to 2 circles. It is seen that the value of a flywheel angular momentum has insignificant impact on the time-response. Slight inclination of the curves indicate that time-response rises when h_0 rise. It is especially clear if one consider the curve corresponding to the value of angular momentum equal to 70% in the Figure 7.

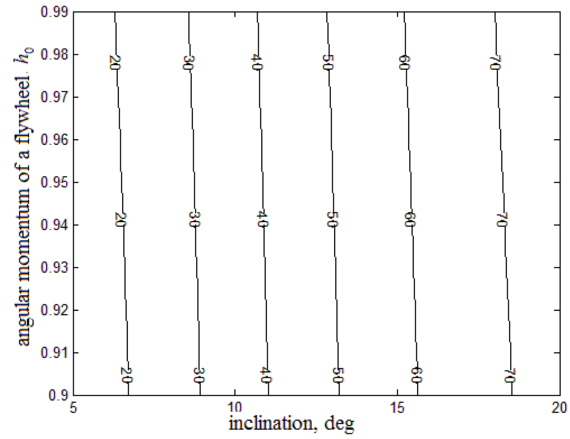

Fig. 6. Angular momentum magnitude after 10 circles

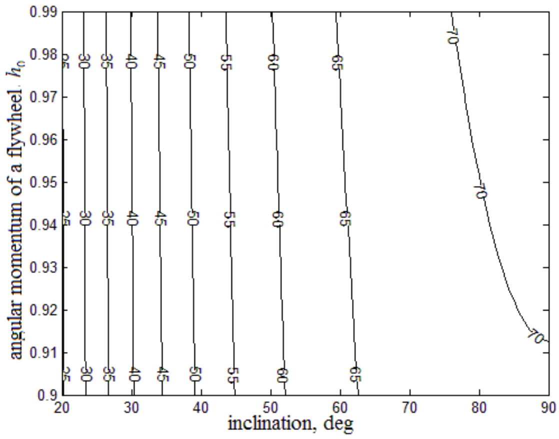

Fig. 7. Angular momentum magnitude after 2 circles

Consider equations (0.24) in the vicinity of equilibrium position. Equilibrium position are $l = h_0, \rho = 0$ and $l = h_0, \rho = \frac{\pi}{2}$ (note that $\sigma = const$). Stability of these positions depends on the sign of η and, therefore, on the orbit inclination. If inclination is less than 45.5° the first position is stable, if greater – the second position is stable. We consider only the first position as the example. Linearized equation for the magnitude of angular momentum is

$$\frac{dx}{du} = -\varepsilon h_0 \sin^2 \Theta x$$

where $x = l - h_0$. For a satellite equipped only with MACS this equation is [12]

$$\frac{dy}{du} = -\varepsilon \lambda_0 \sin^2 \Theta y$$

where $\lambda_0 = \frac{C}{4A} + \frac{C}{4B} + \frac{1}{2}$, $y = l$.

It is seen, that the time response is close for systems with or without a flywheel and raises when the orbit inclination increases. The difference in the time-response rises when expressions $\frac{C}{A}$, $\frac{C}{B}$ decrease. When $C << A, C << B$ the time-response differs by a half since h_0 is close to 1. For both systems we find the time u^* (in circles), that is the time necessary to reduce the angular momentum magnitude by a half. The relation between these parameters for two systems is λ_0 / h_0 and is close to 1 when $h_0 \in [0.9, 1)$ and principal moments of inertia are commeasurable (conditions $A << C$ and $B << C$ leading to $\lambda_0 \to \infty$ are not satisfied). The results for u^* for $\lambda_0 = 1.0247$ are listed in the Table 1. The relation λ_0 / h_0 is close to 1 and the time-response of two systems is close, as it is shown in the linearized case. For all systems $\varepsilon = 0.1$, $\mathbf{J} = diag(3,3.1,3.2)$ kg·m^2, $\rho_0 = 0.1$.

Inclination, °	10	20	30	40	50	60	70	80	90
Without a flywheel	16.58	4.85	2.64	1.86	1.48	1.29	1.18	1.11	1.09
Flywheel $h_0 = 0.9$	16.31	4.92	2.74	1.97	1.56	1.36	1.24	1.17	1.14
Flywheel $h_0 = 0.95$	15.73	4.91	2.73	1.93	1.54	1.34	1.21	1.14	1.12

Table 1. Time-response comparison for systems including MACS and a flywheel and system including only MACS

2.3. Steady-state motion. Gravitational orientation

We now consider the steady-state motion of a satellite when its angular velocity is small and the axes of the frames $Ox_1x_2x_3$ and $OX_1X_2X_3$ virtually coincide. A satellite is subjected to the gravitational and magnetic torques. This motion is called gravitational orientation. Its stability is provided by the gravity-gradient torque. Sufficient conditions of the stability of gravitational orientation are [1]
$A - C > 0$, $\omega_0(B - A) + h > 0$, $4\omega_0(B - C) + h > 0$.

Since the angular momentum of a flywheel is considered big in comparison with the angular momentum of a satellite itself ($h \gg J_{ii}\omega_0$, $i = 1,2,3$), only the condition $A - C > 0$ may be considered. It provides the stability by a pitch angle (rotation in the orbital plane). Stability by roll and yaw angles is provided by a flywheel. MACS implements dipole magnetic moment

$$\mathbf{m} = -k\frac{d\mathbf{B}}{dt}. \tag{0.27}$$

Let's study the influence of the torque $\mathbf{M} = \mathbf{m} \times \mathbf{B}$ on the gravitational orientation. We assume that this torque is small in comparison with gravitational. We rewrite equations (0.8) introducing argument of latitude instead of time

$$\begin{aligned}
\frac{d\Omega_1}{du} &= h_A\Omega_3 + \theta_A(\Omega_2\Omega_3 - 3d_{32}d_{33}) + \frac{1}{A\omega_0^2}M_{1x},\\
\frac{d\Omega_2}{du} &= \theta_B(\Omega_1\Omega_3 - 3d_{31}d_{33}) + \frac{1}{B\omega_0^2}M_{2x},\\
\frac{d\Omega_3}{du} &= -h_C\Omega_1 + \theta_C(\Omega_1\Omega_2 - 3d_{31}d_{32}) + \frac{1}{C\omega_0^2}M_{3x},\\
\frac{d\alpha}{du} &= \frac{1}{\cos\gamma}(\Omega_1\sin\beta + \Omega_2\cos\beta) - 1,
\end{aligned} \tag{0.28}$$

$$\frac{d\beta}{du} = \Omega_3 + \mathrm{tg}\,\gamma\left(\Omega_1 \sin\beta + \Omega_2 \cos\beta\right),$$
$$\frac{d\gamma}{du} = \Omega_1 \cos\beta - \Omega_2 \sin\beta$$

where $h_A = \dfrac{h}{A\omega_0}$, $h_C = \dfrac{h}{C\omega_0}$, $\theta_A = \dfrac{B-C}{A}$, $\theta_B = \dfrac{C-A}{B}$, $\theta_C = \dfrac{A-B}{C}$, $\Omega_i = \omega_i / \omega_0$ $(i = 1,2,3)$.

Equations (0.28) allow the stationary solution $\alpha = \beta = \gamma = 0, \Omega_1 = \Omega_3 = 0, \Omega_2 = 1$ if a satellite is subjected only to the gravitational torque. We try to find solutions, generated from this stationary one in case of small additional magnetic torque. We use Poincare method [10]. Equations (0.28) are of the form

$$\mathbf{x} = \mathbf{f}\left(\mathbf{x}\right) + \varepsilon \mathbf{g}\left(\mathbf{x}\right)$$

where $\mathbf{x} = \left(\Omega_1, \Omega_2, \Omega_3, \alpha, \beta, \gamma\right)$, $\varepsilon = \dfrac{kB_0^2}{B\omega_0}$. We try to find the solution in a form $\mathbf{x} = \mathbf{x}_0 + \varepsilon \mathbf{x}_1 + O\left(\varepsilon^2\right)$ where $\mathbf{x}_0 = \left(0,1,0,0,0,0\right)$ is the stationary solution, $\mathbf{x}_1 = \left(w_1, w_2, w_3, \alpha_1, \beta_1, \gamma_1\right)$. That leads to $\dfrac{d\mathbf{x}_0}{du} + \varepsilon \dfrac{d\mathbf{x}_1}{du} = $ $= \mathbf{f}\left(\mathbf{x}_0\right) + \varepsilon\left(\mathbf{F}\left(\mathbf{x}_0\right)\mathbf{x}_1 + \mathbf{g}\left(\mathbf{x}_0\right)\right) + O\left(\varepsilon^2\right)$ where $F_{ij} = \dfrac{\partial f_i}{\partial x_j}$. In order to find $\mathbf{F}$ we need an explicit form of $d_{32}d_{33}, d_{31}d_{33}, d_{31}d_{32}$:

$$d_{32}d_{33} = \sin\alpha\cos\alpha\sin\beta\cos\gamma + \cos^2\alpha\cos\beta\sin\gamma\cos\gamma,$$
$$d_{31}d_{33} = -\sin\alpha\cos\alpha\cos\beta\cos\gamma + \cos^2\alpha\sin\beta\sin\gamma\cos\gamma,$$
$$d_{31}d_{32} = -\sin^2\alpha\sin\beta\cos\beta - \sin\alpha\cos\alpha\cos^2\beta\sin\gamma + \sin\alpha\cos\alpha\sin^2\beta\sin\gamma +$$
$$+\cos^2\alpha\sin\beta\cos\beta\sin^2\gamma.$$

That leads to

$$\mathbf{F}\left(\mathbf{x}_0\right) = \begin{pmatrix} 0 & 0 & \theta_A + h_A & 0 & 0 & -3\theta_A \\ 0 & 0 & 0 & 3\theta_B & 0 & 0 \\ \theta_C - h_C & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

To find $\mathbf{g}\left(\mathbf{x}_0\right)$ we need the derivative of the geomagnetic induction vector in the $Ox_1x_2x_3$ frame

$$\frac{d\mathbf{B}_x}{dt} = \mathbf{D}^T \frac{d\mathbf{B}_X}{dt} - \boldsymbol{\omega}_{rel} \times \mathbf{B}_x$$

where $\boldsymbol{\omega}_{rel}$ is relative angular velocity of a satellite in the $Ox_1x_2x_3$ frame. Note that the stationary solution $\mathbf{x}_0$ corresponds to zero relative velocity of the frames $Ox_1x_2x_3$ and $OX_1X_2X_3$ and their coincidence. That is $\boldsymbol{\omega}_{rel} = \mathbf{0}$, $\mathbf{D} = \mathbf{E}$ and therefore

$$\frac{d\mathbf{B}_x}{dt} = \frac{d\mathbf{B}_X}{dt}.$$

We use more accurate dipole model instead of averaged here. Geomagnetic induction vector in the frame $OX_1X_2X_3$ is $\mathbf{B} = B_0\left(\sin i \cos u, \cos i, -2\sin i \sin u\right)^T$, so

$$\mathbf{g}(\mathbf{x}_0) = \sin i \cos i \left(-2\frac{B}{A}\cos u, 2\,\mathrm{tg}\, i, \frac{B}{C}\sin u, 0, 0, 0\right)^T.$$

For the $\mathbf{x}_1$ we finally get equations

$$\begin{aligned}
&\frac{dw_1}{du} = (\theta_A + h_A) w_3 - 3\theta_A \gamma_1 - 2\frac{B}{A}\sin i \cos i \cos u, \\
&\frac{dw_2}{du} = 3\theta_B \alpha_1 + 2\sin i \cos i\, \mathrm{tg}\, i, \\
&\frac{dw_3}{du} = (\theta_C - h_C) w_1 + \frac{B}{C}\sin i \cos i \sin u, \\
&\frac{d\alpha_1}{du} = w_2, \ \frac{d\beta_1}{du} = \gamma_1 + w_3, \ \frac{d\gamma_1}{du} = w_1 - \beta_1.
\end{aligned} \tag{0.29}$$

Equations for α_1 and w_2 are separated. General solution for α_1 and w_2 is the oscillation near the stationary solution. Matrix of the homogeneous equations for $\beta_1, \gamma_1, w_1, w_3$ is

$$\mathbf{W} = \begin{pmatrix} 0 & \theta_A + h_A & 0 & -3\theta_A \\ \theta_C - h_C & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

We find its eigen values from the relation

$$\det(\mathbf{W} - \lambda\mathbf{E}) = \lambda^4 + a\lambda^2 + b$$

where $a = 1 + 3\theta_A - (\theta_A + h_A)(\theta_C - h_C)$, $b = -(\theta_C - h_C)(4\theta_A + h_A)$. That leads to

$$\lambda^2 = \frac{1}{2}\left(-a \pm \sqrt{a^2 - 4b}\right).$$

Here we again assume that the angular momentum of a flywheel is times bigger than the angular momentum of a satellite, so $a^2 - 4b = h_A^2 h_C^2 + O\left(h_A^2 h_C + h_C^2 h_A\right) > 0$ and with it $a > 0$. That means that all eigen values are complex and general solution is the oscillation near the stationary solution. Forced solution is of greater interest. It shows the influence of MACS and magnetic torque on the steady-state motion. So we now try to find particular solution of (0.29). For α_1 we have

$$\alpha_1 = -\frac{2\sin^2 i}{3\theta_B},$$

that means that implementing MACS leads to the constant deviation in the orbital plane. We try to find angles β_1, γ_1 in the form

$$\beta_1 = A_1 \sin u + A_2 \cos u,\ \gamma_1 = B_1 \sin u + B_2 \cos u.$$

Therefore

$$w_1 = (A_1 - B_2)\sin u + (A_2 + B_1)\cos u,\ w_3 = -(A_2 + B_1)\sin u + (A_1 - B_2)\cos u.$$

Substituting these expressions to (0.29) and setting equalities between the constants held by $\sin u$ and $\cos u$ we obtain the equations for A_1, A_2, B_1, B_2

$$(1-\theta_A - h_A)A_1 + (-1 + 4\theta_A + h_A)B_2 = -2\frac{B}{A}\sin i \cos i,$$

$$-(1-\theta_A - h_A)A_2 + (-1 + 4\theta_A + h_A)B_1 = 0,$$

$$(-1-\theta_C + h_C)A_2 - (1 + \theta_C - h_C)B_1 = 0,$$

$$(-1-\theta_C + h_C)A_1 + (1 + \theta_C - h_C)B_2 = \frac{B}{C}\sin i \cos i.$$

Solving allows us to find the forced solution of (0.29)

$$\alpha_1 = -\frac{2\sin^2 i}{3\theta_B}$$

$$\beta_1 = \sin i \cos i \left(-2\frac{B}{A}(1 + \theta_C - h_C) + \frac{B}{C}(-1 + 4\theta_A + h_A) \right)\sin u,$$

$$\gamma_1 = \sin i \cos i \left(\frac{B}{C}(1 - \theta_A - h_A) - 2\frac{B}{A}(-1 - \theta_C + h_C) \right)\cos u,$$

$$w_1 = \sin i \cos i \left(-4\frac{B}{A}(1 + \theta_C - h_C) + \frac{B}{C}(5\theta_A + 2h_A - 2) \right)\sin u,$$

$$w_2 = 0,$$

$$w_3 = \sin i \cos i \left(-4\frac{B}{A}(1 + \theta_C - h_C) + \frac{B}{C}(5\theta_A + 2h_A - 2) \right)\cos u.$$

The deviation from the stationary solution induced by MACS is found. Small constant deviation occurs in the orbital plane. Small oscillations with orbital frequency take place for the roll and yaw angles. Note that resonance may occur if eigen value λ of matrix $\mathbf{W}$ is equal to $\pm i$. Parameters leading to these eigen values are found from the expression

$$\lambda^2 = \frac{1}{2}\left(-a \pm \sqrt{a^2 - 4b}\right) = -1,$$

that leads to

$$-a + b + 1 = 0,$$

and finally to

$$h_C - \theta_C - 1 = 0.$$

In the considered case of a flywheel with big angular momentum its value h_C is not commeasurable with θ_C,1 and no resonance can occur.

2.4. Steady-state motion. Arbitrary orientation in the orbital plane

MACS may be used to provide any equilibrium position $\alpha = \alpha_0$ in the orbital plane and its stability. Gravity-gradient torque becomes disturbing one. We assume that $\beta,\gamma \,\square\, 0$, $\Omega_1,\Omega_2 \,\square\, 0$, $\Omega_2 \,\square\, 1$ after the transient motion. Direct cosine matrix (0.7) takes form

$$\mathbf{D} = \begin{pmatrix} \cos\alpha & -\beta\cos\alpha + \gamma\sin\alpha & \sin\alpha \\ \beta & 1 & -\gamma \\ -\sin\alpha & \beta\sin\alpha + \gamma\cos\alpha & \cos\alpha \end{pmatrix}. \tag{0.30}$$

Equations of motion (0.8) taking into account (0.30) and introducing argument of latitude instead of time are

$$\frac{d\Omega_1}{du} = h_A\Omega_3 + \theta_A\left(\Omega_3 - 3\left(\gamma\cos^2\alpha + \beta\sin\alpha\cos\alpha\right)\right) + \frac{1}{A\omega_0^2}M_{1x},$$

$$\frac{d\Omega_2}{du} = 3\theta_B\sin\alpha\cos\alpha + \frac{1}{B\omega_0^2}M_{2x},$$

$$\frac{d\Omega_3}{du} = -h_C\Omega_1 + \theta_C\left(\Omega_1 + 3\left(\gamma\sin\alpha\cos\alpha - \beta\sin^2\alpha\right)\right) + \frac{1}{C\omega_0^2}M_{3x},$$

$$\frac{d\alpha}{du} = \Omega_2 - 1,\ \frac{d\beta}{dt} = \Omega_3 + \gamma,\ \frac{d\gamma}{dt} = \Omega_1 - \beta.$$

Let's assume that M_{2x} doesn't depend on $\beta,\gamma,\Omega_1,\Omega_3$. We will generate M_{2x} so that this requirement is met. Equations for α,Ω_2 are separated. We now consider the problem of reorientation of a satellite to the arbitrary position in the orbital plane and maintaining this orientation. The analysis is based on the equation

$$\frac{d^2\alpha}{du^2} = 3\theta_B\sin\alpha\cos\alpha + \left(M_d + M_r\right) \tag{0.31}$$

where M_d and M_r are damping and restoring components of the torque respectively scaled by $B\omega_0^2$. We introduce new variable $\xi = \alpha - \alpha_0$ and restoring torque $M_r = \kappa^2 \sin(\alpha_0 - \alpha)$. Let gravitational and damping torques be zero. That leads to the equation

$$\frac{d^2\xi}{du^2} + \kappa^2 \sin\xi = 0$$

of a mathematical pendulum oscillations near the position $\xi = 0$. Introducing the damping component of the torque makes this position asymptotically stable. We use "*-Bdot*" algorithm,

$$\mathbf{m}_x = k\left(-\mathbf{D}^T \frac{d\mathbf{B}_X}{du} + \boldsymbol{\omega}_{rel} \times \mathbf{D}^T \mathbf{B}_X \right).$$

Note that omitting the first term in this expression is no longer valid since the angular velocity of a satellite is less or commeasurable with orbital. We assume that the damping component of the torque is negligible with respect to the restoring one, so $\varepsilon / \kappa^2 << 1$ where $\varepsilon = \dfrac{kB_0^2}{B\omega_0}$. That allows us to omit small expressions in (0.30) since their contribution in (0.31) is of the order of $\varepsilon\beta$ and $\varepsilon\gamma$. We neglect Ω_1, Ω_3 with the same argument. Taking into account (0.10) and (0.30) we have

$$\mathbf{m}_x = kB_0\omega_0 \sin i \begin{pmatrix} (\cos\alpha \sin u - \sin\alpha \cos u) + (\Omega_2 - 1)(\sin\alpha \cos u - \cos\alpha \sin u) \\ 0 \\ (\sin\alpha \sin u + \cos\alpha \cos u) + (\Omega_2 - 1)(-\sin\alpha \sin u - \cos\alpha \cos u) \end{pmatrix}.$$

The damping component of the torque is $M_d = \varepsilon \sin^2 i (2 - \Omega_2)$. Equation (0.31) may be rewritten as

$$\frac{d^2\xi}{du^2} + \kappa^2 \sin\xi + \varepsilon \sin^2 i \frac{d\xi}{du} = \varepsilon \sin^2 i .$$

Consider homogeneous equation in the vicinity of the equilibrium position $\xi = 0$. The equation is

$$\frac{d^2\xi}{du^2} + \kappa^2 \xi + \varepsilon \sin^2 i \frac{d\xi}{du} = 0 .$$

Its solution in case $\varepsilon / \kappa^2 << 1$ is

$$\xi = \xi_1 \exp\left(-\frac{1}{2} \varepsilon \sin^2 iu \right) \sin\left(\frac{1}{2} \sqrt{4\kappa^4 - \varepsilon^2 \sin^4 i} \left(u + \xi_0 \right) \right) \qquad (0.32)$$

where ξ_0, ξ_1 are determined by the initial conditions. The solution (0.32) describe the damping oscillations. The rate of damping depends on the orbit inclination. We consider now non-homogeneous equation and introduce the gravity-gradient torque. That leads to the equation

$$\frac{d^2\xi}{du^2}+\left(\kappa^2+\mu\right)\xi+\varepsilon\sin^2 i\frac{d\xi}{du}=\mu$$

where $\mu=\varepsilon\sin^2 i+3\theta_B\sin\alpha_0\cos\alpha_0$. The forced solution is

$$\xi=\frac{2}{\omega}\int_{u_0}^{u}\mu\exp\left(-\frac{1}{2}\varepsilon\sin^2 i(x-u)\right)\sin\left(\frac{1}{2}\omega(u-x)\right)dx$$

where $\omega=\sqrt{4\left(\kappa^2+\mu\right)^2-\varepsilon^2\sin^4 i}$. Assuming $u_0=0$ we have the forced solution

$$\xi=\frac{2\mu}{\omega}\frac{4}{\varepsilon^2\sin^2 i+\omega^2}\left(1-\frac{\varepsilon}{2}\sin\left(\frac{\omega}{2}u\right)-\frac{\omega}{2}\cos\left(\frac{\omega}{2}u\right)\right)$$

And finally the solution of the equation

$$\xi=\xi_1\exp\left(-\frac{1}{2}\varepsilon\sin^2 iu\right)\sin\left(\frac{1}{2}\omega\left(u+\xi_0\right)\right)+$$
$$+\frac{2\mu}{\omega}\frac{4}{\varepsilon^2\sin^2 i+\omega^2}\left(1-\frac{\varepsilon}{2}\sin\left(\frac{\omega}{2}u\right)-\frac{\omega}{2}\cos\left(\frac{\omega}{2}u\right)\right). \quad (0.33)$$

Gravity-gradient torque influence results in the increased deviation from the equilibrium position and small oscillation near this new position. The amplitude of both deviation and oscillation depends on the ratio between the gravity-gradient torque and the restoring component of the torque. Damping component magnitude and the orbit inclination also have impact on the accuracy of orientation since the deviation and amplitude depend not only on the gravitational torque (term $3\theta_B\sin\alpha_0\cos\alpha_0$ in μ) but on the damping component too (term $\varepsilon\sin^2 i$ in μ).

Implementing restoring torque causes extraneous components lying not in the orbital plane since $M_2=m_3B_1-m_1B_3$. We assume, for example, that the restoring torque is generated using the first magnetorquer, so $m_2=m_3=0$. That leads to the torque

$$\mathbf{M}=\left(0,\kappa^2\sin\left(\alpha_0-\alpha\right),-\kappa^2\sin\left(\alpha_0-\alpha\right)\frac{B_{2x}}{B_{3x}}\right)^T. \quad (0.34)$$

The third component is the extraneous one. It is difficult to find a solution for $\beta,\gamma,\Omega_1,\Omega_3$ taking into account gravity-gradient torque, restoring torque (in this case it is extraneous one) and damping torque. If homogeneous system with only gravity-gradient torque is considered its solution is oscillation near the stationary solution $\beta=\gamma=0$, $\Omega_1=\Omega_3=0$. Extraneous torque generates a forced solution that is small oscillation near the stationary solution or near slightly different new stationary solution. This deviation is small because of a flywheel with big angular momentum. Damping torque forces damping of these oscillations. That means that the torque (0.34) with the damping torque allows a satellite to be oriented at arbitrary angle α_0 in the orbital plane. The accuracy is characterized by the solution (0.33).

3. Numerical analysis

Numerical analysis is carried out for the satellite with inertia tensor $\mathbf{J} = (4,5,3)$ kg·m^2. Orbit inclination is 50°. Dipole magnetic moment of magnetorquers in the transient mode is 1 A·m^2. In the nominal motion dipole magnetic moment used for restoring is 1 A·m^2, for damping is 0.1 A·m^2. Flywheel angular momentum is 0.4 N·m·s.

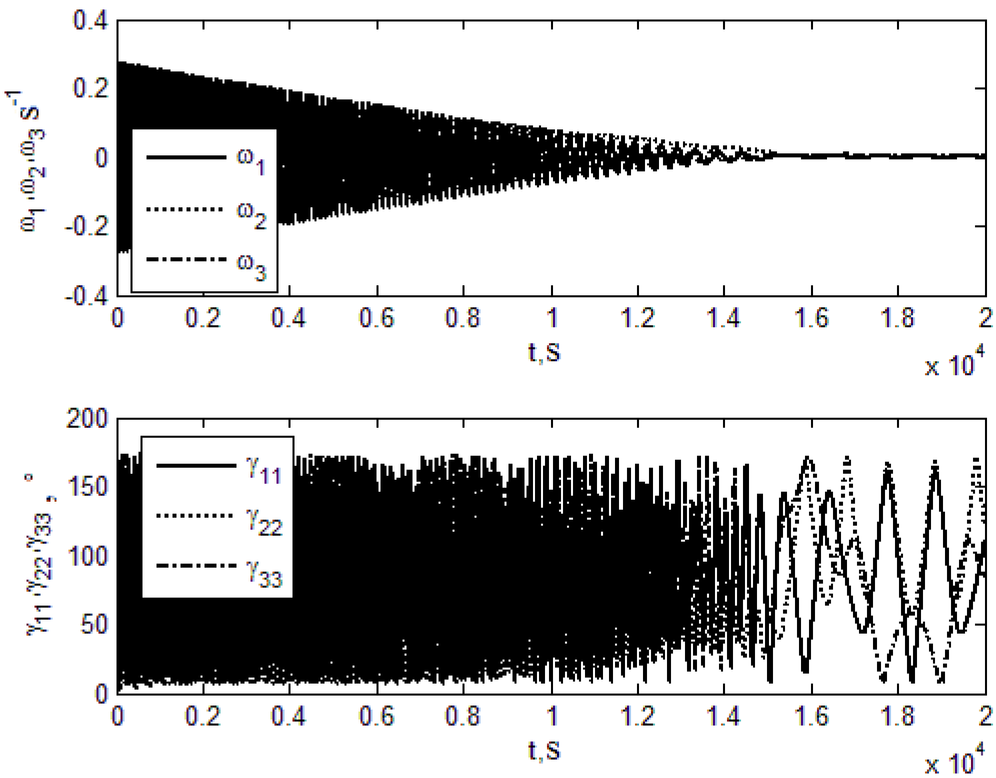

Fig. 8. Transient motion without a flywheel. Initial conditions $\alpha = \beta = \gamma = 30^\circ$, $\omega_1 = \omega_2 = \omega_3 = 10 \,{}^\circ/s$

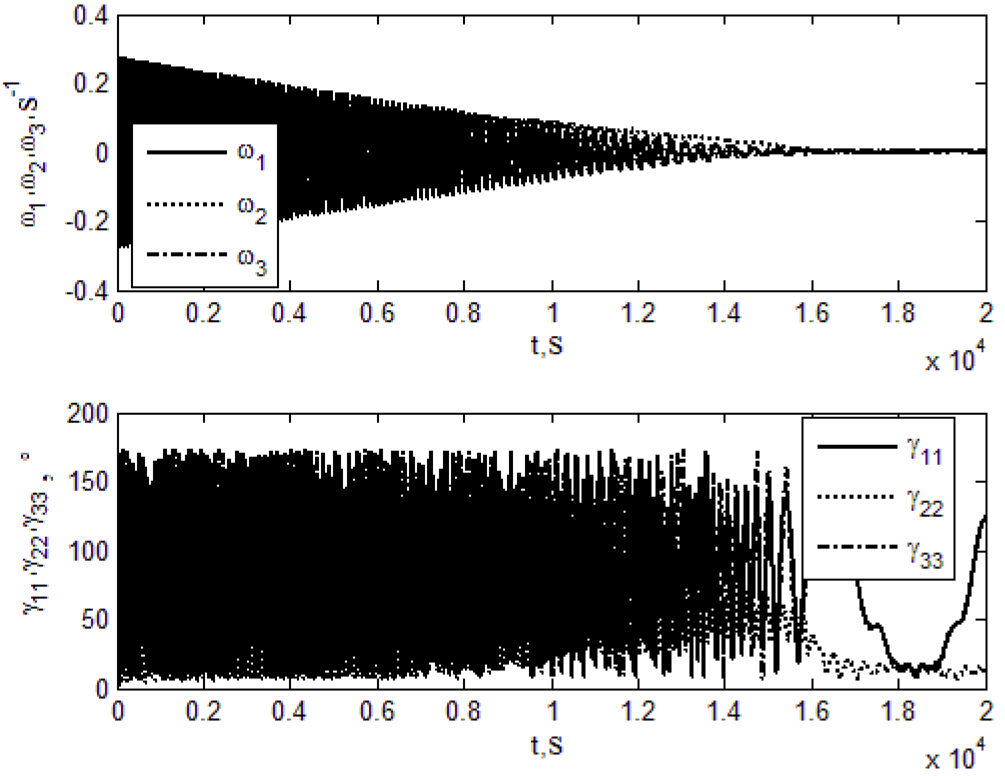

Fig. 9. Transient motion with a flywheel. Initial conditions $\alpha = \beta = \gamma = 30^\circ$, $\omega_1 = \omega_2 = \omega_3 = 10\ {}^\circ\!/_s$

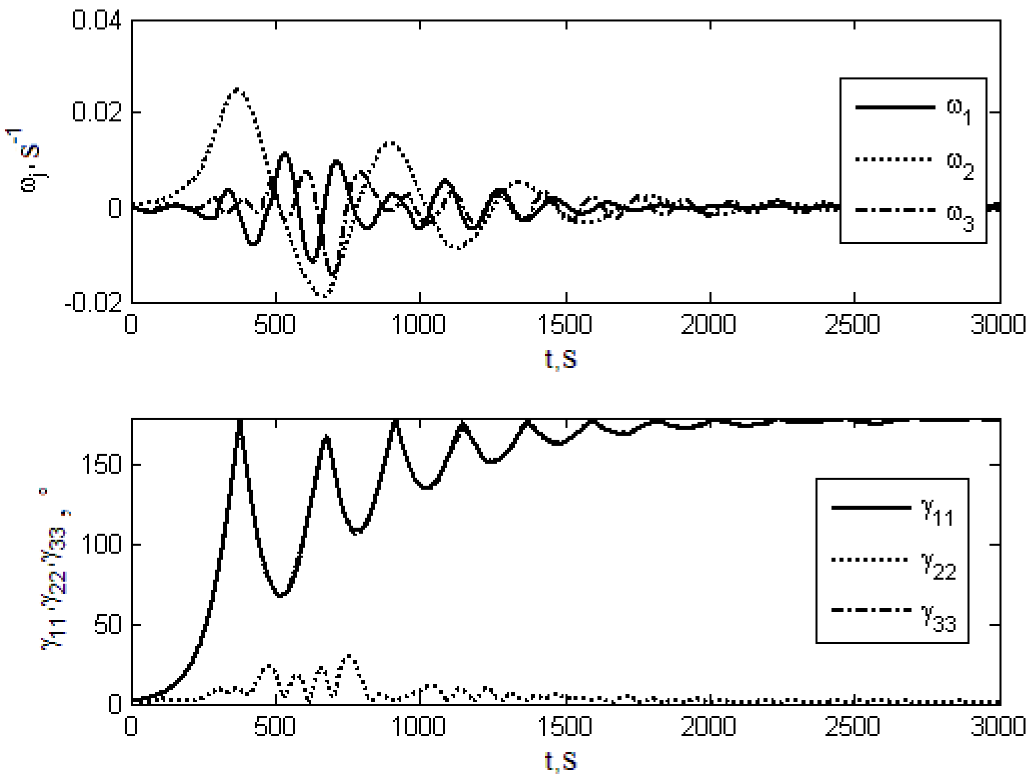

Fig. 10. Reorientation in the orbital plane. Initial conditions $\alpha = \beta = \gamma = 1^\circ$, $\omega_1 = \omega_3 = \omega_2 = 0.03\ {}^\circ\!/_s$,

required attitude $\alpha_0 = 180^\circ$

In these Figures γ_{ii} $(i=1,2,3)$ are the angles between the axes of the $Ox_1x_2x_3$ and the $OX_1X_2X_3$ frames. Figures 8 and 9 show that the time of transient motion is close for systems with or without a flywheel. Figure 10 introduces reorientation by angle 180° in the orbital plane. The axis of a flywheel conserve its orientation perpendicular to the orbital plane.

Conclusion

Transient and steady-state motions of a satellite equipped with MACS and a flywheel are considered. The solution in quadratures for the slow variables in transient motion is found. It is shown that the time-response rises when orbit inclination increases. For steady-state motion small oscillations near the stationary solution in gravity field is found. System time-response is evaluated. The time-response for systems with or without a flywheel differs a little. An algorithm providing arbitrary orientation in the orbital plane in a steady-state motion is proposed. Small oscillations near this position are found. Numerical analysis is carried out.

References

1. Sarychev, V.A., Problems of artificial satellites attitude. In a series "Space survey", Vol. 11, M.:VINITI, 1978, 222 p.
2. Likins, P.W., Attitude stability criteria for dual spin spacecraft // *Journal of Spacecraft and Rockets*, 1967, vol. 4, no. 12, pp. 1638-1643.
3. Sarychev, V.A. and Sazonov, V.V., Optimal Nutation Damping of Spin-stabilized Satellites, *Celestial Mechanics and Dynamical Astronomy*, 1977, vol. 15, no. 2, pp. 225-242.
4. Guelman, M., Waller, R., Ortenberg, F. and Shiryaev, A., The Gurwin-Techsat Microsatellite: Six Years Successful Operation in Space, *Proc. of the 4S Symposium: Small Satellites, Systems and Services, September 2004, La Rochelle*, 62 p.
5. Sakai, S.-I., Fukushima, Y. and Saito, H., Design and on-orbit evaluation of magnetic attitude control system for the 'REIMEI' microsatellite. *Proc. of the 10th IEEE International workshop on advanced motion control, March 2008, Trento*, pp. 584-589.
6. Go, T. H. and Ramnath, R.V., Geomagnetic attitude control of satellites using generalized multiple scales, *Journal of Guidance, Control, and Dynamics*, 1997, vol. 20, no. 4, pp. 690-698.
7. Beletsky, V.V., Artificial satellite motion about center of mass in the gravitational field, M.: MSU publisher, 1975, 308 p.

8. Beletsky, V.V. and Khentov, A.A., Tumbling Motion of a Magnetized Satellite, M.: Nauka, 1985, 288 p.

9. Beletsky, V.V. and Novogrebalsky, A.B., Occurence of Stable Relative Equilibrium of a Satellite in Model Magnetic Field, *Astronomical Journal*, 1973, vol. 50, no. 2, pp. 327-335.

10. Moiseev, N.N., Asymptotic Methods in Non-linear Mechanics, M: Nauka, 1969, 379 p.

11. Stickler, A. C. and Alfriend, K. T., Elementary Magnetic Attitude Control System, *Journal of Spacecraft and Rockets*, 1976, vol. 13, no. 5, pp. 282-287.

12. Ovchinnikov, M. Yu., Pen'kov V.I, Roldugin D.S. and Karpenko S.O., Study of the effectiveness of active magnetic control algorithm, M.: Preprint KIAM RAS, 2010, no. 16. 30 p. http://library.keldysh.ru/preprint.asp?id=2010-16.

13. Ovchinnikov, M. Yu., Pen'kov V.I and Roldugin D.S., Spin-stabilized satellite with three-stage active magnetic attitude control system, M.: Preprint KIAM RAS, 2011, no. 6. 32 p. http://library.keldysh.ru/preprint.asp?id=2011-6.

Contents