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KELDYSH INSTITUTE OF APPLIED MATHEMATICS
RUSSIAN ACADEMY OF SCIENCES

S.O. Karpenko, M.Yu. Ovchinnikov,
D.S. Roldugin, S.S. Tkachev

Synthesis and analysis of geomagnetic control using attitude sensor data. Case of sun sensor and magnetometer use

Moscow, 2011

С.О. Карпенко, М.Ю. Овчинников, Д.С. Ролдугин, С.С. Ткачев

Формирование и анализ алгоритма магнитной ориентации с использованием измерений углового датчика. Случаи солнечного датчика и магнитометра

Рассматривается логика формирования алгоритма активного магнитного управления ориентацией спутника «Чибис-М», реализующего разворот его солнечных панелей на Солнце. При этом используются показания только солнечных датчиков. Уравнения движения интегрируются в квадратурах для осесимметричного аппарата при помощи методов асимптотического анализа. Для несимметричного спутника определяются устойчивые положения равновесия.

Ключевые слова: активная магнитная система ориентации, алгоритм ориентации, осредненная модель магнитного поля Земли, солнечный датчик

S.O. Karpenko, M.Yu. Ovchinnikov, D.S. Roldugin, S.S. Tkachev

Synthesis and analysis of geomagnetic control using attitude sensor data. Case of sun sensor and magnetometer use

An active magnetic control synthesis for attitude guidance of «Chibis-M» microsatellite is considered. The only information required is the data from the sun sensor. Applicability of a control to achieve solar panels sun-pointing is studied. Equations of motions are analytically solved using averaging technique. The behavior of a system with respect to initial conditions and orbit parameters is studied.

Key words: active magnetic control, control algorithm, averaged geomagnetic field model, sun attitude

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Contents

Introduction

Magnetic attitude control systems (MACS) are widely used for attitude control of satellites. They are especially attractive if a satellite has significant cost, mass or energy limitations. So, small satellites almost invariably have MACS. Small satellites often have fewer restrictions on attitude control system accuracy and time-response and MACS may be the only solution for low-power and low-cost spacecraft. MACS may be also used together with other actuators. The most common practice is initial angular velocity damping after the separation from the launch vehicle. This stage may be considered simply but it is necessary for other actuators, maybe with finer accuracy, to have spacecraft angular velocity below some critical value.

Magnetometer is always used as attitude sensor when control is provided by MACS. So, failure of the magnetometer may end up in losing a satellite. In this case an algorithm implementing the data from another sensor for control synthesis is vital. In this paper a universal synthesis scheme is proposed. It is necessary to have an attitude sensor measuring a certain vector in the reference frame of a satellite which is also known in the inertial space via models. Magnetometer is one case of such a sensor, and corresponding algorithm called “-Bdot” is widely used for damping satellite’s angular velocity [1].

A general control scheme is similar to that of “-Bdot” is proposed in this paper. The algorithm implements the sun sensor as attitude sensor. Note that the direction to the Sun is assumed permanent since the time scale of the problems considered is about a day.

1. Problem statement

Here we introduce all necessary reference frames, equations of motion and corresponding variables, geomagnetic field model and general analysis method.

Dealing with MACS one should have a geomagnetic field model. Let us describe the model used in this work. Geomagnetic induction vector in a given point in orbit is often approximated by the Gauss decomposition [2]. This model, however, cannot be used for analytical analysis. So, some simplifications are introduced.

Considering three front terms in the decomposition, one obtains the inclined dipole model. The geomagnetic field is one of the dipole tilted in angle of 168˚26’ to the Earth’s axis and positioned in its center. This model allows rather simple analytical representation but it is still too complicated to obtain the solution of equations of motion in the explicit form. Further simplification called the right dipole model is widely used in analytical and numerical analysis. In this case geomagnetic field is represented as one of the dipole placed in the center of Earth and directed antiparallel to its axis. Geomagnetic field induction vector moves almost uniformly on the near-circular cone when satellite moves along the orbit.

This model still does not allow us to obtain the solution of equations of motion in the explicit form. So, it is logical to make following simplification, modeling the field induction vector moving uniformly on the circular cone. We introduce inertial reference system $O_aY_1Y_2Y_3$, where O_a is the Earth’s center, O_aY_3 axis is directed along the Earth’s axis, O_aY_1 lies in the equatorial plane and is directed to the ascending node of the satellite’s orbit, O_aY_2 is directed so the system to be right-handed. If we now translate the field induction vector to the Earth’s center then the cone is tangent to the O_aY_3 axis and its axis lies in the $O_aY_2Y_3$ plane. The cone half- angle is given by [3]

$$\mathrm{tg}\,\Theta = \frac{3\sin 2i}{2\left(1 - 3\sin^2 i + \sqrt{1 + 3\sin^2 i}\right)} \qquad (1.1)$$

where i is the orbit inclination (Fig.1). Geomagnetic field intensity vector moves uniformly on the cone with double orbital velocity, $\chi = 2\omega_0 t + \chi_0 = 2u + \chi_0'$. Here ω_0 is orbital velocity. Without loss of generality we assume $\chi_0' = 0$.

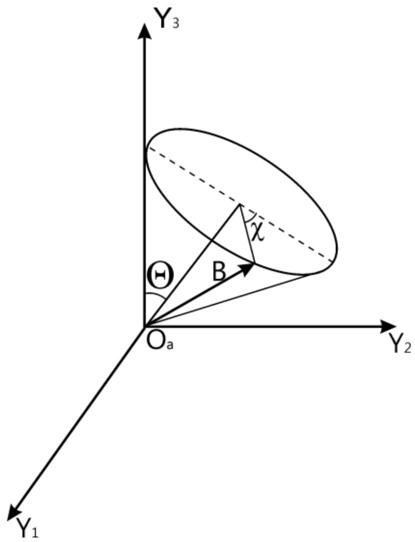

Fig. 1. Averaged geomagnetic field model

This sometimes called averaged model is used in this paper. Though it does not allow us to take into account non-uniformity of geomagnetic induction vector movement (as right dipole model does) and its diurnal change (as inclined dipole model does) it still allows us to describe geomagnetic field with proper accuracy, providing the balance between the authenticity and simplicity of equations. Detailed comparison of the models may be found in [2]. Geomagnetic induction vector in $O_aZ_1Z_2Z_3$ frame takes form

$$\mathbf{B}_Z = B_0\begin{pmatrix}\sin\Theta\sin 2u\\ \sin\Theta\cos 2u\\ \cos\Theta\end{pmatrix}. \qquad (1.2)$$

Let us introduce all necessary reference frames.

$O_aZ_1Z_2Z_3$ is the inertial frame resulted from the rotation of $O_aY_1Y_2Y_3$ around O_aY_1 axis by angle Θ.

$OL_1L_2L_3$ is the frame associated with the angular momentum of a satellite. O is the satellite's center of mass, OL_3 axis is directed along the angular momentum vector, OL_2 is perpendicular to OL_3 and lies in the plane parallel to $O_aZ_1Z_2$ and containing O, OL_1 is directed so the system is right-handed.

$Ox_1x_2x_3$ is fixed reference frame, its axes coincide with principal axes of inertia of a satellite.

Reference frame's mutual orientation will be described in terms of direct cosines matrices $\mathbf{Q},\mathbf{A}$ presented by their elements

$$\begin{array}{cccc} & L_1 & L_2 & L_3 \\ Z_1 & q_{11} & q_{12} & q_{13} \\ Z_2 & q_{21} & q_{22} & q_{23} \\ Z_3 & q_{31} & q_{32} & q_{33} \end{array}, \qquad \begin{array}{cccc} & x_1 & x_2 & x_3 \\ L_1 & a_{11} & a_{12} & a_{13} \\ L_2 & a_{21} & a_{22} & a_{23} \\ L_3 & a_{31} & a_{32} & a_{33} \end{array}.$$

We introduce indices Z, L, x to represent vectors in reference systems $O_aZ_1Z_2Z_3$, $OL_1L_2L_3$ and $Ox_1x_2x_3$. For example, for the first component of torque we write M_{1Z}, M_{1L}, M_{1x} respectively.

We will use the osculating, or Beletsky-Chernousko, variables. These variables are $L, \rho, \sigma, \varphi, \psi, \theta$ where L is the angular momentum magnitude, ρ, σ represent its attitude with respect to the inertial space $O_aZ_1Z_2Z_3$ (Fig. 2). Mutual $Ox_1x_2x_3$ and $OL_1L_2L_3$ frames attitude is expressed via the Euler angles φ, ψ, θ. This variables set was first introduced by Bulgakov [4] representing the gyro motion. Beletsky proposed to use these variables for the axisymmetrical satellite [5], while Chernousko used them for the three-axis satellite [6]. Unperturbed motion in φ, ψ, θ angles was proposed by Wittaker [7], however evolutionary equations were not considered. Direct cosines matrix $\mathbf{Q}$ takes form

$$\mathbf{Q} = \begin{pmatrix} \cos\rho\cos\sigma & -\sin\sigma & \sin\rho\cos\sigma \\ \cos\rho\sin\sigma & \cos\sigma & \sin\rho\sin\sigma \\ -\sin\rho & 0 & \cos\rho \end{pmatrix}. \qquad (1.3)$$

Matrix $\mathbf{A}$ has the form

$$\mathbf{A} = \begin{pmatrix} \cos\varphi\cos\psi - \cos\theta\sin\varphi\sin\psi & -\sin\varphi\cos\psi - \cos\theta\cos\varphi\sin\psi & \sin\theta\sin\psi \\ \cos\varphi\sin\psi + \cos\theta\sin\varphi\cos\psi & -\sin\varphi\sin\psi + \cos\theta\cos\varphi\cos\psi & -\sin\theta\cos\psi \\ \sin\theta\sin\varphi & \sin\theta\cos\varphi & \cos\theta \end{pmatrix}. \qquad (1.4)$$

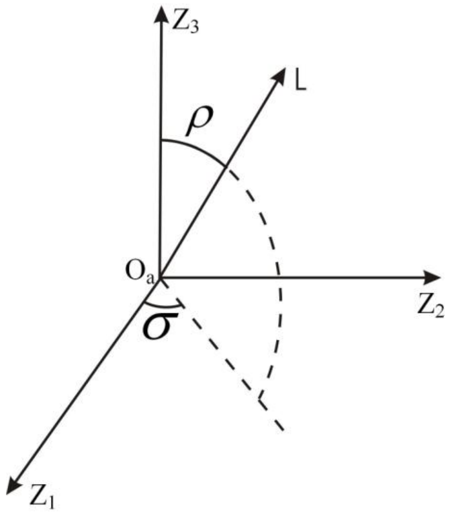

Fig. 2. Angular momentum attitude in the inertial space

Three-axis satellite with inertia tensor $\mathbf{J}_x = diag(A,B,C)$ obey the following equations [8],

$$\frac{dL}{dt}=M_{3L},\ \frac{d\rho}{dt}=\frac{1}{L}M_{1L},\ \frac{d\sigma}{dt}=\frac{1}{L\sin\rho}M_{2L},$$

$$\frac{d\theta}{dt}=L\sin\theta\sin\varphi\cos\varphi\left(\frac{1}{A}-\frac{1}{B}\right)+\frac{1}{L}\left(M_{2L}\cos\psi-M_{1L}\sin\psi\right),$$

$$\frac{d\varphi}{dt}=L\cos\theta\left(\frac{1}{C}-\frac{\sin^2\varphi}{A}-\frac{\cos^2\varphi}{B}\right)+\frac{1}{L\sin\theta}\left(M_{1L}\cos\psi+M_{2L}\sin\psi\right), \qquad (1.5)$$

$$\frac{d\psi}{dt}=L\left(\frac{\sin^2\varphi}{A}+\frac{\cos^2\varphi}{B}\right)-\frac{1}{L}M_{1L}\cos\psi\operatorname{ctg}\theta-\frac{1}{L}M_{2L}\left(\operatorname{ctg}\rho+\sin\psi\operatorname{ctg}\theta\right)$$

where M_{1L}, M_{2L}, M_{3L} are the angular momentum components in $OL_1L_2L_3$ frame.

Axisymmetrical satellite ($\mathbf{J}_x = diag(A,A,C)$) equations of motion are

$$\frac{dL}{dt}=M_{3L},\ \frac{d\rho}{dt}=\frac{1}{L}M_{1L},\ \frac{d\sigma}{dt}=\frac{1}{L\sin\rho}M_{2L},$$

$$\frac{d\theta}{dt}=\frac{1}{L}\left(M_{2L}\cos\psi-M_{1L}\sin\psi\right),$$

$$\frac{d\varphi}{dt}=L\cos\theta\left(\frac{1}{C}-\frac{1}{A}\right)+\frac{1}{L\sin\theta}\left(M_{1L}\cos\psi+M_{2L}\sin\psi\right), \qquad (1.6)$$

$$\frac{d\psi}{dt}=\frac{L}{A}-\frac{1}{L}M_{1L}\cos\psi\operatorname{ctg}\theta-\frac{1}{L}M_{2L}\left(\operatorname{ctg}\rho+\sin\psi\operatorname{ctg}\theta\right).$$

Averaging technique is used for the transient motion analysis [9]. To do so we assume that the control torque is small. Angular momentum change during one orbit revolution and one revolution about the center of mass is small in comparison with the angular momentum itself. In this case small parameter ε may be introduced and equations (1.5) take form

$$\frac{d\mathbf{x}}{dt} = \varepsilon\mathbf{X}(\mathbf{x},\mathbf{y},t), \frac{d\mathbf{y}}{dt} = \mathbf{y}_0(\mathbf{x},\mathbf{y}) + \varepsilon\mathbf{Y}(\mathbf{x},\mathbf{y},t), \qquad (1.7)$$

where $\mathbf{y} = (\varphi,\psi,u,\theta)$ are fast variables while $\mathbf{x} = (l,\rho,\sigma)$ are slow ones. Averaging method may be used for the slow variables evolution analysis. Unperturbed motion is regular precession for the axisymmetrical satellite. This case is studied here and (1.6) are of the form

$$\frac{d\mathbf{x}}{dt} = \varepsilon\mathbf{X}(\mathbf{x},\mathbf{y},t), \frac{d\mathbf{y}}{dt} = \mathbf{y}_0(\mathbf{x}) + \varepsilon\mathbf{Y}(\mathbf{x},\mathbf{y},t). \qquad (1.8)$$

Variable θ becomes slow one. Averaging over the time is identical to the averaging over fast variables. So it is necessary to simply average equations for the slow variables over the fast ones. This leads to the accuracy of the order ε on the time span of the order of $1/\varepsilon$. Using averaged geomagnetic field model and axisymmetrical satellite allows the evolutionary equations to be solved in quadratures.

We need equations (1.6) in the dimensionless form. We introduce the argument of latitude $u = \omega_0(t - t_0)$ instead of time, where t_0 is some fixed moment; dimensionless angular momentum l according to $L = L_0 l$ where L_0 is the initial angular momentum magnitude; and dimensionless control torque $\overline{\mathbf{M}}$. Equations (1.6) are rewritten in the form

$$\frac{dl}{du}=\varepsilon l\overline{M}_{3L},\ \frac{d\rho}{du}=\varepsilon\overline{M}_{1L},\ \frac{d\sigma}{du}=\frac{\varepsilon}{\sin\rho}\overline{M}_{2L},$$

$$\frac{d\varphi}{du}=\eta_1 l\cos\theta+\frac{\varepsilon}{\sin\theta}\left(\overline{M}_{1L}\cos\psi+\overline{M}_{2L}\sin\psi\right),$$

$$\frac{d\psi}{du}=\eta_2 l-\varepsilon\overline{M}_{1L}\cos\psi\operatorname{ctg}\theta-\varepsilon\overline{M}_{2L}\left(\operatorname{ctg}\rho+\sin\psi\operatorname{ctg}\theta\right),$$

$$\frac{d\theta}{du}=\varepsilon\left(\overline{M}_{2L}\cos\psi-\overline{M}_{1L}\sin\psi\right)$$

where $\varepsilon=\frac{kB_0^2}{L_0},\eta_1=\frac{L_0}{\omega_0}\left(\frac{1}{C}-\frac{1}{A}\right),\ \eta_2=\frac{L_0}{A\omega_0}$.

Small control torque leads to ε and $\frac{\varepsilon}{\eta_i}$ being small. Fast variables are (φ,ψ,u) and slow ones are (l,ρ,σ,θ). This allows us to move to new simplified equations

$$\begin{aligned}&\frac{dl}{du}=\varepsilon l\left\langle\overline{M}_{3L}\right\rangle,\ \frac{d\rho}{du}=\varepsilon\left\langle\overline{M}_{1L}\right\rangle,\ \frac{d\sigma}{du}=\frac{\varepsilon}{\sin\rho}\left\langle\overline{M}_{2L}\right\rangle,\\&\left\langle\frac{d\theta}{du}\right\rangle=\varepsilon\left\langle\left\langle\overline{M}_{2L}\right\rangle_u\cos\psi-\left\langle\overline{M}_{1L}\right\rangle_u\sin\psi\right\rangle_\psi\end{aligned}\tag{1.9}$$

where $\langle x\rangle$ corresponds to the value averaged over all fast variables (it is not necessary to average over φ for the axisymmetrical satellite). These equations represent slow angular momentum motion in the inertial space and angle between the axis of symmetry and the angular momentum.

2. Control construction

Let us first consider the control synthesis of well-known “-Bdot” algorithm [1]. We write equations of motion in the form

$$\frac{d\boldsymbol{\omega}}{dt}+\boldsymbol{\omega}\times\mathbf{J}\boldsymbol{\omega}=\mathbf{M},$$

In case of a satellite subjected to the gravity-gradient torque equations of motions admit the Jacobi first integral

$$J=\frac{1}{2}\left(\left(\boldsymbol{\omega},\mathbf{J}\boldsymbol{\omega}\right)+3\omega_0^2\left(\mathbf{e}_3,\mathbf{J}\mathbf{e}_3\right)-\omega_0\left(\mathbf{e}_2,\mathbf{J}\boldsymbol{\omega}\right)\right)$$

where $\mathbf{e}_3, \mathbf{e}_2$ are unit vectors of radius-vector and normal to the orbital plane in the fixed frame.

If a satellite is also subjected to the magnetic torque $\mathbf{M} = \mathbf{m} \times \mathbf{B}$ this expression will no longer be the first integral. Let

$$V = J - 0.5\omega_0^2 (3C - B)$$

be the Lyapunov function candidate. Its derivative according to the equations of motion is

$$\frac{dV}{dt} = \sum_{i=1}^{3} (\Omega_i - \omega_0 d_{2i}) M_i = \mathbf{M}(\mathbf{\Omega} - \omega_0 \mathbf{e}_2).$$

This can be expressed as

$$\frac{dV}{dt} = \mathbf{m}(\mathbf{B} \times \mathbf{\Omega}) \tag{2.1}$$

where $\mathbf{\Omega} = \boldsymbol{\omega} - \omega_0 \mathbf{e}_2$ is relative satellite velocity. Expression (2.1) allows us to find the dipole moment

$$\mathbf{m} = k\mathbf{\Omega} \times \mathbf{B}$$

providing asymptotical stability of orbital attitude, that is the attitude while two axes of fixed frame coincide with radius-vector and normal to the orbit respectively. Here k is a positive coefficient.

In case of high satellite velocity, that is $\omega_0 << |\boldsymbol{\omega}|$, we have $\mathbf{\Omega} \approx \boldsymbol{\omega}$ and the control law may be expressed as

$$\mathbf{m} = k\boldsymbol{\omega} \times \mathbf{B}. \tag{2.2}$$

It can be further transformed to the form

$$\mathbf{m} = -k \frac{d\mathbf{B}}{dt}. \tag{2.3}$$

This can be justified by the relation

$$\mathbf{AQ} \frac{d\mathbf{B}_Z}{dt} = \frac{d\mathbf{B}_x}{dt} + \boldsymbol{\omega} \times \mathbf{B}_x .$$

In case of a high satellite rate velocity the expression in the left side may be neglected and (2.3) became obvious if one considers (2.2).

Let us study control law (2.2). It provides damping of the absolute angular velocity. Suppose we do not have samplings from the magnetometer but instead from any other sensor. In this case we cannot use any of the algorithms mentioned above since they include magnetic field direction. So, we need a control law for magnetorquers which implements the information about any other vector $\mathbf{S}$ rather than about $\mathbf{B}$. The notation $\mathbf{S}$ is chosen because the case of sun sensor is of particular interest in this paper.

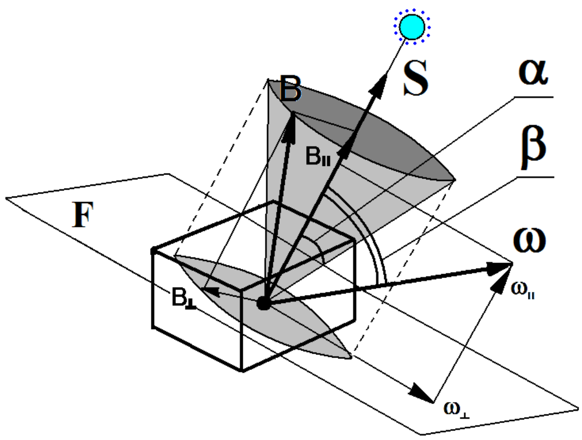

Fig. 3. Control law synthesis

We decompose $\boldsymbol{\omega}$ and $\mathbf{B}$ into components along vector $\mathbf{S}$ and perpendicular to it (Fig. 3)

$\boldsymbol{\omega} = \boldsymbol{\omega}_{\perp} + \boldsymbol{\omega}_{II}$, $\mathbf{B} = \mathbf{B}_{\perp} + \mathbf{B}_{II}$.

So, the control law (2.2) takes form

$$\mathbf{m} = k\left(\boldsymbol{\omega}_{II} \times \mathbf{B}_{\perp}\right) + k\left(\boldsymbol{\omega}_{\perp} \times \mathbf{B}_{II}\right).$$

Let us consider the term

$$\mathbf{m} = k\left(\boldsymbol{\omega}_{\perp} \times \mathbf{B}_{II}\right)$$

only as the control law. In this case we get

$$\mathbf{m} = kB\cos\alpha\left(\boldsymbol{\omega}_{\perp} \times \mathbf{S}\right).$$

Since $\boldsymbol{\omega}_{||} \times \mathbf{S} = 0$ this can be written as

$$\mathbf{m} = kB\cos\alpha\left(\boldsymbol{\omega} \times \mathbf{S}\right). \tag{2.4}$$

In case we have magnetometer readings $\mathbf{S}=\mathbf{B}$ and, therefore, $\alpha=0$, the control law (2.4) takes form (2.2). The control (2.4) may be considered as a general law for the satellite's rate velocity damping while readings available from a sensor providing any unit vector $\mathbf{S}$ in the fixed reference frame. Angle α between $\mathbf{S}$ and $\mathbf{B}$ is computed using models, defining those vectors in the inertial space for any given point in the satellite's orbit. It is not clear whether satellite's velocity is damped to zero value or not. Following control analysis shows that the terminal velocity is not zeroed. Control law (2.4) provides another important feature. It allows the one-axis attitude in the inertial space. This may be useful for the onboard batteries charge. Instead of using main actuators – reaction wheels or thrusters – magnetorquers may be used. This results in lowering power or fuel consumption.

3. Axisymmetrical satellite analysis

3.1. Averaged equations

Consider an axisymmetrical satellite and equations (1.9) for the slow variables evolution. To get these equations we need to average the control torque given by

$$\mathbf{M}=kB_0^2\cos\alpha\left(\boldsymbol{\omega}\times\mathbf{S}\right)\times\mathbf{B}.$$

This can be written in $OL_1L_2L_3$ frame as

$$\mathbf{M}_L=kB_0^2\left[-\cos^2\alpha\boldsymbol{\omega}_L+\mathbf{S}_L\left(\boldsymbol{\omega}_L\mathbf{B}_L\right)\cos\alpha\right].$$

First we find $\langle M_{iL}\rangle$ averaged torque components. Note that $\boldsymbol{\omega}$ should be averaged by ψ only. Therefore $\langle\boldsymbol{\omega}\rangle_\psi$ can be found independently. To do so we write the angular velocity in a form $\boldsymbol{\omega}_x=\mathbf{J}_x^{-1}\mathbf{L}_x$. Angular momentum in the $OL_1L_2L_3$ frame has simple form $\mathbf{L}_L=\left(0,0,L\right)^T$ and $\mathbf{L}_x=\mathbf{A}^T\mathbf{L}_L$ in the bound frame. Specifically, $\mathbf{L}_x=L\left(a_{31},a_{32},a_{33}\right)^T$. This leads to the angular velocity in the bound frame $\boldsymbol{\omega}_x=L\left(\frac{1}{A}a_{31},\frac{1}{A}a_{32},\frac{1}{C}a_{33}\right)^T$. Finally moving to the $OL_1L_2L_3$ frame,

$$\boldsymbol{\omega}_L = \mathbf{A}\boldsymbol{\omega}_x = L\begin{pmatrix} \frac{1}{A}a_{11}a_{31} + \frac{1}{A}a_{12}a_{32} + \frac{1}{C}a_{13}a_{33} \\ \frac{1}{A}a_{21}a_{31} + \frac{1}{A}a_{22}a_{32} + \frac{1}{C}a_{23}a_{33} \\ \frac{1}{A}a_{31}a_{31} + \frac{1}{A}a_{32}a_{32} + \frac{1}{C}a_{33}a_{33} \end{pmatrix}.$$

Averaging angular velocity over fast angle leads to

$$\left\langle \boldsymbol{\omega}_L \right\rangle_\psi = L\left(0,\, 0,\, \frac{1}{C}\cos^2\theta + \frac{1}{A}\sin^2\theta\right).$$

$\cos\alpha$ may be expressed in a form

$$\cos\alpha = B_{1Z}S_{1Z} + B_{2Z}S_{2Z} + B_{3Z}S_{3Z} \tag{3.1}$$

where S_{iZ} $(i = 1,2,3)$ are considered constant since $\mathbf{S}_Z$ is the Sun direction vector in the inertial space. Its movement is very slow in comparison with the satellite movement. $\mathbf{B}_Z$ and $\cos\alpha$ depend on the fast variable u and averaged torque may be written as

$$\left\langle \mathbf{M}_L \right\rangle_{u,\psi} = \varepsilon l\overline{\omega}\begin{pmatrix} S_{1L}\left\langle B_{3L}\cos\alpha\right\rangle_u \\ S_{2L}\left\langle B_{3L}\cos\alpha\right\rangle_u \\ -S_{2L}\left\langle B_{2L}\cos\alpha\right\rangle_u - S_{1L}\left\langle B_{1L}\cos\alpha\right\rangle_u \end{pmatrix} \tag{3.2}$$

where $\overline{\omega} = \cos^2\theta + \frac{C}{A}\sin^2\theta$, $\varepsilon = \frac{kB_0^2 L_0}{C}$ is a small parameter. First we need expressions

$$B_{ij} = \frac{1}{2\pi}\int_0^{2\pi} B_{iZ}B_{jZ}\,du \;\; (i, j = 1,2,3).$$

After some mathematics using (1.2) we get

$$B_{11} = B_{22} = \frac{1}{2}\sin^2\Theta = p\,,\; B_{33} = \cos^2\Theta = q\,,\; B_{12} = B_{23} = B_{13} = 0\,.$$

Expressions in (3.2) can be written as

$$\left\langle B_{3L}\cos\alpha\right\rangle = pS_{1Z}q_{13} + pS_{2Z}q_{23} + qS_{3Z}q_{33}\,,$$

$$\left\langle B_{2L}\cos\alpha\right\rangle = pS_{1Z}q_{12} + pS_{2Z}q_{22} + qS_{3Z}q_{32}\,,$$

$\langle B_{1L}\cos\alpha\rangle = pS_{1Z}q_{11} + pS_{2Z}q_{21} + qS_{3Z}q_{31}$.

Finally averaged control torque components obey following relations,

$$\langle M_{1L}\rangle = \varepsilon\overline{\omega}\sum q_{j1}S_{jZ}\left(pS_{1Z}q_{13} + pS_{2Z}q_{23} + qS_{3Z}q_{33}\right),$$
$$\langle M_{2L}\rangle = \varepsilon\overline{\omega}\sum q_{j2}S_{jZ}\left(pS_{1Z}q_{13} + pS_{2Z}q_{23} + qS_{Z3}q_{33}\right),$$
$$\langle M_{3L}\rangle = -\varepsilon\overline{\omega}\sum q_{j2}S_{jZ}\left(pS_{1Z}q_{12} + pS_{2Z}q_{22} + qS_{3Z}q_{32}\right) -$$
$$-\varepsilon\overline{\omega}\sum q_{j1}S_{jZ}\left(pS_{1Z}q_{11} + pS_{2Z}q_{21} + qS_{3Z}q_{31}\right). \tag{3.3}$$

We need new variable γ instead of σ in order to simplify averaged equations of motion. Furthermore this variable has clear meaning being the angle between $\mathbf{S}$ and $\mathbf{L}$, that is

$$\cos\gamma = S_{1Z}q_{13} + S_{2Z}q_{23} + S_{3Z}q_{33}. \tag{3.4}$$

This variable is introduced in (3.3) to get new equations of motion featuring variable γ, instead of (1.9) featuring σ. $\langle M_{3L}\rangle$ can be expressed in a form $\langle M_{3L}\rangle = \varepsilon\overline{\omega}\left(a\cos^2\Theta + b\sin^2\Theta\right)$. Here

$$a = \left(q_{12}S_{1Z} + q_{22}S_{2Z} + q_{32}S_{3Z}\right)S_{3Z}q_{32} + \left(q_{11}S_{1Z} + q_{21}S_{2Z} + q_{31}S_{3Z}\right)S_{3Z}q_{31} =$$
$$= q_{12}S_{1Z}S_{3Z}q_{32} + q_{22}S_{2Z}S_{3Z}q_{32} + (q_{32}S_{3Z})^2 + q_{11}S_{1Z}S_{3Z}q_{31} + q_{21}S_{2Z}S_{3Z}q_{31} + (S_{3Z}q_{31})^2.$$

Since $q_{11}q_{31} + q_{12}q_{32} + q_{13}q_{33} = 0$ and $q_{21}q_{31} + q_{22}q_{32} + q_{23}q_{33} = 0$ we get

$$a = -S_{1Z}S_{3Z}q_{13}q_{33} - S_{2Z}S_{3Z}q_{33}q_{23} + S_{3Z}^2 - (S_{3Z}q_{33})^2 =$$
$$= S_{3Z}^2 - S_{3Z}q_{33}(S_{1Z}q_{13} + S_{2Z}q_{23} + S_{3Z}q_{33}).$$

Taking into account (3.4) we finally get

$$a = (S_{3Z}^2 - S_{3Z}q_{33}\cos\gamma).$$

Same reasoning leads to

$$b = \frac{1}{2}(S_{1Z}^2 - S_{1Z}q_{13}\cos\gamma + S_{2Z}^2 - S_{2Z}q_{23}\cos\gamma).$$

$\langle M_{3L}\rangle$ may be now rewritten as

$$\langle M_{3L}\rangle = \varepsilon\overline{\omega}p(S_{1Z}^2 - S_{1Z}q_{13}\cos\gamma + S_{2Z}^2 - S_{2Z}q_{23}\cos\gamma) + \varepsilon\overline{\omega}q(S_{3Z}^2 - S_{3Z}q_{33}\cos\gamma).$$

Since $S_{1Z}q_{13}\cos\gamma + S_{2Z}q_{23}\cos\gamma + S_{3Z}q_{33}\cos\gamma = \cos^2\gamma$ it can be further simplified,

$$-\langle M_{3L}\rangle = \varepsilon\overline{\omega}p(1 - S_{3Z}^2 + S_{3Z}q_{33}\cos\gamma - \cos^2\gamma) + \varepsilon\overline{\omega}q(S_{3Z}^2 - S_{3Z}q_{33}\cos\gamma) =$$
$$= \varepsilon\overline{\omega}p\sin^2\gamma + \varepsilon\overline{\omega}(q-p)(S_{3Z}^2 - S_{3Z}q_{33}\cos\gamma).$$

To obtain evolutionary equation for γ we consider the following expression,

$$-\frac{d\gamma}{du}\sin\gamma = (\cos\rho\cos\sigma S_{1Z} + \cos\rho\sin\sigma S_{2Z} - S_{3Z}\sin\rho)\frac{d\rho}{du} +$$

$$+\sin\rho(-\sin\sigma S_{1Z} + \cos\sigma S_{2Z})\frac{d\sigma}{du} = \varepsilon\langle M_{1L}\rangle(\cos\rho\cos\sigma S_{1Z} + \cos\rho\sin\sigma S_{2Z} - S_{2Z}\sin\rho) +$$
$$+\varepsilon\langle M_{2L}\rangle(-\sin\sigma + \cos\sigma) = \varepsilon\langle M_{1L}\rangle S_{1L} + \varepsilon\langle M_{2L}\rangle S_{2L} =$$
$$= \varepsilon\overline{\omega}\left(pS_{1Z}q_{13} + pS_{2Z}q_{23} + qS_{3Z}q_{33}\right)\left(S_{1L}^2 + S_{2L}^2\right).$$

This leads to

$$\frac{d\gamma}{du} = -\frac{\varepsilon\overline{\omega}}{\sin\gamma}\left(pS_{1Z}q_{13} + pS_{2Z}q_{23} + qS_{3Z}q_{33}\right)\left(S_{1L}^2 + S_{2L}^2\right). \tag{3.5}$$

Since

$$\begin{aligned} &pS_{1Z}q_{13} + pS_{2Z}q_{23} + qS_{3Z}q_{33} = p(\cos\gamma - S_{3Z}q_{33}) + qS_{3Z}q_{33} = \\ &= p\cos\gamma + S_{3Z}q_{33}\left(q-p\right) \end{aligned} \tag{3.6}$$

and

$$S_{1L}^2 + S_{2L}^2 = 1 - S_{3L}^2 = 1 - \cos^2\gamma = \sin^2\gamma\,, \tag{3.7}$$

(3.5) takes form

$$\frac{d\gamma}{du} = -\varepsilon\overline{\omega}\sin\gamma\left(p\cos\gamma + S_{3Z}\cos\rho\left(q-p\right)\right).$$

Taking into account (3.6) we also get

$$\frac{d\rho}{du} = \varepsilon\overline{\omega}\left(q_{11}S_{1Z} + q_{21}S_{2Z} + q_{31}S_{3Z}\right)\left(p\cos\gamma + S_{3Z}q_{33}\left(q-p\right)\right).$$

To simplify this expression we implement the following,

$$q_{11}S_{1Z} + q_{21}S_{2Z} + q_{31}S_{3Z} = \frac{1}{q_{31}}\left(q_{11}q_{31}S_{1Z} + q_{21}q_{31}S_{2Z} + q_{31}^2S_{3Z}\right) =$$

$$= \frac{1}{q_{31}}\left(\left(q_{13}S_{1Z} + q_{23}S_{2Z} + q_{33}S_{3Z}\right) - q_{12}q_{32}S_{1Z} - q_{22}q_{32}S_{2Z} + \left(q_{31}^2 + q_{33}^2\right)S_{3Z}\right) =$$

$$=\frac{1}{q_{13}}\left(-q_{33}\cos\gamma+S_{3Z}\left(1-q_{32}^2\right)\right)=\frac{\cos\gamma\cos\rho-S_{3Z}}{\sin\rho}.$$

Finally angular momentum evolution is expressed using equations

$$\frac{d\rho}{du}=\varepsilon\bar{\omega}\frac{\cos\rho\cos\gamma-S_{3Z}}{\sin\rho}\left[p\cos\gamma+S_{3Z}\cos\rho\left(q-p\right)\right],$$

$$\frac{d\gamma}{du}=-\varepsilon\bar{\omega}\sin\gamma\left[p\cos\gamma+S_{3Z}\cos\rho\left(q-p\right)\right], \tag{3.8}$$

$$\frac{dl}{du}=-\varepsilon\bar{\omega}l\left[p\sin^2\gamma+(q-p)(S_{3Z}^2-S_{3Z}\cos\rho\cos\gamma)\right].$$

We need one more equation for θ, so expression

$$\left\langle M_{2L}\cos\psi\right\rangle-\left\langle M_{1L}\sin\psi\right\rangle$$

should be found. Consider its first term

$$\left\langle M_{2L}\cos\psi\right\rangle=\varepsilon l\mu g\left(\Theta\right)+\varepsilon lS_{2L}\left\langle\left(\boldsymbol{\omega}_L\mathbf{B}_L\right)\cos\alpha\right\rangle$$

where $g\left(\Theta\right)=\left\langle\cos^2\alpha\right\rangle=\left(1-S_{3Z}^2\right)p+S_{3Z}^2q$, $\left\langle\omega_{1L}\sin\psi\right\rangle=-\left\langle\omega_{2L}\cos\psi\right\rangle=$

$=L\frac{1}{2}\left(\frac{1}{C}-\frac{1}{A}\right)\sin\theta\cos\theta$ and $\mu=\frac{1}{2}\lambda\sin\theta\cos\theta$, $\lambda=1-C/A$. Next,

$$\left\langle\left(\boldsymbol{\omega}_L\mathbf{B}_L\right)\cos\alpha\right\rangle=-\mu\left\langle B_{2L}\cos\alpha\right\rangle=-\mu\left(q_{12}S_{1Z}+q_{22}S_{2Z}+q_{32}S_{3Z}\right)p=-\mu pS_{2L},$$

since $q_{32}=0$. Therefore

$$\left\langle M_{2L}\cos\psi\right\rangle=\varepsilon l\mu g\left(\Theta\right)-\varepsilon l\mu pS_{2L}^2$$

Для второго слагаемого аналогично имеем

$$\left\langle M_{1L}\sin\psi\right\rangle=-\varepsilon l\mu g\left(\Theta\right)+\varepsilon l\mu S_{1L}\left(pq_{11}S_{1Z}+pq_{21}S_{2Z}+pq_{31}S_{3Z}+(q-p)q_{31}S_{3Z}\right)=$$

$$=-\varepsilon l\mu g\left(\Theta\right)+\varepsilon l\mu S_{1L}\left(pS_{1L}+(q-p)q_{31}S_{3Z}\right).$$

Equation for θ right part takes the form

$$\left\langle M_\theta\right\rangle=2\varepsilon l\mu g\left(\Theta\right)-\varepsilon l\mu p\left(S_{2L}^2+S_{1L}^2\right)-\varepsilon l\mu\left(q-p\right)q_{31}S_{3Z}S_{1L}.$$

Since $q_{32}=0$,

$$q_{31}S_{1L}=q_{31}S_{1L}+q_{32}S_{2L}=S_{3Z}-q_{33}S_{3L}=S_{3Z}-\cos\rho\cos\gamma.$$

This leads to the final equation,

$$\frac{d\theta}{du}=\varepsilon\mu\left[2\left(\left(1-S_{3Z}^2\right)p+S_{3Z}^2q\right)-p\sin^2\gamma+\left(q-p\right)S_{3Z}\left(\cos\gamma\cos\rho-S_{3Z}\right)\right]. \quad (3.9)$$

Merging (3.8) and (3.9) we get evolutionary equations

$$\begin{aligned}
&\frac{d\rho}{du}=\varepsilon\frac{\cos\rho\cos\gamma-S_{3Z}}{\sin\rho}\left[p\cos\gamma+S_{3Z}\cos\rho\left(q-p\right)\right]\left(\cos^2\theta+\frac{C}{A}\sin^2\theta\right),\\
&\frac{d\gamma}{du}=-\varepsilon\sin\gamma\left[p\cos\gamma+S_{3Z}\cos\rho\left(q-p\right)\right]\left(\cos^2\theta+\frac{C}{A}\sin^2\theta\right),\\
&\frac{dl}{du}=-\varepsilon l\left[p\sin^2\gamma+(q-p)(S_{3Z}^2-S_{3Z}\cos\rho\cos\gamma)\right]\left(\cos^2\theta+\frac{C}{A}\sin^2\theta\right), \qquad (3.10)\\
&\frac{d\theta}{du}=\frac{1}{2}\varepsilon\lambda\left[2\left(\left(1-S_{3Z}^2\right)p+S_{3Z}^2q\right)-p\sin^2\gamma+\left(q-p\right)S_{3Z}\left(\cos\gamma\cos\rho-S_{3Z}\right)\right]\sin\theta\cos\theta.
\end{aligned}$$

These equation can be now analyzed.

3.2. Averaged equations first integral

Equations (3.10) allow full set of autonomous first integrals. We divide first equation by the second,

$$\frac{d\rho}{d\gamma}=-\frac{\cos\rho\cos\gamma-S_{3Z}}{\sin\rho\sin\gamma}.$$

This leads to

$$\cos\rho=I_1\sin\gamma+\cos\gamma S_{3Z}$$

and the first integral

$$I_1\sin\gamma=\cos\rho-\cos\gamma S_{3Z}. \qquad (3.11)$$

Next first integral is related to the angular momentum l. We introduce new variable

$$\delta=p\cos\gamma+\left(q-p\right)S_{3Z}\cos\rho.$$

Next,

$$\frac{d\delta}{du}=-p\sin\gamma\frac{d\gamma}{du}-S_{3Z}\sin\rho\left(q-p\right)\frac{d\rho}{du}=$$

$$=\varepsilon\delta p\sin^2\gamma-\varepsilon\delta S_{3Z}(\cos\rho\cos\gamma-S_{3Z})\left(q-p\right)=$$

$$= \varepsilon\delta\left(p\sin^2\gamma + (q-p)(S_{3Z}^2 - S_{3Z}\cos\rho\cos\gamma)\right) = -\delta\frac{1}{l}\frac{dl}{du},$$

therefore

$$\frac{d\delta}{du} + \delta\frac{1}{l}\frac{dl}{du} = 0,$$

$$l = \frac{I_2}{|\delta|} = \frac{I_2}{\left|p\cos\gamma + S_{3Z}(q-p)\cos\rho\right|}.$$

First integral may be expressed as

$$I_2 = l\left|p\cos\gamma + S_{3Z}(q-p)\cos\rho\right|. \tag{3.12}$$

We need one more first integral that introduces relation between θ and γ. To do so we rewrite the last equation in (3.10) taking into account (3.11). Knowing the $\cos\rho$ expression this equation is of the form

$$\frac{d\theta}{du} = \frac{1}{2}\varepsilon\lambda\left[2p + 2S_{3Z}^2(q-p) - p\sin^2\gamma + (q-p)\left(I_1 S_{3Z}\cos\gamma\sin\gamma - S_{3Z}^2\sin^2\gamma\right)\right]\sin\theta\cos\theta,$$

which leads to

$$\frac{d\theta}{du} = \frac{1}{2}\varepsilon\lambda\left[A_1\left(2-\sin^2\gamma\right) + A_2\sin\gamma\cos\gamma\right]\sin\theta\cos\theta \tag{3.13}$$

where $A_1 = p + (q-p)S_{3Z}^2$, $A_2 = I_1 S_{3Z}(q-p)$. Note that $A_1 > 0$. Analogous substitution of $\cos\rho$ in the second equation in (3.10) leads to

$$\frac{d\gamma}{du} = -\varepsilon\sin\gamma\left[p\cos\gamma + S_{3Z}(q-p)\left(I_1\sin\gamma + S_{3Z}\cos\gamma\right)\right]\left(\cos^2\theta + \frac{C}{A}\sin^2\theta\right)$$

and

$$\frac{d\gamma}{du} = -\varepsilon\sin\gamma\left[A_1\cos\gamma + A_2\sin\gamma\right]\left(\cos^2\theta + \frac{C}{A}\sin^2\theta\right). \tag{3.14}$$

Dividing (3.14) by (3.13) we get

$$\frac{A_1\left(2-\sin^2\gamma\right) + A_2\sin\gamma\cos\gamma}{\sin\gamma\left(A_1\cos\gamma + A_2\sin\gamma\right)}d\gamma = -2\frac{\cos^2\theta + \frac{C}{A}\sin^2\theta}{\lambda\sin\theta\cos\theta}d\theta,$$

and first integral

$$I_3 = \ln\frac{1-\cos\gamma}{1+\cos\gamma} - \ln\frac{4\left(A_1\cos\gamma + A_2\sin\gamma\right)}{\left(1+\cos\gamma\right)^2} - \frac{2}{\lambda}\left[\ln\left|\sin\theta\right| - \frac{C}{A}\ln\left|\cos\theta\right|\right].$$

The complete set of autonomous first integrals of equations (3.10) is found.

3.3. Equilibrium position and stability analysis

Satellite's symmetry axis equilibrium positions are of the utmost interest in this paper. Equations (3.10) may be rewritten omitting equations for l,

$$\frac{d\gamma}{du} = -\varepsilon\sin\gamma\left[A_1\cos\gamma + A_2\sin\gamma\right]\left(\cos^2\theta + \frac{C}{A}\sin^2\theta\right),$$

$$\frac{d\theta}{du} = \frac{1}{2}\varepsilon\lambda\left[A_1\left(2-\sin^2\gamma\right) + A_2\sin\gamma\cos\gamma\right]\sin\theta\cos\theta, \qquad (3.15)$$

$$\frac{d\rho}{du} = \varepsilon\frac{\cos\rho\cos\gamma - S_{3Z}}{\sin\rho}\left[A_1\cos\gamma + A_2\sin\gamma\right]\left(\cos^2\theta + \frac{C}{A}\sin^2\theta\right).$$

Equilibrium positions are

1. $\sin\gamma = 0$, $\theta = 0$ or $\theta = \pi/2$, $\cos\rho - S_{3Z} = 0$.
2. $A_1\cos\gamma + A_2\sin\gamma = 0$, $\theta = 0$ or $\theta = \pi/2$.

Consider the second case and linearize equations (3.15) in the vicinity of $\gamma_0 = \operatorname{atan}\left(-\frac{A_1}{A_2}\right)$. Since

$$A_1\cos\left(\gamma+\gamma_0\right) + A_2\sin\left(\gamma+\gamma_0\right) = \left(-A_1\sin\gamma_0 + A_2\cos\gamma_0\right),$$

$$\sin\left(\gamma+\gamma_0\right) = \sin\gamma_0,$$

we get

$$\frac{d\gamma}{du} = -\varepsilon\sin\gamma_0\cos\gamma_0\frac{A_1^2 + A_2^2}{A_2}\overline{\omega}.$$

Note that $A_1 > 0$ and $\operatorname{sign} A_2 = -\operatorname{sign}\left(\sin\gamma_0\cos\gamma_0\right)$. This means that the position $\gamma_0 = \operatorname{atan}\left(-\frac{A_1}{A_2}\right)$ is unstable ($\overline{\omega} > 0$).

Linearized equations for the $\sin\gamma = 0$ equilibrium vicinity are

$$\frac{d\gamma}{du} = -\varepsilon A_1 \gamma \left(\cos^2\theta + \frac{C}{A}\sin^2\theta \right),$$

$$\frac{d\theta}{du} = \varepsilon\lambda A_1 \sin\theta\cos\theta, \qquad (3.16)$$

$$\frac{d\rho}{du} = \varepsilon A_1 \frac{\cos\rho - S_{3Z}}{\sin\rho} \left(\cos^2\theta + \frac{C}{A}\sin^2\theta \right).$$

First equation in (3.16) brings two possibilities for equilibrium position. These positions are the same for the γ angle ($\gamma = 0$ or $\gamma = \pi$) and ρ angle ($\cos\rho - S_{3Z} = 0$). Depending on the inertia tensor of the satellite either $\theta = 0$ (if $\lambda < 0$, that is $C > A$) or $\theta = \pi/2$ (if $C < A$) is stable. In case the angular momentum vector coincides with the Sun direction vector ($\gamma = 0$) the angle between the angular momentum and the averaged geomagnetic model cone axis is equal to the angle between this axis and the Sun direction. This is emphasized by $\cos\rho - S_{3Z} = 0$.

First integral (3.12) allows the terminal angular momentum magnitude estimate,

$$l_{term} = \frac{\left| p\cos\gamma_0 + S_{3Z}\left(q-p\right)\cos\rho_0 \right|}{p + S_{3Z}^2\left(q-p\right)}. \qquad (3.17)$$

Algorithm (2.4) doesn't lead to the asymptotic angular velocity damping. However it provides inertial attitude stability. Stability conditions differ for the non-symmetrical satellite.

4. Non-axisymmetrical satellite stability analysis

Consider slightly non-symmetrical satellite in order to assess the stability conditions for the arbitrary satellite. We introduce small parameter

$$\eta = \frac{B-A}{C} \ll 1.$$

Small parameters η and ε product can be neglected in comparison with ε. Averaging over u and ψ leads to

$$
\begin{aligned}
&\frac{d\gamma}{du} = -\varepsilon \sin\gamma\left[A_1\cos\gamma + A_2\sin\gamma\right]\left(\cos^2\theta + \frac{C}{A}\sin^2\theta\right),\\
&\frac{d\rho}{du} = \varepsilon\frac{\cos\rho\cos\gamma - S_{3Z}}{\sin\rho}\left[A_1\cos\gamma + A_2\sin\gamma\right]\left(\cos^2\theta + \frac{C}{A}\sin^2\theta\right),\\
&\frac{dl}{du} = -\varepsilon l\left[A_1\sin^2\gamma - A_2\cos\gamma\sin\gamma\right]\left(\cos^2\theta + \frac{C}{A}\sin^2\theta\right),\\
&\frac{d\theta}{du} = \frac{L_0}{\omega_0}\frac{C}{A^2}\eta\sin\theta\sin\varphi\cos\varphi + \frac{1}{2}\varepsilon\lambda\left[A_1\left(2-\sin^2\gamma\right) + A_2\sin\gamma\cos\gamma\right]\sin\theta\cos\theta,\\
&\frac{d\varphi}{du} = \frac{L_0}{\omega_0}\lambda l\cos\theta + \frac{L_0}{\omega_0}\frac{C}{A^2}\eta\cos^2\varphi\cos\theta - \varepsilon S_{3Z}\lambda\left(q-p\right)\cos\theta\sin\rho S_{2L}.
\end{aligned}
\tag{4.1}
$$

Three first equations in (4.1) are identical to the equations for the axisymmetrical satellite. Two last equations introduce parameter η. γ, ρ, l equilibrium values hold, and we need to find equilibrium values for θ and φ. Fourth equations in (4.1) leads to the one of the conditions

$$
\left[\begin{aligned}
&\sin\theta = 0,\\
&\frac{L_0}{\omega_0}\frac{C}{A^2}\eta\sin\varphi\cos\varphi + \frac{1}{2}\varepsilon\lambda\left[A_1\left(2-\sin^2\gamma\right) + A_2\sin\gamma\cos\gamma\right]\cos\theta = 0.
\end{aligned}\right.
\tag{4.2}
$$

Fifth equation leads to the one of the conditions

$$
\left[\begin{aligned}
&\frac{L_0}{\omega_0}\lambda l + \frac{L_0}{\omega_0}\frac{C}{A^2}\eta\cos^2\varphi - \varepsilon S_{3Z}\lambda\left(q-p\right)\sin\rho S_{2L} = 0,\\
&\cos\theta = 0.
\end{aligned}\right.
\tag{4.3}
$$

Equilibrium position is determined by one of the relations in (4.2) and one relation from (4.3).

First note that if $\sin\theta = 0$ angle φ becomes fast and no equilibrium for this variable can be studied for the evolutionary equations. Fourth equation in (4.1) may be averaged over φ leading to the second equation in (3.15). Therefore if position $\sin\theta = 0$ is asymptotically stable ($C > A$) previous chapter results are still valid for the slightly non-symmetrical satellite.

If $\cos\theta = 0$ (from (4.3)) then (4.2) leads to $\sin\varphi\cos\varphi = 0$. This equilibrium position is new for the non-symmetrical satellite. Fourth and fifth equations in (4.1) are linearized in the vicinity of the equilibrium $\cos\theta = 0$, $\sin\varphi = 0$ (other variables equilibrium values are the same as in the previous chapter). We consider values $\theta = \pi/2$, $\varphi = 0$ since $\theta = 3\pi/2$ and $\varphi = \pi$ results are completely analogous. We introduce new variable $\beta = \pi/2 - \theta$ and take into account that S_{2L} is of the order of γ or less according to (3.7). Linearized equations are

$$\frac{d\beta}{du} = -\frac{L_0}{\omega_0}\frac{C}{A^2}\eta\varphi - \frac{1}{2}\varepsilon\lambda A_1\beta,$$

$$\frac{d\varphi}{du} = \frac{L_0}{\omega_0}\lambda l\beta + \frac{L_0}{\omega_0}\frac{C}{A^2}\eta\beta.$$

Considered equilibrium is asymptotically stable if $\eta > 0\,(B > A)$ and not stable otherwise. If $\eta < 0\,(B < A)$ asymptotically stable equilibrium positions are $\theta = \pi/2$, $\varphi = \pi/2$ and $\theta = 3\pi/2$, $\varphi = 3\pi/2$.

Maximum moment of inertia axis coincides with the angular momentum vector of the satellite in any case. This goes for the case $\sin\theta = 0$ also.

5. Numerical analysis

Numerical analysis of the satellite motion is present here. The satellite motion is governed by the magnetorquers implementing control law (2.4), gravitational external torque is taken into account also. Inertia tensor is of the “Chibis-M” microsatellite,

$$\mathbf{J} = \begin{pmatrix} 1.0255 & 0.0014 & -0.0724 \\ 0.0014 & 1.5393 & 0.0019 \\ -0.0724 & 0.0019 & 1.8172 \end{pmatrix} \text{ кг}\cdot\text{м}^2.$$

Equilibrium position $\sin\theta = 0$ is stable for this dynamical configuration. The satellite moves along the circular orbit with attitude of about 350 km (orbital velocity $\omega_0 = 10^{-3}$, close to the orbit of the International Space Station); right dipole moment model is used to represent the geomagnetic field; maximum dipole moment of each magnetorquer is 3.2 A·m^2. We assume that satellite attitude is known. Necessary attitude in the inertial space is given by the vector $\mathbf{S}=(5,3,1)$. Fig. 4 brings simulation results, that is angular velocity components and the angle between the third satellite axis and the necessary inertia position.

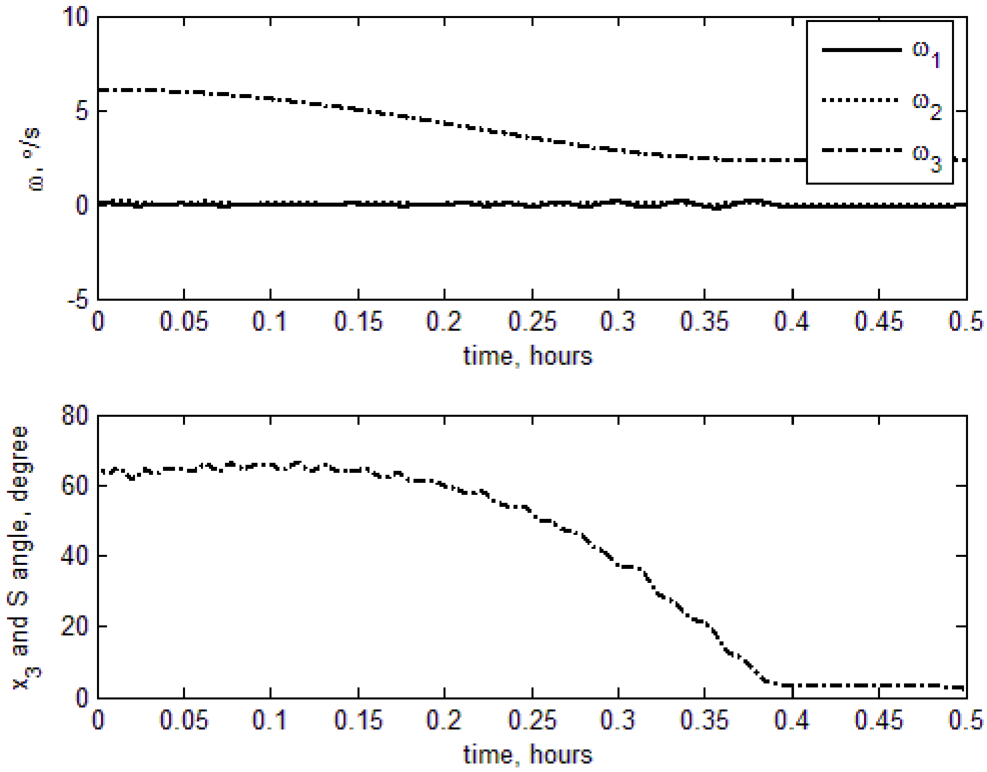

Fig. 4. Numerical analysis results

Equilibrium position $\sin\theta = 0$ is achieved. The satellite is terminally spun about the $\mathbf{S}$ direction. Angular momentum is about 0.08 kg·m^2/s. It is in good accordance with analytical results. This value should be about 0.07 kg·m^2/s according to (3.17). We introduce sun sensors readings error of 1° and sample rate 1 Hz. Magnetorquers sample rate is 5 Hz. Each second allows one sensor readings operation and five control operations. Magnetorquers implement only dipole moments ± 3.2 A·m^2, 0. The torque is constructed on the basis of (2.4) and reduced to the unit vector. If magnetorquer value exceeds 0.5 (or less than -0.5) the torque is implemented by this coil.

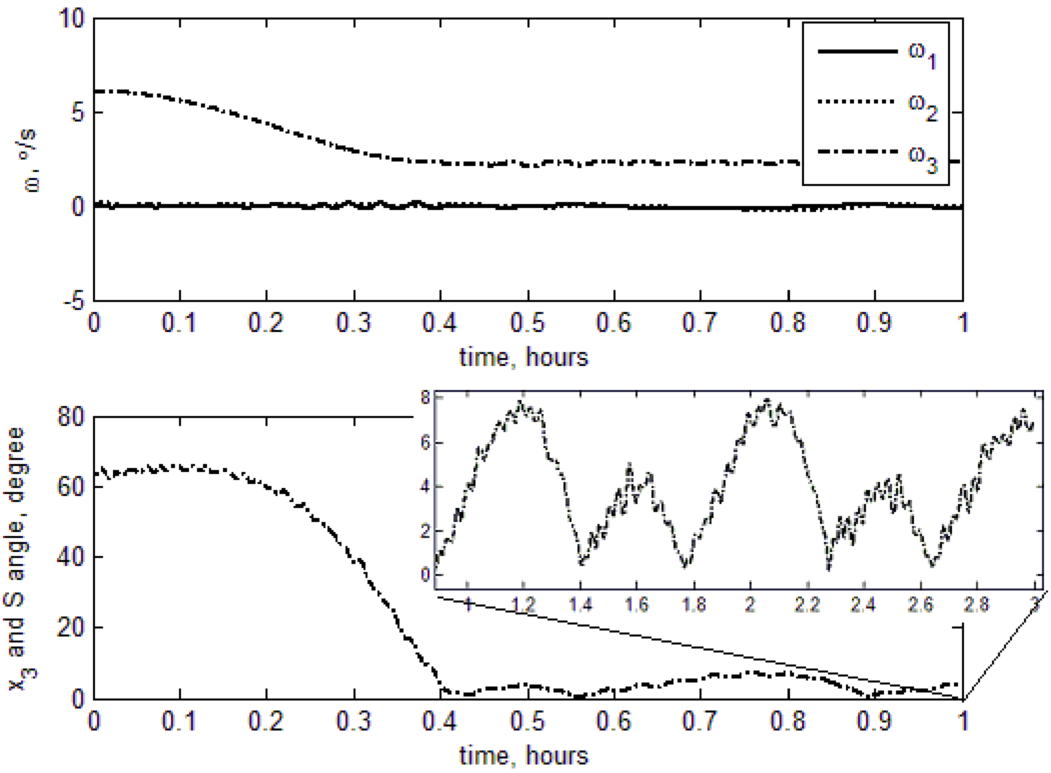

Fig. 5. Numerical analysis with sensors and magnetorquers parameters taken into account

Fig. 5 bring the same time-response of the algorithm but reduced accuracy that is about 8 degrees.

Conclusion

New magnetic control scheme is proposed. Magnetorquers utilize Sun sensors readings only, magnetometer is not used or is unavailable. The control constructed allows the satellite to be damped to the certain rotation value and one-axis inertial space attitude. Evolutionary equations of motion for the axisymmetrical satellite are obtained, full set of autonomous first integrals is present. These integrals bring terminal spinning value. Equilibrium positions are found that bring inertial attitude. Slightly non-symmetrical satellite evolutionary equations are obtained also, stable equilibrium positions depending on the inertia tensor are shown.

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