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Dynamics of gyrostat satellite subject to gravitational torque. Stability analysis

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Динамика спутника-гиростата под действием гравитационного момента. Исследование устойчивости положений равновесия

Исследована динамика спутника-гиростата, движущегося в центральном ньютоновом силовом поле по круговой орбите. Предложен метод определения всех положений равновесия спутника-гиростата в орбитальной системе координат при заданных значениях вектора гиростатического момента и главных центральных моментов инерции, получены условия их существования. Для каждой равновесной ориентации получены достаточные условия устойчивости с использованием в качестве функции Ляпунова обобщенного интеграла энергии. Проведен детальный численный анализ областей выполнения условий устойчивости положений равновесия в зависимости от четырех безразмерных параметров задачи. Показано, что число положений равновесия спутника-гиростата, для которых выполняются достаточные условия устойчивости, в общем случае изменяется при возрастании величины модуля гиростатического момента от 4 до 2.

Ключевые слова: спутник-гиростат, гравитационный момент, положения равновесия, устойчивость.

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Dynamics of gyrostat satellite subject to gravitational torque. Stability analysis

Dynamics of gyrostat satellite moving along a circular orbit in the central Newtonian gravitational field is investigated. A symbolic-numerical method for determining of all equilibrium orientations of gyrostat satellite in the orbital coordinate system with given gyrostatic torque and given principal central moments of inertia is proposed. For each equilibrium orientation sufficient conditions of stability are obtained as a result of analysis of generalized energy integral used as Lyapunov’s function. Investigation of domains where stability conditions take place is provided in detail depending on four dimensionless parameters of the problem. It is shown that the number of stable equilibria of the gyrostat satellite in general case changes from 4 to 2 with the increasing the absolute value of gyrostatic torque.

Key words: gyrostat satellite, gravitational torque, equilibria, stability.

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1. Equations of motion

Let us consider the attitude motion of a gyrostat satellite which is a rigid body with statically and dynamically balanced rotors inside the satellite body. The angular velocities of rotors relative to the satellite body are constant. The center of mass O of gyrostat satellite is situated in a circular orbit.

Let us introduce two right-hand Cartesian coordinate systems with origin in the center of mass O of the gyrostat satellite.

$OXYZ$ is the orbital coordinate system whose OZ axis is directed along the radius vector connecting the centers of mass of the Earth and of the gyrostat satellite; the OX axis is directed along the vector of linear velocity of the center of mass O.

$Oxyz$ is the gyrostat-fixed coordinate system; Ox, Oy, Oz are the principal central axes of inertia of the gyrostat satellite.

Let us define the orientation of the $Oxyz$ coordinate system with respect to the orbital coordinate system by Euler angles ψ, ϑ and φ. Now the direction cosines of Ox, Oy, Oz axes in the orbital coordinate system are represented by the following expressions [1]:

$$
\begin{aligned}
&a_{11} = \cos(x, X) = \cos\psi\cos\varphi - \sin\psi\cos\vartheta\sin\varphi,\\
&a_{12} = \cos(y, X) = -\cos\psi\sin\varphi - \sin\psi\cos\vartheta\cos\varphi,\\
&a_{13} = \cos(z, X) = \sin\psi\sin\vartheta,\\
&a_{21} = \cos(x, Y) = \sin\psi\cos\varphi + \cos\psi\cos\vartheta\sin\varphi,\\
&a_{22} = \cos(y, Y) = -\sin\psi\sin\varphi + \cos\psi\cos\vartheta\cos\varphi,\\
&a_{23} = \cos(z, Y) = -\cos\psi\sin\vartheta,\\
&a_{31} = \cos(x, Z) = \sin\vartheta\sin\varphi,\\
&a_{32} = \cos(y, Z) = \sin\vartheta\cos\varphi,\\
&a_{33} = \cos(z, Z) = \cos\vartheta.
\end{aligned}
\tag{1}
$$

Then, the equations of motion of the gyrostat satellite relative to its center of mass are written in the following form [1, 2]:

$$
\begin{aligned}
&A\dot{p} + (C - B)qr - 3\omega_0^2(C - B)a_{32}a_{33} - \bar{H}_2 r + \bar{H}_3 q = 0,\\
&B\dot{q} + (A - C)rp - 3\omega_0^2(A - C)a_{33}a_{31} - \bar{H}_3 p + \bar{H}_1 r = 0,\\
&C\dot{r} + (B - A)pq - 3\omega_0^2(B - A)a_{31}a_{32} - \bar{H}_1 q + \bar{H}_2 p = 0;
\end{aligned}
\tag{2}
$$

$$
\begin{aligned}
p &= \dot{\psi} a_{31} + \dot{\vartheta}\cos\varphi + \omega_0 a_{21} = \overline{p} + \omega_0 a_{21}, \\
q &= \dot{\psi} a_{32} - \dot{\vartheta}\sin\varphi + \omega_0 a_{22} = \overline{q} + \omega_0 a_{22}, \\
r &= \dot{\psi} a_{33} + \dot{\varphi} + \omega_0 a_{23} = \overline{r} + \omega_0 a_{23}.
\end{aligned} \tag{3}
$$

In equations (2), (3) $\overline{H}_1 = \sum_{k=1}^{n} J_k \alpha_k \dot{\varphi}_k$, $\overline{H}_2 = \sum_{k=1}^{n} J_k \beta_k \dot{\varphi}_k$, $\overline{H}_3 = \sum_{k=1}^{n} J_k \gamma_k \dot{\varphi}_k$; J_k is the axial moment of inertia of k-th rotor; $\alpha_k, \beta_k, \gamma_k$ are the constant direction cosines of the symmetry axis of the k-th rotor in the coordinate system $Oxyz$; $\dot{\varphi}_k$ is the constant angular velocity of the k-th rotor relative to the gyrostat; A, B, C are the principal central moments of inertia of the gyrostat; p, q, r are the projections of the absolute angular velocity of the gyrostat satellite onto the Ox, Oy, Oz axes; ω_0 is the angular velocity of motion of the center of mass of the gyrostat satellite along a circular orbit. Dots designate differentiation with respect to time t.

Further it will be more convenient to use parameters

$$
H_1 = \overline{H}_1 / \omega_0, \; H_2 = \overline{H}_2 / \omega_0, \; H_3 = \overline{H}_3 / \omega_0. \tag{4}
$$

For the systems of equations (2) and (3) the generalized energy integral exists in the form

$$
\begin{aligned}
&\frac{1}{2}(A\overline{p}^2 + B\overline{q}^2 + C\overline{r}^2) + \frac{3}{2}\omega_0^2[(A-C)a_{31}^2 + (B-C)a_{32}^2] + \\
&+\frac{1}{2}\omega_0^2[(B-A)a_{21}^2 + (B-C)a_{23}^2] - \omega_0^2(H_1 a_{21} + H_2 a_{22} + H_3 a_{23}) = const.
\end{aligned} \tag{5}
$$

2. Equilibrium orientations

Setting in (2) and (3) $\psi = \psi_0 = const$, $\vartheta = \vartheta_0 = const$, $\varphi = \varphi_0 = const$, we obtain at $A \neq B \neq C$ the equations

$$
\begin{aligned}
(C-B)(a_{22}a_{23} - 3a_{32}a_{33}) - H_2 a_{23} + H_3 a_{22} = P = 0, \\
(A-C)(a_{23}a_{21} - 3a_{33}a_{31}) - H_3 a_{21} + H_1 a_{23} = Q = 0, \\
(B-A)(a_{21}a_{22} - 3a_{31}a_{32}) - H_1 a_{22} + H_2 a_{21} = R = 0,
\end{aligned} \tag{6}
$$

allowing us to determine the gyrostat satellite equilibria in the orbital coordinate system. Actually, it is more convenient to use in subsequent investigation the equivalent system

$$
\begin{aligned}
&Pa_{11} + Qa_{12} + Ra_{13} = 0,\\
&Pa_{21} + Qa_{22} + Ra_{23} = 0,\\
&Pa_{31} + Qa_{32} + Ra_{33} = 0.
\end{aligned} \tag{7}
$$

System (6) depends on four dimensionless parameters

$$
h_1 = \frac{H_1}{B-C},\ h_2 = \frac{H_2}{B-C},\ h_3 = \frac{H_3}{B-C},\ \nu = \frac{B-A}{B-C}. \tag{8}
$$

Equations (7) are equivalent to equations (6) and can be rewritten in the form

$$
\begin{aligned}
&4(Aa_{21}a_{31} + Ba_{22}a_{32} + Ca_{23}a_{33}) + (H_1 a_{31} + H_2 a_{32} + H_3 a_{33}) = 0,\\
&Aa_{11}a_{31} + Ba_{12}a_{32} + Ca_{13}a_{33} = 0,\\
&(Aa_{11}a_{21} + Ba_{12}a_{22} + Ca_{13}a_{23}) + (H_1 a_{11} + H_2 a_{12} + H_3 a_{13}) = 0
\end{aligned} \tag{9}
$$

or using dimensionless parameters (8) in the form

$$
\begin{aligned}
&-4(\nu a_{21}a_{31} + a_{23}a_{33}) + (h_1 a_{31} + h_2 a_{32} + h_3 a_{33}) = 0,\\
&\nu a_{11}a_{31} + a_{13}a_{33} = 0,\\
&\nu a_{11}a_{21} + a_{13}a_{23} - (h_1 a_{11} + h_2 a_{12} + h_3 a_{13}) = 0.
\end{aligned} \tag{10}
$$

Taking into account expressions (1), system (6) or system (9) can be considered as a system of three equations with unknowns $\psi_0, \vartheta_0, \varphi_0$. The second more convenient method to close equations (9) consists in adding six conditions of orthogonality for the direction cosines (1)

$$
\begin{aligned}
&a_{11}^2 + a_{12}^2 + a_{13}^2 = 1,\\
&a_{21}^2 + a_{22}^2 + a_{23}^2 = 1,\\
&a_{31}^2 + a_{32}^2 + a_{33}^2 = 1,\\
&a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} = 0,\\
&a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} = 0,\\
&a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} = 0.
\end{aligned} \tag{11}
$$

Further, we will study the equilibrium orientations of the gyrostat satellite using systems (9) and (11).

As it was shown in [1, 2], the system of second equation in (9) and first, second, fourth, fifth and sixth equations in (11) can be solved for $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}$ if $A \neq B \neq C$, using dimensionless parameters (8) in the form:

$$a_{11} = \frac{-4a_{32}a_{33}}{h_1a_{31} + h_2a_{32} + h_3a_{33}}, \qquad a_{21} = \frac{4[\nu a_{32}^2 - (1-\nu)a_{33}^2]a_{31}}{h_1a_{31} + h_2a_{32} + h_3a_{33}},$$
$$a_{12} = \frac{4(1-\nu)a_{33}a_{31}}{h_1a_{31} + h_2a_{32} + h_3a_{33}}, \qquad a_{22} = \frac{-4(\nu a_{31}^2 + a_{33}^2)a_{32}}{h_1a_{31} + h_2a_{32} + h_3a_{33}}, \qquad (12)$$
$$a_{13} = \frac{4\nu a_{31}a_{32}}{h_1a_{31} + h_2a_{32} + h_3a_{33}}, \qquad a_{23} = \frac{4[(1-\nu)a_{31}^2 + a_{32}^2]a_{33}}{h_1a_{31} + h_2a_{32} + h_3a_{33}};$$

Substituting equations (12) in the first and third equations (9) and adding the third equation (11) we get three equations

$$\begin{aligned}
&16[a_{32}^2a_{33}^2 + (1-\nu)^2 a_{33}^2a_{31}^2 + \nu^2 a_{31}^2a_{32}^2] = (h_1a_{31} + h_2a_{32} + h_3a_{33})^2(a_{31}^2 + a_{32}^2 + a_{33}^2),\\
&4\nu(1-\nu)a_{31}a_{32}a_{33} + [h_1a_{32}a_{33} - h_2(1-\nu)a_{33}a_{31} - h_3\nu a_{31}a_{32}]\times\\
&\times(h_1a_{31} + h_2a_{32} + h_3a_{33}) = 0,\\
&a_{31}^2 + a_{32}^2 + a_{33}^2 = 1.
\end{aligned} \qquad (13)$$

for determination of direction cosines a_{31}, a_{32}, a_{33}. If system (13) will be solved then relations (12) allow us to find the other six direction cosines. In (13) the right part of first equation was multiplied by $a_{31}^2 + a_{32}^2 + a_{33}^2 = 1$.

Let us introduce the values $x = \frac{a_{31}}{a_{33}},\ y = \frac{a_{32}}{a_{33}}$ and divide all terms of first equation in (13) by a_{33}^4 and second equation by a_{33}^3. Then we will have the system of two equations with unknown values x, y:

$$\begin{aligned}
&16[y^2 + (1-\nu)^2 x^2 + \nu^2 x^2 y^2] = (h_1x + h_2y + h_3)^2(1 + x^2 + y^2),\\
&4\nu(1-\nu)xy + [h_1y - h_2(1-\nu)x - h_3\nu xy](h_1x + h_2y + h_3) = 0.
\end{aligned} \qquad (14)$$

Now substituting expressions $a_{31} = xa_{33},\ a_{32} = ya_{33}$ in the last equation of the system (13), we receive

$$a_{33}^2 = \frac{1}{1 + x^2 + y^2}. \qquad (15)$$

The system of equations (14) can be presented in such form:

$$\begin{aligned}
&a_0y^2 + a_1y + a_2 = 0,\\
&b_0y^4 + b_1y^3 + b_2y^2 + b_3y + b_4 = 0.
\end{aligned} \qquad (16)$$

Here

$$
\begin{aligned}
&a_0 = h_2(h_1 - \nu h_3 x),\\
&a_1 = h_1 h_3 + [4\nu(1-\nu) + h_1^2 - (1-\nu)h_2^2 - \nu h_3^2]x - \nu h_1 h_3 x^2,\\
&a_2 = -(1-\nu)h_2(h_1 x + h_3)x,\\
&b_0 = h_2^2,\\
&b_1 = 2h_2(h_1 x + h_3),\\
&b_2 = (h_2^2 + h_3^2 - 16) + 2h_1 h_3 x + (h_1^2 + h_2^2 - 16\nu^2)x^2,\\
&b_3 = 2h_2(h_1 x + h_3)(1 + x^2),\\
&b_4 = (h_1 x + h_3)^2(1 + x^2) - 16(1-\nu)^2 x^2.
\end{aligned}
\tag{17}
$$

Using the resultant concept, we eliminate variable y from the equations (16) with the help of Mathematica symbolic matrix function. The equation in x then has the form [3]:

$$
\begin{aligned}
&p_0 x^{12} + p_1 x^{11} + p_2 x^{10} + p_3 x^9 + p_4 x^8 + p_5 x^7 + p_6 x^6 +\\
&+ p_7 x^5 + p_8 x^4 + p_9 x^3 + p_{10} x^2 + p_{11} x + p_{12} = 0,
\end{aligned}
\tag{18}
$$

where

$$p_0 = -h_1^4 h_3^4 \nu^6,$$

$$p_1 = 2h_1^3 h_3^3 \nu^5 \left[2h_1^2 - h_2^2(\nu - 1) - 2\nu\left(h_3^2 + 2\nu - 2\right)\right],$$

$$\ldots$$

$$p_{11} = -2h_1^3 h_3^3 \left\{2h_1^2 - h_2^2(\nu - 1) - 2\nu\left(h_3^2 + 2\nu - 2\right)\right\},$$

$$p_{12} = -h_1^4 h_3^4.$$

Substituting the value of a real root of equation (18) into the equations (16), we can find roots of these equations. For each solution of the system (15) one can determine two values of a_{33} from equation (15), and then the values a_{31} and a_{32}. Thus, each real root of the algebraic equation (18) corresponds to two sets of values $a_{31}, a_{32}, a_{33},$ which, by virtue of (12), uniquely determine the remaining direction cosines $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}$. It follows from these considerations that the gyrostat satellite in general case $(h_1 \neq 0,\ h_2 \neq 0,\ h_3 \neq 0)$ may have no more than 24 equilibrium orientations in the orbital coordinate system.

3. Stability analysis of equilibria

Using equation (18), it is possible to determine numerically all equilibrium orientations of the gyrostat satellite in the orbital coordinate system and analyze their stability. In [3] dependence of the number of real solutions of equation (18) from the parameters was analyzed numerically. Evolution of domains in the space of parameters that correspond to various numbers of equilibria was carried out in detail. The numerical calculations in [3] were made for the case when $B > A > C$ $(0 < \nu < 1)$ and nonzero parameters $h_1,\ h_2,\ h_3$.

In this work we will investigate the stability of equilibrium solutions satisfying equations (10) and (11).

Let us use the generalized energy integral (4) as Lyapunov's function in order to obtain sufficient conditions of stability of the equilibrium orientations of the gyrostat satellite. The generalized energy integral (4) can be rewritten in such form:

$$\frac{1}{2}(A\overline{p}^2 + B\overline{q}^2 + C\overline{r}^2) + \frac{1}{2}(B-C)\omega_0^2\{3[(1-\nu)a_{31}^2 + a_{32}^2] + \\ +(\nu a_{21}^2 + a_{23}^2) - 2(h_1 a_{21} + h_2 a_{22} + h_3 a_{23})\} = const. \tag{19}$$

Let us present angles $\psi,\ \vartheta,\ \varphi$ as

$$\psi = \psi_0 + \overline{\psi},\ \vartheta = \vartheta_0 + \overline{\vartheta},\ \varphi = \varphi_0 + \overline{\varphi}, \tag{20}$$

where $\overline{\psi}, \overline{\vartheta}, \overline{\varphi}$ are small deviations from the equilibrium position $\psi = \psi_0 = const$, $\vartheta = \vartheta_0 = const$, $\varphi = \varphi_0 = const$, satisfying the system of equations (10). Then, energy integral (19) can be presented in the form

$$\frac{1}{2}(A\overline{p}^2 + B\overline{q}^2 + C\overline{r}^2) + \frac{1}{2}(B-C)\omega_0^2(A_{\psi\psi}\overline{\psi}^2 + A_{\vartheta\vartheta}\overline{\vartheta}^2 + A_{\varphi\varphi}\overline{\varphi}^2 + \\ +2A_{\psi\vartheta}\overline{\psi}\overline{\vartheta} + 2A_{\psi\varphi}\overline{\psi}\overline{\varphi} + 2A_{\vartheta\varphi}\overline{\vartheta}\overline{\varphi}) + \Sigma = const. \tag{21}$$

where the symbol Σ designates the terms of higher than the second order of smallness relative to $\overline{\psi}, \overline{\vartheta}, \overline{\varphi}$ and

$$
\begin{aligned}
&A_{\psi\psi}=\nu(a_{11}^2-a_{21}^2)+(a_{13}^2-a_{23}^2)+h_1a_{21}+h_2a_{22}+h_3a_{23},\\
&A_{\vartheta\vartheta}=(3+\cos^2\psi_0)(1-\nu\sin^2\varphi_0)\cos 2\vartheta_0-\\
&-\frac{1}{4}\nu\sin 2\psi_0\cos\vartheta_0\sin 2\varphi_0+(h_1\sin\varphi_0+h_2\cos\varphi_0)\cos\psi_0\cos\vartheta_0+h_3a_{23},\\
&A_{\varphi\varphi}=\nu[(a_{22}^2-a_{21}^2)-3(a_{32}^2-a_{31}^2)]+h_1a_{21}+h_2a_{22}, \qquad (22)\\
&A_{\psi\vartheta}=-\frac{1}{2}\sin 2\psi_0\sin 2\vartheta_0+\nu(a_{11}a_{23}+a_{13}a_{21})-\sin\psi_0(h_1a_{31}+h_2a_{32}+h_3a_{33}),\\
&A_{\psi\varphi}=\nu(a_{11}a_{22}+a_{12}a_{21})-h_1a_{12}+h_2a_{11},\\
&A_{\vartheta\varphi}=-\frac{3}{2}\nu\sin 2\vartheta_0\sin 2\varphi_0+\nu(a_{21}\cos\varphi_0+a_{22})a_{23}-(h_1\cos\varphi_0-h_2\sin\varphi_0)a_{23}.
\end{aligned}
$$

It follows from the Lyapunov's theorem that the equilibrium solution is stable if the quadratic form

$$
\begin{aligned}
&\frac{1}{2}(A\bar{p}^2+B\bar{q}^2+C\bar{r}^2)+\frac{1}{2}(B-C)\omega_0^2(A_{\psi\psi}\bar{\psi}^2+A_{\vartheta\vartheta}\bar{\vartheta}^2+A_{\varphi\varphi}\bar{\varphi}^2+ \qquad (23)\\
&+2A_{\psi\vartheta}\bar{\psi}\bar{\vartheta}+2A_{\psi\varphi}\overline{\psi\varphi}+2A_{\vartheta\varphi}\bar{\vartheta}\bar{\varphi})
\end{aligned}
$$

for this solution is positive definite. The sufficient conditions of stability will be fulfilled if the following inequalities take place:

$$
\begin{aligned}
&A_{\psi\psi}>0,\\
&A_{\psi\psi}A_{\vartheta\vartheta}-(A_{\psi\vartheta})^2>0, \qquad (24)\\
&A_{\psi\psi}A_{\vartheta\vartheta}A_{\varphi\varphi}+2A_{\psi\vartheta}A_{\psi\varphi}A_{\vartheta\varphi}-A_{\psi\psi}(A_{\vartheta\varphi})^2-A_{\vartheta\vartheta}(A_{\psi\varphi})^2-A_{\varphi\varphi}(A_{\psi\vartheta})^2>0.
\end{aligned}
$$

Substituting the expressions for $A_{\psi\psi}, A_{\vartheta\vartheta}, A_{\varphi\varphi}, A_{\psi\vartheta}, A_{\psi\varphi}, A_{\vartheta\varphi}$ from (22) for the corresponding equilibrium solution into (24), we will obtain the sufficient conditions for stability of this solution.

4. Numerical investigation of stability of equilibria

Using integral (21) and conditions (24) we can analyze numerically the sufficient conditions of stability of the equilibrium solutions.

For each set of the values of four system parameters $\nu,\ h_1,\ h_2,\ h_3$ we have defined real roots x of equation (18) using Mathematica numeric package for algebraic equations.

As it was mentioned above, to each real root x_1 of equation (18) corresponds real root y_1 of equations (16).

For the solution x_1, y_1 we can determine two values of a_{33} from the equation (15), and then the values $a_{31} = x_1 a_{33}$ and $a_{32} = y_1 a_{33}$.

Therefore, it is possible to define two sets of direction cosines a_{31}, a_{32}, a_{33} corresponding to each real root x_1 of equation (18). To obtain the remaining direction cosines $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}$ it is necessary to use relationships (12).

We will present the results of calculations by Euler angles (1) for clear understanding of their physical sense. Then the roots of the equations (18), (16) will have the following view:

$$\mathrm{tg}\varphi = \frac{x_1}{y_1} = \frac{a_{31}}{a_{32}} \text{ and}$$

$$\varphi = arctg\frac{a_{31}}{a_{32}} \; (0 \le \varphi < 2\pi).$$

The angle ϑ $(0 \le \vartheta < \pi)$ can be defined from (15) $(\cos\vartheta = a_{33})$. The angle ψ $(0 \le \psi < 2\pi)$ is uniquely determined with the help of relations

$$\sin\psi = \frac{a_{13}}{\sin\vartheta}; \; \cos\psi = -\frac{a_{23}}{\sin\vartheta}.$$

After that we can uniquely determine the orientation angles $\psi_0, \vartheta_0, \varphi_0$, calculate coefficients of quadratic form (23) and check the conditions of its positive definiteness (24).

Due to the fact that $0 \le \varphi < 2\pi$, to each real root $\mathrm{tg}\varphi = x_1/y_1$ corresponds two values of angle φ (φ_1 and $\varphi_2 = \varphi_1 + \pi$). From the features of the quadratic form coefficients (22) it follows that the sufficient stability conditions (24) for the values φ_1 and φ_2 are equal. In addition, it is possible to prove that conditions (24) do not depend on the sign of the parameters h_1, h_2, h_3. Hence, the numerical analysis of the sufficient stability conditions of equilibrium solutions of the equations (10) possible to do with positive values of h_1, h_2, h_3, $0 < \nu < 1$ condition and single value of φ (φ_1 or φ_2) corresponding to each real root of equations (16), (18).

The numerical calculations were made for fixed values of ν, h_2 and h_3. The figures 1-87 were calculated for the values ν, h_2 and h_3, indicated in the Table 1.

In the figures 1-87 the dependence of φ from h_1 with the fixed values of ν, h_3 and with the different values of h_2 is presented. Dashed lines indicate curves where stability conditions (24) are valid.

Since the sufficient stability conditions (24) for the values φ_1 and $\varphi_1 + \pi$ $(0 \le \varphi < 2\pi)$ are equal, the numerical results in the figures 1-87 are presented for the φ range $0 \le \varphi < \pi$.

Calculations were made for the inertia parameters $\nu = 0.01$ (near limit value $\nu = 0$ $(A = B)$), $\nu = 0.1$, $\nu = 0.2$, $\nu = 0.3$, $\nu = 0.4$, $\nu = 0.5$, $\nu = 0.6$, $\nu = 0.7$, $\nu = 0.8$, $\nu = 0.9$ and $\nu = 0.99$ (near limit value $\nu = 1$ $(A = C)$) see Table 1. The results of calculations are presented in the Figures 1-87.

From the analysis of all calculations for the indicated above parameters it follows that for the h_3 parameter values less then $1-\nu$ and for small h_1, h_2 there are 24 equilibrium solution curves and for 4 equilibrium curves stability conditions (24) are valid (Figs.3, 8, 10, 17, 23, 24, 26, 27, 37, 42, 43, 46, 51, 52, 55, 59, 62, 69, 72, 75, 80, 85). There are also 4 stable equilibria for $\nu > 0.5$ and $h_3 \geq 1-\nu$ (Figs.66, 68, 74, 79, 83).

When the values of parameter h_1 increase, there are a sequential mergers of equilibrium curves at points that correspond to the points of intersection of the straight lines $h_2 = \text{const}$ with the borders of regions with the fixed number of equilibria which was calculated in [3]. For example, at the Fig. 25 ($\nu = 0.2,\ h_3 = 0.4$) there are 4 points of intersection of the straight line $h_2 = 0.1$ with the borders of regions with fixed number of equilibria $h_1 = 0.039$, $h_1 = 0.17$, $h_1 = 0.531$ and $h_1 = 2.077$; at the Fig. 27 ($\nu = 0.2,\ h_2 = 0.1,\ h_3 = 0.4$) at these points equilibrium curves are merging.

When the values of parameters h_1, h_2, h_3 of the gyrostatic torque are greater or equal 4 there are 8 equilibrium solutions (Figs.7, 14, 22, 34, 35, 41, 50, 58, 65, 87) and only 2 of them are stable. For the big values of parameters h_1, h_2, h_3 the equilibrium values of φ are close to the trivial solutions, where some axis of the orbital coordinate system and some axes of the body coordinate system coincide.

Conclusions

In this work the attitude motion of a gyrostat satellite under the action of gravitational torque in a circular orbit has been investigated. The main attention was given to investigation of stability conditions of the satellite's equilibrium orientations in general case when $A \neq B \neq C$ and $h_1 \neq 0,\ h_2 \neq 0,\ h_3 \neq 0$.

Using the Lyapunov theorem, sufficient conditions for stability of the equilibrium orientations are obtained in the form of set inequalities.

Sufficient conditions of stability are investigated numerically for each equilibrium orientation using generalized integral of energy as a Lyapunov's function.

It was shown that the number of stable equilibria of the gyrostat satellite in general case changes from 4 to 2 with the increasing the absolute value of gyrostatic torque. For the h_3 parameter values less then $1-\nu$ and for small h_1, h_2 there are always four stable equilibria.

The obtained results can be used in the stage of preliminary projecting of the satellite with gravitational stabilization system.

References

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2. Sarychev V.A., Gutnik S.A. Investigations of the relative equilibria of a gyrostat satellite // *Preprint Keldysh Institute for Applied Mathematics*, 1990, No.84. 31p.
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Table 1.

Parameter values for stability analysis

Gyrostatic parameters / ν	h_3	h_2	h_3	h_2
0.01	**0.01**	**0.5**	**0.01**	**2.0**
0.1	**0.01** **0.01** **0.01** **0.01** **0.01** **0.2** **0.2**	**0.05** **0.2** **0.5** **2.0** **4.0** **0.05** **0.1**	**0.45** **0.45** **0.45** **0.45** **0.45** **1.0** **2.0**	**0.05** **0.2** **0.5** **2.0** **4.0** **0.5** **0.5**
0.2	**0.01** **0.01** **0.01** **0.01** **0.01** **0.01** **0.2** **0.2** **0.4** **0.4**	**0.05** **0.2** **0.5** **0.85** **2.0** **4.0** **0.05** **0.1** **0.05** **0.1**	**0.4** **0.4** **0.4** **0.4** **0.4** **0.4** **0.4** **0.4** **1.0**	**0.2** **0.3** **0.5** **0.6** **1.0** **2.0** **3.0** **4.0** **0.5**
0.3	**0.01** **0.01** **0.01** **0.01** **0.01**	**0.05** **0.5** **1.0** **2.0** **4.0**	**0.2** **0.2** **1.0** **2.0**	**0.05** **0.1** **0.5** **0.5**
0.4	**0.01** **0.01** **0.01** **0.01** **0.01**	**0.05** **0.5** **1.0** **2.0** **4.0**	**0.2** **0.2** **1.0** **2.0**	**0.05** **0.1** **0.5** **0.5**

Gyrostatic parameters / ν	h_3	h_2	h_3	h_2
0.5	**0.01** **0.01** **0.01** **0.01**	**0.05** **0.5** **2.0** **4.0**	**0.2** **1.0** **2.0**	**0.1** **0.5** **0.5**
0.6	**0.01** **0.01** **0.01** **0.01**	**0.05** **0.8** **2.0** **4.0**	**0.4** **1.0** **2.0**	**0.1** **0.5** **0.5**
0.7	**0.01** **0.01** **0.01**	**0.05** **0.8** **2.0**	**0.2** **1.0** **2.0**	**0.1** **0.5** **0.5**
0.8	**0.01** **0.01** **0.01**	**0.05** **0.8** **2.5**	**1.0** **1.629**	**0.5** **0.5**
0.9	**0.01** **0.01** **0.01**	**0.05** **0.9** **3.6**	**1.0** **2.0**	**0.5** **0.5**
0.99	**0.005** **0.005**	**0.5** **2.0**	**0.005**	**4.0**

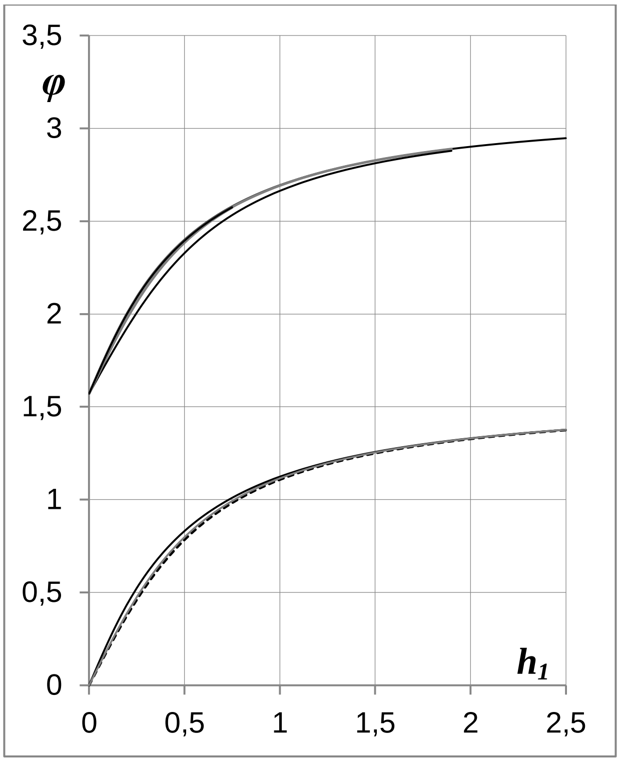

Fig. 1. v=0.01, h_2 = 0.5, h_3 = 0.01
(16 equilibria, 2 stable)

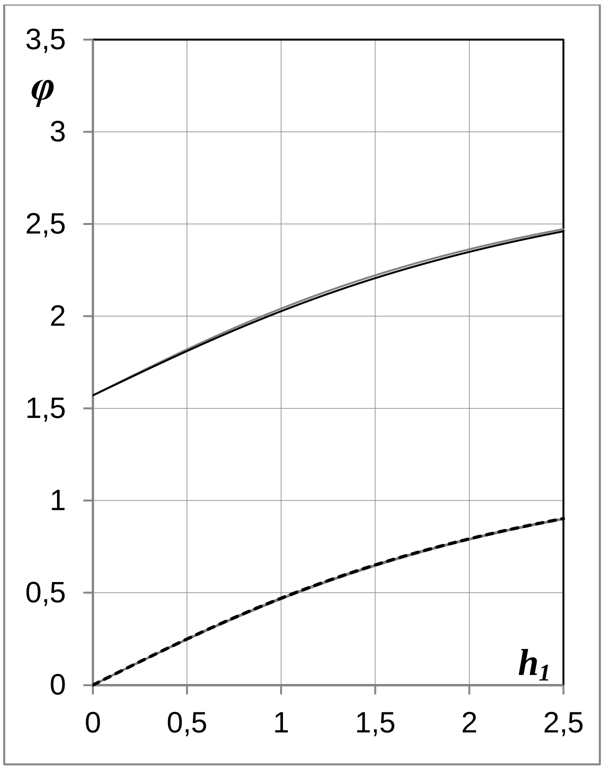

Fig. 2. v=0.01, h_2 = 2.0, h_3 = 0.01
(12 equilibria, 2 stable)

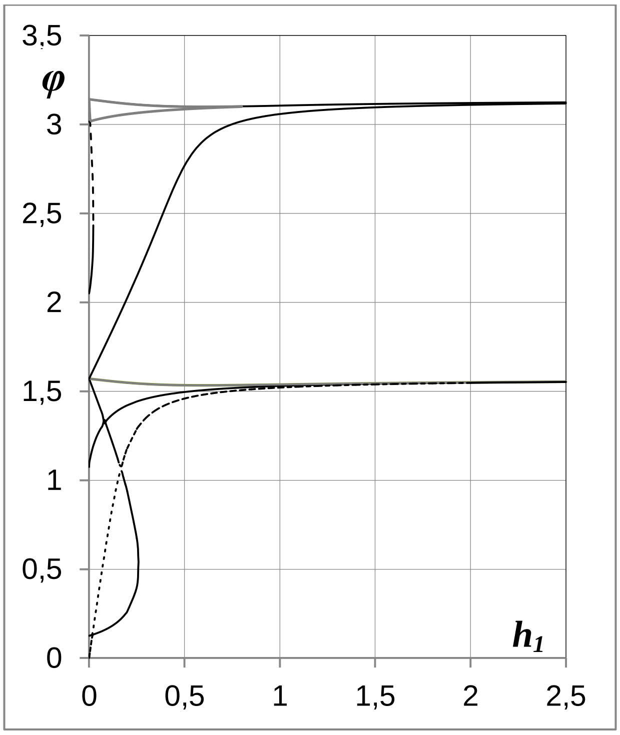

Fig. 3. v=0.1, h_2 = 0.05, h_3 = 0.01
(24 equilibria, 4 stable)

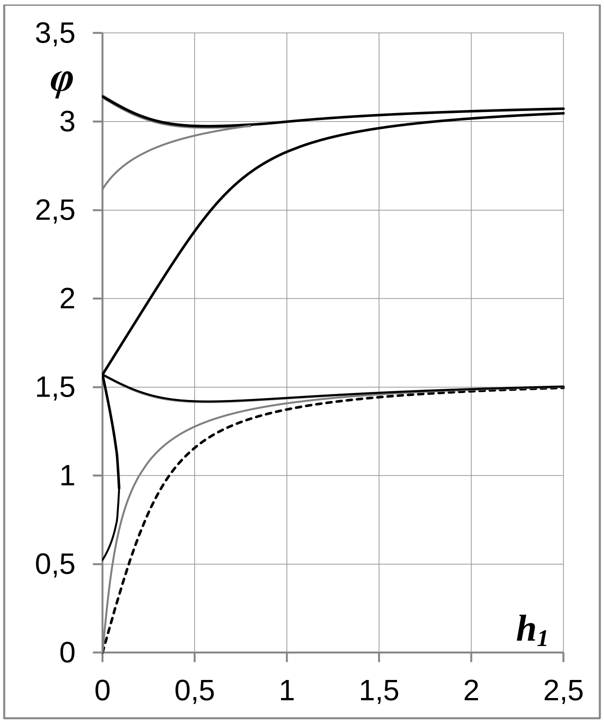

Fig. 4. v=0.1, h_2 = 0.2, h_3 = 0.01
(20 equilibria, 2 stable)

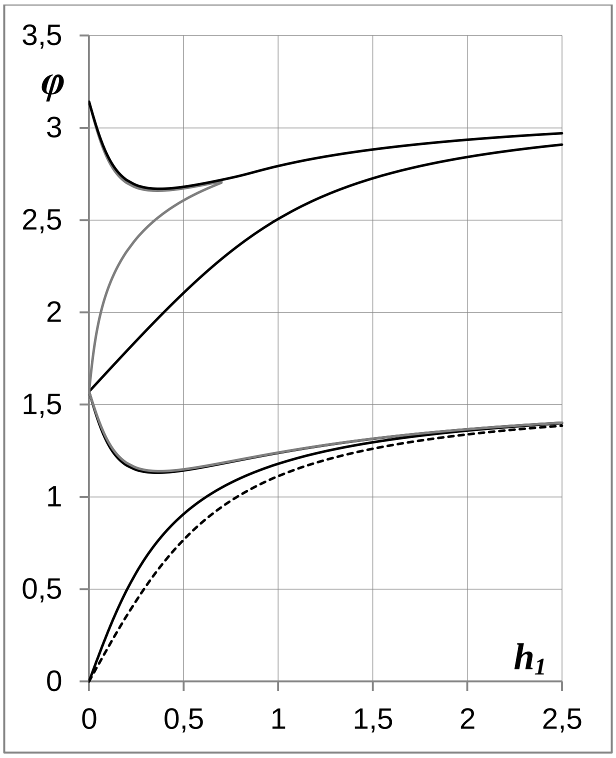

Fig. 5. v=0.1, h_2 = 0.5, h_3 = 0.01
(16 equilibria, 2 stable)

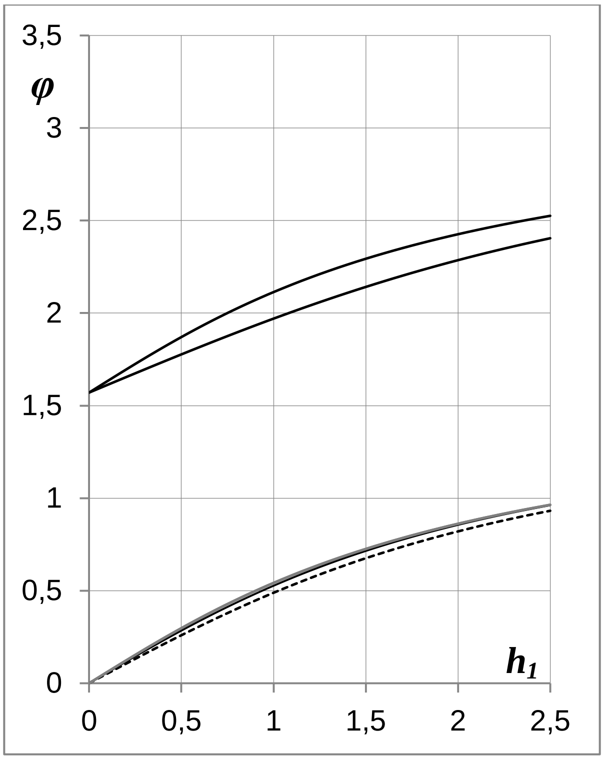

Fig. 6. v=0.1, h_2 = 2.0, h_3 = 0.01
(12 equilibria, 2 stable)

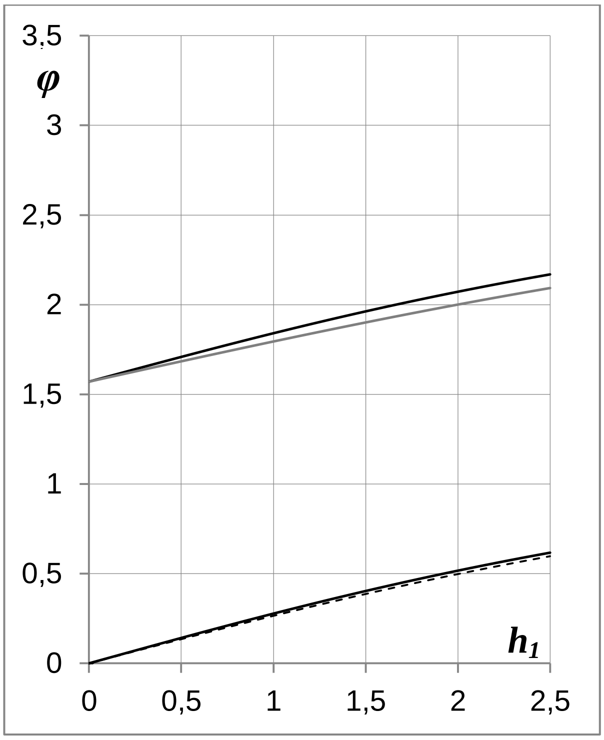

Fig. 7. v=0.1, h_2 = 4.0, h_3 = 0.01
(8 equilibria, 2 stable)

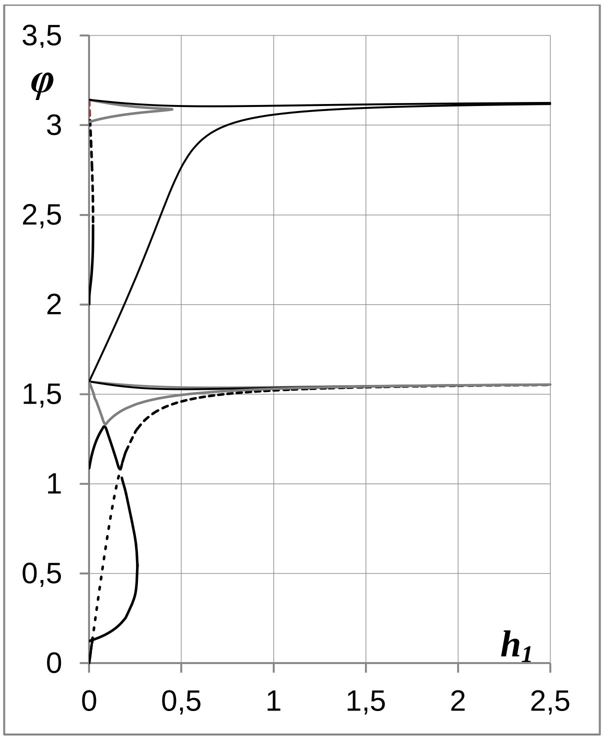

Fig. 8. v=0.1, h_2 = 0.05, h_3 = 0.2
(24 equilibria, 4 stable)

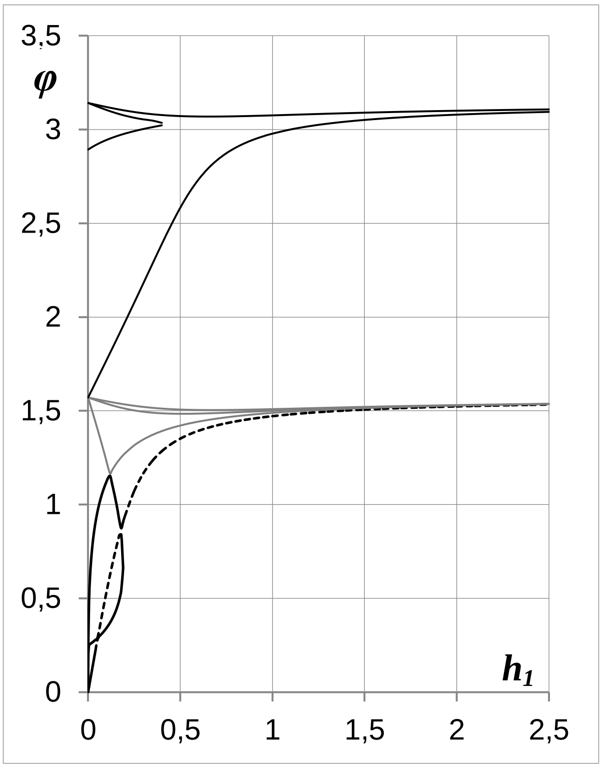

Fig. 9. v=0.1, h_2 = 0.1, h_3 = 0.2
(20 equilibria, 2 stable)

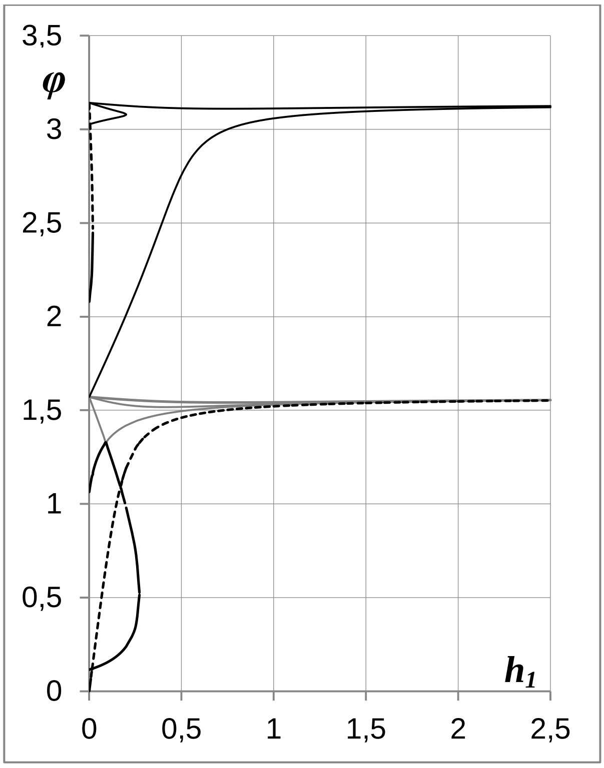

Fig. 10. v=0.1, h_2 = 0.05, h_3 = 0.45
(24 equilibria, 4 stable)

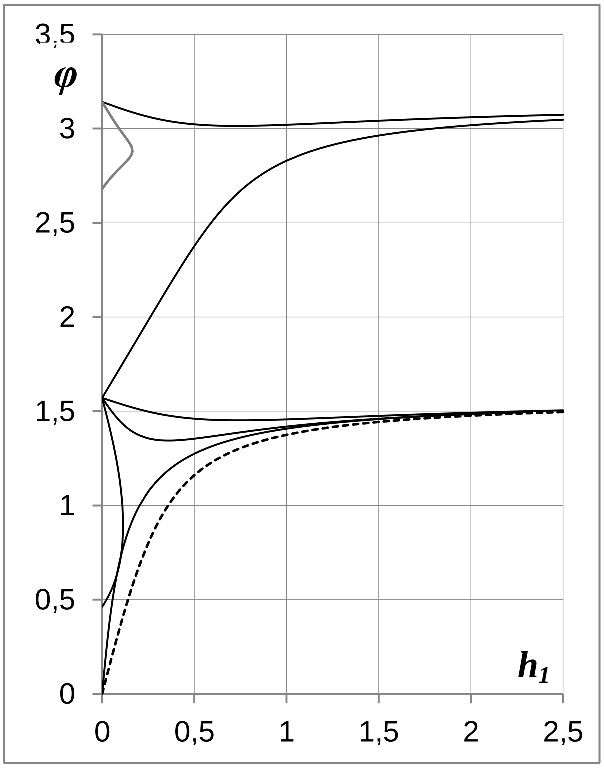

Fig. 11. v=0.1, h_2 = 0.2, h_3 = 0.45
(20 equilibria, 2 stable)

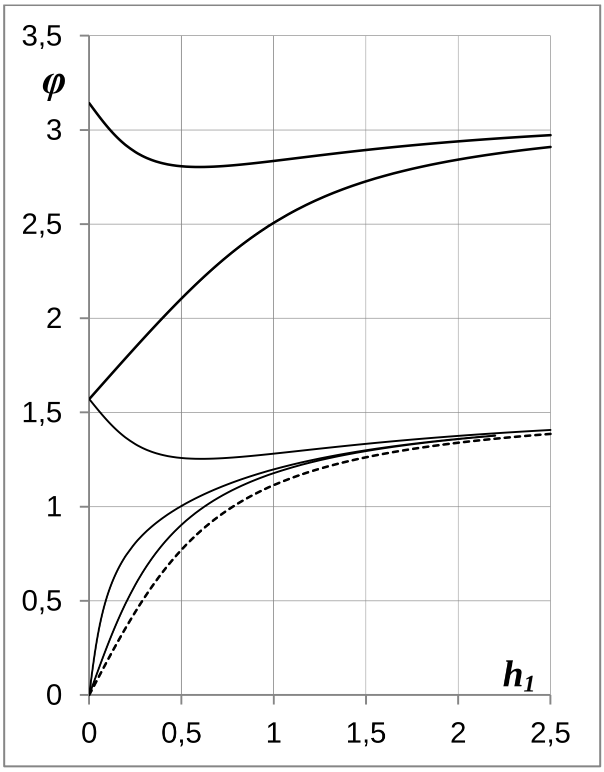

Fig. 12. v=0.1, h_2 = 0.5, h_3 = 0.45
(12 equilibria, 2 stable)

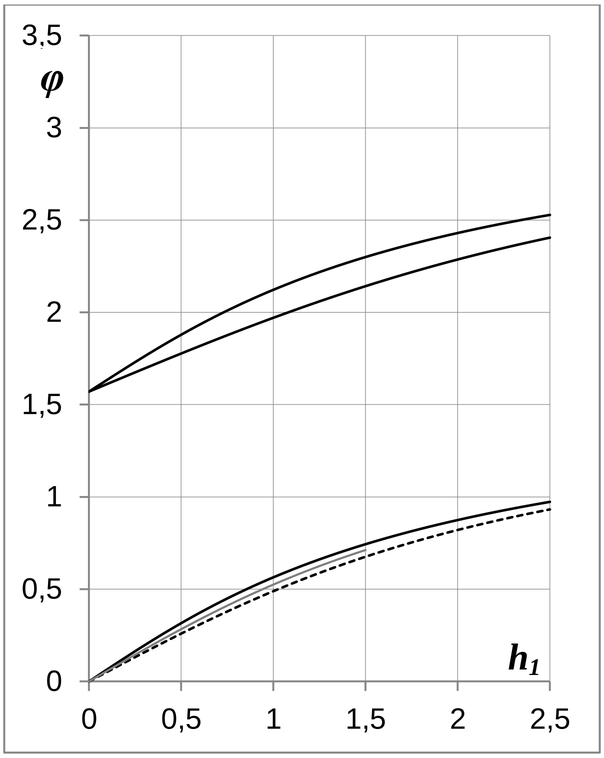

Fig. 13. ν=0.1, h_2 = 2.0, h_3 = 0.45
(12 equilibria, 2 stable)

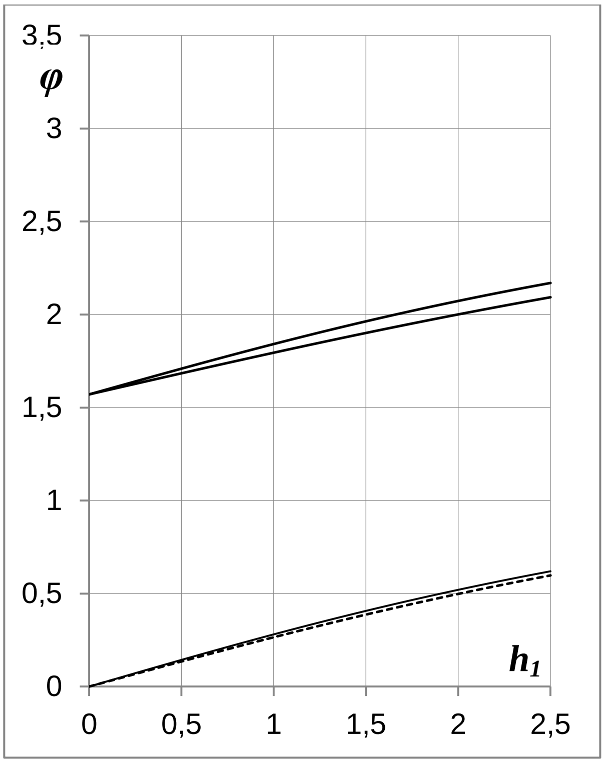

Fig. 14. ν=0.1, h_2 = 4.0, h_3 = 0.45
(8 equilibria, 2 stable)

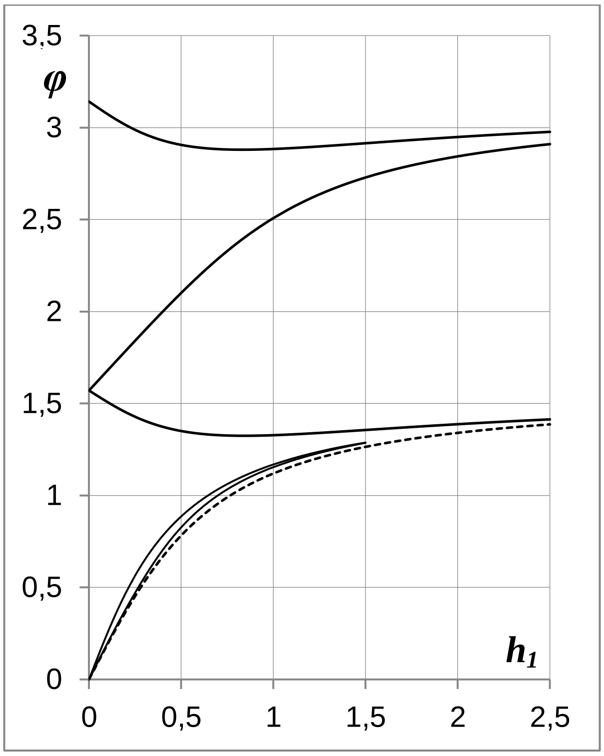

Fig. 15. ν=0.1, h_2 = 0.5, h_3 = 1.0
(12 equilibria, 2 stable)

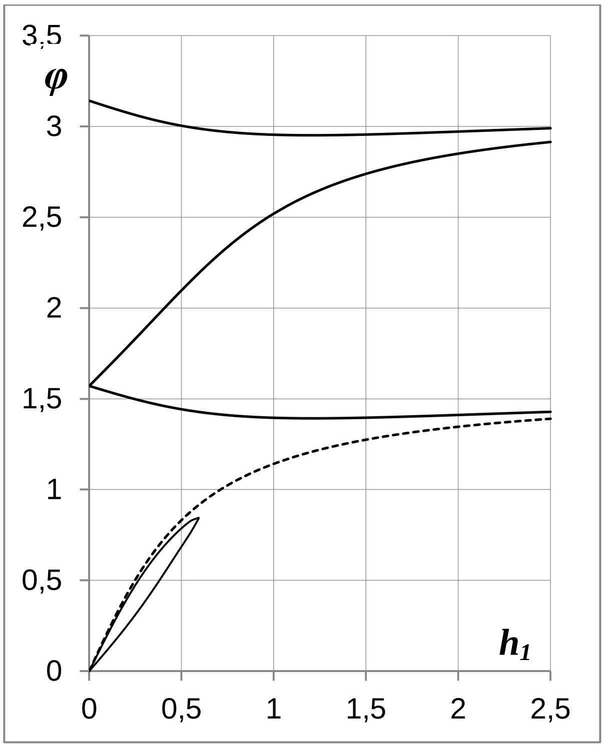

Fig. 16. ν=0.1, h_2 = 0.5, h_3 = 2.0
(12 equilibria, 2 stable)

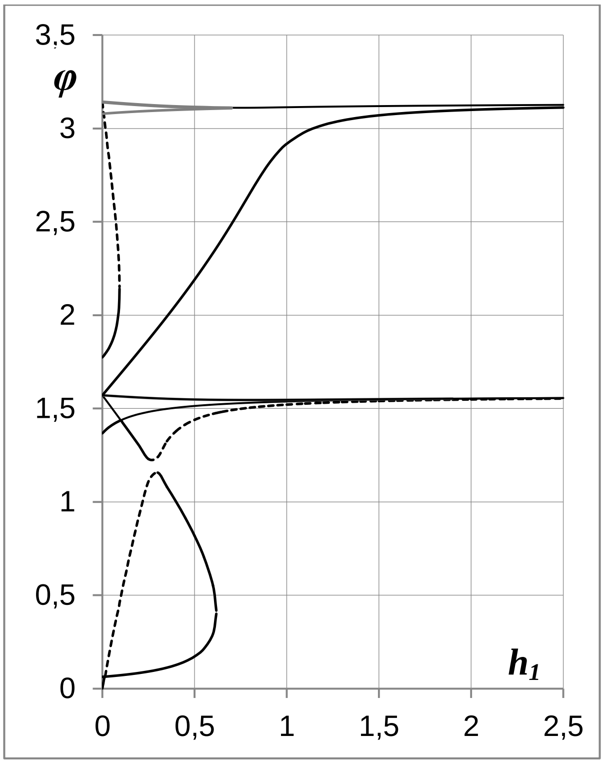

Fig. 17. v=0.2, h_2 = 0.05, h_3 = 0.01
(24 equilibria, 4 stable)

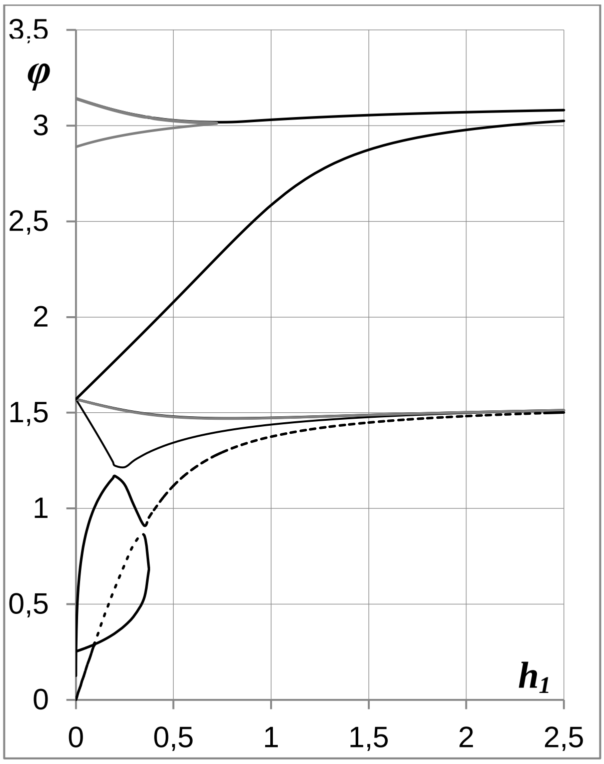

Fig. 18. v=0.2, h_2 = 0.2, h_3 = 0.01
(20 equilibria, 2 stable)

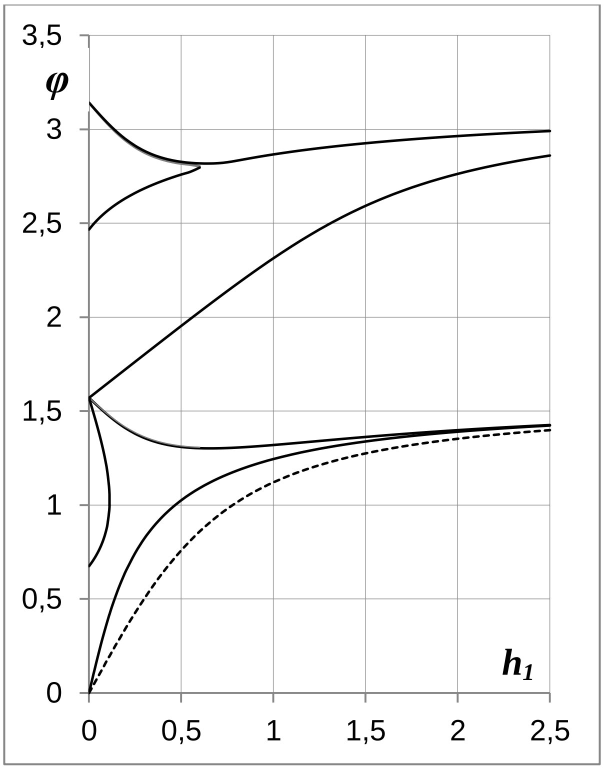

Fig. 19. v=0.2, h_2 = 0.5, h_3 = 0.01
(20 equilibria, 2 stable)

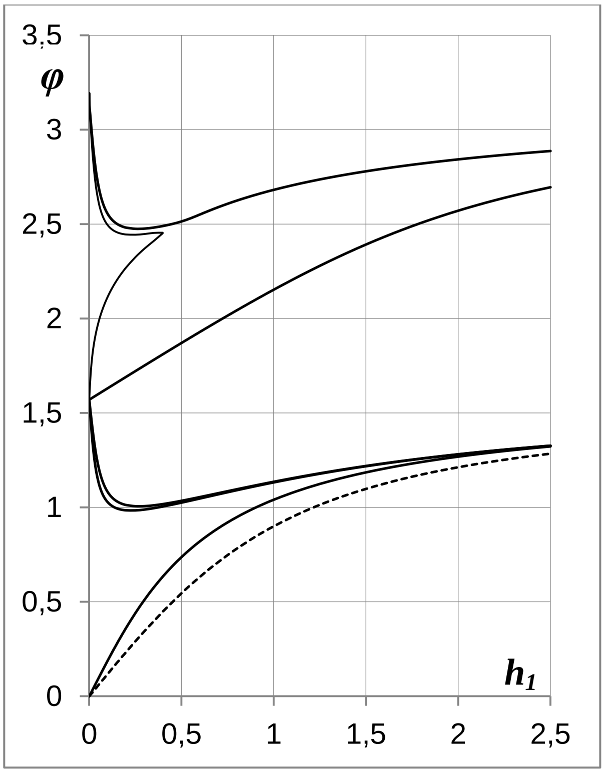

Fig. 20. v=0.2, h_2 = 0.85, h_3 = 0.0
(16 equilibria, 2 stable)

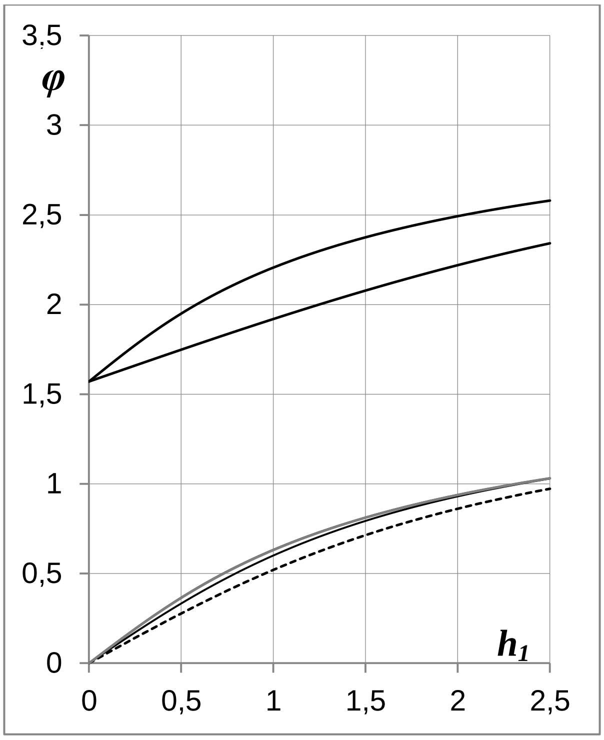

Fig. 21. v=0.2, h_2 = 2.0, h_3 = 0.01
(12 equilibria, 2 stable)

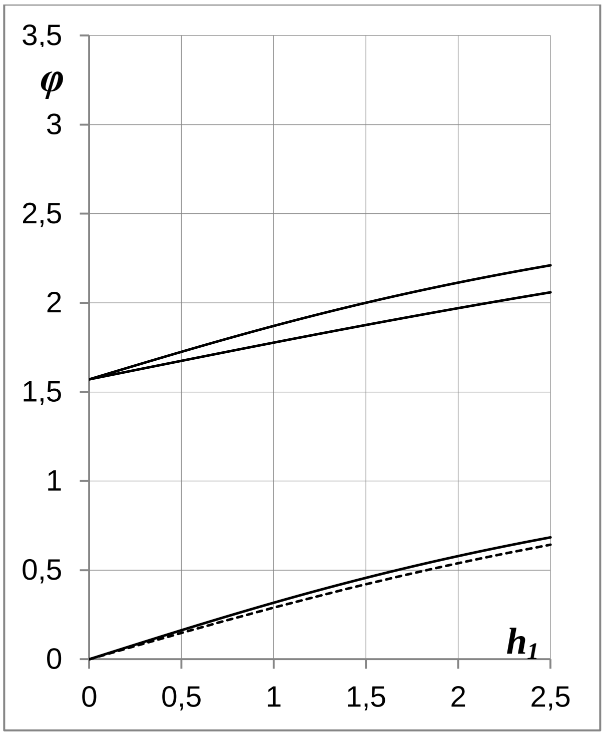

Fig. 22. v=0.2, h_2 = 4.0, h_3 = 0.01
(8 equilibria, 2 stable)

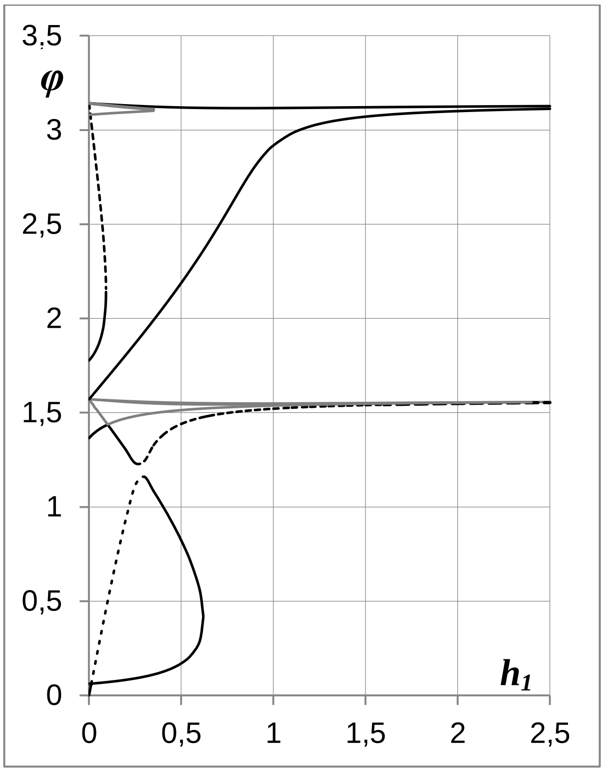

Fig. 23. v=0.2, h_2 = 0.05, h_3 = 0.2
(24 equilibria, 4 stable)

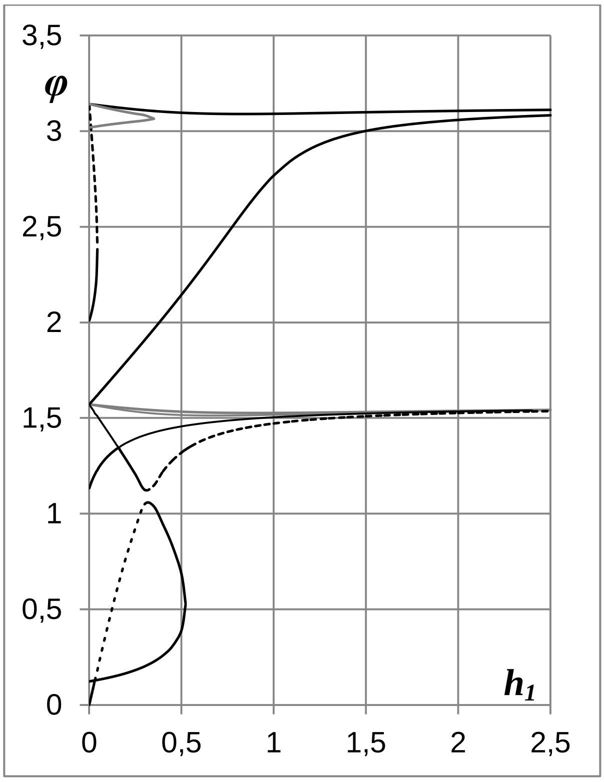

Fig. 24. v=0.2, h_2 = 0.1, h_3 = 0.2
(24 equilibria, 4 stable)

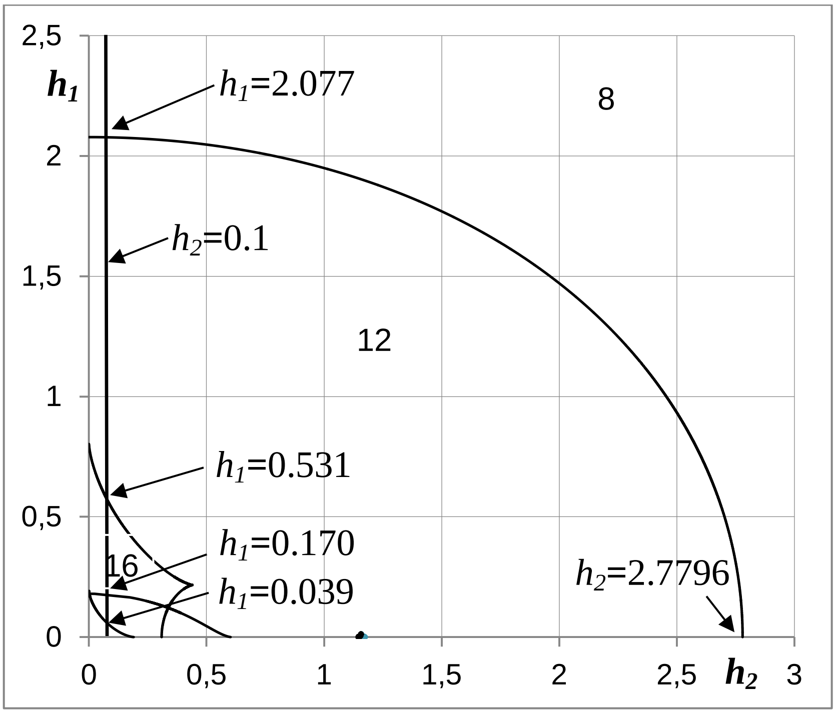

Fig. 25. v=0.2, h_3 = 0.4

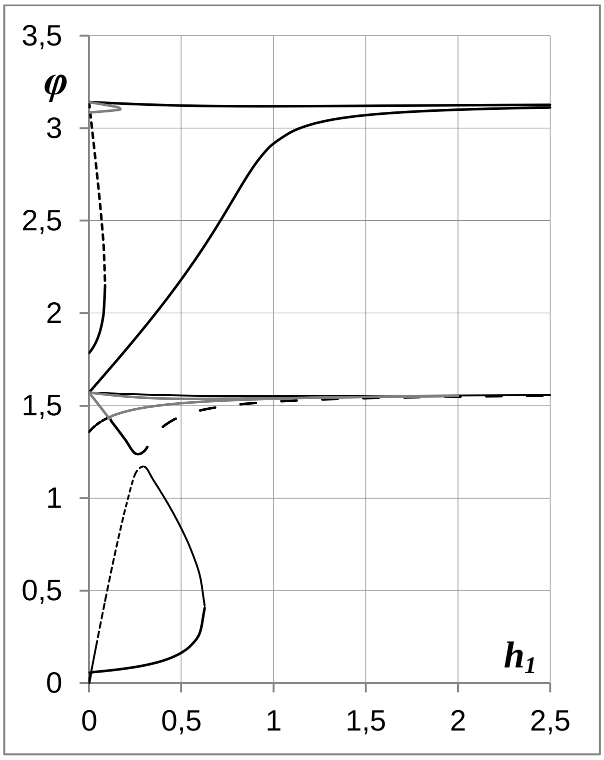

Fig. 26. v=0.2, h_2 = 0.05, h_3 = 0.4
(24 equilibria, 4 stable)

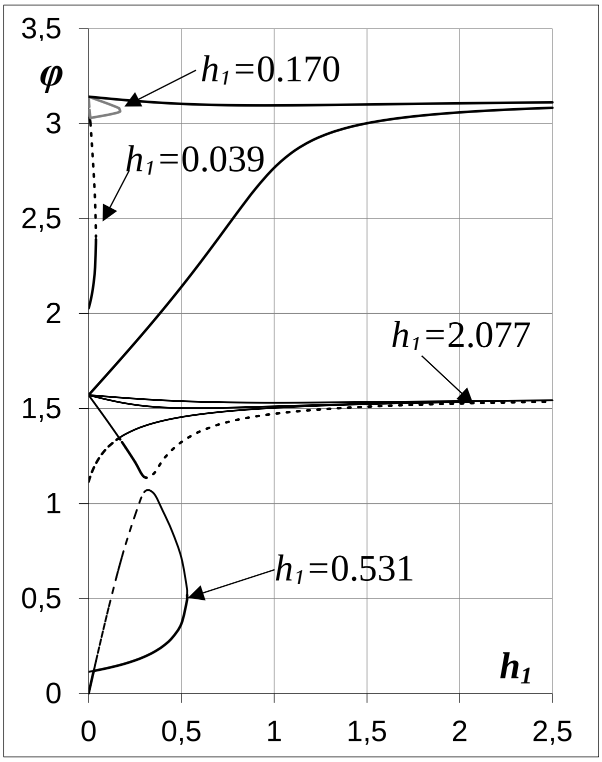

Fig. 27. v=0.2, h_2 = 0.1, h_3 = 0.4
(24 equilibria, 4 stable)

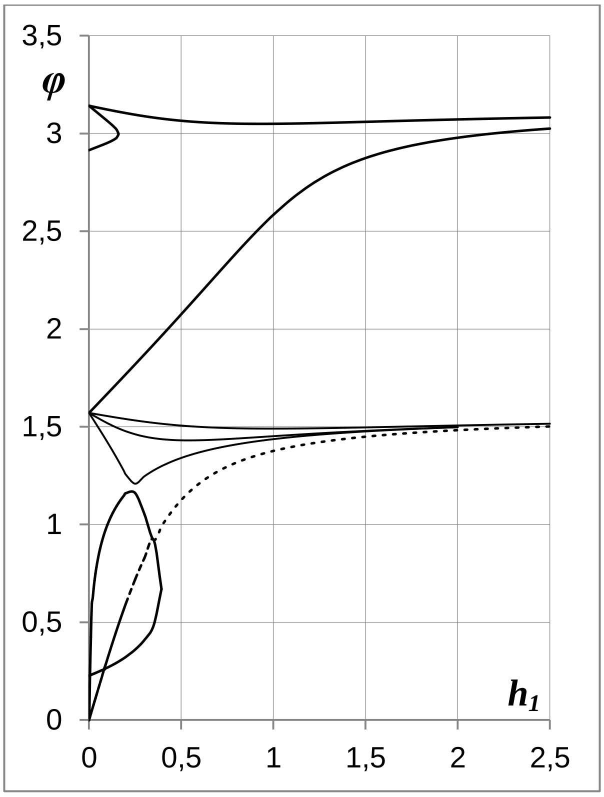

Fig. 28. v=0.2, h_2 = 0.2, h_3 = 0.4
(20 equilibria, 2 stable)

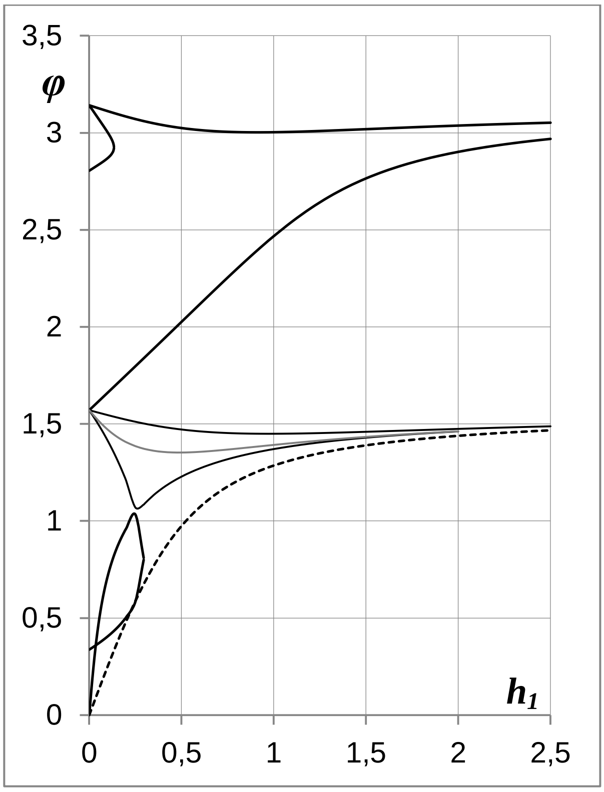

Fig. 29. v=0.2, h_2 = 0.3, h_3 = 0.4
(20 equilibria, 2 stable)

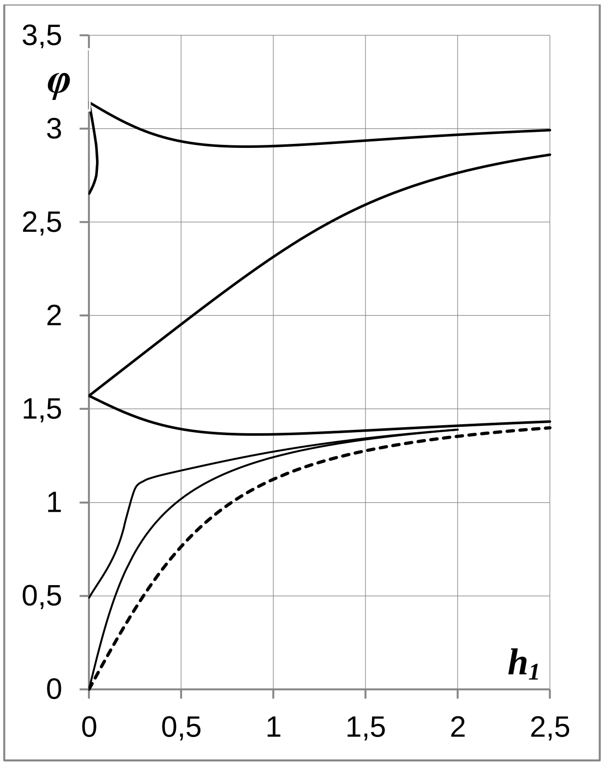

Fig. 30. v=0.2, h_2 = 0.5, h_3 = 0.4
(16 equilibria, 2 stable)

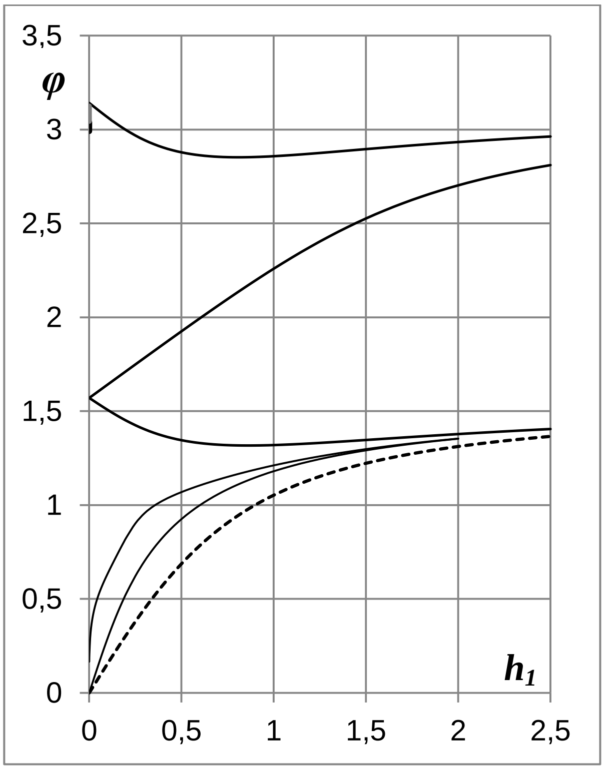

Fig. 31. v=0.2, h_2 = 0.6, h_3 = 0.4
(12 equilibria, 2 stable)

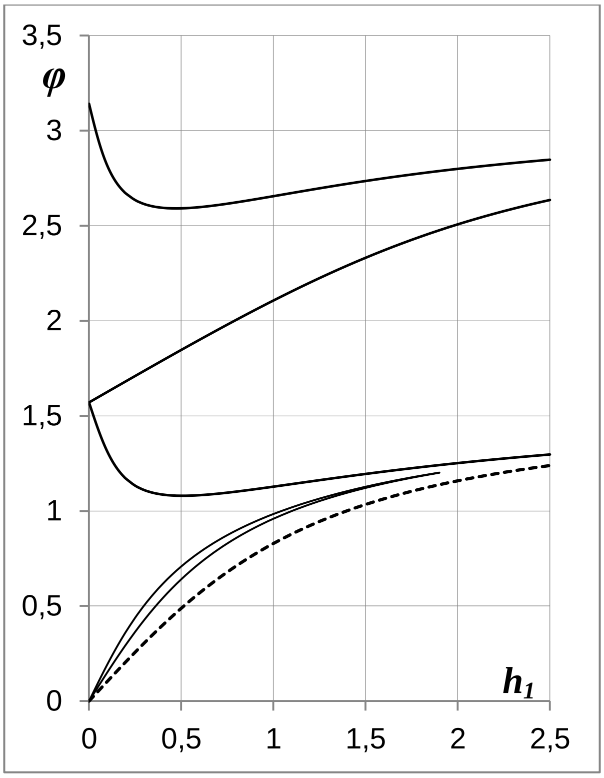

Fig. 32. *ν*=0.2, h_2 = 1.0, h_3 = 0.4
(12 equilibria, 2 stable)

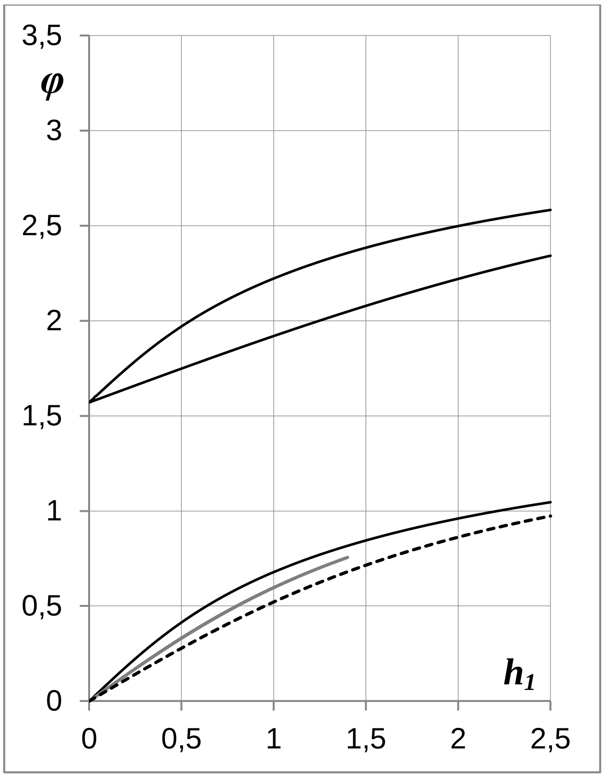

Fig. 33. *ν*=0.2, h_2 = 2.0, h_3 = 0.4
(12 equilibria, 2 stable)

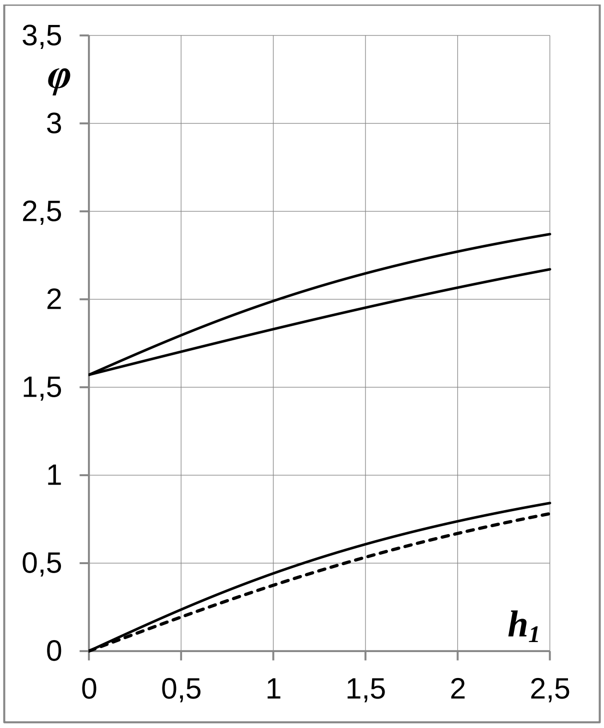

Fig. 34. *ν*=0.2, h_2 = 3.0, h_3 = 0.4
(8 equilibria, 2 stable)

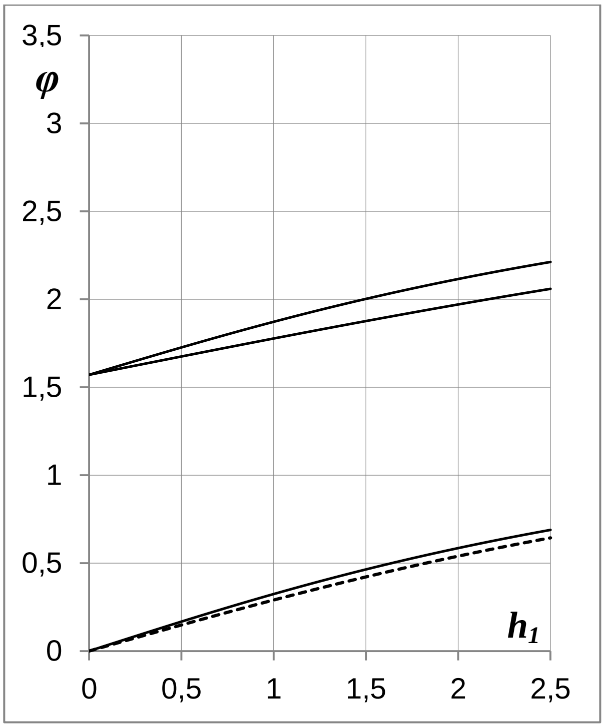

Fig. 35. *ν*=0.2, h_2 = 4.0, h_3 = 0.4
(8 equilibria, 2 stable)

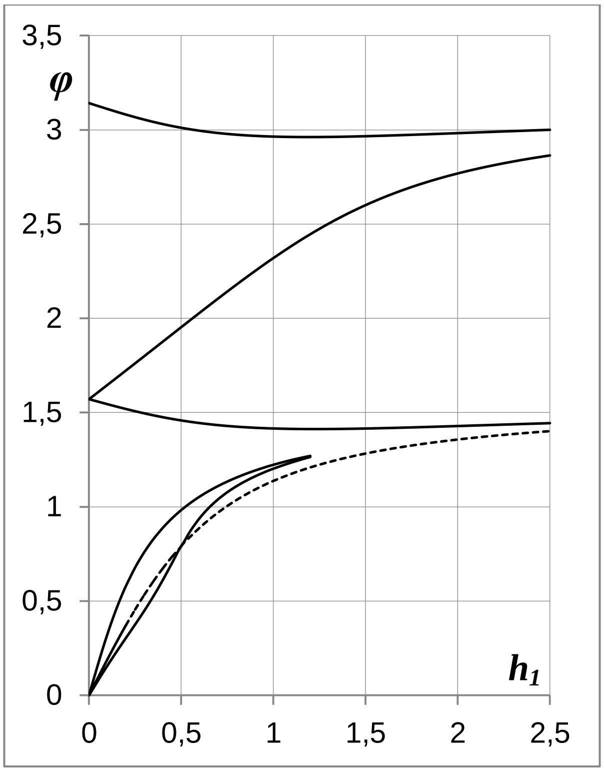

Fig. 36. v=0.2, h_2 = 0.5, h_3 = 1.0
(12 equilibria, 2 stable)

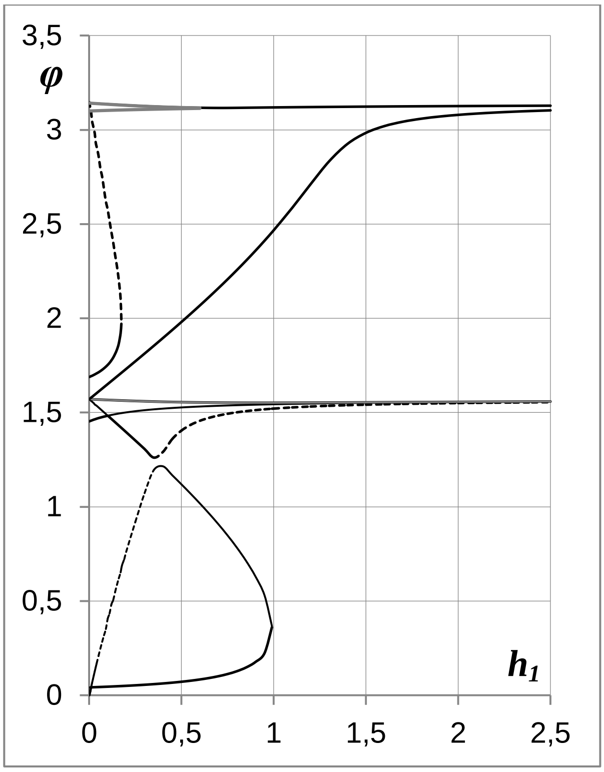

Fig. 37. v=0.3, h_2 = 0.05, h_3 = 0.01
(24 equilibria, 4 stable)

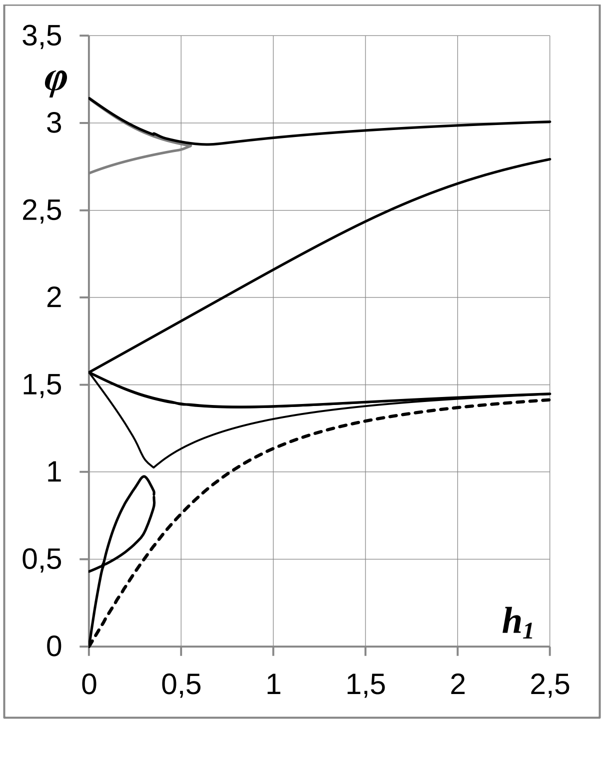

Fig. 38. v=0.3, h_2 = 0.5, h_3 = 0.01
(20 equilibria, 2 stable)

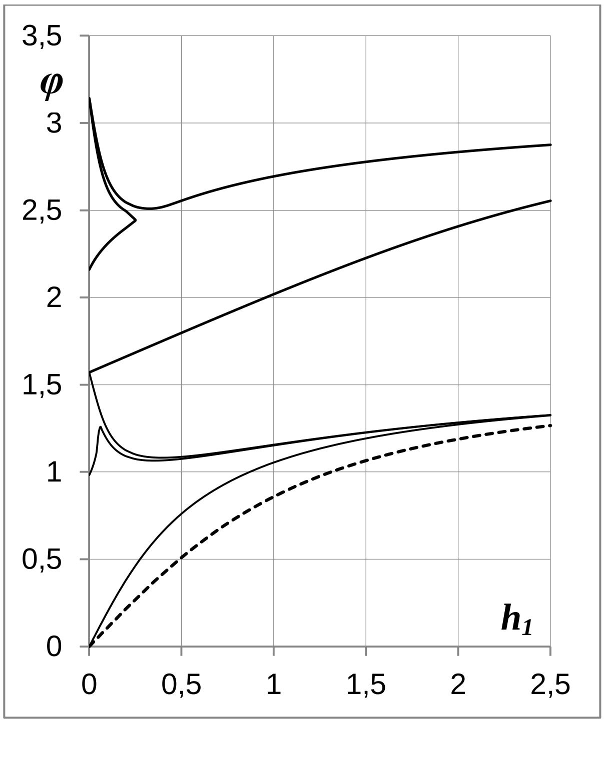

Fig. 39. v=0.3, h_2 = 1.0, h_3 = 0.01
(16 equilibria, 2 stable)

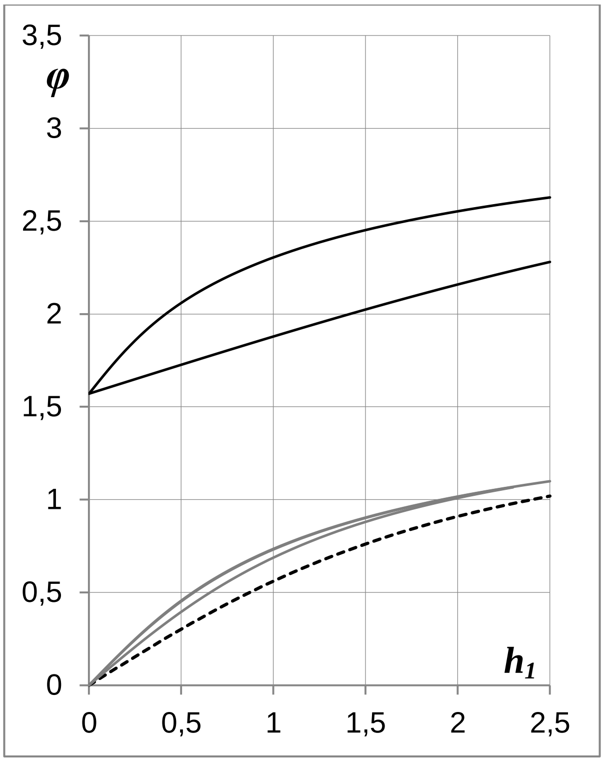

Fig. 40. ν=0.3, h_2 = 2.0, h_3 = 0.01
(12 equilibria, 2 stable)

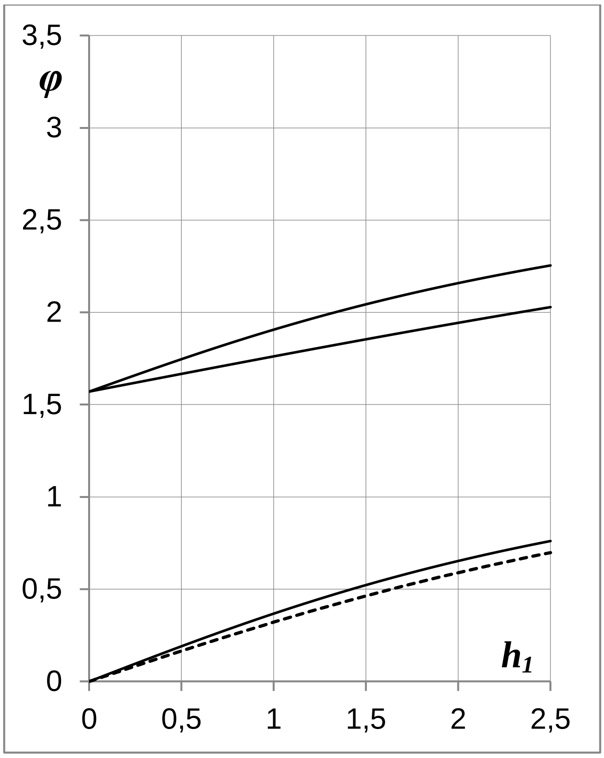

Fig. 41. ν=0.3, h_2 = 4.0, h_3 = 0.01
(8 equilibria, 2 stable)

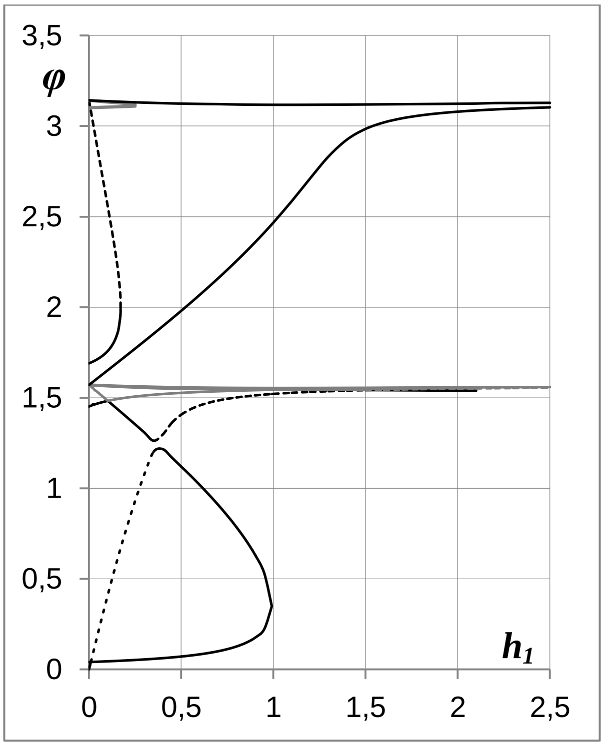

Fig. 42. ν=0.3, h_2 = 0.05, h_3 = 0.2
(24 equilibria, 4 stable)

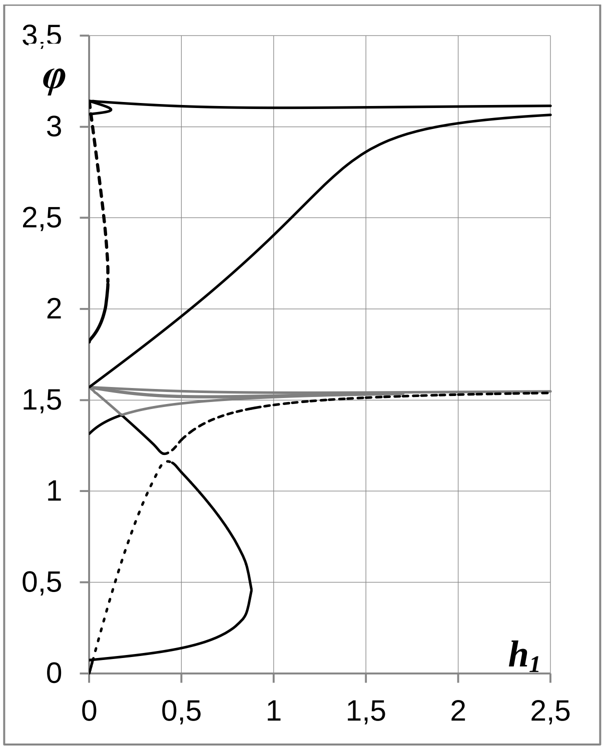

Fig. 43. ν=0.3, h_2 = 0.1, h_3 = 0.2
(24 equilibria, 4 stable)

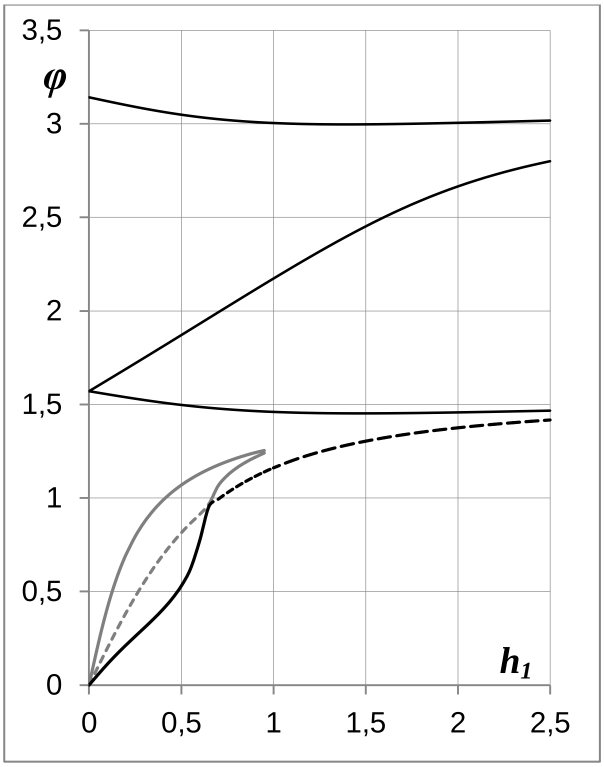

Fig. 44. v=0.3, h_2 = 0.5, h_3 = 1.0
(16 equilibria, 2 stable)

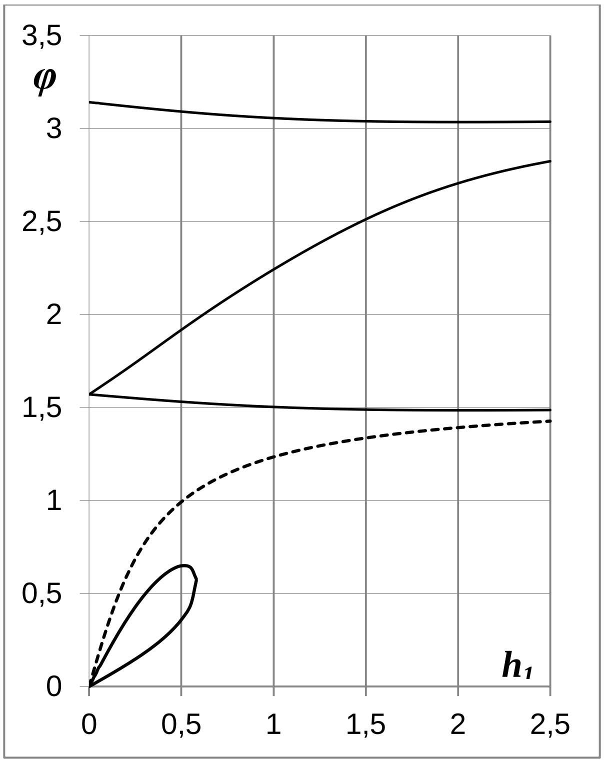

Fig. 45. v=0.3, h_2 = 0.5, h_3 = 2.0
(12 equilibria, 2 stable)

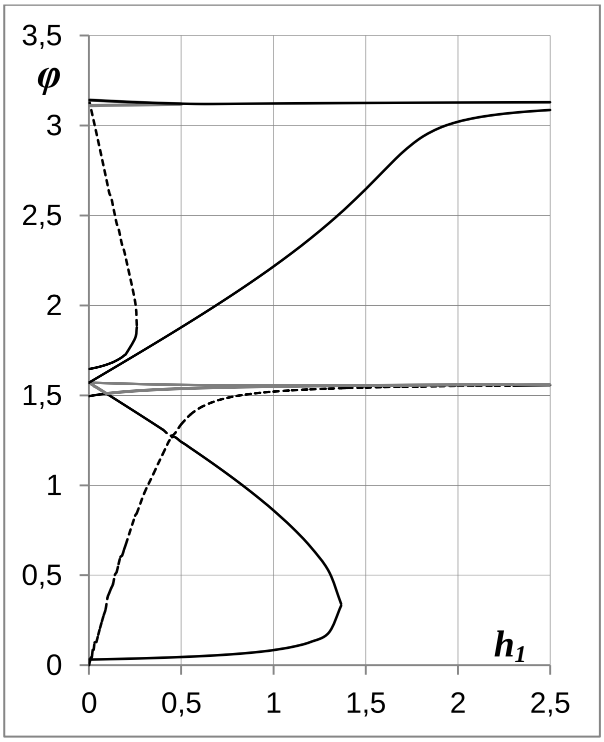

Fig. 46. v=0.4, h_2 = 0.05, h_3 = 0.01
(24 equilibria, 4 stable)

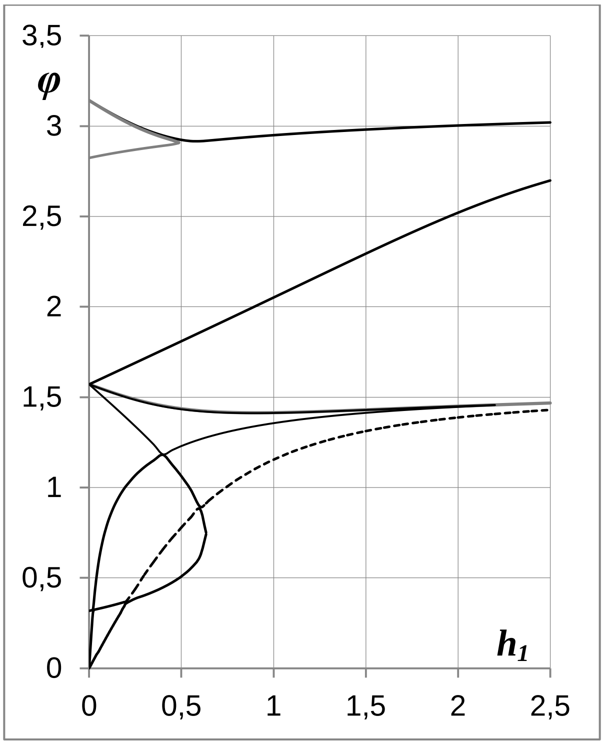

Fig. 47. v=0.4, h_2 = 0.5, h_3 = 0.01
(20 equilibria, 2 stable)

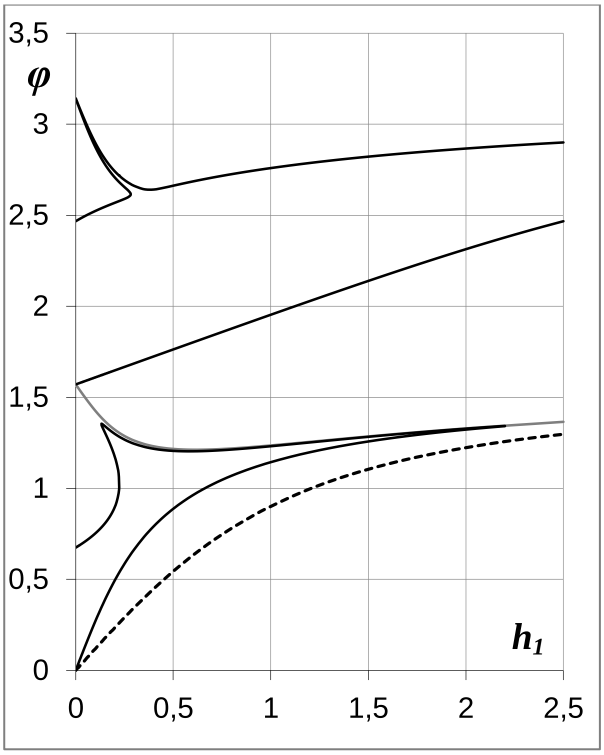

Fig. 48. v=0.4, h_2 = 1.0, h_3 = 0.01
(20 equilibria, 2 stable)

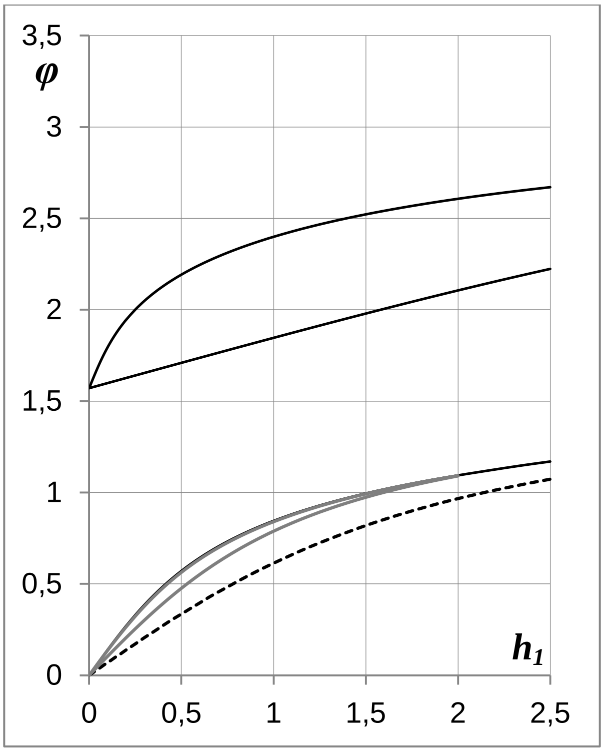

Fig. 49. v=0.4, h_2 = 2.0, h_3 = 0.01
(12 equilibria, 2 stable)

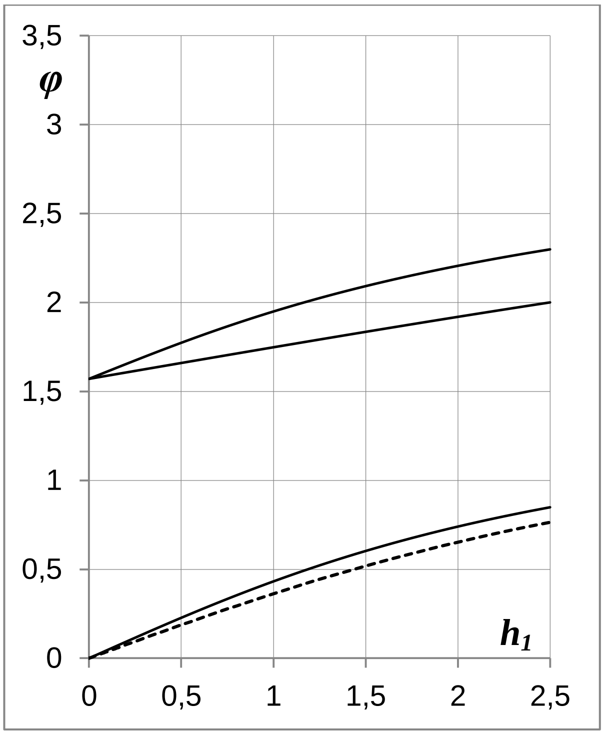

Fig. 50. v=0.4, h_2 = 4.0, h_3 = 0.01
(8 equilibria, 2 stable)

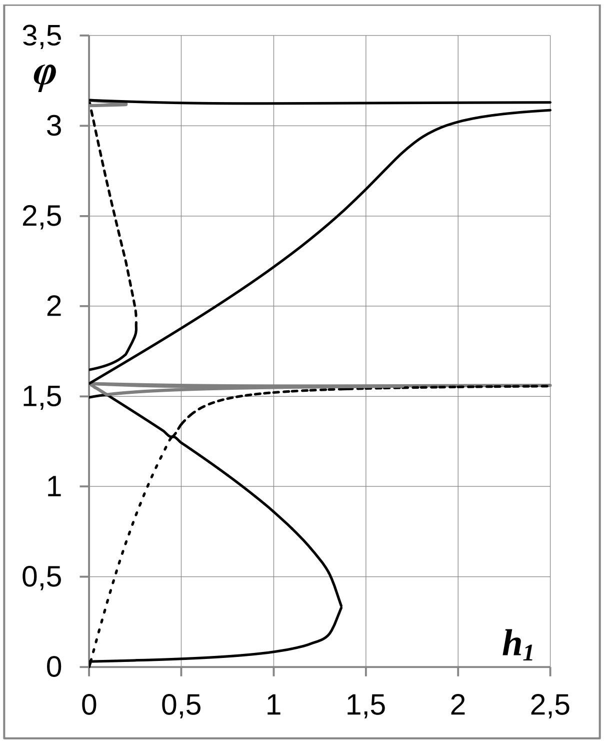

Fig. 51. v=0.4, h_2 = 0.05, h_3 = 0.2
(24 equilibria, 4 stable)

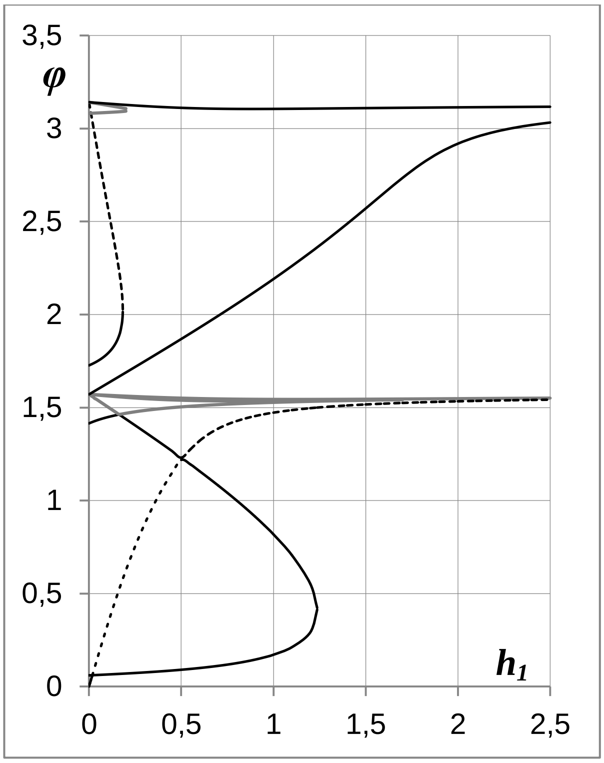

Fig. 52. v=0.4, h_2 = 0.1, h_3 = 0.2
(24 equilibria, 4 stable)

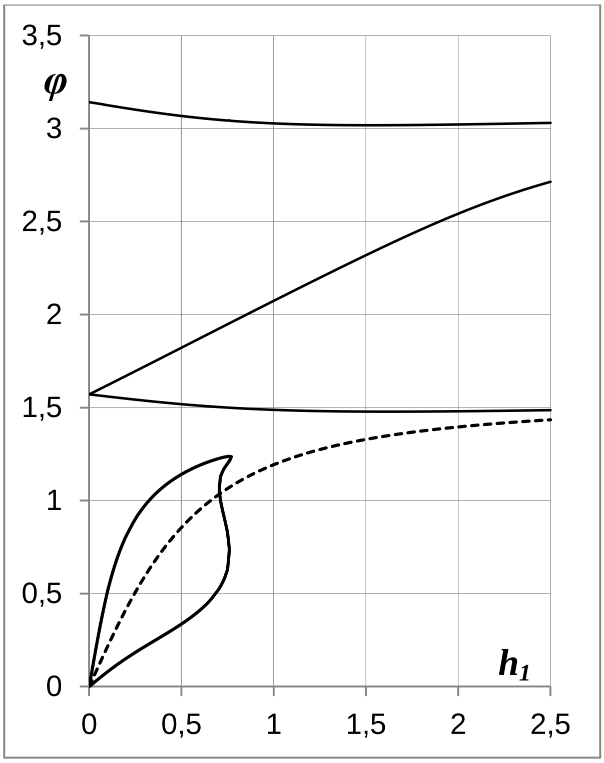

Fig. 53. v=0.4, h_2 = 0.5, h_3 = 1.0
(12 equilibria, 2 stable)

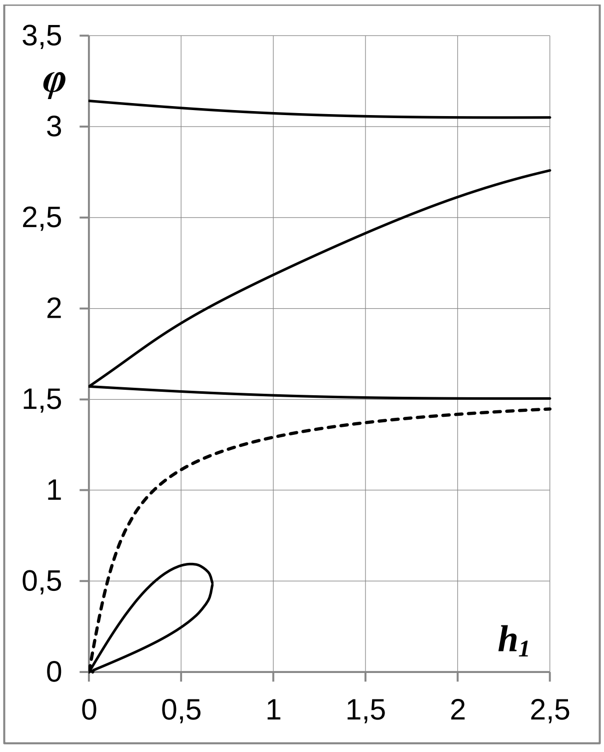

Fig. 54. v=0.4, h_2 = 0.5, h_3 = 2.0
(12 equilibria, 2 stable)

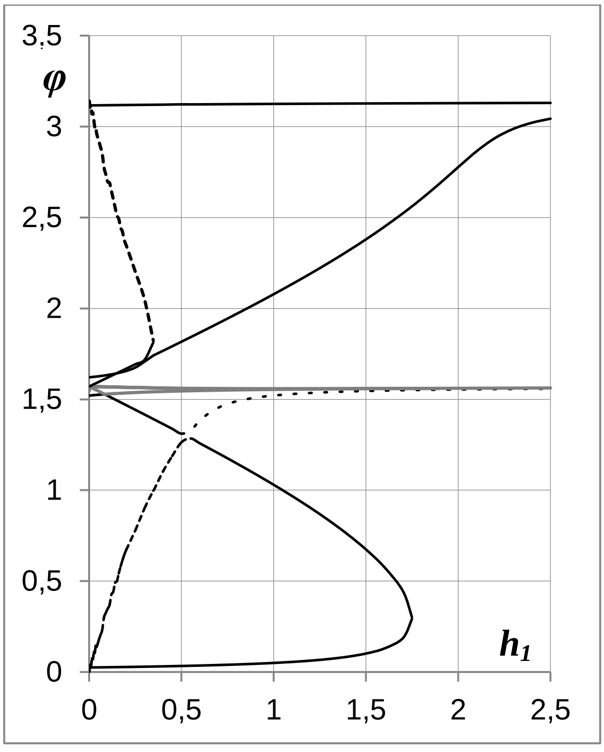

Fig. 55. v=0.5, h_2 = 0.05, h_3 = 0.01
(24 equilibria, 4 stable)

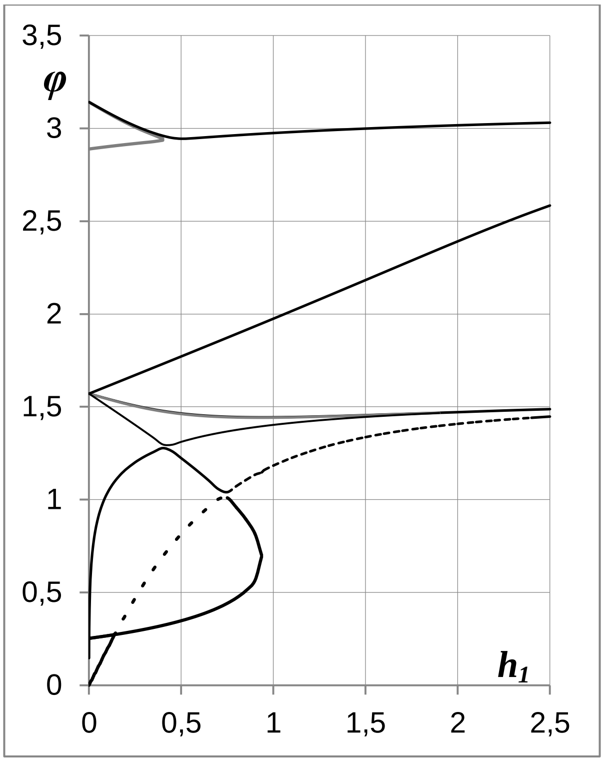

Fig. 56. v=0.5, h_2 = 0.5, h_3 = 0.01
(20 equilibria, 2 stable)

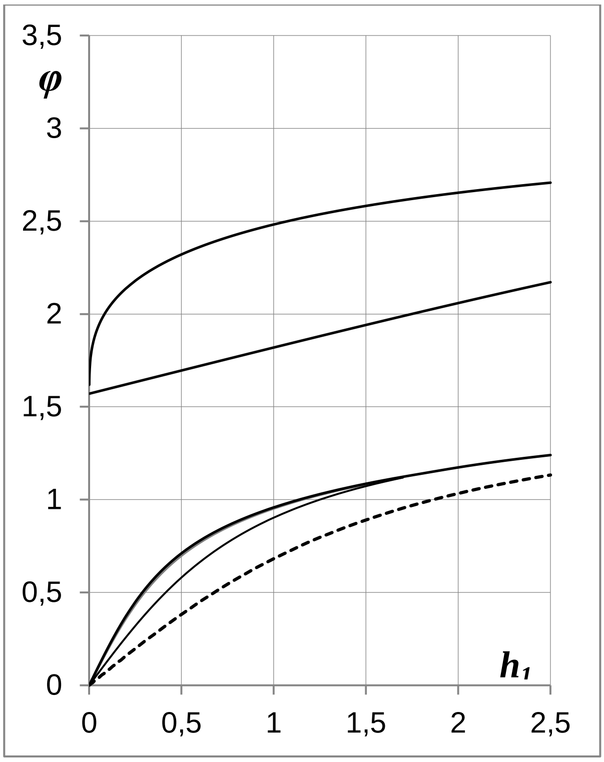

Fig. 57. v=0.5, h_2 = 2.0, h_3 = 0.01
(12 equilibria, 2 stable)

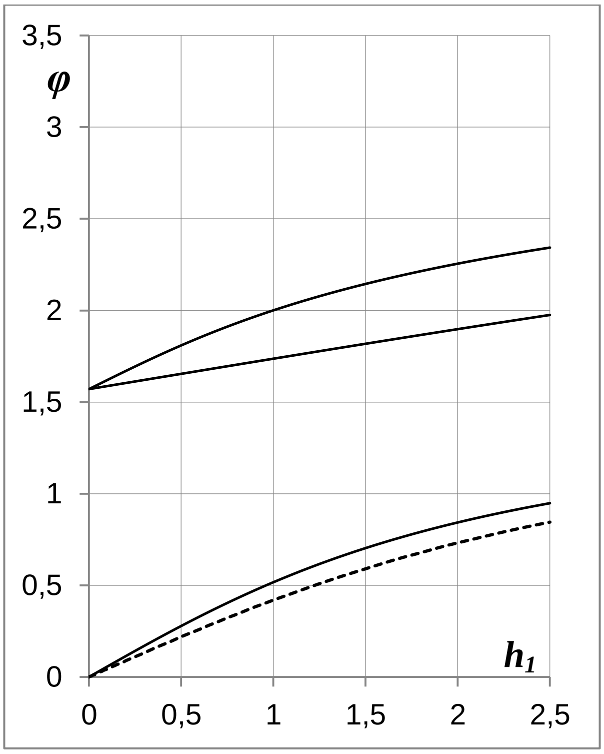

Fig. 58. v=0.5, h_2 = 4.0, h_3 = 0.01
(8 equilibria, 2 stable)

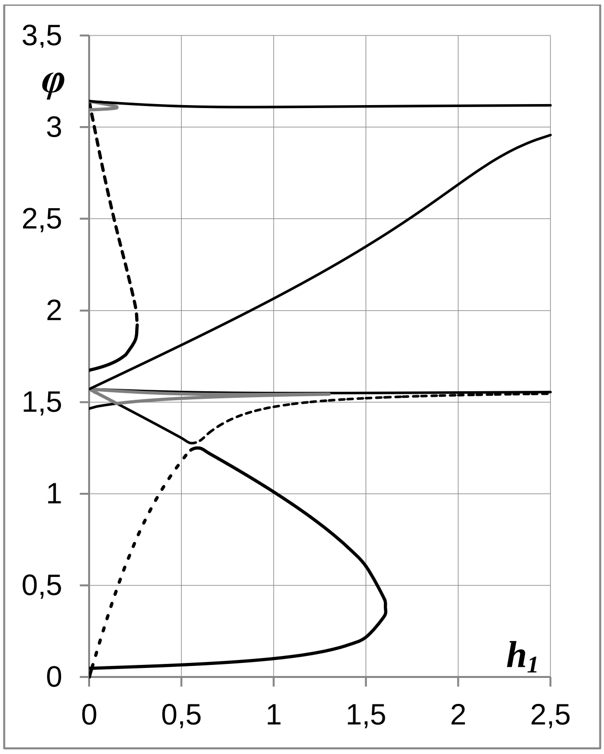

Fig. 59. v=0.5, h_2 = 0.1, h_3 = 0.2
(24 equilibria, 4 stable)

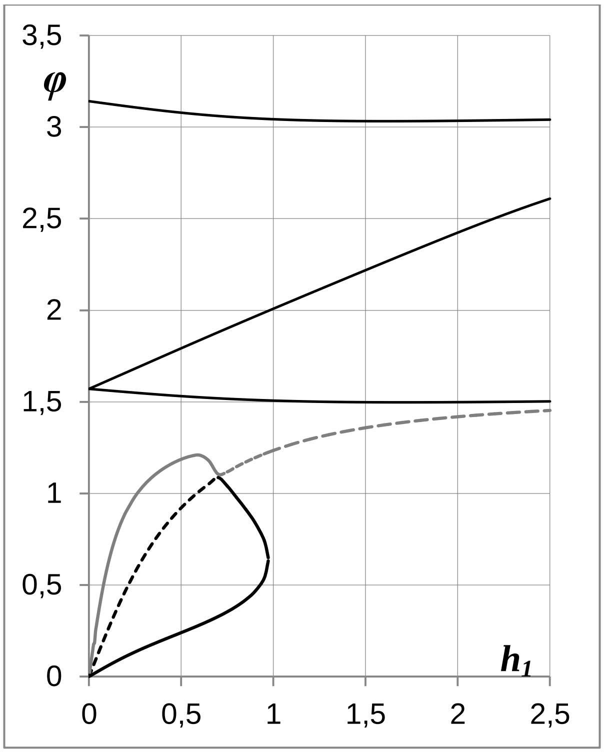

Fig. 60. v=0.5, h_2 = 0.5, h_3 = 1.0
(12 equilibria, 2 stable)

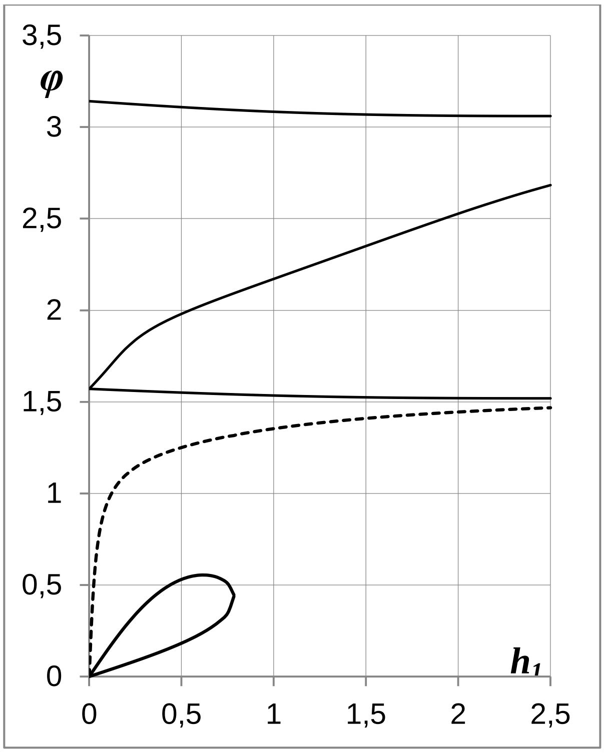

Fig. 61. v=0.5, h_2 = 0.5, h_3 = 2.0
(12 equilibria, 2 stable)

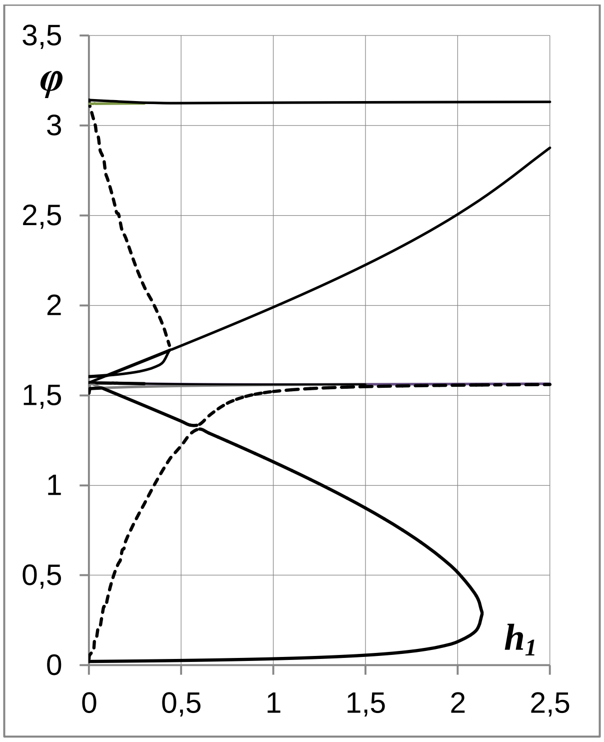

Fig. 62. v=0.6, h_2 = 0.05, h_3 = 0.01
(24 equilibria, 4 stable)

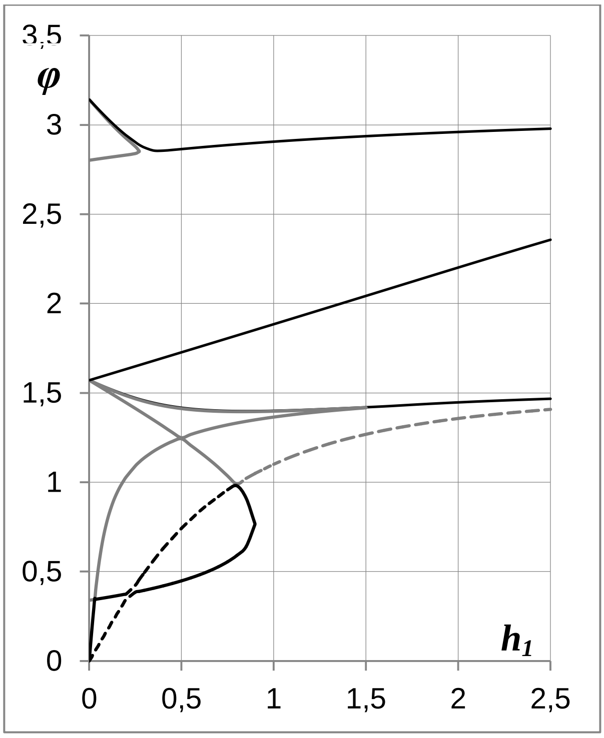

Fig. 63. v=0.6, h_2 = 0.8, h_3 = 0.01
(20 equilibria, 2 stable)

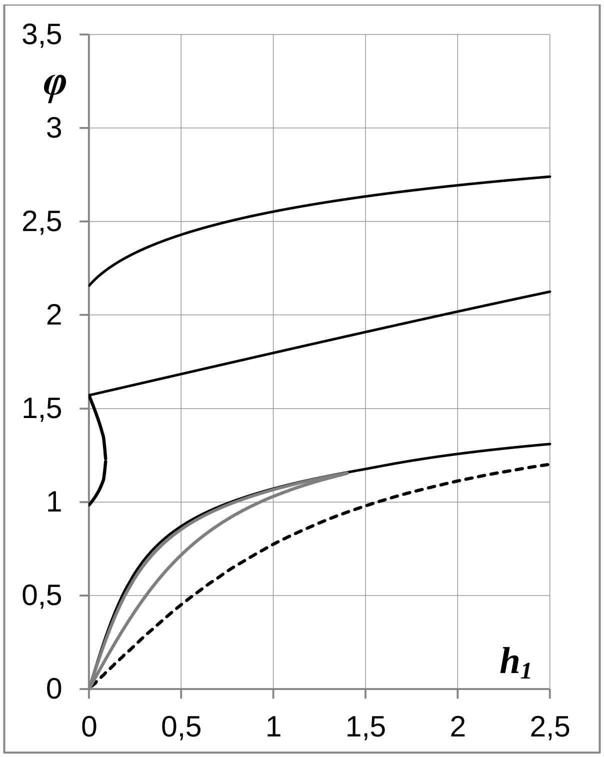

Fig. 64. ν=0.6, h_2 = 2.0, h_3 = 0.01
(16 equilibria, 2 stable)

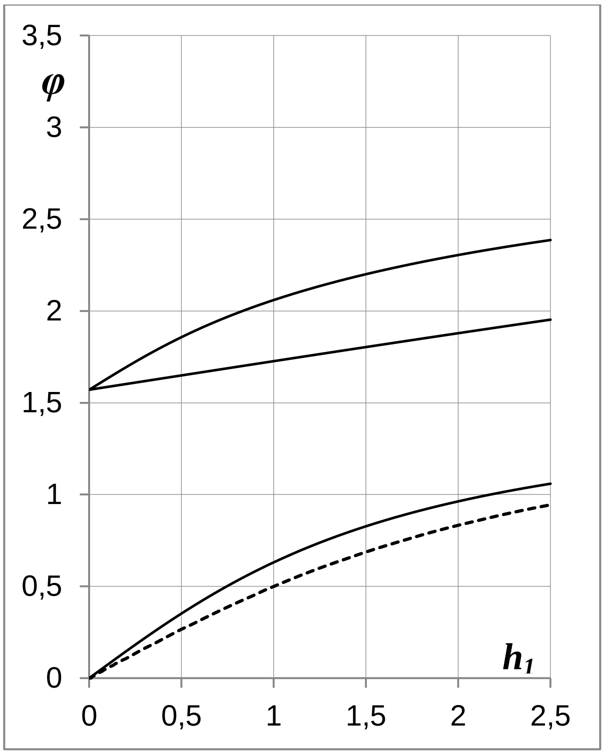

Fig. 65. ν=0.6, h_2 = 4.0, h_3 = 0.01
(8 equilibria, 2 stable)

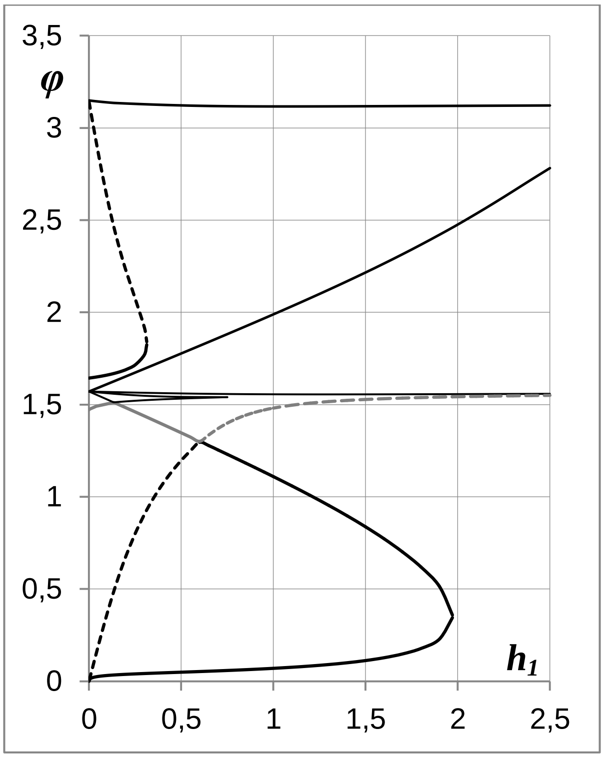

Fig. 66. ν=0.6, h_2 = 0.1, h_3 = 0.4
(20 equilibria, 4 stable)

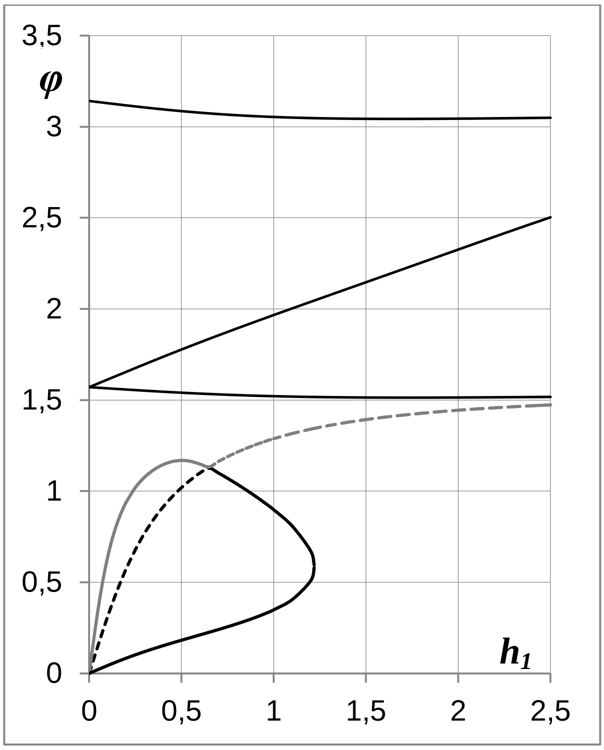

Fig. 67. ν=0.6, h_2 = 0.5, h_3 = 1.0
(12 equilibria, 2 stable)

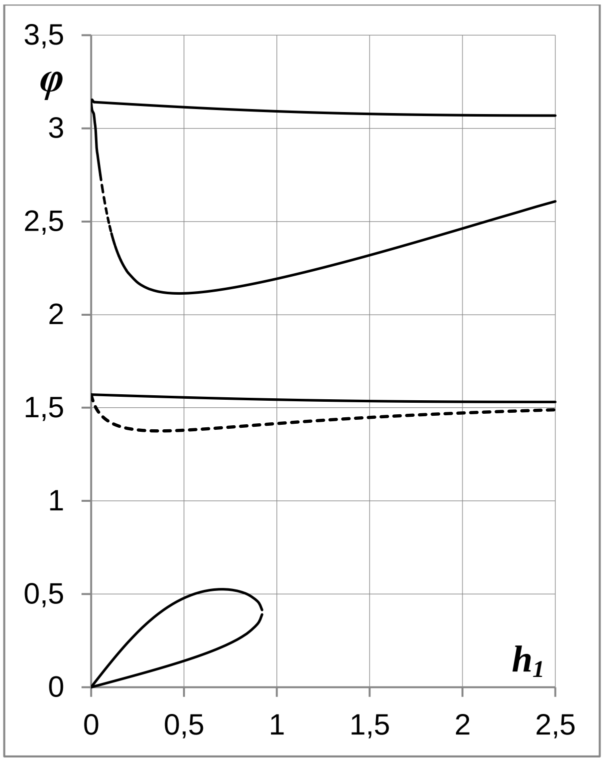

Fig. 68. v=0.6, h_2 = 0.5, h_3 = 2.0
(12 equilibria, 4 stable)

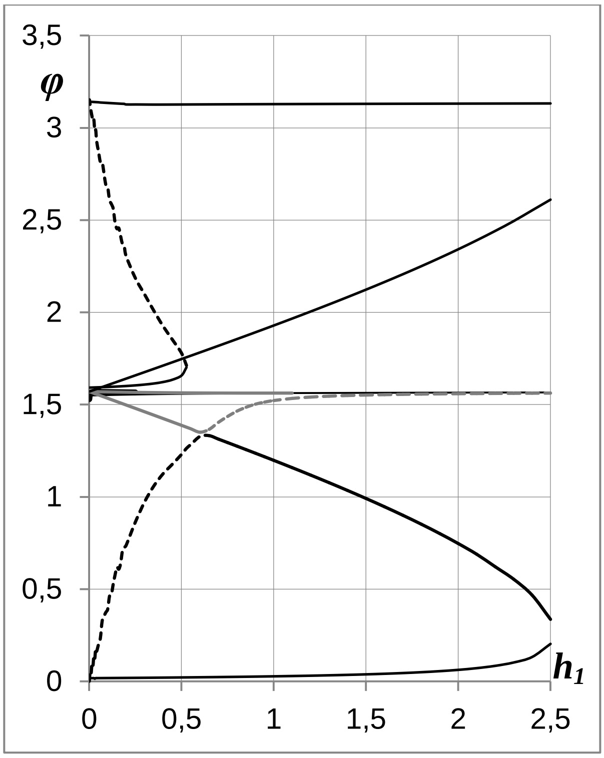

Fig. 69. v=0.7, h_2 = 0.05, h_3 = 0.01
(24 equilibria, 4 stable)

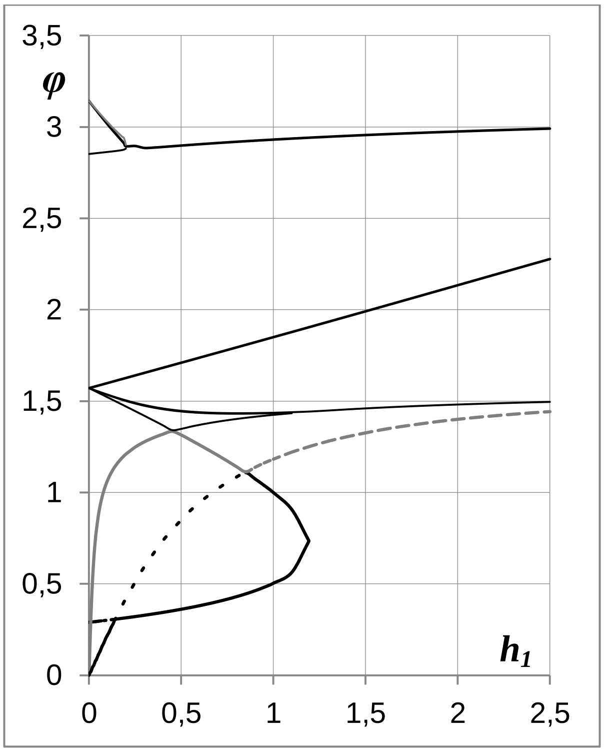

Fig. 70. v=0.7, h_2 = 0.8, h_3 = 0.01
(20 equilibria, 2 stable)

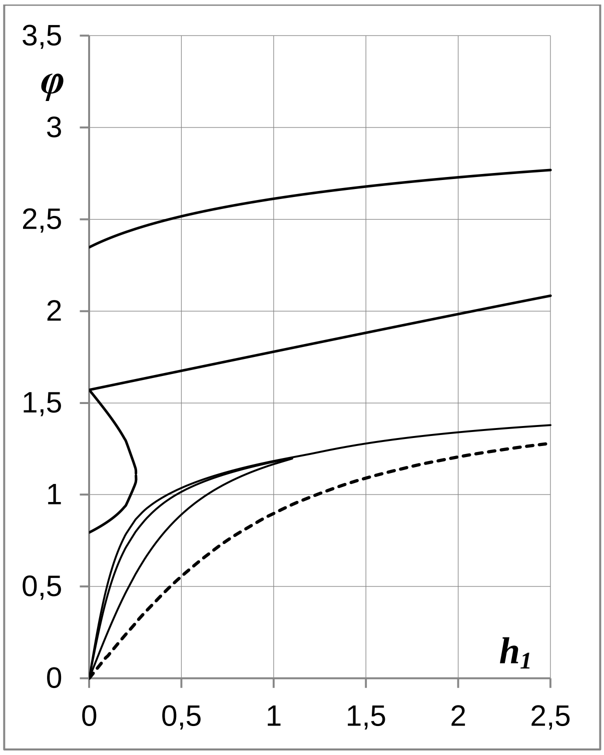

Fig. 71. v=0.7, h_2 = 2.0, h_3 = 0.01
(16 equilibria, 2 stable)

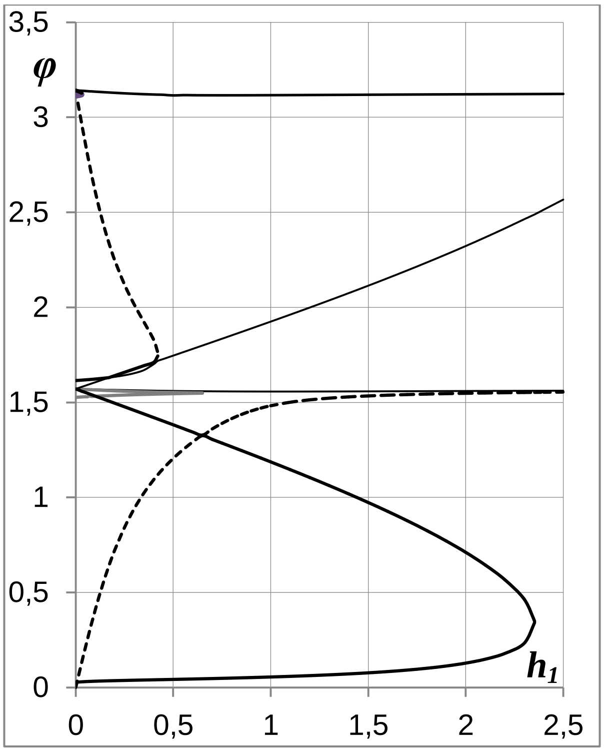

Fig. 72. v=0.7, h_2 = 0.1, h_3 = 0.2
(24 equilibria, 4 stable)

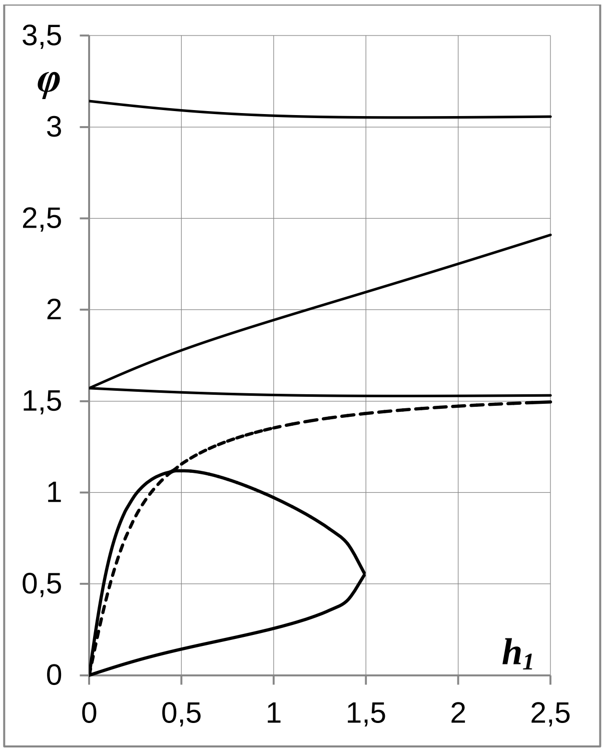

Fig. 73. v=0.7, h_2 = 0.5, h_3 = 1.0
(12 equilibria, 2 stable)

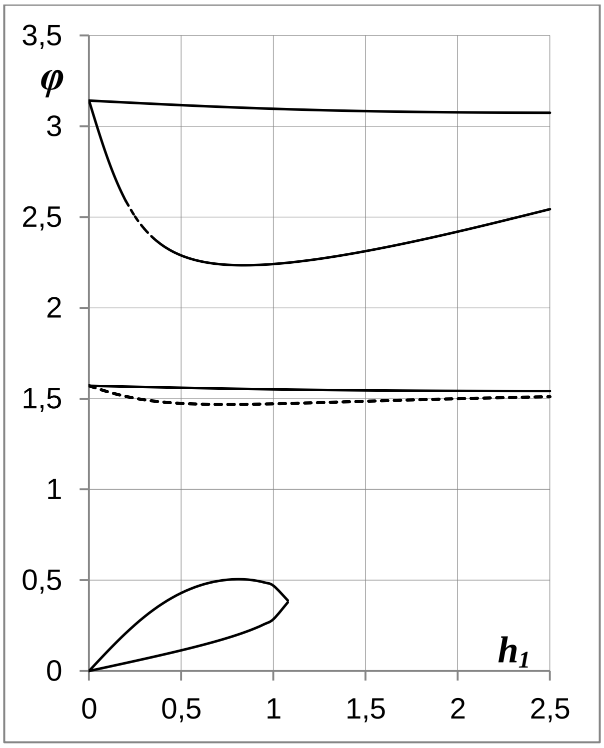

Fig. 74. v=0.7, h_2 = 0.5, h_3 = 2.0
(12 equilibria, 4 stable)

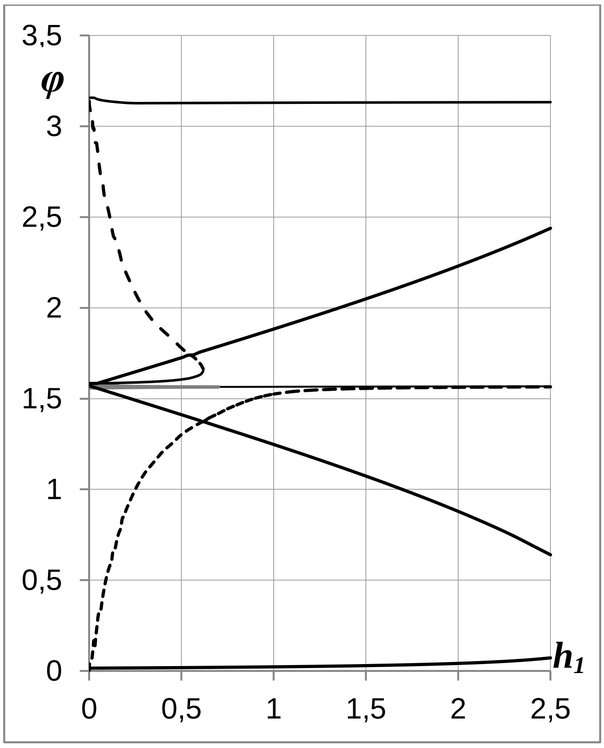

Fig. 75. v=0.8, h_2 = 0.05, h_3 = 0.01
(24 equilibria, 4 stable)

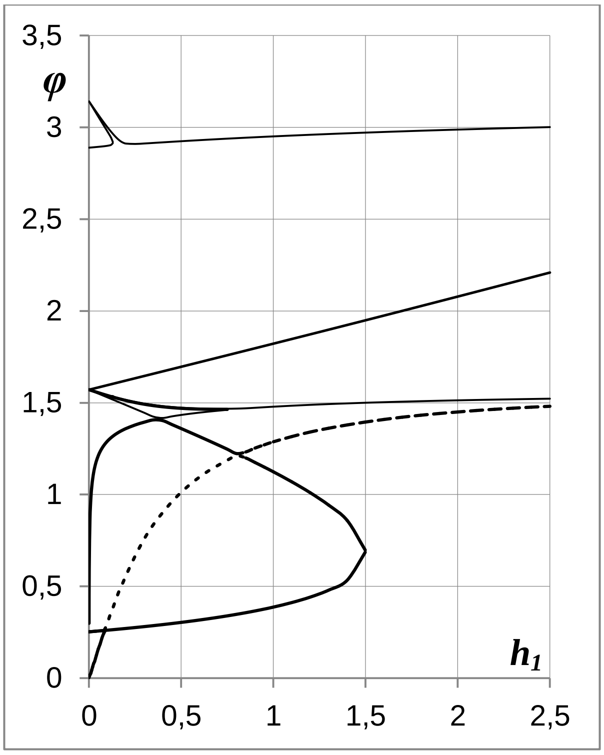

Fig. 76. v=0.8, h_2 = 0.8, h_3 = 0.01
(20 equilibria, 2 stable)

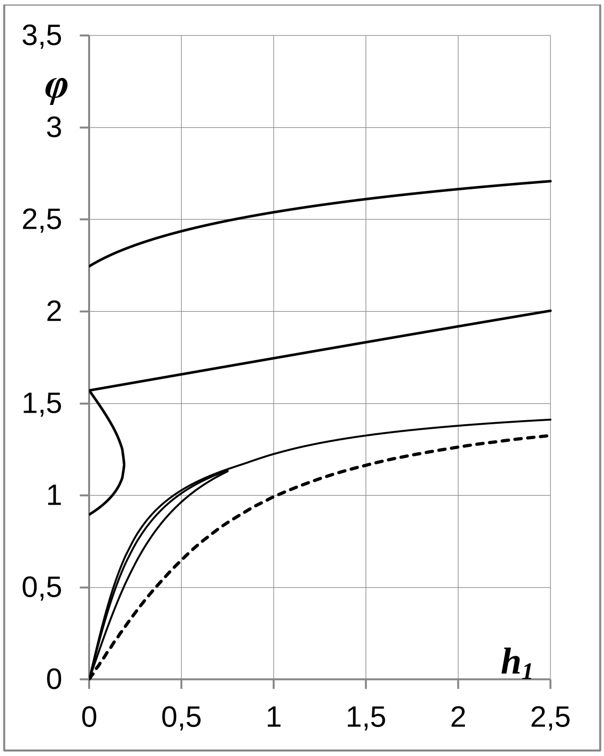

Fig. 77. v=0.8, h_2 = 2.5, h_3 = 0.01
(16 equilibria, 2 stable)

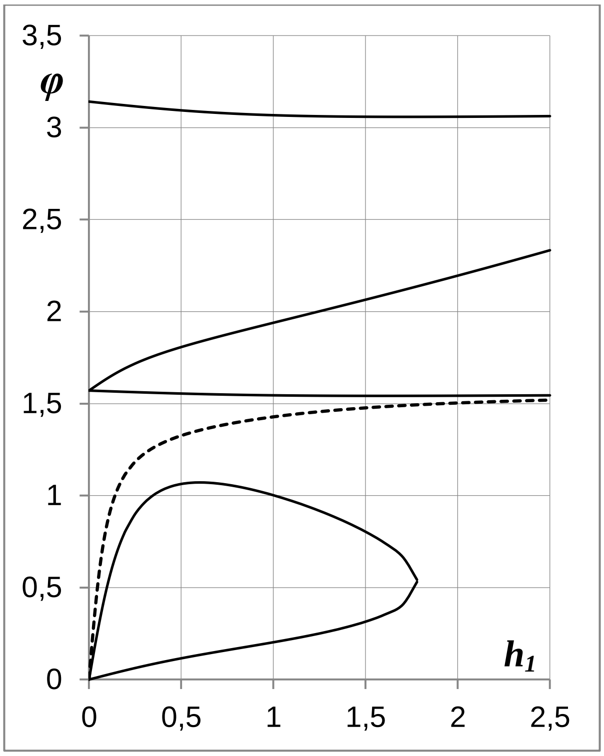

Fig. 78. v=0.8, h_2 = 0.5, h_3 = 1.0
(12 equilibria, 2 stable)

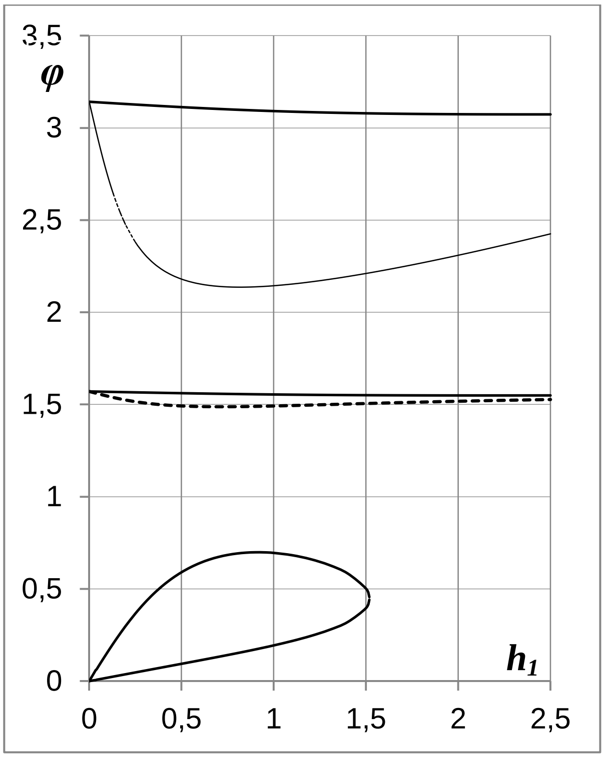

Fig. 79. v=0.8, h_2 = 0.5, h_3 = 1.629
(12 equilibria, 4 stable)

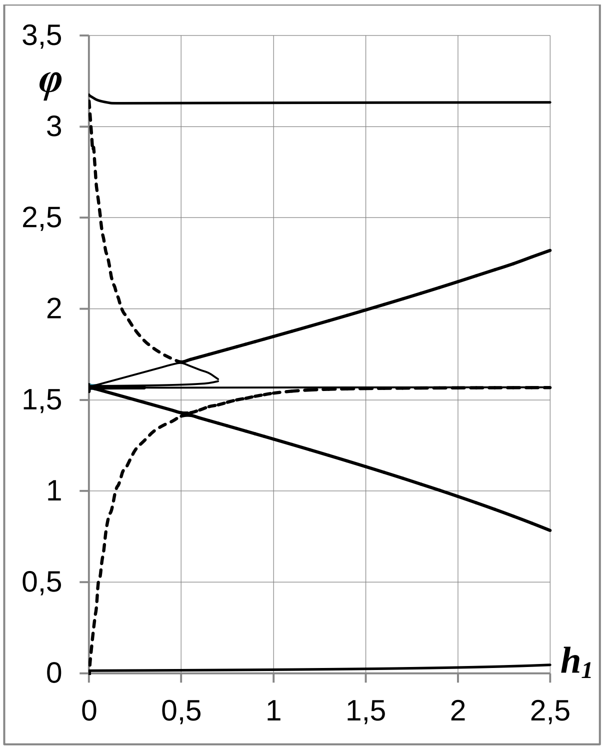

Fig. 80. v=0.9, h_2 = 0.05, h_3 = 0.01
(24 equilibria, 4 stable)

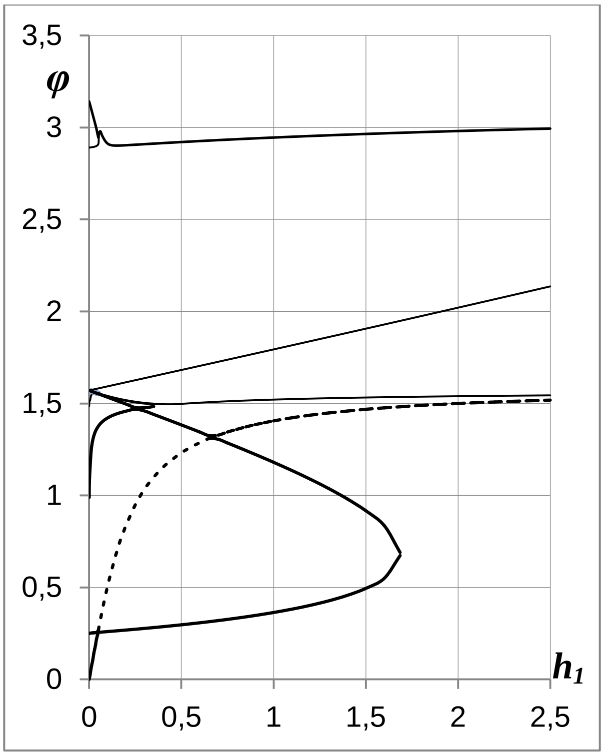

Fig. 81. v=0.9, h_2 = 0.9, h_3 = 0.01
(20 equilibria, 2 stable)

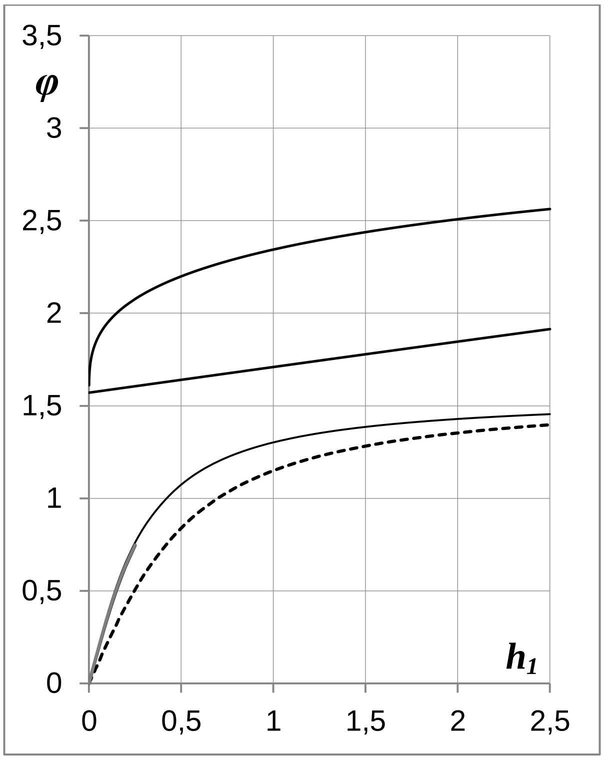

Fig. 82. v=0.9, h_2 = 3.6, h_3 = 0.01
(12 equilibria, 2 stable)

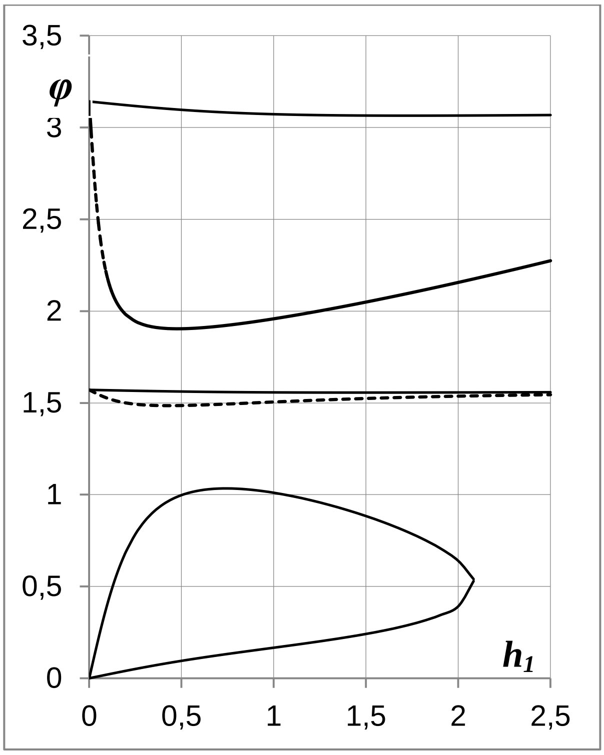

Fig. 83. v=0.9, h_2 = 0.5, h_3 = 1.0
(12 equilibria, 4 stable)

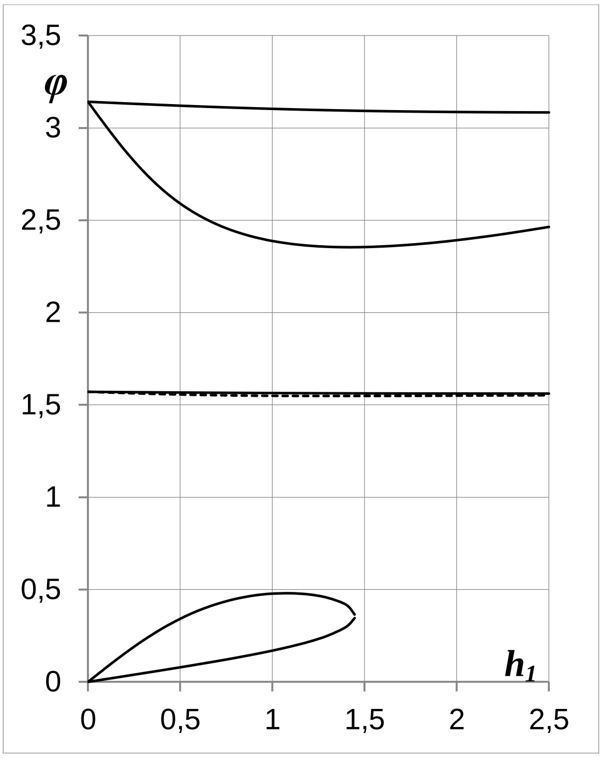

Fig. 84. v=0.9, h_2 = 0.5, h_3 = 2.0
(12 equilibria, 2 stable)

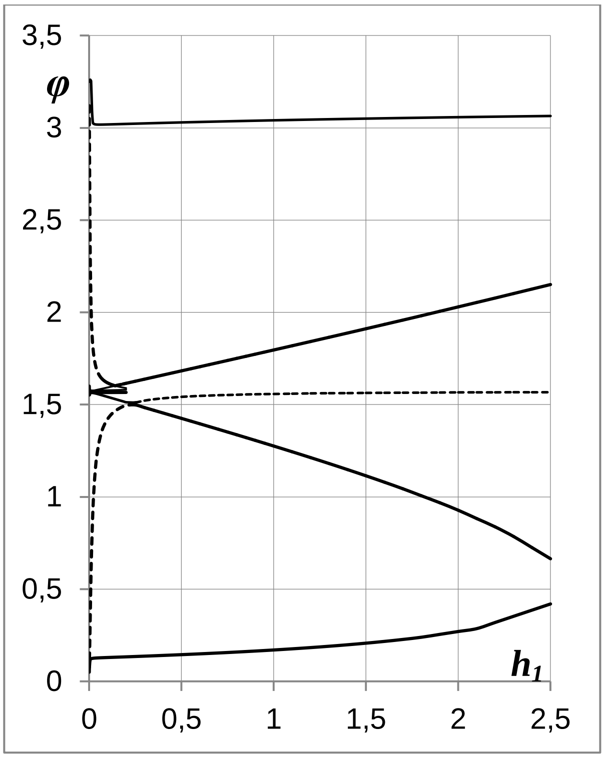

Fig. 85. v=0.99, h_2 = 0.5, h_3 =0.005
(24 equilibria, 4 stable)

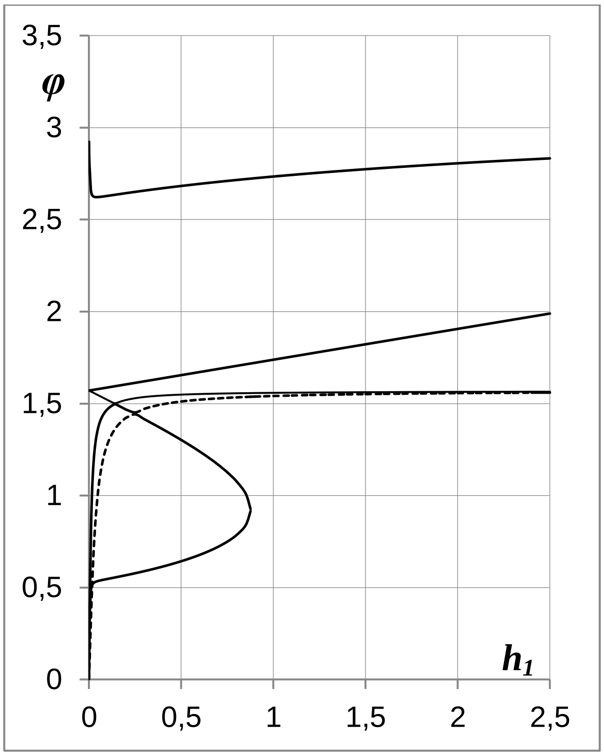

Fig. 86. v=0.99, h_2 = 2.0, h_3 = 0.005
(12 equilibria, 2 stable)

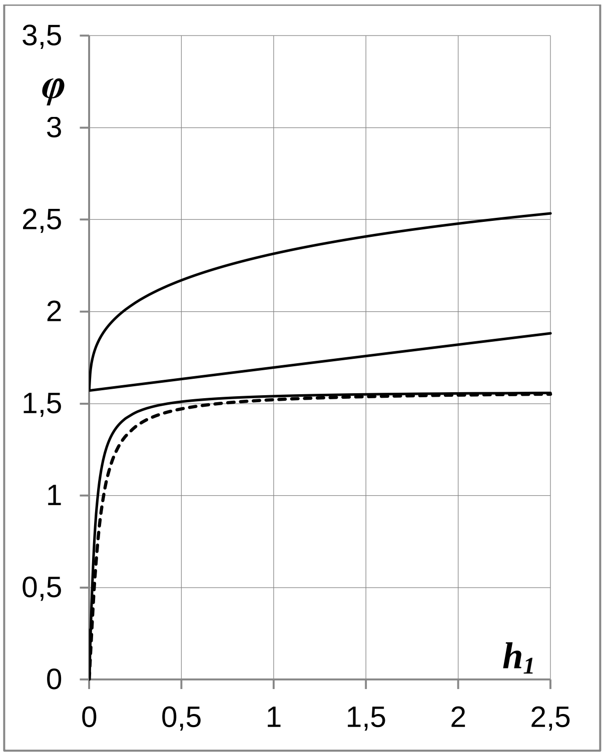

Fig. 87. v=0.99, h_2 = 4.0, h_3 = 0.005
(8 equilibria, 2 stable)