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РОССИЙСКАЯ АКАДЕМИЯ НАУК
ИНСТИТУТ ПРИКЛАДНОЙ МАТЕМАТИКИ
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Fejer convolutions for an extremal problem in the Steklov class

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Свёртки Фейера для одной экстремальной задачи в классе Стеклова

Аннотация. Известная проблема В.А. Стеклова тесно связана со следующей экстремальной задачей. Ищется максимум многочлена (при фиксированной степени), ортонормированного по мере из класса Стеклова (т.е. класса вероятностных мер на единичной окружности с плотностью, отграниченной от нуля в каждой лебеговой точке). Мы исследуем асимптотику некоторых тригонометрических многочленов, определяемых с помощью свёрток Фейера. Эти многочлены могут использоваться при построении асимптотических решений этой экстремальной задачи.

Ключевые слова. Проблема Стеклова; ортогональные многочлены на окружности; свёртки Фейера.

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Fejer convolutions for an extremal problem in the Steklov class [1]

Abstract. The famous problem of V.A. Steklov is intimately related with the following extremal problem. Fix degree and find a maximum of the orthonormal polynomial with respect to measure from the Steklov class (i.e. class of probability measures on the unit circle, such that its density is bounded away from zero at every Lebesgue point. We study asymptotics of certain trigonometric polynomials defined by the Fejer convolutions. These polynomials can be used to construct asymptotical solutions of the above extremal problem.

Key words. Steklov problem; orthogonal polynomials on the circle; Fejer convolution.

Contents

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1 Introduction

Let $\{\phi_n\}$ be a sequence of polynomials of $z = e^{\,i\theta}$ orthonormal on the unit circle

$$\int_0^{2\pi} \phi_n \,\overline{\phi}_m \, d\sigma(\theta) = \delta_{n,m} \ , \quad n, m = 0, 1, 2 \ldots \ , \tag{1.1}$$

with respect to measures from the Steklov class S_δ defined as the class of probability measures σ on the unit circle satisfying

$$\sigma' \geqslant \delta/(2\pi) \tag{1.2}$$

at every Lebesgue point. The famous conjecture of V.A. Steklov (see [1], [2]) stated that polynomials ϕ_n generated by a measure from the Steklov class have to be (uniformely in n) bounded on the support of the orthogonality measure. This conjecture was disproved by E.A. Rakhmanov [3]. In [4] Rakhmanov also has raised a problem about possible growth of orthogonal polynomials whose weight is bounded away from zero. An important role in this problem is played by the following extremal problem. For a fixed n, define

$$M_{n,\delta} = \sup_{\sigma \in S_\delta} \|\phi_n(z;\sigma)\|_{L^\infty(\mathbb{T})} \tag{1.3}$$

The trivial estimate from above is

$$M_{n,\,\delta} \leqslant \sqrt{\frac{n+1}{\delta}} \ , \quad n \in \mathbb{N} \, . \tag{1.4}$$

In [4] Rakhmanov proved

$$C \sqrt{\frac{n+1}{\delta \ \ln^3 n}} \leqslant M_{n,\,\delta} \ , \quad C > 0 \, . \tag{1.5}$$

Thus the Rakhmanov's result left very narrow gap where the magnitude of $M_{n,\,\delta}$ can live. The main result of our paper [5] is

Theorem 1.1 *For any $\delta \in (0, \delta_0)$ with δ_0 sufficiently small, we have*

$$M_{n,\delta} > C(\delta)\sqrt{n} \, . \tag{1.6}$$

We recall, that in the frame of proof of Theorem 1.1 in [5] we introduced an explicit form of the asymptotically extremal polynomial ϕ_n. The polynomial ϕ_n was defined by means of its $*$-polynomial, i.e.

$$\phi_n^* = z^n \overline{\phi_n(1/\overline{z})} \, .$$

Let ϕ_n^* be as follows

$$\phi_n^*(z) = C_n f_n(z), \qquad f_n(z) = P_m(z) + Q_m(z) + Q_m^*(z)\,, \tag{1.7}$$

where P_m and Q_m are certain polynomials of degrees $2m-1$ and $m-1$ correspondingly, and $m = [\delta_1 n]$ where $\delta_1 > 0$ is small enough. Notice here that Q_m^* is defined by applying the n–th order star operation. The constant C_n should be chosen in such a way that

$$\int_{-\pi}^{\pi} |\phi_n^*|^{-2} d\theta = 2\pi\,,$$

(i.e. a measure of orthogonality corresponding to ϕ_n is probabilistic). One of the main technical difficulty in the proof of the Theorem 1.1 in [5] was verification of

$$C_n = \left(\int_{-\pi}^{\pi} |f_n|^{-2} d\theta \right)^{1/2} \sim 1\,, \tag{1.8}$$

uniformly in n.

Here in order to diversify approaches for overcoming the technical difficulties of the proof of the Theorem 1.1 we introduce another (than in [5]) explicit form for polynomial Q_m in (1.7) (see Section 3) and we prove estimates of Q_m and its derivatives which are used for obtaining (1.8) (see Sections 4, 5, 6).

2 Structure of the asymptotically extremal polynomial

We explain assignment of the different terms constituting the polynomial ϕ_n in (1.7). If all zeros of Q_m are outside from the unit disk $\mathbb{D}$ then all (exactly n) zeros of $Q_m + Q_m^*$ are on the unit circle $\mathbb{T}$. Since the polynomial ϕ defined by (1.7) expected to be orthogonal on $\mathbb{T}$ then zeros of $Q_m + Q_m^*$ in (1.7) have to be pushed away out from $\mathbb{D}$, by means of a polynomial P_m chosen by an appropriate way. This "pushing" polynomial P_m has no any other assignments and it has a small modulus. Thus the main contribution in the polynomial ϕ_n on the unit circle is performed by $Q_m + Q_m^*$. Let us find an appropriate representation for this term. We have

$$\begin{aligned} Q_m + Q_m^* &= |Q_m| \exp(i \operatorname{Arg}(Q_m)) + |Q_m| \exp(in\theta - i \operatorname{Arg}(Q_m)) \\ &= 2|Q_m| \exp\left(\tfrac{in\theta}{2}\right) \cos\left(\tfrac{n\theta}{2} - \operatorname{Arg}(Q_m)\right) \\ &= \exp\left(\tfrac{in\theta}{2}\right) 2\sqrt{\mathcal{A}(\theta)} \cdot \cos\left((n/2 - m + 1)\theta + \Phi(\theta)\right). \end{aligned}$$

Here we have denoted

$$\mathcal{A}(\theta) := |Q(e^{i\theta})|^2, \qquad \Phi(\theta) := \operatorname{Arg} Q_m^{*[m-1]}(e^{i\theta})\,, \quad \theta \in (0, 2\pi)\,.$$

In order to control this term from below we need to obtain convenient expressions for the argument of Q_m and its derivative. We have

$$\Phi(\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\mathcal{A}'(\varphi)}{\mathcal{A}(\varphi)} \ln\left|\sin\frac{\varphi-\theta}{2}\right| d\varphi\,, \tag{2.1}$$

and

$$\Phi'(\theta) = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{\mathcal{A}'(\varphi)}{\mathcal{A}(\varphi)}\right)' \ln\left|\sin\frac{\varphi-\theta}{2}\right| d\varphi\,. \tag{2.2}$$

In order to prove (2.1) we have

$$\begin{aligned}\Phi(\theta) &= (m-1)\,\theta - \operatorname{Arg}\left(Q_m(e^{i\theta})\right) = \frac{\theta}{2}\,(m-1) + \frac{1}{2i}\ln\frac{Q_m^{*[m-1]}(e^{i\theta})}{Q_m(e^{i\theta})} = \\ &= \text{v.p.}\,\frac{-1}{4\pi}\int_{|z|=1} \frac{(z+e^{i\theta})dz}{z(z-e^{i\theta})}\ \ln|Q_m(z)|^2 \\ &= \text{v.p.}\,\frac{-1}{4\pi}\int_0^{2\pi} \ln|Q_m(e^{i\varphi})|^2\cot\frac{\varphi-\theta}{2}d\varphi.\end{aligned}$$

Since $\cot\dfrac{\varphi-\theta}{2}$ is the odd function with respect to φ, then

$$\text{v.p.}\int_0^{2\pi} \text{Const}\ \cot\frac{\varphi-\theta}{2}d\varphi = 0\,, \quad \forall\ \text{Const},$$

and we can continue

$$\text{v.p.}\int_0^{2\pi} \ln\,\mathcal{A}(\varphi)\ \cot\frac{\varphi-\theta}{2}d\varphi = \int_0^{2\pi} \ln\,\frac{\mathcal{A}(\varphi)}{\mathcal{A}(\theta)}\ \cot\frac{\varphi-\theta}{2}d\varphi =$$

$$= \ln\frac{\mathcal{A}(\varphi)}{\mathcal{A}(\theta)}\,2\ln\left|\sin\frac{\varphi-\theta}{2}\right|\Bigg|_0^{2\pi} - 2\int_0^{2\pi}\frac{\mathcal{A}'(\varphi)}{\mathcal{A}(\varphi)}\ \ln\left|\sin\frac{\varphi-\theta}{2}\right| d\varphi\,.$$

The first term is equal zero and we arrive to (2.1). Representation (2.2) can be proven in a similar way.

Thus we see from this representations, that to control $Q_m + Q_m^*$ we need to have good estimates form above and below for $|Q_m|^2$ and its two derivatives.

3 Definitions for $|Q_m|^2$ and statements of the results

In [5] the polynomial Q_m was defined as an analytic polynomial without zeroes in $\mathbb{D}$ which gives Fejer-Riesz factorization of

$$|Q_m(z)|^2 := \mathcal{G}_m(\theta) + |R_{(m,\alpha/2)}(e^{i\theta})|^2, \tag{3.1}$$

$$\mathcal{G}_m(\theta) = \mathcal{F}_m(\theta) + \frac{1}{2}\mathcal{F}_m\left(\theta - \frac{\pi}{m}\right) + \frac{1}{2}\mathcal{F}_m\left(\theta + \frac{\pi}{m}\right), \tag{3.2}$$

where $R_{(k,\alpha)}$ is the Taylor approximation to the function $(1-z)^{-\alpha}$, i.e.

$$R_{(k,\alpha)}(z) = c_0 + \sum_{j=1}^{k} c_j z^j, \tag{3.3}$$

and $\mathcal{F}_m$ is the Fejer kernel

$$\mathcal{F}_m(\varphi) = \frac{\sin^2 \frac{m\varphi}{2}}{m \sin^2 \frac{\varphi}{2}}, \qquad \mathcal{F}_m(0) = m, \qquad \int_0^{2\pi} \mathcal{F}_m(\varphi) d\varphi = 2\pi. \tag{3.4}$$

We see that the first term $\mathcal{G}_m$ in (3.1) (we shall call it "shapochka") provides a desirable growth of the orthonormal polynomial

$$|\phi_n(1)| \sim \sqrt{n}.$$

In our construction the trigonometric polynomial Q_m have to keep a large modulus on the interval of order $1/m$. We need it to keep bounded derivative of the polynomial and therefore to have a reasonably smooth Szego function to provide the Steklov condition for othogonality measure. Since the Fejer kernel decays very fast we meet this requirement taking a sum of the kernels (see (3.2)). We also need that outside of this interval the polynomial has a controlled decay, which is again has to be not so fast as the decay of the Fejer kernel. In (3.1) it is an assignment for polynomial $R_{(m,\alpha/2)}$ (we shall call it "wings").

Estimates from above and below for the trigonometric polynomial Q_m (defined in (3.1), – (3.3)) and estimates from above for its first and second derivatives have been done in Appendices A and B of [5] (see Lemmas 5.2, 5.3 and proof of Lemma 6.1). These rather technical estimates lead to the key Lemma 6.1 of [5] (required to prove (1.8)), which controls Φ, the phase of $Q_m(e^{i\theta})$, for $|\theta| < \upsilon$, where υ is some small positive and fixed number

$$|\Phi'(\theta)| \lesssim m\,. \tag{3.5}$$

Here, preserving the same general requirements for the polynomial Q_m as described above, we introduce another constructions for the parts of Q_m than (3.2) and (3.3). We define

$$\mathcal{A}(\theta) := |Q(e^{i\theta})|^2 := (q \otimes \mathcal{F}_m)(\theta), \tag{3.6}$$

where $\otimes$ is convolution with Fejer kernel $\mathcal{F}_m$ – (3.4) of the function:

$$q(\theta) = me^{-m^2 \sin\theta/2} + (m^{-2} + \sin^2(\theta/2))^{-\alpha/2} =: q_1 + q_2, \tag{3.7}$$

which is split in two terms (in accordance with (3.1)), i.e. q_1 "shapochka" and q_2 "wings". Our aim now is to obtain the estimations corresponding to those from the Appendices of [5], which are necessary to get (3.5) and eventually to check (1.8) in new settings (3.6), (3.7). In this section we state the result obtained.

We start with the "shapochka"

$$\mathcal{A}_1(x) := (q_1 \otimes \mathcal{F}_m)(x) := \int_{-\pi}^{\pi} q_1(t)\, \mathcal{F}_m(x-t)\, dt\ , \qquad q_1(t) = me^{-m^2 \sin^2 \frac{t}{2}}\ .$$

Before to state a result about asymptotics of the derivatives $\mathcal{A}_1^{(p)}(x)$, for $p = 0, 1, 2$, as $m \to \infty$, we introduce a sequence of entire functions $\{E_l\}$:

$$E_0(r) := (r-1)\, e^{-r^2}, \quad E_1(r) := r(r-1)\, e^{-r^2}, \quad E_2(r) := (r-1)(2r^2-1)\, e^{-r^2}\ .$$

Denote $\{c_j^{(l)}\}$ coefficients of the power series expansion of $E_l(r)$ at $r = 1$:

$$E_l(r) = \sum_{j=1}^{\infty} c_j^{(l)} (1-r)^j\ .$$

Then we form two another sets of entire functions

$$\widetilde{C}_l(t) := \sum_{\nu=0}^{\infty} (-1)^\nu \tilde{c}_\nu^{(l)} t^{2\nu}\ ; \quad \widetilde{S}_l(t) := \sum_{\nu=0}^{\infty} (-1)^\nu \tilde{s}_\nu^{(l)} t^{2\nu+1}$$

with coefficients

$$\tilde{c}_\nu^{(l)} := \sum \frac{c_j^{(l)} j!}{(j+2\nu+1)!}\ , \quad \tilde{s}_\nu^{(l)} := \sum \frac{c_j^{(l)} j!}{(j+2\nu+2)!}\ .$$

Lemma 3.1 *For any $\varepsilon > 0$, when $m \to \infty$ we have*

$$\mathcal{A}_1(x) = \begin{cases} m\dfrac{\sqrt{\pi}x^2}{\sin^2 \dfrac{x}{2}}\, \widetilde{C}_0(mx) + O(mx)\,, & |x|\, m \leqslant \dfrac{1}{\varepsilon}\,, \\[2ex] \dfrac{1}{m}\dfrac{\sqrt{\pi}\left(1 - \dfrac{\cos mx}{e}\right)}{\sin^2 \dfrac{x}{2}} + O\left(\dfrac{1}{m^2x^2}\right)\,, & |x|\, m \geqslant \dfrac{1}{\varepsilon}\,, \end{cases}$$

and for the derivatives

$$\mathcal{A}_1^{(1)}(x) = \begin{cases} m^2 \dfrac{-\sqrt{\pi}x^2}{\sin^2 \dfrac{x}{2}}\, \widetilde{S}_1(mx) + O(m)\,, & |x|\, m \leqslant \dfrac{1}{\varepsilon}\,, \\ \\ \dfrac{\sqrt{\pi}}{e} \dfrac{\sin mx}{\sin^2 \dfrac{x}{2}} + O\left(\dfrac{1}{mx^2}\right)\,, & |x|\, m \geqslant \dfrac{1}{\varepsilon}\,, \end{cases}$$

and

$$\mathcal{A}_1^{(2)}(x) = \begin{cases} m^3 \dfrac{-\sqrt{\pi}x^2}{2\sin^2 \dfrac{x}{2}}\, (\widetilde{C}_0(mx) + \widetilde{C}_2(mx)) + O(m^{7/3})\,, & |x|\, m \leqslant \dfrac{1}{\varepsilon}\,, \\ \\ m \dfrac{\sqrt{\pi}}{e} \dfrac{\cos mx}{\sin^2 \dfrac{x}{2}} + O\left(\dfrac{1}{mx^3}\right)\,, & |x|\, m \geqslant \dfrac{1}{\varepsilon}\,. \end{cases}$$

Moreover, the function $\widetilde{C}_0(\xi)$ *is even and*

$$\widetilde{C}_0(\xi) > 0, \quad \forall \xi \in \mathbb{R} \tag{3.8}$$

Then we pass to the "wings"

$$\mathcal{A}_2 := q_2 \otimes \mathfrak{F}_m := \int_{-\pi}^{\pi} q_2(t)\, \mathfrak{F}_m(\theta - t)\, dt.$$

Lemma 3.2

$$\mathcal{A}_2(\theta) \asymp \max\left(\frac{1}{m}, |\theta|\right)^{-\alpha}, \qquad \theta \in [-\pi, \pi]. \tag{3.9}$$

For the derivatives $\mathcal{A}_2' = q_2' \otimes \mathfrak{F}_m$, $\mathcal{A}_2'' = q_2'' \otimes \mathfrak{F}_m$ we have obtained just estimation from above (compare with Lemma 5.2 from [5])

Lemma 3.3

$$|\mathcal{A}_2'(\theta)| \lesssim m^{\alpha+1} \min\left(1, \frac{1}{m\theta}\right)^{\alpha+1}.$$

and

$$|\mathcal{A}_2''(\theta)| \lesssim m^{\alpha+2} \min\left(1, \frac{1}{m\theta}\right)^{2}.$$

4 Asymptotics for the "shapochka" and its derivatives

Here we prove the Lemma 3.1 about asymptotics of

$$\mathcal{A}_1^{(p)}(x) := \int_{-\pi}^{\pi} q_1^{(p)}(t)\, \mathcal{F}_m(x-t)\, dt\,, \quad p = 0,1,2, \tag{4.1}$$

where we recall

$$q_1(t) = m e^{-m^2 \sin^2 \frac{t}{2}}, \qquad \mathcal{F}_m(\varphi) = \frac{\sin^2 \frac{m\varphi}{2}}{m \sin^2 \frac{\varphi}{2}}.$$

We start with a general approach to estimations of (4.1) for arbitrary p. Then we specify the general result for $p = 0,1,2$. The general approach consists on the following steps.

1. We split the integral (4.1) in two parts

$$\mathcal{A}_1^{(p)} = \int_{-m^{-2/3}}^{m^{-2/3}} + \int_{[-\pi,\pi]\backslash[-m^{-2/3},m^{-2/3}]} =: \widetilde{\mathcal{A}_{1p}} + \widetilde{\widetilde{\mathcal{A}_{1p}}}\,,$$

where the second integral is estimated as

$$\widetilde{\widetilde{\mathcal{A}_{1p}}} = O\left(m^{2+2p} e^{-\frac{m^{2/3}}{4}}\right)\,. \tag{4.2}$$

2. We denote

$$S_m(x,t) := \exp\left(-\frac{m^2t^2}{4}\right) \frac{\sin^2 \frac{m}{2}(x-t)}{\left(\frac{x-t}{2}\right)^2}\,, \quad f_p(x,t) := \frac{q_1^{(p)}(t)\left(\frac{x-t}{2}\right)^2}{m\, e^{-\frac{m^2t^2}{4}} \sin^2\left(\frac{x-t}{2}\right)}\,.$$

Thus

$$\widetilde{\mathcal{A}_{1p}} = \int_{-m^{-2/3}}^{m^{-2/3}} f_p(x,t)\, S_m(x,t)\, dt\,. \tag{4.3}$$

We take an expansion

$$f_p(x,t) = \sum_{j=0}^{\infty} \widetilde{F}_{p,j}(x,m)\, t^j\,, \tag{4.4}$$

and we note that coefficients $F_{p,j}$ are bounded for $x \in (-\pi,\pi)$. Now we keep the first N terms in the Taylor expansion and estimate the rest of the sum by using the bound $|t| < m^{-2/3}$:

$$f_p(x,t) = \sum_{j \le N} \widetilde{F}_{p,j}(x,m)\, t^j + \sum_{j>N} O(m^{k_j})\, t^{l_j}\,. \tag{4.5}$$

The sharpness of the asymptotics we get will depend on N.

3. Next we substitute (4.5) in (4.3) and using estimation like in (4.2) we extend the interval of integration in (4.3) from $[-m^{-2/3}, m^{-2/3}]$ to $[-\infty, \infty]$:

$$\mathcal{A}_1^{(p)} = \sum_{j\le N} \widetilde{F}_{p,j}(x,m)\, J_j(x,m) + \sum_{j>N} O(m^{k_j})\, \tilde{J}_{l_j}(x,m)\ , \tag{4.6}$$

where

$$J_j(x,m) := \int_{-\infty}^{\infty} t^j S_m(x,t)\, dt\ , \qquad \tilde{J}_j(x,m) := \int_{-\infty}^{\infty} |t|^j S_m(x,t)\, dt\ .$$

This representation easily implies

$$J_{2k}(x,m) = \tilde{J}_{2k}(x,m) > 0, \quad x \in \mathbb{R},\ \ k \in \mathbb{N}. \tag{4.7}$$

4. Then we proceed with the integrals J_j. We use an identity

$$\frac{\sin^2\left(\frac{m(x-t)}{2}\right)}{\left(\frac{x-t}{2}\right)^2} = 2m^2 \int_0^1 \int_0^s \cos(rm(x-t))\, dr\, ds = 2m^2 \int_0^1 \int_r^1 \cos(rm(x-t))\, ds\, dr\ .$$

It gives us

$$J_j(x,m) = 2m^2 \int_0^1 \int_{-\infty}^{\infty} t^j e^{-\frac{m^2t^2}{4}} \int_r^1 \cos(rm(x-t))\, ds\, dt\, dr\ .$$

Evaluating the integrals in s and t we get

$$\begin{aligned} J_{2k} &= \frac{(-1)^{k+1}\sqrt{\pi}}{m^{2k-1}}\, 2^{k+2} \int_0^1 \cos(rmx)\, E_{2k}(r)\, dr\ ; \\ J_{2k+1} &= \frac{(-1)^{k+1}\sqrt{\pi}}{m^{2k}}\, 2^{k+3} \int_0^1 \sin(rmx)\, E_{2k+1}(r)\, dr\ , \end{aligned} \tag{4.8}$$

where $E_l(r)$ are entire functions:

$$E_0 := (r-1)\, e^{-r^2}\ , \quad E_1 = r(r-1)\, e^{-r^2}\ , \quad E_2 := (r-1)(2r^2-1)\, e^{-r^2}\ , \quad \ldots$$

Thus to conclude the description of the general approach we have to explain how to get asymptotics of $J_0(x,m)$, when $m \to \infty$. We shall do it in two regions:

$$|x|m \leqslant \frac{1}{\varepsilon} \quad \text{and} \quad |x|m \geqslant \frac{1}{\varepsilon} \quad \text{for } \forall\varepsilon > 0\ .$$

5. For bounded $|x|m$, we take the power series expansion of the entire function $E_l(r)$ at the point $t=1$:

$$E_l(r)=\sum_{j=1}^{\infty}c_j^{(l)}(1-r)^j \ ,$$

and substitute it in (4.8). Expanding the obtained integrals

$$\int_0^1\cos(rmx)\,(1-r)^j dr=\sum_{\nu=0}^{\infty}\frac{(-1)^\nu j!(mx)^{2\nu}}{(j+2\nu+1)!}$$

$$\int_0^1\sin(rmx)\,(1-r)^j dr=\sum_{\nu=0}^{\infty}\frac{(-1)^\nu j!(mx)^{2\nu+1}}{(j+2\nu+2)!} \ ,$$

we construct another entire functions

$$\widetilde{C}_l(t):=\sum_{\nu=0}^{\infty}(-1)^\nu\tilde{c}_\nu^{(l)}t^{2\nu} \ ; \quad \widetilde{S}_l(t):=\sum_{\nu=0}^{\infty}(-1)^\nu\tilde{s}_\nu^{(l)}t^{2\nu+1}$$

with coefficients

$$\tilde{c}_\nu^{(l)}:=\sum_{j=1}^{\infty}\frac{c_j^{(l)}j!}{(j+2\nu+1)!} \ , \quad \tilde{s}_\nu^{(l)}:=\sum_{j=1}^{\infty}\frac{c_j^{(l)}j!}{(j+2\nu+2)!} \ .$$

Thus for finite $|x|m$ we get the following representation for the integrals (4.8)

$$J_{2k}(x,m)=\frac{(-1)^{k+1}\sqrt{\pi}2^{k+2}}{m^{2k-1}}\,\widetilde{C}_{2k}(mx), \ J_{2k+1}(x,m)=\frac{(-1)^{k+1}\sqrt{\pi}2^{k+3}}{m^{2k}}\,\widetilde{S}_{2k+1}(mx) \ . \tag{4.9}$$

If $k=0$, the first formula above along with (4.7) give (3.8).

6. For growing $|x|m$ we perform integration by parts in (4.8). We have

$$J_0(x,m)=-4\sqrt{\pi}m\left\{\left(\frac{\cos mx}{e}-1\right)(mx)^{-2}+\frac{4}{e}\sin mx(mx)^{-3}+\right.$$

$$\left.+(mx)^{-4}\left[\frac{6}{e}(1-\cos mx)+4\int_0^1(\cos rmx-1)\mathcal{P}_5(r)\,e^{-r^2}dr\right]\right\} \ ,$$

where $\mathcal{P}_5(r)$ is some polynomial $\deg\mathcal{P}_5=5$. Thus

$$J_0(x,m)=4\sqrt{\pi}m\left(1-\frac{\cos mx}{e}\right)\frac{1}{(mx)^2}+O\left(\frac{1}{m^2x^3}\right) \ . \tag{4.10}$$

Analogously

$$J_1(x,m) = -\frac{8\sqrt{\pi}}{e}\sin mx\,\frac{1}{(mx)^2} + O\left(\frac{1}{m^3x^3}\right)\,, \tag{4.11}$$

and so on:

$$J_2(x,m) = \frac{8\sqrt{\pi}}{m}\left(\frac{\cos mx}{e}+1\right)\frac{1}{(mx)^2} + O\left(\frac{1}{m^4x^3}\right)\,,$$

$$J_3(x,m) = \frac{16\sqrt{\pi}}{m^2}\left(\frac{\sin mx - mx}{e}\right)\frac{1}{(mx)^2} + O\left(\frac{1}{m^5x^3}\right)\,.$$

Now we are prepared to analyse the special cases of $\mathcal{A}_1^p$, for $p = 0, 1, 2$. We shall use the notations:

$$a(x) := \left(\frac{x/2}{\sin(x/2)}\right)^2, \quad b(x) := \frac{x^2\cos(x/2) - 2x\sin(x/2)}{4\sin^3(x/2)}\,.$$

The case $p = 0$**.** We have

$$f_0(x,t) = \exp\left\{\frac{m^2t^2}{4} - m^2\sin^2\left(\frac{t}{2}\right)\right\}\cdot a(x-t) = a(x) + b(x)\,t + O(1)\,t^2 +$$

$$+ O(1)\,t^3 + \left(\frac{1}{48}a(x)\,m^2 + O(1)\right)t^4 + \ldots$$

We note, that exponents in this series have the following periodic structure

$$O(m^{2k})\,t^{4k}\,,\quad O(m^{2k})\,t^{4k+1}\,,\quad O(m^{2k})\,t^{4k+2}\,,\quad O(m^{2k})\,t^{4k+3}\,,\quad k\in\mathbb{N}\,.$$

Thus

$$f_0(x,t) = a(x) + O(1)\,t\,,\quad t\in[-m^{-2/3}, m^{-2/3}]\,.$$

It gives us

$$\mathcal{A}_1 = a(x)\,J_0(x,m) + O(1)\,\tilde{J}_1(x,m)\,,$$

where $O(1)$ is bounded as function of x and m. From here using (4.9) and (4.10) we arrive to the statement of the Lemma 3.1 for $p = 0$.

The case $p = 1$**.** We have

$$f_1(x,t) = -\frac{m^2}{2}\exp\left\{\frac{m^2t^2}{4} - m^2\sin^2(t/2)\right\}\cdot\sin t\cdot a(x-t) = -m^2\frac{a(x)}{2}\,t -$$

$$-m^2\frac{b(x)}{2}\,t^2 + O(1)\,m^2t^3 + O(1)\,m^2t^4 - \left(\frac{m^4a(x)}{96} + O(1)\,m^2\right)t^5 + \ldots$$

Thus

$$f_1(x,t) = -m^2 \frac{a(x)}{2}\, t + O(m^2)\, t^2 \,, \quad t \in [-m^{-2/3}, m^{-2/3}] \,.$$

It gives us

$$\mathcal{A}_1^{(1)} = -m^2 \frac{a(x)}{2}\, J_1(x,m) + O(m^2)\, J_2(x,m) \,,$$

From here using (4.9) and (4.11) we arrive to the statement of the Lemma 3.1 for $p = 1$.

The case $p = 2$**.** We have

$$f_2(x,t) = -\frac{m^2}{4} \exp\left\{ \frac{m^2 t^2}{4} - m^2 \sin^2(t/2) \right\} \cdot (2\cos t - m^2 \sin^2 t) \cdot a(x-t) \,.$$

Expansion by t^ν implies

$$f_2(x,t) = -m^2 \frac{a(x)}{2} \left(1 - \frac{m^2 t^2}{2}\right) - m^2 \frac{b(x)}{2} \left(t - \frac{m^2 t^3}{2}\right) +$$

$$+ \frac{m^6 a(x)}{192}\, t^6 + O(m^{4/3}) \,, \quad t \in [-m^{-2/3}, m^{-2/3}] \,.$$

It gives us

$$\mathcal{A}_1^{(2)} = -m^2 \frac{a(x)}{2} \left(J_0 - \frac{m^2}{2} J_2\right) - m^2 \frac{b(x)}{2} \left(J_1 - \frac{m^2}{2} J_3\right) + O(m^{4/3})\, J_0 \,.$$

Performing integration by parts we get

$$J_0 - \frac{m^2}{2} J_2 = -8\sqrt{\pi}\, m \int_0^1 \cos(rmx)\, r^2 (r-1)\, e^{-r^2} dr =$$

$$= -\frac{8\sqrt{\pi}}{e}\, m\, \cos mx\, \frac{1}{(mx)^2} + O\left(\frac{1}{m^3 x^4}\right) \,,$$

and

$$J_1 - \frac{m^2}{2} J_3 = -16\sqrt{\pi}\, m \int_0^1 \sin(rmx)\, r(r+1)(r-1)^2\, e^{-r^2} dr =$$

$$= \frac{32\sqrt{\pi}}{e} (2\cos mx + e)\, \frac{1}{(mx)^3} + O\left(\frac{1}{m^4 x^4}\right) \,,$$

>From here and (4.9)we obtain the statement of the Lemma 3.1 for $p = 2$.

Lemma 3.1 is proved.

5 Asymptotics for the "wings"

Here we prove the Lemma 3.2 about asymptotics of

$$\mathcal{A}_2 := q_2 \otimes \mathcal{F}_m := \int_{-\pi}^{\pi} q_2(t)\, \mathcal{F}_m(\theta - t)\, dt.$$

We start with estimations from below. We have for the Fejer kernel

$$\mathcal{F}_n(t) \geqslant \begin{cases} \dfrac{4n}{\pi^2}\,, & |t| < \dfrac{\pi}{n} \\[2ex] 0\,, & |t| > \dfrac{\pi}{n} \end{cases} .$$

Therefore (due to positivity of q_2)

$$(q_2 \otimes \mathcal{F}_n)(\theta) \geqslant \int_{\theta-\frac{\pi}{n}}^{\theta+\frac{\pi}{n}} \frac{4n}{\pi^2}\, q_2(t)\, dt \geqslant \int_{\theta-\frac{\pi}{n}}^{\theta+\frac{\pi}{n}} \frac{4n}{\pi^2} \left(\frac{1}{n} + \frac{|t|}{2}\right)^{-\alpha} dt\,,$$

where we used the fact that $\alpha > 0$. There are two options:

$$|\theta| < \frac{\pi}{n} \quad \Rightarrow \quad |t| \lesssim \frac{1}{n} \quad \Rightarrow \quad \left(\frac{1}{n} + \frac{|t|}{2}\right)^{-\alpha} \asymp n^{\alpha}$$

or

$$|\theta| \geqslant \frac{\pi}{n} \quad \Rightarrow \quad \frac{1}{n} + \frac{|t|}{2} \asymp \frac{1}{n} + \frac{|\theta|}{2} \asymp |\theta|\,.$$

Thus the estimate from below in (3.9) is proven.

Now we proceed with estimations from above. We have for the Fejer kernel

$$\mathcal{F}_n(t) \leqslant \begin{cases} n\,, & |t| < \dfrac{\pi}{n} \\[2ex] \dfrac{\pi^2}{nt^2}\,, & |t| > \dfrac{\pi}{n} \end{cases} . \tag{5.1}$$

Therefore

$$(q_2 \otimes \mathcal{F}_n)(\theta) \leqslant \int_{\theta-\frac{\pi}{n}}^{\theta+\frac{\pi}{n}} n \left(\frac{|t|}{\pi}\right)^{-\alpha} dt + \int_{\theta+\frac{\pi}{n}}^{\pi} \frac{\pi^2}{n} \left(\frac{|t|}{\pi}\right)^{-\alpha} \frac{dt}{(\theta - t)^2} +$$

$$+ \int_{-\pi}^{\theta-\frac{\pi}{n}} \frac{\pi^2}{n} \left(\frac{|t|}{\pi}\right)^{-\alpha} \frac{dt}{(\theta - t)^2} \; =: \; I_1 + I_2 + I_3\,.$$

Using symmetry we can suppose $\theta \in (0, \pi/2)$. The case $\theta \in (\pi/2, \pi)$ can be handled similarly. For the first integral we have

$$|\theta| \leqslant \frac{\pi}{n} \quad \Rightarrow \quad h := \frac{n}{\pi}|\theta| \leqslant 1 \quad \Rightarrow$$

$$I_1 = \frac{\pi n^\alpha}{1-\alpha}\left((1+h)^{1-\alpha} + (1-h)^{1-\alpha}\right) \asymp n^\alpha$$

where we used the fact that $\alpha < 1$, and

$$|\theta| > \frac{\pi}{n} \quad \Rightarrow \quad h > 1 \quad \Rightarrow$$

$$I_1 = \frac{\pi n^\alpha}{1-\alpha}\left((h+1)^{1-\alpha} - (h-1)^{1-\alpha}\right) \asymp n^\alpha h^{-\alpha} = |\theta|^{-\alpha} .$$

To proceed with I_2 we change the variables

$$u \ : \quad \frac{1}{\theta - t} = \frac{-u}{\theta} .$$

Then

$$I_2 = \int\limits_{\frac{\theta}{\pi-\theta}}^{\frac{n\theta}{\pi}} \left(\frac{\pi^2}{n\theta}\right)\left(\frac{\theta}{\pi}\,\frac{u+1}{u}\right)^{-\alpha} du = \frac{\pi^2}{n}\pi^{-\alpha}\theta^{-\alpha-1} \int\limits_{\frac{\theta}{\pi-\theta}}^{\frac{n\theta}{\pi}} \left(\frac{u+1}{u}\right)^{-\alpha} du .$$

We note, that integrand $\left(\frac{u+1}{u}\right)^{-\alpha} \in (0,1)$, therefore

$$|\theta| > \frac{\pi}{n} \quad \Rightarrow \quad I_2 = O\left(n^{-1}\theta^{-\alpha-1} n\theta\right) = O(\theta^{-\alpha})$$

or

$$|\theta| \leqslant \frac{\pi}{n} \quad \Rightarrow \quad I_2 < \frac{\pi^2}{n}\pi^{-\alpha}\theta^{-\alpha-1} \int\limits_{0}^{\frac{n\theta}{\pi}} u^\alpha du = O(n^\alpha) .$$

To proceed with I_3 we change the variables

$$u \ : \ \frac{1}{\theta - t} = \frac{u}{\theta} .$$

We have

$$I_3 = \int\limits_{\frac{\theta}{\pi+\theta}}^{\frac{n\theta}{\pi}} \left(\frac{\pi^2}{n\theta}\right)\left(\frac{\theta}{\pi}\left|\frac{1-u}{u}\right|\right)^{-\alpha} du = \frac{\pi^{2+\alpha}}{n\theta^{1+\alpha}} \int\limits_{\frac{\theta}{\pi+\theta}}^{\frac{n\theta}{\pi}} u^\alpha |1-u|^{-\alpha} du .$$

We estimate this integral in the three cases. The first case is

$$\frac{n\theta}{\pi} < \frac{1}{2} \quad \Rightarrow \quad |1-u|^{-\alpha} < 2^\alpha \quad \Rightarrow \quad I_3 < \frac{2^\alpha \pi^{2+\alpha}}{n\theta^{1+\alpha}} \int\limits_0^{\frac{n\theta}{\pi}} u^\alpha du \sim n^\alpha \, .$$

The second case is

$$\frac{1}{2} \leqslant \frac{n\theta}{\pi} < 1 \quad \Rightarrow$$

$$I_3 < \pi n^\alpha \left(\frac{n\theta}{\pi}\right)^{-\alpha-1} \int\limits_0^1 u^\alpha (1-u)^{-\alpha} du = \pi n^\alpha 2^{\alpha+1} \frac{\pi\alpha}{\sin \pi\alpha} \sim n^\alpha \, .$$

In the third case we split the integral in the three parts:

$$1 \leqslant \frac{n\theta}{\pi} \quad \Rightarrow \quad I_3 < \frac{\pi^{2+\alpha}}{n\theta^{1+\alpha}} \left(\int\limits_0^1 u^\alpha (1-u)^{-\alpha} du + \int\limits_1^\infty \left[\left(\frac{u}{u-1}\right)^\alpha - 1 - \frac{\alpha}{u} \right] du \right.$$

$$\left. + \int\limits_1^{\frac{n\theta}{\pi}} \left(1 - \frac{\alpha}{u}\right) du \right) = \frac{\pi^{2+\alpha}}{n\theta^{1+\alpha}} \left(\mathrm{Const}(\alpha) + \frac{n\theta}{\pi} + \alpha \ln \frac{n\theta}{\pi} \right) \sim \theta^{-\alpha} \, .$$

Lemma 3.2 is proved.

6 Bounds for the derivatives of the "wings"

Here we prove the Lemma 3.3 about upper bounds for

$$\mathcal{A}_2' = q_2' \otimes \mathcal{F}_m, \quad \mathcal{A}_2'' = q_2'' \otimes \mathcal{F}_m, \quad \text{where} \quad q_2 = \left(\frac{1}{m^2} + \sin^2 \frac{t}{2}\right)^{-\frac{\alpha}{2}} .$$

We have

$$q_2' = -\frac{\alpha}{4} \sin t \left(\frac{1}{m^2} + \sin^2 \frac{t}{2}\right)^{-\frac{\alpha}{2}-1} ,$$

and

$$q_2'' = \frac{\alpha(\alpha+2)}{16} \sin^2 t \left(\frac{1}{m^2} + \sin^2 \frac{t}{2}\right)^{-\frac{\alpha}{2}-2} - \frac{\alpha}{4} \cos t \left(\frac{1}{m^2} + \sin^2 \frac{t}{2}\right)^{-\frac{\alpha}{2}-1} .$$

Using inequality (some kind of inequality between geometrical and arithmetical means):

$$a \left(\frac{1}{m^2} + a^2\right)^{\frac{-\beta-1}{2}} = \frac{a}{\sqrt{\frac{1}{m^2} + a^2}} \left(\frac{\frac{1}{m^2\beta}}{\frac{1}{m^2} + a^2}\right)^{\beta/2} (\beta m^2)^{\beta/2} \leqslant$$

$$\leqslant (\beta m^2)^{\beta/2}\left(\frac{\frac{a^2}{\frac{1}{m^2}+a^2}+\beta\frac{\frac{1}{m^2\beta}}{\frac{1}{m^2}+a^2}}{1+\beta}\right)^{\frac{\beta+1}{2}} = (\beta m^2)^{\beta/2}(\beta+1)^{-\frac{\beta+1}{2}} \ ,$$

we can bound $q_2'(t)$ and $q_2''(t)$ from above for $t \in [-\pi,\pi]$:

$$|q_2'| \leqslant \frac{\alpha}{2}\sin\frac{t}{2}\left(\frac{1}{m^2}+\sin^2\frac{t}{2}\right)^{-\frac{\alpha}{2}-1} \leqslant \frac{\alpha}{2}\sqrt{\frac{(\alpha+1)^{\alpha+1}}{(\alpha+2)^{\alpha+2}}}\, m^{\alpha+1} \ .$$

and

$$|q_2''| \leqslant \frac{\alpha(\alpha+2)}{4}\sin^2\frac{t}{2}\left(\frac{1}{m^2}+\sin^2\frac{t}{2}\right)^{-\frac{\alpha}{2}-2} + \frac{\alpha}{4}\left(\frac{1}{m^2}+\sin^2\frac{t}{2}\right)^{-\frac{\alpha}{2}-1} \leqslant$$

$$\leqslant \frac{\alpha}{4}\left(2+\left(\frac{\alpha+2}{\alpha+4}\right)^{\frac{\alpha+4}{2}}\right) m^{\alpha+2} \ .$$

We also can make an estimations from above using powers of t

$$|q_2'(t)| < \frac{\alpha}{4}|t|\left(\left(\frac{|t|}{\pi}\right)^2\right)^{-\frac{\alpha}{2}-1} = \frac{\alpha\pi^{\alpha+2}}{4}|t|^{-\alpha-1},$$

$$|q_2''(t)| < \frac{\alpha(\alpha+2)}{16}t^2\left(\left(\frac{|t|}{\pi}\right)^2\right)^{-\frac{\alpha}{2}-2} + \frac{\alpha}{4}\left(\left(\frac{|t|}{\pi}\right)^2\right)^{-\frac{\alpha}{2}-1} =$$

$$= \frac{\alpha\pi^{\alpha+2}}{4}\left(1+(\alpha+2)\frac{\pi^2}{4}\right)|t|^{-\alpha-2} \ .$$

Thus

$$|q_2'(t)| \lesssim \min\left(m, \frac{\pi}{|t|}\right)^{\alpha+1}, \quad |q_2''(t)| \lesssim \min\left(m, \frac{\pi}{|t|}\right)^{\alpha+2} \ .$$

Now we can proceed with $\mathcal{A}_2'$ and $\mathcal{A}_2''$ on $[-\pi,\pi]$. We split the integral in two parts

$$\mathcal{A}_2'(\theta) = \int_{\mathbb{T}} q_2'(t)\,\mathcal{F}_m(\theta-t)dt = \int_{\theta/2-\pi}^{\theta/2} q_2'(t)\,\mathcal{F}_m(\theta-t)dt + \int_{\theta/2}^{\theta/2+\pi} q_2'(t)\,\mathcal{F}_m(\theta-t)dt \ ,$$

and we have

$$|\mathcal{A}_2'(\theta)| \leqslant \int_{\theta/2-\pi}^{\theta/2} |q_2'(t)|\,dt \cdot \max_{t\in[\theta/2,\theta/2+\pi]} |\mathcal{F}_m(t)| + \int_{\theta/2-\pi}^{\theta/2} |\mathcal{F}_m(t)|\,dt \cdot \max_{t\in[\theta/2,\theta/2+\pi]} |q_2'(t)| \ .$$

We use periodicity of $\mathcal{F}, q_2^{(j)}, \mathcal{A}_2^{(j)}$ and their symmetry with respect to zero. Now we recall (5.1)

$$|\mathcal{F}_m(t)| \leqslant \frac{1}{m} \min\left(m, \frac{\pi}{|t|}\right)^2, \quad t \in [-\pi, \pi] \, .$$

Then we fix $\theta > 0$, fix period such that $\theta/2$ is in middle of the period, and we continue

$$|\mathcal{A}_2'(\theta)| \lesssim \frac{1}{m} \min\left(m, \frac{2\pi}{|\theta|}\right)^2 \cdot \int_{-\pi}^{\pi} \min\left(m, \frac{\pi}{|t|}\right)^{\alpha+1} dt +$$

$$+ \min\left(m, \frac{2\pi}{|\theta|}\right)^{\alpha+1} \cdot \int_{-\pi}^{\pi} \frac{1}{m} \min\left(m, \frac{\pi}{|t|}\right)^2 dt \lesssim$$

$$\lesssim m^{\alpha+1} \min\left(1, \frac{1}{m\theta}\right)^2 + m^{\alpha+1} \min\left(1, \frac{1}{m\theta}\right)^{\alpha+1} \, .$$

Since

$$\min\left(1, \frac{1}{m\theta}\right)^2 \lesssim \min\left(1, \frac{1}{m\theta}\right)^{\alpha+1} \, ,$$

we can keep here just the second term.

For $\mathcal{A}_2''$ we do the same with change of the degrees $\alpha + 1$ for $\alpha + 2$.

$$|\mathcal{A}_2''(\theta)| \lesssim m^{\alpha+2} \min\left(1, \frac{1}{m\theta}\right)^2 + m^{\alpha+2} \min\left(1, \frac{1}{m\theta}\right)^{\alpha+2} \, ,$$

but for this case we have to keep the first term.

Lemma 3.1 is proved.

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