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K E L D Y S H I N S T I T U T E O F A P P L I E D M A T H E M A T I C S
R U S S I A N A C A D E M Y O F S C I N E C E S

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Динамическая модель спутника с солнечной панелью в двухстепенном шарнире

Рассматривается спутник, имеющий одну солнечную панель, закрепленную в управляемом шарнире. Мотор шарнира позволяет вращать панель вокруг двух направлений. Панель и спутник считаются твердыми телами. В работе получены уравнения движения такой системы.

Ключевые слова: сложная структура, солнечная панель, двухстепенной подвес, динамические уравнения

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Dynamical model of a satellite with 2DOF solar panel

Satellite with 2DOF solar panel is considered. The panel hinge consists of a motor capable of panel attitude control. Satellite and solar panel are rigid bodies both. The work is aimed at the satellite-panel system attitude dynamical model construction.

Key words: complex structure, solar panel, 2DOF hinge, equations of motion

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Contents

Introduction

Presence of a movable solar array attached to the satellite body with a controlled hinge leads to significant complication of the satellite dynamics. A mathematical model and corresponding software for numerical investigation of the satellite arbitrary movements are necessary. However one can estimate steady-state motion of the satellite at specified relative motion of the array using asymptotic analysis methods for a mathematical model of the satellite motion about its center of mass.

A key point of the mathematical model formulation is introduction of the properties of the satellite elements which consists of the satellite bus (hereinafter referred to as bus) and the solar array (SA). The bus and array are connected with two-degree-of-freedom hinge and are affected by external forces and torques. The forces determine the satellite orbital motion and the torques applied to the bus and SA determine their angular motion. The SA mass is 7% of the bus mass, it measures less than 2.5 meters, the structure is rigid while the Formosat-7 satellite is considered as a prototype. It gives reason to consider the SA a rigid body for the dynamics analysis. Examination of the system consisting of the bus, SA and hinge helps to ascertain the main contribution of the articulated structure in comparison with the satellite approximation by a rigid body.

The problem of a complex dynamical model is considered in detail both in theoretical papers and applied projects. Nevertheless, for the dynamics analysis of a specific satellite we should develop a model which would realize specified, analyzable geometric and dynamical topology of the system. So, one needs to derive the motion equations and introduce appropriate variables to describe the phase state of the satellite bus and SA, and necessary reference frames.

1. Generic dynamical model

There is a number of approaches applicable for the equations derivation. The paper [1] contains the basic methods of derivation of motion equations for a satellite supplied with a solar array or other movable elements. In particular, there are some approaches (described in detail in [2; 3]) based on momentum and angular momentum variation laws, d'Alembert principle, Lagrange, Hamilton, Boltzmann-Homel, Gibbs equations and using particular linear and angular velocities [1]. In papers [4; 5] a general approach for motion equations derivation, including those for flexible constructions, is presented. However, the application of these equations is limited by their unhandiness caused by large generality of the system under consideration. We use a method applying the basic dynamic equation (d’Alembert principle) for a system with ideal constraints

$$\sum_{\mu}\left(m_\mu \ddot{\mathbf{r}}_\mu - \mathbf{F}_\mu\right)\delta\mathbf{r}_\mu = 0.$$

Here $\delta\mathbf{r}_\mu$ is elementary displacement of μ^{th} particle, $\mathbf{F}_\mu$ is the resultant of all active forces affecting this particle. It has the same advantage as Lagrange equations: the ideal constraints reactions are not present in the final motion equations. The coordinates which the motion equations are written for, can be easily interpreted.

Dynamic equations should be supplemented by kinematic relations for parameters which specify the satellite attitude. Euler angles and directional cosines matrix are convenient for the analytical analysis, while quaternions are better suited for the numerical study. Let quaternion $\mathbf{q} = (\boldsymbol{\lambda}, \lambda_0)$ specify the satellite position, then the kinematic relations take the form

$$\dot{\mathbf{q}} = \frac{1}{2}\mathbf{\Omega}\mathbf{q}, \tag{1.1}$$

where

$$\mathbf{\Omega} = \begin{bmatrix} 0 & \omega_3 & -\omega_2 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{bmatrix}.$$

Relationship of the matrix of directional cosines and the quaternions elements can be written as follows:

$$\mathbf{D}=\begin{bmatrix} 1-2\lambda_2^2-2\lambda_3^2 & 2(\lambda_1\lambda_2+\lambda_0\lambda_3) & 2(\lambda_1\lambda_3-\lambda_0\lambda_2) \\ 2(\lambda_1\lambda_2-\lambda_0\lambda_3) & 1-2\lambda_1^2-2\lambda_3^2 & 2(\lambda_2\lambda_3+\lambda_0\lambda_1) \\ 2(\lambda_1\lambda_3+\lambda_0\lambda_2) & 2(\lambda_2\lambda_3-\lambda_0\lambda_1) & 1-2\lambda_1^2-2\lambda_2^2 \end{bmatrix}. \qquad (1.2)$$

Kinematic equations (1.1) must be integrated together with the dynamic equations as the former ones include the angular velocity components and the latter ones contain the elements of the directional cosines matrix which can be calculated when using quaternions by means of (1.2).

Determination of the satellite steady motion is also of special interest. At given relative motion of the SA it can be determined using asymptotic analysis methods for the mathematical model of the satellite motion about the center of mass. If determined, the steady motion can also be used for the model verification. Steady-state motions and equilibrium positions of the system under consideration in gravitational field is studied in many papers. The ones of V. Sarychev [6–8] cover determination of the equilibrium positions of satellite–pendulum system at a circular orbit. There is a double pendulum considered in [6], satellite–asymmetric pendulum system in [7] and satellite and asymmetrical pendulum with an arbitrary inertia tensor in [8]. In all cases, under some additional conditions, all equilibrium positions are determined.

The papers of M. Lavagna and A.E. Finzi [9; 10] cover the analysis of systems made up of three bodies bound by hinges. Equilibrium positions are determined for this configuration and their stability is examined.

2. Assumptions, reference frames, equations of motion

Consider rather general kinematic scheme of the satellite body connection with SA (Fig. 1.1)

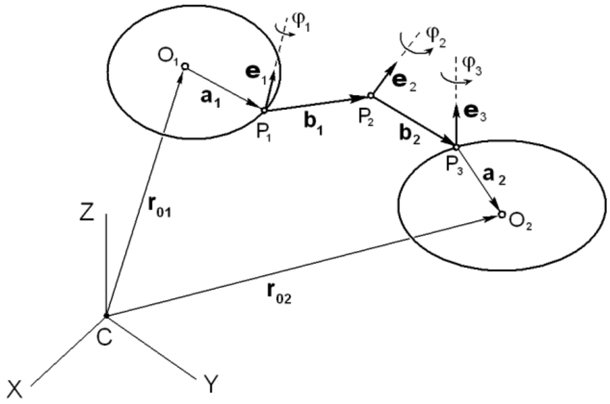

Fig. 1.1. Geometry of two bodies bound by hinges

The satellite (body with center of mass at point O_1) and SA (body with center of mass at point O_2) are bound by two weightless absolutely rigid rods connected together with hinges bodies $P_i\left(i=1,2,3\right)$, each with one degree of freedom. Superposing the hinges at one point one can implement a hinge connecting the satellite to a SA with two or three degrees of freedom.

To derive motion equations we represent the mechanical system in a form of material particles set. For each particle the following equation

$$m_\mu \ddot{\mathbf{r}}_\mu = \mathbf{F}_\mu + \mathbf{R}_\mu$$

is satisfied where $\mathbf{F}_\mu$ and $\mathbf{R}_\mu$ are resultant factors of active (external) forces and reactions affecting the particle. The constraints imposed on the system are considered ideal. Then at any virtual displacement $\delta\mathbf{r}_\mu$ compatible with the constraints the following relation holds

$$\sum_\mu \mathbf{R}_\mu \delta\mathbf{r}_\mu = 0$$

and we obtain general dynamics equation

$$\sum_\mu \left(m_\mu \ddot{\mathbf{r}}_\mu - \mathbf{F}_\mu\right)\delta\mathbf{r}_\mu = 0 .$$

Sum separately over particles of the first body $\left(\sum^{(1)}\right)$, the second one $\left(\sum^{(2)}\right)$ and connecting hinges $\left(\sum^{(3)}\right)$. Taking into account zero mass of the connecting links, obtain

$$\sum_{\mu}^{(1)}\left(m_\mu \ddot{\mathbf{r}}_\mu - \mathbf{F}_\mu\right)\delta\mathbf{r}_\mu + \sum_{\mu}^{(2)}\left(m_\mu \ddot{\mathbf{r}}_\mu - \mathbf{F}_\mu\right)\delta\mathbf{r}_\mu - \sum_{\mu}^{(3)}\mathbf{F}_\mu \delta\mathbf{r}_\mu = 0. \qquad (2.1)$$

Introduce following notations:

$m_{Oi} = \sum^{(i)} m_\mu$ for the bodies masses ($i = 1,2$);

$\mathbf{r}_{Oi} = \dfrac{1}{m_{Oi}}\sum^{(i)} m_\mu \mathbf{r}_\mu$ for radius-vectors of the bodies centers of mass ($i = 1,2$);

$\mathbf{a}_1 = \mathbf{O}_1\mathbf{P}_1$, $\mathbf{b}_1 = \mathbf{P}_1\mathbf{P}_2$, $\mathbf{b}_2 = \mathbf{P}_2\mathbf{P}_3$, $\mathbf{a}_2 = \mathbf{P}_3\mathbf{O}_2$;

$\mathbf{e}_i$ for unit vectors along the hinge axis P_i ($i = 1,2,3$);

$\delta\varphi_i$ for virtual changes of the slewing angles in the hinge P_i.

Then virtual displacement of the μ^{th} particle belonging to the first body (satellite) is written as follows

$$\delta\mathbf{r}_\mu = \delta\mathbf{r}_{O1} + \delta\boldsymbol{\theta}_1 \times \left(\mathbf{r}_\mu - \mathbf{r}_{O1}\right)$$

where $\delta\boldsymbol{\theta}_1$ is the satellite virtual rotation. For virtual displacement of μ^{th} particle belonging to the second body (SA) obtain

$$\begin{aligned}
\delta\mathbf{r}_\mu = \delta\mathbf{r}_{O1} + \delta\boldsymbol{\theta}_1 \times \mathbf{a}_1 + \left(\delta\boldsymbol{\theta}_1 + \mathbf{e}_1\delta\varphi_1\right)\times \mathbf{b}_1 +& \\
+\left(\delta\boldsymbol{\theta}_1 + \mathbf{e}_1\delta\varphi_1 + \mathbf{e}_2\delta\varphi_2\right)\times \mathbf{b}_2 +& \\
+\left(\delta\boldsymbol{\theta}_1 + \mathbf{e}_1\delta\varphi_1 + \mathbf{e}_2\delta\varphi_2 + \mathbf{e}_3\delta\varphi_3\right)\times\left(\mathbf{a}_2 + \mathbf{r}_\mu - \mathbf{r}_{O2}\right) =& \\
= \delta\mathbf{r}_{O1} + \delta\boldsymbol{\theta}_1 \times \left(\mathbf{a}_1 + \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{a}_2 + \mathbf{r}_\mu - \mathbf{r}_{O2}\right) +& \\
+\mathbf{e}_1 \times \left(\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{a}_2 + \mathbf{r}_\mu - \mathbf{r}_{O2}\right)\delta\varphi_1 +& \\
+\mathbf{e}_2 \times \left(\mathbf{b}_2 + \mathbf{a}_2 + \mathbf{r}_\mu - \mathbf{r}_{O2}\right)\delta\varphi_2 +& \\
+\mathbf{e}_3 \times \left(\mathbf{a}_2 + \mathbf{r}_\mu - \mathbf{r}_{O2}\right)\delta\varphi_3.&
\end{aligned}$$

Denote also

$\mathbf{F}_i = \sum^{(i)}\mathbf{F}_\mu$ as resultant vectors of external forces applied to the bodies ($i = 1,2$);

$\mathbf{M}_i = \sum^{(i)} \left(\mathbf{r}_\mu - \mathbf{r}_{\mathrm{O}i}\right) \times \mathbf{F}_\mu$ as resultant torques of external forces about the center of mass of corresponding body ($i = 1, 2$);

$\mathbf{K}_i = \sum^{(i)} m_\mu \left(\mathbf{r}_\mu - \mathbf{r}_{\mathrm{O}i}\right) \times \left(\dot{\mathbf{r}}_\mu - \dot{\mathbf{r}}_{\mathrm{O}i}\right)$ as the bodies angular moments about the center of mass of corresponding body ($i = 1, 2$);

Substituting $\delta\mathbf{r}_\mu$ in (2.1) and taking into account that the connecting links are affected only by control torques in the hinges M_{ui} ($i = 1, 2, 3$), that is

$$\sum_\mu^{(3)} \mathbf{F}_\mu \delta\mathbf{r}_\mu = \sum_{i=1}^{3} M_{ui} \delta\varphi_i ,$$

obtain

$$\sum_\mu^{(1)} \left(m_\mu \ddot{\mathbf{r}}_\mu - \mathbf{F}_\mu\right)\left[\delta\mathbf{r}_{\mathrm{O1}} + \delta\boldsymbol{\theta}_1 \times \left(\mathbf{r}_\mu - \mathbf{r}_{\mathrm{O1}}\right)\right] + \sum_\mu^{(2)} \left(m_\mu \ddot{\mathbf{r}}_\mu - \mathbf{F}_\mu\right)\delta\mathbf{r}_{\mathrm{O1}} +$$
$$+ \sum_\mu^{(2)} \left(m_\mu \ddot{\mathbf{r}}_\mu - \mathbf{F}_\mu\right)\left[\delta\boldsymbol{\theta}_1 \times \left(\mathbf{a}_1 + \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{a}_2 + \mathbf{r}_\mu - \mathbf{r}_{\mathrm{O2}}\right)\right] +$$
$$+ \sum_\mu^{(2)} \left(m_\mu \ddot{\mathbf{r}}_\mu - \mathbf{F}_\mu\right)\left[\mathbf{e}_1 \times \left(\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{a}_2 + \mathbf{r}_\mu - \mathbf{r}_{\mathrm{O2}}\right)\right]\delta\varphi_1$$
$$+ \sum_\mu^{(2)} \left(m_\mu \ddot{\mathbf{r}}_\mu - \mathbf{F}_\mu\right)\left[\mathbf{e}_2 \times \left(\mathbf{b}_2 + \mathbf{a}_2 + \mathbf{r}_\mu - \mathbf{r}_{\mathrm{O2}}\right)\right]\delta\varphi_2 +$$
$$+ \sum_\mu^{(2)} \left(m_\mu \ddot{\mathbf{r}}_\mu - \mathbf{F}_\mu\right)\left[\mathbf{e}_3 \times \left(\mathbf{a}_2 + \mathbf{r}_\mu - \mathbf{r}_{\mathrm{O2}}\right)\right]\delta\varphi_3 - \sum_{i=1}^{3} M_{ui} \delta\varphi_i = 0$$

and after transformation

$$\left(m_{\mathrm{O1}} \ddot{\mathbf{r}}_{\mathrm{O1}} + m_{\mathrm{O2}} \ddot{\mathbf{r}}_{\mathrm{O2}} - \mathbf{F}_1 - \mathbf{F}_2\right)\delta\mathbf{r}_{\mathrm{O1}} +$$
$$+ \left[\dot{\mathbf{K}}_1 + \dot{\mathbf{K}}_2 - \mathbf{M}_1 - \mathbf{M}_2 + \left(\mathbf{a}_1 + \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{a}_2\right) \times \left(m_{\mathrm{O2}} \ddot{\mathbf{r}}_{\mathrm{O2}} - \mathbf{F}_2\right)\right]\delta\boldsymbol{\theta}_1 +$$
$$+ \left\{\left[\dot{\mathbf{K}}_2 - \mathbf{M}_2 + \left(\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{a}_2\right) \times \left(m_{\mathrm{O2}} \ddot{\mathbf{r}}_{\mathrm{O2}} - \mathbf{F}_2\right)\right]\mathbf{e}_1 - M_{u1}\right\}\delta\varphi_1 + \qquad (2.2)$$
$$+ \left\{\left[\dot{\mathbf{K}}_2 - \mathbf{M}_2 + \left(\mathbf{b}_2 + \mathbf{a}_2\right) \times \left(m_{\mathrm{O2}} \ddot{\mathbf{r}}_{\mathrm{O2}} - \mathbf{F}_2\right)\right]\mathbf{e}_2 - M_{u2}\right\}\delta\varphi_2 +$$
$$+ \left\{\left[\dot{\mathbf{K}}_2 - \mathbf{M}_2 + \mathbf{a}_2 \times \left(m_{\mathrm{O2}} \ddot{\mathbf{r}}_{\mathrm{O2}} - \mathbf{F}_2\right)\right]\mathbf{e}_3 - M_{u3}\right\}\delta\varphi_3 = 0.$$

As magnitudes $\delta\mathbf{r}_{\mathrm{O1}}$, $\delta\boldsymbol{\theta}_1$, $\delta\varphi_1$, $\delta\varphi_2$, $\delta\varphi_3$ are independent, relation (2.2) is true only when

$$
\begin{aligned}
&m_{O1}\ddot{\mathbf{r}}_{O1}+m_{O2}\ddot{\mathbf{r}}_{O2}-\mathbf{F}_1-\mathbf{F}_2=0,\\
&\dot{\mathbf{K}}_1+\dot{\mathbf{K}}_2-\mathbf{M}_1-\mathbf{M}_2+\left(\mathbf{a}_1+\mathbf{b}_1+\mathbf{b}_2+\mathbf{a}_2\right)\times\left(m_{O2}\ddot{\mathbf{r}}_{O2}-\mathbf{F}_2\right)=0,\\
&\left[\dot{\mathbf{K}}_2-\mathbf{M}_2+\left(\mathbf{b}_1+\mathbf{b}_2+\mathbf{a}_2\right)\times\left(m_{O2}\ddot{\mathbf{r}}_{O2}-\mathbf{F}_2\right)\right]\mathbf{e}_1-M_{u1}=0,\\
&\left[\dot{\mathbf{K}}_2-\mathbf{M}_2+\left(\mathbf{b}_2+\mathbf{a}_2\right)\times\left(m_{O2}\ddot{\mathbf{r}}_{O2}-\mathbf{F}_2\right)\right]\mathbf{e}_2-M_{u2}=0,\\
&\left[\dot{\mathbf{K}}_2-\mathbf{M}_2+\mathbf{a}_2\times\left(m_{O2}\ddot{\mathbf{r}}_{O2}-\mathbf{F}_2\right)\right]\mathbf{e}_3-M_{u3}=0.
\end{aligned}
\tag{2.3}
$$

The obtained equations must be supplemented by kinematic relations

$$
\begin{aligned}
&\mathbf{r}_{O2}=\mathbf{r}_{O1}+\mathbf{a}_1+\mathbf{b}_1+\mathbf{b}_2+\mathbf{a}_2,\\
&\dot{\mathbf{r}}_{O2}=\dot{\mathbf{r}}_{O1}+\boldsymbol{\omega}_1\times\mathbf{a}_1+\left(\boldsymbol{\omega}_1+\mathbf{e}_1\dot{\varphi}_1\right)\times\mathbf{b}_1+\left(\boldsymbol{\omega}_1+\mathbf{e}_1\dot{\varphi}_1+\mathbf{e}_2\dot{\varphi}_2\right)\times\mathbf{b}_2+\\
&\qquad+\left(\boldsymbol{\omega}_1+\mathbf{e}_1\dot{\varphi}_1+\mathbf{e}_2\dot{\varphi}_2+\mathbf{e}_3\dot{\varphi}_3\right)\times\mathbf{a}_2,\\
&\ddot{\mathbf{r}}_{O2}=\ddot{\mathbf{r}}_{O1}+\dot{\boldsymbol{\omega}}_1\times\mathbf{a}_1+\boldsymbol{\omega}_1\times\left(\boldsymbol{\omega}_1\times\mathbf{a}_1\right)+\\
&\qquad+\left(\dot{\boldsymbol{\omega}}_1+\mathbf{e}_1\ddot{\varphi}_1+\boldsymbol{\omega}_1\times\mathbf{e}_1\dot{\varphi}_1\right)\times\mathbf{b}_1+\left(\boldsymbol{\omega}_1+\mathbf{e}_1\dot{\varphi}_1\right)\times\left[\left(\boldsymbol{\omega}_1+\mathbf{e}_1\dot{\varphi}_1\right)\times\mathbf{b}_1\right]+\\
&\qquad+\left[\dot{\boldsymbol{\omega}}_1+\mathbf{e}_1\ddot{\varphi}_1+\boldsymbol{\omega}_1\times\mathbf{e}_1\dot{\varphi}_1+\mathbf{e}_2\ddot{\varphi}_2+\left(\boldsymbol{\omega}_1+\mathbf{e}_1\dot{\varphi}_1\right)\times\mathbf{e}_2\dot{\varphi}_2\right]\times\mathbf{b}_2+\\
&\qquad+\left(\boldsymbol{\omega}_1+\mathbf{e}_1\dot{\varphi}_1+\mathbf{e}_2\dot{\varphi}_2\right)\times\left[\left(\boldsymbol{\omega}_1+\mathbf{e}_1\dot{\varphi}_1+\mathbf{e}_2\dot{\varphi}_2\right)\times\mathbf{b}_2\right]+\\
&\qquad+\left[\dot{\boldsymbol{\omega}}_1+\mathbf{e}_1\ddot{\varphi}_1+\boldsymbol{\omega}_1\times\mathbf{e}_1\dot{\varphi}_1+\mathbf{e}_2\ddot{\varphi}_2+\left(\boldsymbol{\omega}_1+\mathbf{e}_1\dot{\varphi}_1\right)\times\mathbf{e}_2\dot{\varphi}_2+\mathbf{e}_3\ddot{\varphi}_3+\right.\\
&\qquad\left.+\left(\boldsymbol{\omega}_1+\mathbf{e}_1\dot{\varphi}_1+\mathbf{e}_2\dot{\varphi}_2\right)\times\mathbf{e}_3\dot{\varphi}_3\right]\times\mathbf{a}_2+\\
&\qquad+\left(\boldsymbol{\omega}_1+\mathbf{e}_1\dot{\varphi}_1+\mathbf{e}_2\dot{\varphi}_2+\mathbf{e}_3\dot{\varphi}_3\right)\times\left[\left(\boldsymbol{\omega}_1+\mathbf{e}_1\dot{\varphi}_1+\mathbf{e}_2\dot{\varphi}_2+\mathbf{e}_3\dot{\varphi}_3\right)\times\mathbf{a}_2\right],
\end{aligned}
\tag{2.4}
$$

where $\boldsymbol{\omega}_1=\dot{\boldsymbol{\theta}}_1$ is absolute angular velocity of the first body (the bus).

It is more convenient to use the radius-vector of the system center of mass $\mathbf{r}_O$ instead of the radius-vector of the bus center of mass $\mathbf{r}_{O1}$. To do this use evident relation

$$\left(m_{O1}+m_{O2}\right)\mathbf{r}_O=m_{O1}\mathbf{r}_{O1}+m_{O2}\mathbf{r}_{O2}.$$

Then

$$
\begin{aligned}
&\mathbf{r}_{O1}=\mathbf{r}_O-\frac{m_{O2}}{m_{O1}+m_{O2}}\left(\mathbf{a}_1+\mathbf{b}_1+\mathbf{b}_2+\mathbf{a}_2\right),\\
&\mathbf{r}_{O2}=\mathbf{r}_O+\frac{m_{O1}}{m_{O1}+m_{O2}}\left(\mathbf{a}_1+\mathbf{b}_1+\mathbf{b}_2+\mathbf{a}_2\right).
\end{aligned}
\tag{2.5}
$$

The first equation of (2.3) takes the form of

$$\left(m_{O1}+m_{O2}\right)\ddot{\mathbf{r}}_O-\mathbf{F}_1-\mathbf{F}_2=0,$$

where

$$\mathbf{F}_i=\sum\nolimits^{(i)}\mathbf{F}_\mu=-fM\sum\nolimits^{(i)}m_\mu\frac{\mathbf{r}_\mu}{r_\mu^3}=-fM\sum\nolimits^{(i)}m_\mu\frac{\mathbf{r}_O+\left(\mathbf{r}_\mu-\mathbf{r}_O\right)}{\left|\mathbf{r}_O+\left(\mathbf{r}_\mu-\mathbf{r}_O\right)\right|^3}=$$

$$=-fM\frac{1}{r_O^2}\sum\nolimits^{(i)}m_\mu\frac{\dfrac{\mathbf{r}_O}{r_O}+\dfrac{\mathbf{r}_\mu-\mathbf{r}_O}{r_O}}{\left|\dfrac{\mathbf{r}_O}{r_O}+\dfrac{\mathbf{r}_\mu-\mathbf{r}_O}{r_O}\right|^3}\approx-fMm_{oi}\frac{\mathbf{r}_o}{r_o^3}.$$

Here as usually we take advantage of the fact that characteristic linear dimension of the system is much smaller than the distance between the system mass center and Earth center, that is $\left|\mathbf{r}_\mu-\mathbf{r}_O\right|\ll r_O$. Consequently, forces $\mathbf{F}_1$ and $\mathbf{F}_2$ can be substituted by usual equations for gravitational forces attracting particle masses,

$$\mathbf{F}_i=-\mu_G m_{oi}\frac{\mathbf{r}_o}{r_o^3}, \qquad (2.6)$$

$\mu_G=fM$ is the Earth gravitational parameter, f is the universal gravitational constant, M is the Earth mass. The first equation of (2.3) takes the final form

$$\ddot{\mathbf{r}}_O+\mu_G\frac{\mathbf{r}_O}{r_O^3}=0. \qquad (2.7)$$

It results from this that the system center of mass moves in Keplerian orbit.

If different perturbing factors such as non-spherical Earth gravitational field, the atmosphere resistance, Sun and Moon influence and solar radiation pressure are taken into account, equation (2.7) takes the form

$$\ddot{\mathbf{r}}_O+\mu_G\frac{\mathbf{r}_O}{r_O^3}=\mathbf{F}_{pert}, \qquad (2.8)$$

where $\mathbf{F}_{pert}$ is disturbing acceleration. The orbit of the system center of mass will not be Keplerian one.

For the satellite body angular momentum write

$\mathbf{K}_1 = \mathbf{I}_1 \boldsymbol{\omega}_1$,

therefore,

$\dot{\mathbf{K}}_1 = \mathbf{I}_1 \dot{\boldsymbol{\omega}}_1 + \boldsymbol{\omega}_1 \times \mathbf{I}_1 \boldsymbol{\omega}_1$

where $\mathbf{I}_1$ is the bus tensor of inertia. Similarly, for the second body of the system (SA) we get

$\dot{\mathbf{K}}_2 = \mathbf{I}_2 \dot{\boldsymbol{\omega}}_2 + \boldsymbol{\omega}_2 \times \mathbf{I}_2 \boldsymbol{\omega}_2$,

where

$\boldsymbol{\omega}_2 = \boldsymbol{\omega}_1 + \mathbf{e}_1 \dot{\varphi}_1 + \mathbf{e}_2 \dot{\varphi}_2 + \mathbf{e}_3 \dot{\varphi}_3$

is second body angular velocity, $\mathbf{I}_2$ is the array inertia tensor.

Equations (2.3) are given in a vector-matrix form. So, all the vectors there must be given in the same reference frame. However, it is not always convenient in practice. Further the following reference frames will be used:

– $CXYZ$, Earth-centered inertial frame;

– $Oxyz$, orbital frame with origin in the system center of mass;

– $O_1x_1y_1z_1$, satellite-fixed reference frame (its axes are directed along the central principal axes of inertia of the bus);

– $O_2x_2y_2z_2$, array-fixed reference frame (its axes are directed along the central principal axes of inertia of SA);

– $P_i\xi_i\eta_i\varsigma_i$ ($i = 1,2,3$), the i^{th} connecting link fixed reference frame (in this context consider the second body as the third connecting link).

Matrix of rotation from inertial frame $CXYZ$ to orbital frame $Oxyz$ denotes by $\mathbf{C}$; matrix of rotation from the orbital frame to frame $O_1x_1y_1z_1$ denotes by $\mathbf{A}_1$; matrix of rotation from the orbital frame to frame $O_2x_2y_2z_2$ denotes by $\mathbf{A}_2$; matrix of rotation from frame $O_1x_1y_1z_1$ to frame $P_1\xi_1\eta_1\varsigma_1$ denotes by $\mathbf{B}_1$; similarly $\mathbf{B}_2$ и $\mathbf{B}_3$ denote matrixes of rotation form frame $P_1\xi_1\eta_1\varsigma_1$ to the one $P_2\xi_2\eta_2\varsigma_2$ and from frame $P_2\xi_2\eta_2\varsigma_2$ to the one $P_3\xi_3\eta_3\varsigma_3$. Note that frame $P_3\xi_3\eta_3\varsigma_3$ is fixed in the second body

(SA). However, it is convenient to introduce it separately from frame $O_2x_2y_2z_2$ where the body inertia tensor has a diagonal form and frame $P_3\xi_3\eta_3\varsigma_3$ one of the axes of which is directed along the hinge axis P_3. Constant matrix of rotation from frame $P_3\xi_3\eta_3\varsigma_3$ to frame $O_2x_2y_2z_2$ denotes by $\mathbf{D}$. Obviously

$$\mathbf{A}_2 = \mathbf{D}\mathbf{B}_3\mathbf{B}_2\mathbf{B}_1\mathbf{A}_1. \tag{2.9}$$

Assume vectors $\mathbf{r_0}, \mathbf{r_{01}}, \mathbf{r_{02}}, \mathbf{F_1}, \mathbf{F_2}$ are specified in frame $CXYZ$; vectors $\boldsymbol{\omega}_1, \mathbf{M_1}, \mathbf{a_1}, \mathbf{e_1}$ are specified in frame $O_1x_1y_1z_1$; vectors $\boldsymbol{\omega}_2, \mathbf{M}_2$ are specified in frame $O_2x_2y_2z_2$; vectors $\mathbf{b_1}, \mathbf{e_2}$ are specified in frame $P_1\xi_1\eta_1\varsigma_1$; vectors $\mathbf{b}_2, \mathbf{e}_3$ are specified in frame $P_2\xi_2\eta_2\varsigma_2$; vector $\mathbf{a}_2$ is specified in frame $P_3\xi_3\eta_3\varsigma_3$. Then equations (2.3) take the form

$$\begin{gathered}
\ddot{\mathbf{r}}_{\mathrm{O}} + \mu_G \frac{\mathbf{r}_{\mathrm{O}}}{r_{\mathrm{O}}^3} = 0, \\
\left(\mathbf{I}_1\dot{\boldsymbol{\omega}}_1 + \boldsymbol{\omega}_1 \times \mathbf{I}_1\boldsymbol{\omega}_1 - \mathbf{M}_1\right) + \mathbf{B}_1^T\mathbf{B}_2^T\mathbf{B}_3^T\mathbf{D}^T\left(\mathbf{I}_2\dot{\boldsymbol{\omega}}_2 + \boldsymbol{\omega}_2 \times \mathbf{I}_2\boldsymbol{\omega}_2 - \mathbf{M}_2\right) + \\
+\left(\mathbf{a}_1 + \mathbf{B}_1^T\mathbf{b}_1 + \mathbf{B}_1^T\mathbf{B}_2^T\mathbf{b}_2 + \mathbf{B}_1^T\mathbf{B}_2^T\mathbf{B}_3^T\mathbf{a}_2\right) \times \mathbf{A}_1\mathbf{C}\left(m_{\mathrm{O2}}\ddot{\mathbf{r}}_{\mathrm{O2}} - \mathbf{F}_2\right) = 0, \\
\left[\mathbf{B}_1^T\mathbf{B}_2^T\mathbf{B}_3^T\mathbf{D}^T\left(\mathbf{I}_2\dot{\boldsymbol{\omega}}_2 + \boldsymbol{\omega}_2 \times \mathbf{I}_2\boldsymbol{\omega}_2 - \mathbf{M}_2\right) + \left(\mathbf{B}_1^T\mathbf{b}_1 + \mathbf{B}_1^T\mathbf{B}_2^T\mathbf{b}_2 + \mathbf{B}_1^T\mathbf{B}_2^T\mathbf{B}_3^T\mathbf{a}_2\right) \times \right. \\
\left. \times \mathbf{A}_1\mathbf{C}\left(m_{\mathrm{O2}}\ddot{\mathbf{r}}_{\mathrm{O2}} - \mathbf{F}_2\right)\right]\mathbf{e}_1 - M_{u1} = 0, \\
\left[\mathbf{B}_1^T\mathbf{B}_2^T\mathbf{B}_3^T\mathbf{D}^T\left(\mathbf{I}_2\dot{\boldsymbol{\omega}}_2 + \boldsymbol{\omega}_2 \times \mathbf{I}_2\boldsymbol{\omega}_2 - \mathbf{M}_2\right) + \left(\mathbf{B}_1^T\mathbf{B}_2^T\mathbf{b}_2 + \mathbf{B}_1^T\mathbf{B}_2^T\mathbf{B}_3^T\mathbf{a}_2\right) \times \right. \\
\left. \times \mathbf{A}_1\mathbf{C}\left(m_{\mathrm{O2}}\ddot{\mathbf{r}}_{\mathrm{O2}} - \mathbf{F}_2\right)\right]\mathbf{B}_1^T\mathbf{e}_2 - M_{u2} = 0, \\
\left[\mathbf{B}_1^T\mathbf{B}_2^T\mathbf{B}_3^T\mathbf{D}^T\left(\mathbf{I}_2\dot{\boldsymbol{\omega}}_2 + \boldsymbol{\omega}_2 \times \mathbf{I}_2\boldsymbol{\omega}_2 - \mathbf{M}_2\right) + \mathbf{B}_1^T\mathbf{B}_2^T\mathbf{B}_3^T\mathbf{a}_2 \times \right. \\
\left. \times \mathbf{A}_1\mathbf{C}\left(m_{\mathrm{O2}}\ddot{\mathbf{r}}_{\mathrm{O2}} - \mathbf{F}_2\right)\right]\mathbf{B}_1^T\mathbf{B}_2^T\mathbf{e}_3 - M_{u3} = 0.
\end{gathered} \tag{2.10}$$

Note that expression $m_{\mathrm{O2}}\ddot{\mathbf{r}}_{\mathrm{O2}} - \mathbf{F}_2$ appearing in equations (2.10) can be rewritten in the form

$$m_{\mathrm{O2}}\ddot{\mathbf{r}}_{\mathrm{O2}} - \mathbf{F}_2 \equiv m_{\mathrm{O2}}\left(\ddot{\mathbf{r}}_{\mathrm{O2}} - \ddot{\mathbf{r}}_{\mathrm{O}}\right) + m_{\mathrm{O2}}\ddot{\mathbf{r}}_{\mathrm{O}} - \mathbf{F}_2. \tag{2.11}$$

Taking into account (2.7) and (2.8) we get

$$m_{O2}\ddot{\mathbf{r}}_O - \mathbf{F}_2 = m_{O2}\left(\ddot{\mathbf{r}}_O + \mu_G \frac{\mathbf{r}_O}{r_O^3}\right) = 0.$$

Finally taking into account (2.5) in expression (2.11), the latter takes the form

$$\begin{aligned} m_{O2}\ddot{\mathbf{r}}_{O2} - \mathbf{F}_2 = m_{O2}\left(\ddot{\mathbf{r}}_{O2} - \ddot{\mathbf{r}}_O\right) = \frac{m_{O1}m_{O2}}{m_{O1}+m_{O2}}\mathbf{C}^T\mathbf{A}_1^T\{\dot{\boldsymbol{\omega}}_1 \times \mathbf{a}_1 + \boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_1 \times \mathbf{a}_1) + \\ + (\dot{\boldsymbol{\omega}}_1 + \mathbf{e}_1\ddot{\varphi}_1 + \boldsymbol{\omega}_1 \times \mathbf{e}_1\dot{\varphi}_1) \times \mathbf{b}_1 + (\boldsymbol{\omega}_1 + \mathbf{e}_1\dot{\varphi}_1) \times \left[(\boldsymbol{\omega}_1 + \mathbf{e}_1\dot{\varphi}_1) \times \mathbf{b}_1\right] + \\ + \left(\dot{\boldsymbol{\omega}}_1 + \mathbf{e}_1\ddot{\varphi}_1 + \mathbf{e}_2\ddot{\varphi}_2 + \boldsymbol{\omega}_1 \times \mathbf{e}_1\dot{\varphi}_1 + (\boldsymbol{\omega}_1 + \mathbf{e}_1\dot{\varphi}_1) \times \mathbf{e}_2\dot{\varphi}_2\right) \times \mathbf{b}_2 + \\ + (\boldsymbol{\omega}_1 + \mathbf{e}_1\dot{\varphi}_1 + \mathbf{e}_2\dot{\varphi}_2) \times \left[(\boldsymbol{\omega}_1 + \mathbf{e}_1\dot{\varphi}_1 + \mathbf{e}_2\dot{\varphi}_2) \times \mathbf{b}_2\right] + \\ + \left(\dot{\boldsymbol{\omega}}_1 + \mathbf{e}_1\ddot{\varphi}_1 + \mathbf{e}_2\ddot{\varphi}_2 + \mathbf{e}_3\ddot{\varphi}_3 + \boldsymbol{\omega}_1 \times \mathbf{e}_1\dot{\varphi}_1 + (\boldsymbol{\omega}_1 + \mathbf{e}_1\dot{\varphi}_1) \times \mathbf{e}_2\dot{\varphi}_2 + (\boldsymbol{\omega}_1 + \mathbf{e}_1\dot{\varphi}_1 + \mathbf{e}_2\dot{\varphi}_2) \times \mathbf{e}_3\dot{\varphi}_3\right) \times \mathbf{a}_2 + \\ + (\boldsymbol{\omega}_1 + \mathbf{e}_1\dot{\varphi}_1 + \mathbf{e}_2\dot{\varphi}_2 + \mathbf{e}_3\dot{\varphi}_3) \times \left[(\boldsymbol{\omega}_1 + \mathbf{e}_1\dot{\varphi}_1 + \mathbf{e}_2\dot{\varphi}_2 + \mathbf{e}_3\dot{\varphi}_3) \times \mathbf{a}_2\right]\}. \end{aligned} \tag{2.12}$$

Present expressions for the resultant torques of the external forces affecting the bus ($\mathbf{M}_1$) and SA ($\mathbf{M}_2$). Here disturbing torques are taken into account for both bus and SA and a control torque acting on a satellite.

3. Adaptation of the motion equations to a particular satellite configuration

In this chapter equations system (2.10) is adapted for the configuration of satellite Formosat-7. So, one can specify vector parameters (the hinges sizes and their position), some matrix constants depending on construction and even reduce the equations system order. We also choose a method for determination of the bus, SA and hinges attitude which seems to be the most evident, the one using plane angles (however quaternion-based kinematics is used). Specify all the frames mentioned in the previous chapter.

Inertial frame $CXYZ$ has the origin in Earth center, its first axis is directed to the vernal equinox point, the third one is directed along Earth spin axis, the second supplements the frame to the right-hand one.

Orbital frame $Oxyz$ has the origin in the system center of mass, its third axis is directed along the normal to the orbit plane, the second one is directed along the satellite radius-vector, the first one supplements the frame to the right-hand one.

To define the matrix of transition between the inertial and orbital frames, introduce intermediate frame $CX_1Y_1Z_1$. Its origin is situated in the Earth center, the first axis is directed to the orbit pericenter, the third one is directed along the normal to the orbit plane, the second one supplements the frame to the right-hand one. The matrix of transition from the auxiliary frame to the inertial one has the form

$$\mathbf{D}_1 = \begin{pmatrix} \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i & -\cos\Omega\sin\omega - \sin\Omega\cos\omega\cos i & \sin\Omega\sin i \\ \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i & -\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos i & -\cos\Omega\sin i \\ \sin\omega\sin i & \cos\omega\sin i & \cos i \end{pmatrix}$$

where Ω is the ascending node longitude, ω is the argument of pericenter, i is the orbit inclination. Consider these parameters known from the satellite orbital motion (the first equation (2.10)). The matrix of transition from the auxiliary frame to the orbital one has the form

$$\mathbf{D}_2 = \begin{pmatrix} \cos u & \sin u & 0 \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where u is the argument of latitude. Then the matrix of transition from the inertial frame to the orbital one has the form

$$\mathbf{C} = \mathbf{D}_2\mathbf{D}_1^T .$$

Specify the matrix of transition from the inertial frame to the one $O_1x_1y_1z_1$ using plane angles α, β, γ with rotations sequence 2-3-1,

$$\mathbf{A} = \begin{pmatrix} \cos\alpha\cos\beta & \sin\beta & -\sin\alpha\cos\beta \\ -\cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma & \cos\beta\cos\gamma & \sin\alpha\sin\beta\cos\gamma + \cos\alpha\sin\gamma \\ \cos\alpha\sin\beta\sin\gamma + \sin\alpha\cos\gamma & -\cos\beta\sin\gamma & -\sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma \end{pmatrix}. \quad (3.1)$$

This choice of the transition matrix will be used later, so in these cases we will write $\mathbf{A}(\alpha, \beta, \gamma)$.

Tensor of inertia of the satellite bus, point of the hinge fastening and torques applied to the satellite are specified in the frame $O_1x_1y_1z_1$. The velocity is to be determined, the torque is considered known, point of the hinge fastening is specified by vector

$$\mathbf{a}_1 = (0.0094, -0.4489, -0.1268) \text{ m}, \qquad (3.2)$$

inertia tensor

$$\mathbf{I}_1 = \begin{pmatrix} 38.57 & 0 & 0 \\ 0 & 29.05 & 0 \\ 0 & 0 & 33.96 \end{pmatrix} \text{ kg}\cdot\text{m}^2.$$

Choose $P_1\xi_1\eta_1\varsigma_1$ axes (Fig. 3.1) so that the spin axis will be $P_1\eta_1$, as according to (3.2) the hinge is positioned practically along the second axis. The matrix of transition between frames $O_1x_1y_1z_1$ and $P_1\xi_1\eta_1\varsigma_1$ depends on the hinge position in the satellite bus and its rotation through angle φ_1. Assume that in initial position ($\varphi_1 = 0$) the orientation of the first hinge in the satellite bus is determined by plane angles $\alpha_1, \beta_1, \gamma_1$ which are known from the satellite design. Then the transition matrix is determined by expression $\mathbf{B}_1(\alpha_1 + \varphi_1, \beta_1, \gamma_1)$ similarly to (3.1). It allows to write the vector of axis of the first hinge rotation in the frame $O_1x_1y_1z_1$, $\mathbf{e}_1 = \mathbf{B}_1^T(0,1,0)^T$, vector $\mathbf{e}_1$ is constant in the frame $O_1x_1y_1z_1$.

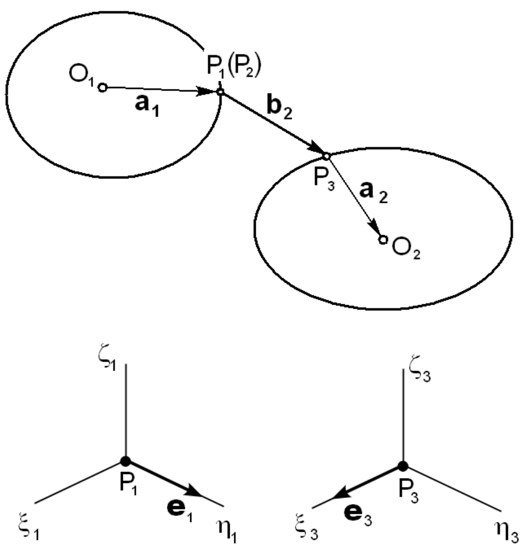

Fig.3.1. Particular satellite configuration

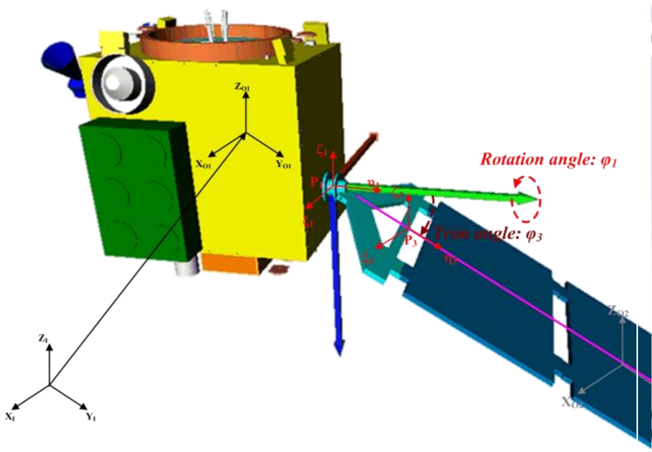

Fig.3.2. Frames of the Bus and SA

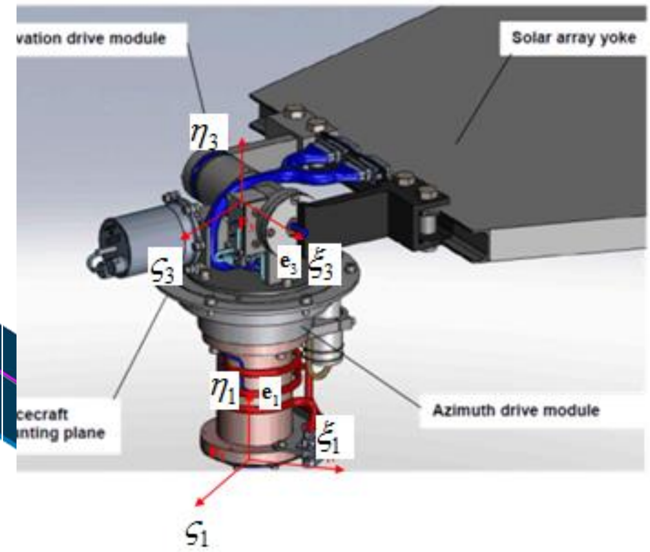

Fig.3.3. Frames of the SA and yoke

As the SA has two degrees of freedom, one of the hinges in model (2.10) should be frozen. Let it be the second hinge and the frame $P_2\xi_2\eta_2\varsigma_2$ fixed to it coincides with $P_1\xi_1\eta_1\varsigma_1$, the hinge rotation angle φ_2, its rotation velocity $\dot{\varphi}_2$, applied control torque M_{u2} are zero (Fig.3.2-3.3). Transition matrix $\mathbf{B}_2$ is a unit one. The matrix of transition from the frame $P_2\xi_2\eta_2\varsigma_2$ (and, therefore, from the frame $P_1\xi_1\eta_1\varsigma_1$) to $P_3\xi_3\eta_3\varsigma_3$ is determined by rotation about the third hinge axis. Assume that the first hinge ensures rotation about the second axis and the third one ensures rotation about the first axis. Then the transition matrix is determined by expression $\mathbf{B}_3(0,0,\varphi_3)$. In this case for the vector of the third hinge rotation direction we get

$\mathbf{e}_3 = \mathbf{B}_3^T (1,0,0)^T$.

Transition matrix $\mathbf{D}$ between the frame linked with the third part of the system and the one fixed to the SA is determined from the array design and hinges and is a function of three constant angles. As it is more convenient to prescribe the orientation of the hinge in the frame linked with SA and not vice versa, $\mathbf{D} = \mathbf{D}^T(\alpha_2, \beta_2, \gamma_2)$. Matrix $\mathbf{A}_2$ is determined from relation (2.9). In frame $O_2x_2y_2z_2$ the tensor of inertia of the array is specified

$$\mathbf{I}_2 = \begin{pmatrix} 5.549 & 0 & 0 \\ 0 & 1.757 & 0 \\ 0 & 0 & 7.304 \end{pmatrix} \text{ kg}\cdot\text{m}^2.$$

Vector of the point of the hinge fastening into the SA $\mathbf{a}_2$ can be written in the frame $P_3\xi_3\eta_3\varsigma_3$ as $\mathbf{a}_2' = (0, 1.175, 0)$ m. It prescribes the method of choice of the frame $P_3\xi_3\eta_3\varsigma_3$ and the vector of fastening in the frame $O_2x_2y_2z_2$ $\mathbf{a}_2 = \mathbf{D}\mathbf{a}_2'$.

The connecting links are short so that $\mathbf{b}_1 = \mathbf{b}_2 = \mathbf{0}$.

Thereby, all the constant vectors appearing in (2.10), (2.12) are determined. These are $\mathbf{e}_i$, $\mathbf{a}_i$ and $\mathbf{b}_i$ as well as the transition matrices expressed either by constant angles, or by the motion parameters (the angles of the satellite orientation and the hinges rotation, the orbit position in the inertial space). The satellite bus and array masses entering into (2.12) equal to m_{O1}=249 kg and m_{O2}=17.52 kg respectively.

Angular velocity of the array rotation velocity is determined by expression

$$\boldsymbol{\omega}_2 = \boldsymbol{\omega}_1 + \mathbf{e}_1\dot{\varphi}_1 + \mathbf{e}_3\dot{\varphi}_3. \tag{3.3}$$

Its derivative

$$\dot{\boldsymbol{\omega}}_2 = \dot{\boldsymbol{\omega}}_1 + \mathbf{e}_1\ddot{\varphi}_1 + \mathbf{e}_3\ddot{\varphi}_3 + \boldsymbol{\omega}_1 \times \mathbf{e}_1\dot{\varphi}_1 + \left(\boldsymbol{\omega}_1 + \mathbf{e}_1\dot{\varphi}_1\right) \times \mathbf{e}_3\dot{\varphi}_3. \tag{3.4}$$

Introduce variables ψ_i as the hinges rotation velocities $\dot{\varphi}_i$ and write (3.3) and (3.4) in the frame $O_2x_2y_2z_2$,

$$\boldsymbol{\omega}_2 = \mathbf{D}\mathbf{B}_3\mathbf{B}_1\left(\boldsymbol{\omega}_1 + \mathbf{e}_1\psi_1 + \mathbf{B}_1^T\mathbf{e}_3\psi_3\right) \tag{3.5}$$

$$\dot{\boldsymbol{\omega}}_2 = \mathbf{D}\mathbf{B}_3\mathbf{B}_1\left(\dot{\boldsymbol{\omega}}_1 + \mathbf{e}_1\dot{\psi}_1 + \mathbf{B}_1^T\mathbf{e}_3\dot{\psi}_3 + \boldsymbol{\omega}_1 \times \mathbf{e}_1\psi_1 + \left(\boldsymbol{\omega}_1 + \mathbf{e}_1\psi_1\right) \times \mathbf{B}_1^T\mathbf{e}_3\psi_3\right). \tag{3.6}$$

Introduce denotations

$$\mathbf{w}_2 = \mathbf{w}_2\left(\psi_1, \psi_3, \varphi_1\right) = \boldsymbol{\omega}_1 + \mathbf{e}_1\psi_1 + \mathbf{B}_1^T\left(\varphi_1\right)\mathbf{e}_3\psi_3, \tag{3.7}$$

$$\dot{\mathbf{w}}_2 = \dot{\boldsymbol{\omega}}_1 + \mathbf{e}_1\dot{\psi}_1 + \mathbf{B}_1^T\mathbf{e}_3\dot{\psi}_3 + \boldsymbol{\omega}_1 \times \mathbf{e}_1\psi_1 + \left(\boldsymbol{\omega}_1 + \mathbf{e}_1\psi_1\right) \times \mathbf{B}_1^T\mathbf{e}_3\psi_3. \tag{3.8}$$

Taking into account the recorded vectors, transition matrix and expressions for the second body angular velocity (3.5) and its derivative (3.6) rewrite motion equations (2.10),

$$\ddot{\mathbf{r}}_{\mathrm{O}} + \mu_G \frac{\mathbf{r}_{\mathrm{O}}}{r_{\mathrm{O}}^3} = 0,$$

$$\left(\mathbf{I}_1\dot{\boldsymbol{\omega}}_1 + \boldsymbol{\omega}_1 \times \mathbf{I}_1\boldsymbol{\omega}_1 - \mathbf{M}_1\right) + \left(\mathbf{J}_2\dot{\mathbf{w}}_2 + \mathbf{w}_2 \times \mathbf{J}_2\mathbf{w}_2 - \mathbf{B}_1^T\mathbf{B}_3^T\mathbf{D}^T\mathbf{M}_2\right) +$$
$$+ \left(\mathbf{a}_1 + \mathbf{B}_1^T\mathbf{B}_3^T\mathbf{D}\mathbf{a}_2'\right) \times \mathbf{A}_1\mathbf{C}\mathbf{c} = 0,$$
$$\left[\left(\mathbf{J}_2\dot{\mathbf{w}}_2 + \mathbf{w}_2 \times \mathbf{J}_2\mathbf{w}_2 - \mathbf{B}_1^T\mathbf{B}_3^T\mathbf{D}^T\mathbf{M}_2\right) + \mathbf{B}_1^T\mathbf{B}_3^T\mathbf{D}\mathbf{a}_2' \times \mathbf{A}_1\mathbf{C}\mathbf{c}\right]\mathbf{e}_1 - M_{u1} = 0, \qquad (3.9)$$
$$\left[\left(\mathbf{J}_2\mathbf{w}_2 + \mathbf{w}_2 \times \mathbf{J}_2\mathbf{w}_2 - \mathbf{B}_1^T\mathbf{B}_3^T\mathbf{D}^T\mathbf{M}_2\right) + \mathbf{B}_1^T\mathbf{B}_3^T\mathbf{D}\mathbf{a}_2' \times \mathbf{A}_1\mathbf{C}\mathbf{c}\right]\mathbf{B}_1^T\mathbf{e}_3 - M_{u3} = 0,$$

where

$$\mathbf{c} = \frac{m_{\mathrm{O1}}m_{\mathrm{O2}}}{m_{\mathrm{O1}} + m_{\mathrm{O2}}}\mathbf{C}^T\mathbf{A}_1^T\left\{\dot{\boldsymbol{\omega}}_1 \times \mathbf{a}_1 + \boldsymbol{\omega}_1 \times \left(\boldsymbol{\omega}_1 \times \mathbf{a}_1\right) + \dot{\mathbf{w}}_2 \times \mathbf{B}_1^T\mathbf{B}_3^T\mathbf{D}\mathbf{a}_2' + \mathbf{w}_2 \times \left[\mathbf{w}_2 \times \mathbf{B}_1^T\mathbf{B}_3^T\mathbf{D}\mathbf{a}_2'\right]\right\},$$

$$\mathbf{J}_2 = \mathbf{B}_1^T\mathbf{B}_3^T\mathbf{D}^T\mathbf{I}_2\mathbf{D}\mathbf{B}_3\mathbf{B}_1.$$

The second body angular velocity is specified by expression (3.7), its derivative is determined by (3.8). Equations (3.9) are supplemented by kinematic relations for the satellite (1.1) and for the hinges having the form of

$$\dot{\varphi}_i = \psi_i. \qquad (3.10)$$

Equations (3.9), (3.10) and any kinematic relations for the satellite are the full set of equations for determination of the satellite orientation angles through necessary parameters, its velocities $\omega_{1x}, \omega_{1y}, \omega_{1z}$, the hinges rotation angles φ_1, φ_3 and their rotation velocities ψ_1, ψ_3.

To integrate numerically we need to solve equation (3.9) for higher order derivatives ($\dot{\boldsymbol{\omega}}$, $\dot{\psi}_1$, $\dot{\psi}_3$). Introduce notations

$$\mathbf{f}_1 = \boldsymbol{\omega}_1 \times \mathbf{I}_1\boldsymbol{\omega}_1 - \mathbf{M}_1,$$

$$\mathbf{f}_2 = \mathbf{w}_2 \times \mathbf{J}_2\mathbf{w}_2 - \mathbf{B}_1^T\mathbf{B}_3^T\mathbf{D}^T\mathbf{M}_2 + \mathbf{J}_2\mathbf{f}_3$$

$$\mathbf{f}_3 = \boldsymbol{\omega}_1 \times \mathbf{e}_1\psi_1 + \left(\boldsymbol{\omega}_1 + \mathbf{e}_1\psi_1\right) \times \mathbf{B}_1^T\mathbf{e}_3\psi_3,$$

$$\mathbf{f}_4 = \mathbf{f}_3 \times \boldsymbol{\alpha}_2 + \boldsymbol{\omega}_1 \times \left(\boldsymbol{\omega}_1 \times \mathbf{a}_1\right) + \mathbf{w}_2 \times \left(\mathbf{w}_2 \times \boldsymbol{\alpha}_2\right),$$

$$m = \frac{m_{\mathrm{O1}}m_{\mathrm{O2}}}{m_{\mathrm{O1}} + m_{\mathrm{O2}}}.$$

Here $\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{f}_3$, $\mathbf{f}_4$ are functions not containing higher order derivatives, $\boldsymbol{\alpha}_2 = \mathbf{B}_1^T \mathbf{B}_3^T \mathbf{D} \mathbf{a}_2'$ is vector $\mathbf{a}_2$ written in the bus-fixed reference frame. Substituting the introduces notations in equations (3.9), we get

$$\begin{aligned}
&\left(\mathbf{I}_1\dot{\boldsymbol{\omega}}_1 + \mathbf{f}_1\right) + \left(\mathbf{J}_2\left(\dot{\boldsymbol{\omega}}_1 + \mathbf{e}_1\dot{\psi}_1 + \mathbf{B}_1^T\mathbf{e}_3\dot{\psi}_3\right) + \mathbf{f}_2\right) + \left(\mathbf{a}_1 + \boldsymbol{\alpha}_2\right) \times \mathbf{A}_1\mathbf{C}\mathbf{c} = 0, \\
&\left[\left(\mathbf{J}_2\left(\dot{\boldsymbol{\omega}}_1 + \mathbf{e}_1\dot{\psi}_1 + \mathbf{B}_1^T\mathbf{e}_3\dot{\psi}_3\right) + \mathbf{f}_2\right) + \boldsymbol{\alpha}_2 \times \mathbf{A}_1\mathbf{C}\mathbf{c}\right]\mathbf{e}_1 - M_{u1} = 0, \\
&\left[\left(\mathbf{J}_2\left(\dot{\boldsymbol{\omega}}_1 + \mathbf{e}_1\dot{\psi}_1 + \mathbf{B}_1^T\mathbf{e}_3\dot{\psi}_3\right) + \mathbf{f}_2\right) + \boldsymbol{\alpha}_2 \times \mathbf{A}_1\mathbf{C}\mathbf{c}\right]\mathbf{B}_1^T\mathbf{e}_3 - M_{u3} = 0,
\end{aligned} \qquad (3.11)$$

where

$$\mathbf{c} = m\mathbf{C}^T\mathbf{A}_1^T\left\{\dot{\boldsymbol{\omega}}_1 \times \mathbf{a}_1 + \boldsymbol{\omega}_1 \times \left(\boldsymbol{\omega}_1 \times \mathbf{a}_1\right) + \dot{\mathbf{w}}_2 \times \boldsymbol{\alpha}_2 + \mathbf{w}_2 \times \left[\mathbf{w}_2 \times \boldsymbol{\alpha}_2\right]\right\}, \qquad (3.12)$$

and expression (3.8) takes the form

$$\dot{\mathbf{w}}_2 = \dot{\boldsymbol{\omega}}_1 + \mathbf{e}_1\dot{\psi}_1 + \mathbf{B}_1^T\mathbf{e}_3\dot{\psi}_3 + \mathbf{f}_3.$$

Then (3.12) subject to notation $\mathbf{f}_4$ is written in the form

$$\mathbf{c} = m\mathbf{C}^T\mathbf{A}_1^T\left(\dot{\boldsymbol{\omega}}_1 \times \left(\mathbf{a}_1 + \boldsymbol{\alpha}_2\right) + \left(\mathbf{e}_1\dot{\psi}_1 + \mathbf{B}_1^T\mathbf{e}_3\dot{\psi}_3\right) \times \boldsymbol{\alpha}_2 + \mathbf{f}_4\right). \qquad (3.13)$$

Write (3.11) subject to (3.13)

$$\begin{aligned}
&\left(\mathbf{I}_1\dot{\boldsymbol{\omega}}_1 + \mathbf{f}_1\right) + \left(\mathbf{J}_2\left(\dot{\boldsymbol{\omega}}_1 + \mathbf{e}_1\dot{\psi}_1 + \mathbf{B}_1^T\mathbf{e}_3\dot{\psi}_3\right) + \mathbf{f}_2\right) + \\
&\qquad + m\left(\mathbf{a}_1 + \boldsymbol{\alpha}_2\right) \times \left(\dot{\boldsymbol{\omega}}_1 \times \left(\mathbf{a}_1 + \boldsymbol{\alpha}_2\right) + \left(\mathbf{e}_1\dot{\psi}_1 + \mathbf{B}_1^T\mathbf{e}_3\dot{\psi}_3\right) \times \boldsymbol{\alpha}_2 + \mathbf{f}_4\right) = 0, \\
&\left[\left(\mathbf{J}_2\left(\dot{\boldsymbol{\omega}}_1 + \mathbf{e}_1\dot{\psi}_1 + \mathbf{B}_1^T\mathbf{e}_3\dot{\psi}_3\right) + \mathbf{f}_2\right)\right. \\
&\qquad \left. + m\boldsymbol{\alpha}_2 \times \left(\dot{\boldsymbol{\omega}}_1 \times \left(\mathbf{a}_1 + \boldsymbol{\alpha}_2\right) + \left(\mathbf{e}_1\dot{\psi}_1 + \mathbf{B}_1^T\mathbf{e}_3\dot{\psi}_3\right) \times \boldsymbol{\alpha}_2 + \mathbf{f}_4\right)\right]\mathbf{e}_1 - M_{u1} = 0, \\
&\left[\left(\mathbf{J}_2\left(\dot{\boldsymbol{\omega}}_1 + \mathbf{e}_1\dot{\psi}_1 + \mathbf{B}_1^T\mathbf{e}_3\dot{\psi}_3\right) + \mathbf{f}_2\right)\right. \\
&\qquad \left. + m\boldsymbol{\alpha}_2 \times \left(\dot{\boldsymbol{\omega}}_1 \times \left(\mathbf{a}_1 + \boldsymbol{\alpha}_2\right) + \left(\mathbf{e}_1\dot{\psi}_1 + \mathbf{B}_1^T\mathbf{e}_3\dot{\psi}_3\right) \times \boldsymbol{\alpha}_2 + \mathbf{f}_4\right)\right]\mathbf{B}_1^T\mathbf{e}_3 - M_{u3} = 0.
\end{aligned}$$

Introduce the triple vector product matrix as follows:

$$\mathbf{a} \times \left(\mathbf{y} \times \mathbf{b}\right) = \mathbf{K}(\mathbf{a},\mathbf{b})\mathbf{y}$$

where

$$\mathbf{K}(\mathbf{a},\mathbf{b})=\begin{pmatrix} a_2b_2+a_3b_3 & -a_2b_1 & -a_3b_1 \\ -a_1b_2 & a_1b_1+a_3b_3 & -a_3b_2 \\ -a_1b_3 & -a_2b_3 & a_1b_1+a_2b_2 \end{pmatrix}.$$

In this case equations (3.11) will be rewritten in the form

$$\begin{gathered}
\left(\mathbf{I}_1+\mathbf{J}_2+m\mathbf{K}(\mathbf{a}_1+\boldsymbol{\alpha}_2,\mathbf{a}_1+\boldsymbol{\alpha}_2)\right)\dot{\boldsymbol{\omega}}_1+\left(\mathbf{J}_2+m\mathbf{K}(\mathbf{a}_1+\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_2)\right)\mathbf{e}_1\dot{\psi}_1+ \\
+\left(\mathbf{J}_2+m\mathbf{K}(\mathbf{a}_1+\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_2)\right)\mathbf{B}_1^T\mathbf{e}_3\dot{\psi}_3=-\mathbf{f}_1-\mathbf{f}_2-m\left(\mathbf{a}_1+\boldsymbol{\alpha}_2\right)\times\mathbf{f}_4, \\
\left(\left(\mathbf{J}_2+m\mathbf{K}(\boldsymbol{\alpha}_2,\mathbf{a}_1+\boldsymbol{\alpha}_2)\right)^T\mathbf{e}_1,\dot{\boldsymbol{\omega}}_1\right)+\left(\left(\mathbf{e}_1,\mathbf{J}_2\mathbf{e}_1\right)+m\left|\mathbf{e}_1\times\boldsymbol{\alpha}_2\right|^2\right)\dot{\psi}_1 \\
+\left(\left(\mathbf{e}_1,\mathbf{J}_2\mathbf{B}_1^T\mathbf{e}_3\right)+m\left(\mathbf{e}_1,\mathbf{B}_1^T\mathbf{e}_3\right)\alpha_2^2-m\left(\mathbf{e}_1,\boldsymbol{\alpha}_2\right)\left(\mathbf{B}_1^T\mathbf{e}_3,\boldsymbol{\alpha}_2\right)\right)\dot{\psi}_3=M_{u1}-\left(\mathbf{f}_2+m\boldsymbol{\alpha}_2\times\mathbf{f}_4,\mathbf{e}_1\right), \\
\left(\left(\mathbf{J}_2+m\mathbf{K}(\boldsymbol{\alpha}_2,\mathbf{a}_1+\boldsymbol{\alpha}_2)\right)^T\mathbf{B}_1^T\mathbf{e}_3,\dot{\boldsymbol{\omega}}_1\right)+\left(\left(\mathbf{J}_2\mathbf{e}_1,\mathbf{B}_1^T\mathbf{e}_3\right)+m\left(\mathbf{e}_1,\mathbf{B}_1^T\mathbf{e}_3\right)\alpha_2^2-m\left(\mathbf{e}_1,\boldsymbol{\alpha}_2\right)\left(\mathbf{B}_1^T\mathbf{e}_3,\boldsymbol{\alpha}_2\right)\right)\dot{\psi}_1 \\
+\left(\left(\mathbf{B}_1^T\mathbf{e}_3,\mathbf{J}_2\mathbf{B}_1^T\mathbf{e}_3\right)+m\left|\mathbf{B}_1^T\mathbf{e}_3\times\boldsymbol{\alpha}_2\right|^2\right)\dot{\psi}_3=M_{u3}-\left(\mathbf{f}_2+m\boldsymbol{\alpha}_2\times\mathbf{f}_4,\mathbf{B}_1^T\mathbf{e}_3\right),
\end{gathered}$$

where $\left(\mathbf{a},\mathbf{b}\right)=a_1b_1+a_2b_2+a_3b_3$ denotes scalar product of two vectors. Take into account that the hinges are mutually orthogonal

$$\begin{gathered}
\left(\mathbf{I}_1+\mathbf{J}_2+m\mathbf{K}(\mathbf{a}_1+\boldsymbol{\alpha}_2,\mathbf{a}_1+\boldsymbol{\alpha}_2)\right)\dot{\boldsymbol{\omega}}_1+\left(\mathbf{J}_2+m\mathbf{K}(\mathbf{a}_1+\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_2)\right)\mathbf{e}_1\dot{\psi}_1+ \\
+\left(\mathbf{J}_2+m\mathbf{K}(\mathbf{a}_1+\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_2)\right)\mathbf{B}_1^T\mathbf{e}_3\dot{\psi}_3=-\mathbf{f}_1-\mathbf{f}_2-m\left(\mathbf{a}_1+\boldsymbol{\alpha}_2\right)\times\mathbf{f}_4, \\
\left(\left(\mathbf{J}_2+m\mathbf{K}(\boldsymbol{\alpha}_2,\mathbf{a}_1+\boldsymbol{\alpha}_2)\right)^T\mathbf{e}_1,\dot{\boldsymbol{\omega}}_1\right)+\left(\left(\mathbf{e}_1,\mathbf{J}_2\mathbf{e}_1\right)+m\left|\mathbf{e}_1\times\boldsymbol{\alpha}_2\right|^2\right)\dot{\psi}_1 \\
+\left(\left(\mathbf{e}_1,\mathbf{J}_2\mathbf{B}_1^T\mathbf{e}_3\right)-m\left(\mathbf{e}_1,\boldsymbol{\alpha}_2\right)\left(\mathbf{B}_1^T\mathbf{e}_3,\boldsymbol{\alpha}_2\right)\right)\dot{\psi}_3=M_{u1}-\left(\mathbf{f}_2+m\boldsymbol{\alpha}_2\times\mathbf{f}_4,\mathbf{e}_1\right), \\
\left(\left(\mathbf{J}_2+m\mathbf{K}(\boldsymbol{\alpha}_2,\mathbf{a}_1+\boldsymbol{\alpha}_2)\right)^T\mathbf{B}_1^T\mathbf{e}_3,\dot{\boldsymbol{\omega}}_1\right)+\left(\left(\mathbf{J}_2\mathbf{e}_1,\mathbf{B}_1^T\mathbf{e}_3\right)-m\left(\mathbf{e}_1,\boldsymbol{\alpha}_2\right)\left(\mathbf{B}_1^T\mathbf{e}_3,\boldsymbol{\alpha}_2\right)\right)\dot{\psi}_1 \\
+\left(\left(\mathbf{B}_1^T\mathbf{e}_3,\mathbf{J}_2\mathbf{B}_1^T\mathbf{e}_3\right)+m\left|\mathbf{B}_1^T\mathbf{e}_3\times\boldsymbol{\alpha}_2\right|^2\right)\dot{\psi}_3=M_{u3}-\left(\mathbf{f}_2+m\boldsymbol{\alpha}_2\times\mathbf{f}_4,\mathbf{B}_1^T\mathbf{e}_3\right).
\end{gathered}\tag{3.14}$$

Solving equations (3.14) for higher order derivatives, obtain

$$\begin{pmatrix}\dot{\boldsymbol{\omega}}_1\\ \dot{\psi}_1\\ \dot{\psi}_3\end{pmatrix}=\mathbf{S}^{-1}\begin{pmatrix}-\mathbf{f}_1-\mathbf{f}_2-m\left(\mathbf{a}_1+\boldsymbol{\alpha}_2\right)\times\mathbf{f}_4\\ M_{u1}-\left(\mathbf{f}_2+m\boldsymbol{\alpha}_2\times\mathbf{f}_4,\mathbf{e}_1\right)\\ M_{u3}-\left(\mathbf{f}_2+m\boldsymbol{\alpha}_2\times\mathbf{f}_4,\mathbf{B}_1^T\mathbf{e}_3\right)\end{pmatrix}. \tag{3.15}$$

Here matrix $\mathbf{S}$ has the form

$$\mathbf{S}=\begin{pmatrix} \mathbf{I}_1+\mathbf{J}_2+m\mathbf{K}(\mathbf{a}_1+\boldsymbol{\alpha}_2,\mathbf{a}_1+\boldsymbol{\alpha}_2) & \left(\mathbf{J}_2+m\mathbf{K}(\mathbf{a}_1+\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_2)\right)\mathbf{e}_1 & \left(\mathbf{J}_2+m\mathbf{K}(\mathbf{a}_1+\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_2)\right)\mathbf{B}_1^T\mathbf{e}_3 \\ \mathbf{e}_1^T\left(\mathbf{J}_2+m\mathbf{K}(\boldsymbol{\alpha}_2,\mathbf{a}_1+\boldsymbol{\alpha}_2)\right) & \left(\mathbf{e}_1,\mathbf{J}_2\mathbf{e}_1\right)+m\left|\mathbf{e}_1\times\boldsymbol{\alpha}_2\right|^2 & \left(\mathbf{e}_1,\mathbf{J}_2\mathbf{B}_1^T\mathbf{e}_3\right)-m\left(\mathbf{e}_1,\boldsymbol{\alpha}_2\right)\left(\mathbf{B}_1^T\mathbf{e}_3,\boldsymbol{\alpha}_2\right) \\ \mathbf{e}_3^T\mathbf{B}_1\left(\mathbf{J}_2+m\mathbf{K}(\boldsymbol{\alpha}_2,\mathbf{a}_1+\boldsymbol{\alpha}_2)\right) & \left(\mathbf{J}_2\mathbf{e}_1,\mathbf{B}_1^T\mathbf{e}_3\right)-m\left(\mathbf{e}_1,\boldsymbol{\alpha}_2\right)\left(\mathbf{B}_1^T\mathbf{e}_3,\boldsymbol{\alpha}_2\right) & \left(\mathbf{B}_1^T\mathbf{e}_3,\mathbf{J}_2\mathbf{B}_1^T\mathbf{e}_3\right)+m\left|\mathbf{B}_1^T\mathbf{e}_3\times\boldsymbol{\alpha}_2\right|^2 \end{pmatrix}$$

Equations (3.15) supplemented by kinematic relations form a closed set of the equations of motion about the center of mass.

Right terms (3.15) depend on the system parameters, the satellite orientation, its angular velocity $\boldsymbol{\omega}_1$, angles and angular velocity in the hinges (φ_1, φ_3, ψ_1, ψ_3), total torque of the external forces affecting the satellite and array $\mathbf{M}_1$ and $\mathbf{M}_2$, as well as torques in the hinges axes M_{u1} and M_{u3}.

These equations, however, may be slightly simplified.

4. Equations adaptation for numerical methods

It can be seen from equations (3.15) that numerical procedure demands inverse of 5x5 matrix. This section considers simplification of the equations (3.15).

Introduce notations

$$\mathbf{J}=\mathbf{I}_1+\mathbf{J}_2+m\mathbf{K}(\mathbf{a}_1+\boldsymbol{\alpha}_2,\mathbf{a}_1+\boldsymbol{\alpha}_2) \tag{4.1}$$

$$\mathbf{c}_1=-\mathbf{f}_1-\mathbf{f}_2-m\left(\mathbf{a}_1+\boldsymbol{\alpha}_2\right)\times\mathbf{f}_4, \tag{4.2}$$

$$c_2=M_{u1}-\left(\mathbf{f}_2+m\boldsymbol{\alpha}_2\times\mathbf{f}_4,\mathbf{e}_1\right),$$

$$c_3=M_{u3}-\left(\mathbf{f}_2+m\boldsymbol{\alpha}_2\times\mathbf{f}_4,\mathbf{B}_1^T\mathbf{e}_3\right),$$

$$\mathbf{N}=\mathbf{J}_2+m\mathbf{K}(\mathbf{a}_1+\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_2). \tag{4.3}$$

Note that

$$\left(\mathbf{J}_2+m\mathbf{K}(\boldsymbol{\alpha}_2,\mathbf{a}_1+\boldsymbol{\alpha}_2)\right)^T=\mathbf{J}_2^T+m\mathbf{K}^T(\boldsymbol{\alpha}_2,\mathbf{a}_1+\boldsymbol{\alpha}_2)=\mathbf{J}_2+m\mathbf{K}(\mathbf{a}_1+\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_2)=\mathbf{N}.$$

In this case equations (3.15) take a form

$$\mathbf{J}\dot{\boldsymbol{\omega}}_1+\mathbf{L}\mathbf{e}_1\dot{\psi}_1+\mathbf{L}\mathbf{B}_1^T\mathbf{e}_3\dot{\psi}_3=\mathbf{c}_1,$$

$$\begin{aligned}&\left(\mathbf{N}\mathbf{e}_1,\dot{\boldsymbol{\omega}}_1\right)+\left(\left(\mathbf{e}_1,\mathbf{J}_2\mathbf{e}_1\right)+m\left|\mathbf{e}_1\times\boldsymbol{\alpha}_2\right|^2\right)\dot{\psi}_1+\\ &\qquad+\left(\left(\mathbf{e}_1,\mathbf{J}_2\mathbf{B}_1^T\mathbf{e}_3\right)-m\left(\mathbf{e}_1,\boldsymbol{\alpha}_2\right)\left(\mathbf{B}_1^T\mathbf{e}_3,\boldsymbol{\alpha}_2\right)\right)\dot{\psi}_3=c_2,\end{aligned} \tag{4.4}$$

$$\left(\mathbf{N}\mathbf{B}_1^T\mathbf{e}_3,\dot{\boldsymbol{\omega}}_1\right)+\left(\left(\mathbf{J}_2\mathbf{e}_1,\mathbf{B}_1^T\mathbf{e}_3\right)-m\left(\mathbf{e}_1,\boldsymbol{\alpha}_2\right)\left(\mathbf{B}_1^T\mathbf{e}_3,\boldsymbol{\alpha}_2\right)\right)\dot{\psi}_1+$$
$$+\left(\left(\mathbf{B}_1^T\mathbf{e}_3,\mathbf{J}_2\mathbf{B}_1^T\mathbf{e}_3\right)+m\left|\mathbf{B}_1^T\mathbf{e}_3\times\boldsymbol{\alpha}_2\right|^2\right)\dot{\psi}_3=c_3.$$

From the first equation of (4.4)

$$\dot{\boldsymbol{\omega}}_1=\mathbf{J}^{-1}\left(\mathbf{c}_1-\mathbf{N}\mathbf{e}_1\dot{\psi}_1-\mathbf{N}\mathbf{B}_1^T\mathbf{e}_3\dot{\psi}_3\right).$$

Substitute it to the second and third equations of (4.4)

$$\left(\left(\mathbf{e}_1,\left(\mathbf{J}_2-\mathbf{N}^T\mathbf{J}^{-1}\mathbf{N}\right)\mathbf{e}_1\right)+m\left|\mathbf{e}_1\times\boldsymbol{\alpha}_2\right|^2\right)\dot{\psi}_1+$$
$$+\left(\left(\mathbf{e}_1,\left(\mathbf{J}_2-\mathbf{N}^T\mathbf{J}^{-1}\mathbf{N}\right)\mathbf{B}_1^T\mathbf{e}_3\right)-m\left(\mathbf{e}_1,\boldsymbol{\alpha}_2\right)\left(\mathbf{B}_1^T\mathbf{e}_3,\boldsymbol{\alpha}_2\right)\right)\dot{\psi}_3=c_2-\left(\mathbf{L}\mathbf{e}_1,\mathbf{J}^{-1}\mathbf{c}_1\right),$$

$$\left(\left(\mathbf{e}_1,\left(\mathbf{J}_2-\mathbf{N}^T\mathbf{J}^{-1}\mathbf{N}\right)\mathbf{B}_1^T\mathbf{e}_3\right)-m\left(\mathbf{e}_1,\boldsymbol{\alpha}_2\right)\left(\mathbf{B}_1^T\mathbf{e}_3,\boldsymbol{\alpha}_2\right)^2\right)\dot{\psi}_1+$$
$$+\left(\left(\mathbf{B}_1^T\mathbf{e}_3,\left(\mathbf{J}_2-\mathbf{N}^T\mathbf{J}^{-1}\mathbf{N}\right)\mathbf{B}_1^T\mathbf{e}_3\right)+m\left|\mathbf{B}_1^T\mathbf{e}_3\times\boldsymbol{\alpha}_2\right|^2\right)\dot{\psi}_3=c_3-\left(\mathbf{L}\mathbf{B}_1^T\mathbf{e}_3,\mathbf{J}^{-1}\mathbf{c}_1\right).$$

Denote

$$\mathbf{S}_1=\begin{pmatrix}\left(\mathbf{e}_1,\left(\mathbf{J}_2-\mathbf{N}^T\mathbf{J}^{-1}\mathbf{N}\right)\mathbf{e}_1\right)+m\left|\mathbf{e}_1\times\boldsymbol{\alpha}_2\right|^2 & \left(\mathbf{e}_1,\left(\mathbf{J}_2-\mathbf{N}^T\mathbf{J}^{-1}\mathbf{N}\right)\mathbf{B}_1^T\mathbf{e}_3\right)-m\left(\mathbf{e}_1,\boldsymbol{\alpha}_2\right)\left(\mathbf{B}_1^T\mathbf{e}_3,\boldsymbol{\alpha}_2\right)\\ \left(\mathbf{e}_1,\left(\mathbf{J}_2-\mathbf{N}^T\mathbf{J}^{-1}\mathbf{N}\right)\mathbf{B}_1^T\mathbf{e}_3\right)-m\left(\mathbf{e}_1,\boldsymbol{\alpha}_2\right)\left(\mathbf{B}_1^T\mathbf{e}_3,\boldsymbol{\alpha}_2\right) & \left(\mathbf{B}_1^T\mathbf{e}_3,\left(\mathbf{J}_2-\mathbf{N}^T\mathbf{J}^{-1}\mathbf{N}\right)\mathbf{B}_1^T\mathbf{e}_3\right)+m\left|\mathbf{B}_1^T\mathbf{e}_3\times\boldsymbol{\alpha}_2\right|^2\end{pmatrix}. \quad (4.5)$$

In this case

$$\begin{pmatrix}\dot{\psi}_1\\ \dot{\psi}_3\end{pmatrix}=\mathbf{S}_1^{-1}\begin{pmatrix}c_2-\left(\mathbf{e}_1,\mathbf{N}^T\mathbf{J}^{-1}\mathbf{c}_1\right)\\ c_3-\left(\mathbf{B}_1^T\mathbf{e}_3,\mathbf{N}^T\mathbf{J}^{-1}\mathbf{c}_1\right)\end{pmatrix} \quad (4.6)$$

and

$$\dot{\boldsymbol{\omega}}_1=\mathbf{J}^{-1}\left(\mathbf{c}_1-\left(\mathbf{N}\mathbf{e}_1\ \ \mathbf{N}\mathbf{B}_1^T\mathbf{e}_3\right)\mathbf{S}_1^{-1}\begin{pmatrix}c_2-\left(\mathbf{e}_1,\mathbf{N}^T\mathbf{J}^{-1}\mathbf{c}_1\right)\\ c_3-\left(\mathbf{B}_1^T\mathbf{e}_3,\mathbf{N}^T\mathbf{J}^{-1}\mathbf{c}_1\right)\end{pmatrix}\right), \quad (4.7)$$

where $\left(\mathbf{N}\mathbf{e}_1\ \ \mathbf{N}\mathbf{B}_1^T\mathbf{e}_3\right)$ is 3x2 matrix.

So (4.6) and (4.7) is the solution of (4.4). It is the same as (3.15). However, in this case we should find the inverse of the 3x3 matrix and 2x2 instead of inversing the 5x5 matrix in the (3.15). This is considered as a huge benefit in terms of computational complexity.

The equations (4.6) and (4.7) are complemented by the kinematic equations

$$\dot{\varphi}_1 = \psi_1,$$
$$\dot{\varphi}_2 = \psi_2, \qquad (4.8)$$
$$\dot{\mathbf{q}} = \frac{1}{2}\mathbf{\Omega}\mathbf{q}$$

where

$$\mathbf{\Omega} = \begin{bmatrix} 0 & \omega_3 & -\omega_2 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{bmatrix}.$$

The set of (4.6)-(4.8) is the full set of equations that describes the behavior of the system.

Conclusion

Variables describing the satellite with a rigid solar array are chosen. The reference frames are introduced and the mathematical model for the satellite with 2DOF solar panel is developed. Equations are resolved with respect to the higher-order derivatives. The equation adaptation for numerical methods is performed.

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