ISSN 2071-2898 (Print)
ISSN 2071-2901 (Online)

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Recommended form of bibliographic references: Ovchinnikov M. Y., Roldugin D. S. Dual-spin satellite angular motion in magnetic and gravitational fields // Keldysh Institute Preprints. 2015. No. 22. 20 p.
URL: http://library.keldysh.ru/preprint.asp?id=2015-22&lg=e

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Dual-spin satellite angular motion in magnetic and gravitational fields

Moscow — 2015

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Движение спутника с двойным вращением в магнитном и гравитационном полях

Рассматривается спутник, оснащенный магнитной системой ориентации и тангажным маховиком. Исследуется быстродействие системы в переходном режиме. Находятся приближенные значения характеристических показателей Ляпунова. В установившемся режиме гравитационной ориентации рассматриваются малые движения в окрестности положения равновесия. Исследуется точность ориентации. Исследуется алгоритм произвольной, но заданной ориентации спутника в плоскости орбиты. Проводится численное моделирование.

Ключевые слова: магнитная система ориентации, тангажный маховик

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Dual-spin satellite angular motion in magnetic and gravitational fields

Attitude motion of a satellite equipped with single flywheel and active magnetic attitude control system is considered. Time-response of the system in transient mode is studied. Characteristic exponent approximations values are obtained. Small steady-state motion near the gravitational attitude is considered. Accuracy and time-response is studied. An algorithm of arbitrary but given attitude in the orbital plane is proposed and studied. Numerical analysis is carried out.

Key words: magnetic attitude control system, flywheel, dual spin

Work was supported by Russian Scientific Foundation grant № 14-11-00621.

Contents

Introduction

Magnetic control systems are extensively used for attitude stabilization of small satellites. The main problem with this control system is underactuation. Control torque is always directed perpendicular to the local magnetic induction vector. In this paper dual-spin satellite is considered. Satellite is equipped with a flywheel having high angular rate and/or mass. This allows the satellite to acquire inherent and ideally constant angular momentum. The flywheel is designed for this angular momentum to prevail over the satellite's one. The whole system behaves like a gyro in inertial space maintaining flywheel attitude. Dual-spin satellite has particular stable equilibrium positions in gravitational field. Flywheel axis tends to align with orbital plane normal. Energy dissipation devices transform this position to an asymptotically stable one [1]. Passive nutation damping devices may be used to utilize this advantage [2]. Passive damping however doesn't allow on-orbit time-response tuning. Furthermore, only gravitationally stable position in orbital plane is available. Gravitational torque value may also be insufficient to be considered as the main restoring one for small satellites without gravity-gradient boom. Active magnetic control system allows time-response tuning and may be even used to provide arbitrary in-plane attitude.

Present paper deals with transient motion first. Magnetic attitude control system is used for angular velocity damping purpose. Flywheel is considered in operational mode all the time, since its spinning phase may be omitted and arising angular rate dealt with instead. Time-response is characterized with approximate Lyapunov characteristic exponent values. Nominal gravitational attitude accuracy is assessed. Control algorithm for arbitrary in-plane attitude is proposed and relevant accuracy is estimated. This paper is aimed to replace our obsolete work [3] that contains a number of mistakes.

1. Problem statement

Angular motion of a satellite is considered. Satellite deformations are not taken into account. Satellite moves along circular LEO. Gravitational and geomagnetic fields influence is considered. Satellite is equipped with a flywheel and three mutually orthogonal magnetorquers. They are capable of producing any dipole moment within given range. Satellite attitude is known.

Two reference frames are used:

- orbital frame $OX_1X_2X_3$, O is satellite's center of mass, OX_3 is directed along satellite's radius-vector, OX_2 is directed along the orbital plane normal, OX_1 makes the whole frame right-handed (it is directed along satellite's orbital velocity in case of circular orbit);

- bound frame $Ox_1x_2x_3$, its axes coincide with principal axes of inertia of the satellite.

Satellite attitude is represented with Euler angles α,β,γ (rotation sequence 2-3-1) and angular velocity components, either in inertial space $\omega_1,\omega_2,\omega_3$ or relative to orbital frame $\Omega_1,\Omega_2,\Omega_3$. Quaternion is used for numerical simulations. Transition matrix between orbital and bound reference frames is

$$\mathbf{A}=\begin{pmatrix} \cos\alpha\cos\beta & \sin\beta & -\sin\alpha\cos\beta \\ -\cos\alpha\sin\beta\cos\gamma+\sin\alpha\sin\gamma & \cos\beta\cos\gamma & \sin\alpha\sin\beta\cos\gamma+\cos\alpha\sin\gamma \\ \sin\alpha\cos\gamma+\cos\alpha\sin\beta\sin\gamma & -\cos\beta\sin\gamma & -\sin\alpha\sin\beta\sin\gamma+\cos\alpha\cos\gamma \end{pmatrix}. \qquad (1.1)$$

Dynamical equations of motion for a three-axial satellite with inertia tensor $\mathbf{J}=diag(A,B,C)$ are

$$\mathbf{J}\frac{d\boldsymbol{\omega}}{dt}+\boldsymbol{\omega}\times\mathbf{J}\boldsymbol{\omega}+\boldsymbol{\omega}\times\mathbf{h}=\mathbf{M}_{gr}+\mathbf{M}_{ctrl} \qquad (1.2)$$

where $\mathbf{h}=(0,h,0)$ is flywheel's angular momentum, $\mathbf{M}_{gr}$, $\mathbf{M}_{ctrl}$ are gravitational and control magnetic torques. Absolute angular velocity $\boldsymbol{\omega}$ is related to the relative one $\boldsymbol{\Omega}$ according to

$$\boldsymbol{\omega}=\boldsymbol{\Omega}+\mathbf{A}\boldsymbol{\omega}_{orb}$$

where $\boldsymbol{\omega}_{orb}=(0,\omega_0,0)$ is orbital reference frame angular rate. Dynamical equations of motion may be written using relative angular velocity,

$$\mathbf{J}\frac{d\boldsymbol{\Omega}}{dt}+\boldsymbol{\Omega}\times\mathbf{J}\boldsymbol{\Omega}+\boldsymbol{\Omega}\times\mathbf{h}=-\mathbf{A}\boldsymbol{\omega}_{orb}\times\mathbf{h}+\mathbf{M}_{rel}+\mathbf{M}_{gr}+\mathbf{M}_{ctrl} \qquad (1.3)$$

where $\mathbf{M}_{rel}=-\mathbf{JWA}\boldsymbol{\omega}_{orb}-\boldsymbol{\Omega}\times\mathbf{JA}\boldsymbol{\omega}_{orb}-\mathbf{A}\boldsymbol{\omega}_{orb}\times\mathbf{J}\left(\boldsymbol{\Omega}+\mathbf{A}\boldsymbol{\omega}_{orb}\right)$,

$$\mathbf{W}=\begin{pmatrix} 0 & \Omega_3 & -\Omega_2 \\ -\Omega_3 & 0 & \Omega_1 \\ \Omega_2 & -\Omega_1 & 0 \end{pmatrix}.$$

Dynamical equations are supplemented with kinematic relations

$$\frac{d\alpha}{dt}=\frac{1}{\cos\beta}\left(\Omega_2\cos\gamma-\Omega_3\sin\gamma\right),$$

$$\frac{d\beta}{dt}=\Omega_2\sin\gamma+\Omega_3\cos\gamma, \qquad (1.4)$$

$$\frac{d\gamma}{dt}=\Omega_1-\mathrm{tg}\,\beta\left(\Omega_2\cos\gamma-\Omega_3\sin\gamma\right).$$

Geomagnetic field is represented using direct dipole model. Induction vector in orbital reference frame is

$$\mathbf{B} = B_0\left(\cos u \sin i,\ \cos i,\ -2\sin u \sin i\right)^T = B_0\left(B_1, B_2, B_3\right)^T$$

where i is orbit inclination, u is argument of latitude.

2. Transient motion

Transient motion is first studied in order to assess control system time-response. We assume relations

$$A-C>0,\ B-A+h/\omega_0>0,\ 4\left(B-C\right)+h/\omega_0>0 \qquad (2.1)$$

for the satellite moving in gravitational field. This ensures stable equilibrium corresponding to orbital and bound reference frames coincidence. Massive or fast flywheel satisfies two last conditions regardless of satellite inertia moments. This means that flywheel axis (Ox_2 in our case) tends to align along the orbital plane normal (OX_2 axis). First relation in (2.1) represents orbital and bound frames first axis coincidence. In-plane stability is provided with gravitational torque only.

Transient motion is mainly characterized by flywheel axis stabilization time. In-plane stabilization using gravitational or magnetic torques corresponds to nominal operations mode. Control system time-response in transient motion may be represented using characteristic exponents of linearized equations of motion (1.3)-(1.4). Dimensionless form for these equations is

$$\begin{gathered}\ddot{\gamma}+\dot{\beta}-\left(\lambda_A+h_A\right)\left(\dot{\beta}-\gamma\right)=\frac{1}{A\omega_0^2}M_1,\\ \ddot{\alpha}=\frac{1}{B\omega_0^2}M_2, \qquad (2.2)\\ \ddot{\beta}-\dot{\gamma}+\left(\lambda_C+h_C\right)\left(\dot{\gamma}+\beta\right)=\frac{1}{C\omega_0^2}M_3\end{gathered}$$

where $\lambda_A=\dfrac{B-C}{A}$, $\lambda_C=\dfrac{B-A}{C}$, $h_A=\dfrac{h}{A\omega_0}$, $h_C=\dfrac{h}{C\omega_0}$. Dot represents derivative with respect to argument of latitude (dimensionless time). We should take into account gravitational torque

$$\mathbf{M}_{gr}=3\omega_0^2\mathbf{e}_3\times\mathbf{J}\mathbf{e}_3=3\omega_0^2\begin{pmatrix}-\left(B-C\right)\gamma\\ \left(A-C\right)\alpha\\ 0\end{pmatrix}$$

and magnetic control torque. We use angular velocity damping algorithm in transient motion,

$$\mathbf{M}_{ctrl} = \mathbf{m} \times \mathbf{B} = k\left(\mathbf{\Omega} \times \mathbf{B}\right) \times \mathbf{B} = k\omega_0 B_0^2 \begin{pmatrix} \dot{\beta} B_1 B_3 - \dot{\gamma}\left(B_2^2 + B_3^2\right) + \dot{\alpha} B_1 B_2 \\ \dot{\gamma} B_1 B_2 - \dot{\alpha}\left(B_1^2 + B_3^2\right) + \dot{\beta} B_2 B_3 \\ \dot{\alpha} B_2 B_3 - \dot{\beta}\left(B_2^2 + B_1^2\right) + \dot{\gamma} B_1 B_3 \end{pmatrix}. \quad (2.3)$$

Linearized equations of motion are finally

$$\begin{aligned} &\ddot{\gamma} + \dot{\beta} - \left(\lambda_A + h_A\right)\left(\dot{\beta} - \gamma\right) + 3\lambda_A \gamma = \varepsilon\xi\left(\dot{\beta} B_1 B_3 - \dot{\gamma}\left(B_2^2 + B_3^2\right) + \dot{\alpha} B_1 B_2\right), \\ &\ddot{\alpha} = \varepsilon \frac{C}{B}\left(\dot{\gamma} B_1 B_2 - \dot{\alpha}\left(B_1^2 + B_3^2\right) + \dot{\beta} B_2 B_3\right), \\ &\ddot{\beta} - \dot{\gamma} + \left(\lambda_C + h_C\right)\left(\dot{\gamma} + \beta\right) = \varepsilon\left(\dot{\alpha} B_2 B_3 - \dot{\beta}\left(B_2^2 + B_1^2\right) + \dot{\gamma} B_1 B_3\right) \end{aligned} \quad (2.4)$$

where $\xi = \dfrac{C}{A}$, $\varepsilon = \dfrac{kB_0^2}{C\omega_0}$.

Transient motion corresponds to the flywheel axis tendency towards orbital plane normal. Therefore angles β and γ are of interest. Equations (2.4) should be simplified by separating in-plane motion. In order to do this we need to get rid of $\dot{\alpha}$ in the first and third equations in (2.4). This may be achieved when $B_2 = 0$ and therefore $i = 90^\circ$. Further analysis is performed for the satellite moving on polar orbit. This analysis is also valid for near polar orbits. Introducing another small parameter does not change the solution since only higher-order small terms are added in (2.4). This allows practically important sun-synchronous orbits to be cover with following results. Out of plane motion equations are

$$\begin{aligned} &\ddot{\beta} + \varepsilon \cos^2 u \dot{\beta} + \left(2\varepsilon \sin u \cos u - 1 + \theta_C\right)\dot{\gamma} + \theta_C \beta = 0, \\ &\ddot{\gamma} + \left(2\varepsilon\xi \sin u \cos u + 1 - \theta_A\right)\dot{\beta} + 4\varepsilon\xi \sin^2 u \dot{\gamma} + \left(3\lambda_A + \theta_A\right)\gamma = 0 \end{aligned} \quad (2.5)$$

where $\theta_A = h_A + \lambda_A, \theta_C = h_C + \lambda_C$. Designate $\mathbf{x} = \left(\dot{\beta}, \dot{\gamma}, \beta, \gamma\right)$. Equations (2.5) has a form

$$\dot{\mathbf{x}} = \mathbf{A}_0 \mathbf{x} + \varepsilon \mathbf{A}_1(u)\mathbf{x} \quad (2.6)$$

where

$$\mathbf{A}_0 = \begin{pmatrix} 0 & 1-\theta_C & -\theta_C & 0 \\ \theta_A - 1 & 0 & 0 & -3\lambda_A - \theta_A \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\mathbf{A}_1=\begin{pmatrix} -\cos^2 u & -2\sin u\cos u & 0 & 0\\ -2\xi\sin u\cos u & -4\xi\sin^2 u & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

Consider equation

$$\dot{\mathbf{x}}_0=\mathbf{A}_0\mathbf{x}_0$$

representing satellite motion in gravitational field with installed flywheel but without energy dissipation mechanism. Characteristic exponents equation is

$$\lambda^4+\left(3\lambda_A+1+\theta_A\theta_C\right)\lambda^2+\left(3\lambda_A+\theta_A\right)\theta_C=0 \qquad (2.7)$$

leading to

$$\lambda_{1,2}=\pm i\sqrt{\frac{1}{2}\left(3\lambda_A+1+\theta_A\theta_C+\sqrt{\left(3\lambda_A+1+\theta_A\theta_C\right)^2+12\lambda_A\theta_C\left(\theta_A-1\right)}\right)},$$

$$\lambda_{3,4}=\pm i\sqrt{\frac{1}{2}\left(-3\lambda_A-1-\theta_A\theta_C+\sqrt{\left(3\lambda_A+1+\theta_A\theta_C\right)^2+12\lambda_A\theta_C\left(\theta_A-1\right)}\right)}$$

where i is imaginary unit. Eigenvectors corresponding to λ_k are

$$\boldsymbol{\varphi}_k=A_k\begin{pmatrix}\lambda_k & \dfrac{\theta_C+\lambda_k^2}{1-\theta_C} & 1 & \dfrac{\theta_C+\lambda_k^2}{\lambda_k\left(1-\theta_C\right)}\end{pmatrix}^T$$

where A_k are arbitrary constants. Equations (2.5) solution in case $\varepsilon=0$ is

$$\begin{aligned}\beta_0&=A_1\exp\left(\lambda_1 u\right)+A_2\exp\left(-\lambda_1 u\right)+A_3\exp\left(\lambda_3 u\right)+A_4\exp\left(-\lambda_3 u\right),\\ \gamma_0&=B_1\exp\left(\lambda_1 u\right)+B_2\exp\left(-\lambda_1 u\right)+B_3\exp\left(\lambda_3 u\right)+B_4\exp\left(-\lambda_3 u\right)\end{aligned} \qquad (2.8)$$

where $B_j=\chi A_j$, $\chi=\dfrac{\theta_C+\lambda_k^2}{\lambda_k\left(1-\theta_C\right)}$. Solution (2.8) may be refined to take into account angular velocity damping. Represent equations (2.6) solution as

$$\begin{aligned}\mathbf{x}&=\sum_{k=1}^{4}\left(\boldsymbol{\varphi}_k+\sum_{j=1}^{n}\varepsilon^j\boldsymbol{\psi}_{kj}\left(u\right)+O\left(\varepsilon^{n+1}\right)\right)\exp\left(\lambda_k+\sum_{j=1}^{n}\varepsilon^j\mu_{kj}+O\left(\varepsilon^{n+1}\right)\right)u=\\ &=\sum_{k=1}^{4}\left(\boldsymbol{\varphi}_k+\sum_{j=1}^{n}\varepsilon^j\boldsymbol{\psi}_{kj}\left(u\right)\right)\exp\left(\lambda_k+\sum_{j=1}^{n}\varepsilon^j\mu_{kj}\right)u+O\left(\varepsilon^{n+1}\right).\end{aligned} \qquad (2.9)$$

Substituting to (2.6) and grouping terms with similar ε power,

$$\varepsilon=0 \qquad \dot{\boldsymbol{\varphi}}_k+\left(\lambda_k\mathbf{E}-\mathbf{A}_0\right)\boldsymbol{\varphi}_k=0,$$

$$\varepsilon=1 \qquad \dot{\boldsymbol{\psi}}_{k1}+\left(\lambda_k\mathbf{E}-\mathbf{A}_0\right)\boldsymbol{\psi}_{k1}=-\mu_{k1}\boldsymbol{\varphi}_k+\mathbf{A}_1\boldsymbol{\varphi}_k,$$

$$\varepsilon=2 \qquad \dot{\boldsymbol{\psi}}_{k2}+\left(\lambda_k\mathbf{E}-\mathbf{A}_0\right)\boldsymbol{\psi}_{k2}=-\mu_{k1}\boldsymbol{\psi}_{k1}-\mu_{k2}\boldsymbol{\varphi}_k+\mathbf{A}_1\boldsymbol{\psi}_{k1},$$

$\varepsilon = j \qquad \dot{\boldsymbol{\psi}}_{kj} + \left(\lambda_k \mathbf{E} - \mathbf{A}_0\right)\boldsymbol{\psi}_{kj} = \mathbf{f}_j\left(u, \mu_{k1}, ..., \mu_{kj-1}, \boldsymbol{\varphi}_k, \boldsymbol{\psi}_{k1}, ..., \boldsymbol{\psi}_{kj-1}\right).$

Note that $\boldsymbol{\varphi}_k + \sum_{j=1}^{n} \varepsilon^j \boldsymbol{\psi}_{kj}(u)$ is not an eigenvector since $\mathbf{A}_1$ depends on time. Every next approximation $\boldsymbol{\psi}_{kj}$ may be found after corresponding differential equation is solved. These equations are not necessarily easier than initial equations of motion. Lyapunov characteristic exponents are found from periodicity condition $\boldsymbol{\psi}_{kj}(u) = \boldsymbol{\psi}_{kj}(u + 2\pi)$. In some cases we can state [4] that these periodic solutions can be found and series (2.9) converges. Sufficient condition is $\lambda_{kj} - \lambda_{kl} \neq im$ for $j \neq l$ where m is an integer. These differences are eigenvalues of $\left(\lambda_k \mathbf{E} - \mathbf{A}_0\right)$. This is homogeneous part matrix in equations for $\boldsymbol{\psi}_{kj}$. This means that general solution period differs from 2π. Partial solution should ensure proper periodicity for $\boldsymbol{\psi}_{kj}$. Generating solution characteristic exponents satisfy sufficient condition so we can find characteristic exponent approximations for equations (2.6). Equations (2.5) solution corresponding to k^{th} characteristic exponent approximation is

$$\beta = \left(A_k + \varepsilon\tau_k(u)\right)\exp\left(\lambda_k + \varepsilon\sigma_k\right)u,$$
$$\gamma = \left(B_k + \varepsilon\vartheta_k(u)\right)\exp\left(\lambda_k + \varepsilon\sigma_k\right)u.$$

Substituting to (2.5) provides

$$\varepsilon\ddot{\tau}_k + \left(A_k + \varepsilon\tau_k\right)\left(\lambda_k + \varepsilon\sigma_k\right)^2 + \varepsilon\cos^2 u\left[\varepsilon\dot{\tau}_k + \left(A_k + \varepsilon\tau_k\right)\left(\lambda_k + \varepsilon\sigma_k\right)\right] +$$
$$+\left(2\varepsilon\sin u\cos u - 1 + \theta_C\right)\left[\varepsilon\dot{\vartheta}_k + \left(B_k + \varepsilon\vartheta_k\right)\left(\lambda_k + \varepsilon\sigma_k\right)\right] + \theta_C\left(A_k + \varepsilon\tau_k\right) = 0,$$
$$\varepsilon\ddot{\vartheta}_k + \left(B_k + \varepsilon\vartheta_k\right)\left(\lambda_k + \varepsilon\sigma_k\right)^2 + \left(2\varepsilon\xi\sin u\cos u + 1 - \theta_A\right)\left[\varepsilon\dot{\tau}_k + \left(A_k + \varepsilon\tau_k\right)\left(\lambda_k + \varepsilon\sigma_k\right)\right] +$$
$$+4\varepsilon\xi\sin^2 u\left[\varepsilon\dot{\vartheta}_k + \left(B_k + \varepsilon\vartheta_k\right)\left(\lambda_k + \varepsilon\sigma_k\right)\right] + \left(3\lambda_A + \theta_A\right)\left(B_k + \varepsilon\vartheta_k\right) = 0.$$

Grouping terms with ε^0 leads to

$$\begin{aligned} &\left(\lambda_k^2 + \theta_C\right)A_k + \left(\theta_C - 1\right)\lambda_k B_k = 0, \\ &\left(1 - \theta_A\right)\lambda_k A_k + \left(\lambda_k^2 + 3\lambda_A + \theta_A\right)B_k = 0. \end{aligned} \qquad (2.10)$$

Consider (2.10) as equations with respect to solution parameters. Its determinant coincides with the one of characteristic equation (2.7) and equals zero. Grouping terms with ε provides

$$\ddot{\tau}_k + 2A_k\lambda_k\sigma_k + \lambda_k^2\tau_k + \cos^2 u A_k\lambda_k + 2\sin u\cos u B_k\lambda_k +$$
$$+(\theta_C - 1)B_k\lambda_k + (\theta_C - 1)\lambda_k\vartheta_k + \theta_C\tau_k = 0,$$
$$\ddot{\vartheta}_k + 2B_k\lambda_k\sigma_k + \lambda_k^2\vartheta_k + 2\xi\sin u\cos u A_k\lambda_k + (1-\theta_A)\dot{\tau}_k + \tag{2.11}$$
$$+(1-\theta_A)A_k\lambda_k + (1-\theta_A)\lambda_k\tau_k + 4\xi\sin^2 u B_k\lambda_k + (3\lambda_A + \theta_A)\vartheta_k = 0.$$

These equations may be written as

$$F_1 - f_1 + C_1\sigma_k = 0,$$
$$F_2 - f_2 + C_2\sigma_k = 0 \tag{2.12}$$

where

$$F_1 = \ddot{\tau}_k + (\theta_C - 1)\dot{\vartheta}_k + (\theta_C - 1)\lambda_k\vartheta_k + (\theta_C + \lambda_k^2)\tau_k,$$
$$F_2 = \ddot{\vartheta}_k + (1-\theta_A)\dot{\tau}_k + (1-\theta_A)\lambda_k\tau_k + (3\lambda_A + \theta_A + \lambda_k^2)\vartheta_k,$$
$$f_1(u) = -\cos^2 u A_k\lambda_k - 2\sin u\cos u B_k\lambda_k,$$
$$f_2(u) = -4\xi\sin^2 u B_k\lambda_k - 2\xi\sin u\cos u A_k\lambda_k,$$
$$C_1 = 2A_k\lambda_k + (\theta_C - 1)B_k,$$
$$C_2 = 2B_k\lambda_k + (1-\theta_A)A_k.$$

First equation in (2.12) is multiplied with $\lambda_k^2 + \theta_C$, second is multiplied with $-(\theta_C - 1)\lambda_k$ and the result is summed. Similarly they are multiplied with $1-\theta_A$ and $-(\lambda_k^2 + 3\lambda_A + \theta_A)$ and then summed. Resulting equations are

$$(\lambda_k^2 + \theta_C)(F_1 - f_1 + C_1\sigma_k) - (\theta_C - 1)\lambda_k(F_2 - f_2 + C_2\sigma_k) = 0,$$
$$(1-\theta_A)\lambda_k(F_1 - f_1 + C_1\sigma_k) - (\lambda_k^2 + 3\lambda_A + \theta_A)(F_2 - f_2 + C_2\sigma_k) = 0. \tag{2.13}$$

Equations (2.13) determinant coincides with determinant for (2.10) and equals zero. Equations (2.13) have non-zero solution. Specifically for each non-zero $F_2 - f_2 + C_2\sigma_k$ we can find $F_1 - f_1 + C_1\sigma_k$ satisfying

$$(\lambda_k^2 + \theta_C)(F_1 - f_1 + C_1\sigma_k) - (\theta_C - 1)\lambda_k(F_2 - f_2 + C_2\sigma_k) = 0. \tag{2.14}$$

This relation can be used to find first order characteristic exponent approximation σ_k. First order approximation of corresponding eigenvector is of no importance here. However we will use its properties. Functions τ_k and ϑ_k and their derivatives are periodic. This property should be ensured by proper integration constants of (2.11) and characteristic exponent approximation σ_k. Integration of (2.14) from 0 to 2π leads to

$$(\lambda_k^2 + \theta_C)(-\overline{f}_1 + 2\pi C_1\sigma_k) - (\theta_C - 1)\lambda_k(-\overline{f}_2 + 2\pi C_2\sigma_k) = 0$$

where

$$\overline{f}_1 = \int_0^{2\pi} f_1(u)\,du = \int_0^{2\pi} \left[-\frac{1}{2} A_k \lambda_k (1+\cos 2u) - B_k \lambda_k \sin 2u \right] du = -\pi A_k \lambda_k,$$

$$\overline{f}_2 = \int_0^{2\pi} f_2(u)\,du = \int_0^{2\pi} \left[2B_k \lambda_k (-1+\cos 2u) - A_k \lambda_k \sin 2u \right] \xi du = -4\pi\xi\chi A_k \lambda_k.$$

This allows first-order k^{th} characteristic exponent approximation to be found as

$$\sigma_k = \frac{1}{2} \frac{4\xi\chi\lambda_k^2 (\theta_C - 1) - \lambda_k \left(\lambda_k^2 + \theta_C\right)}{\left(\lambda_k^2 + \theta_C\right)\left[2\lambda_k + \chi(\theta_C - 1)\right] - \lambda_k (\theta_C - 1)\left[2\chi\lambda_k + (1-\theta_A)\right]}. \qquad (2.15)$$

Time-response is represented with real part of σ_k. Denoting $\lambda_k = i\eta_k$ and $\chi = \dfrac{\theta_C - \eta_k^2}{\eta_k (\theta_C - 1)} i = \chi_r i$ leads (2.15) to be rewritten as

$$\sigma_k = \frac{1}{2} \frac{4\xi\chi_r\eta_k^2 (1-\theta_C) + \eta_k \left(\eta_k^2 - \theta_C\right)}{\left(\theta_C - \eta_k^2\right)\left[2\eta_k + \chi_r (\theta_C - 1)\right] + \eta_k (\theta_C - 1)\left[2\chi_r\eta_k + (\theta_A - 1)\right]}$$

and $\sigma_1 = \sigma_2$, $\sigma_3 = \sigma_4$. Characteristic exponent approximations are real. Finally first-order characteristic exponent approximations for equations (2.5) are

$$\begin{aligned}
\nu_1 &= i\eta_1 + \frac{1}{2}\varepsilon(1+4\xi) \frac{\eta_1^2\left(\eta_1^2 - \theta_C\right)}{-3\eta_1^4 + \left[2\theta_C + (\theta_A - 1)(\theta_C - 1)\right]\eta_1^2 + \theta_C^2},\\
\nu_2 &= -i\eta_1 + \frac{1}{2}\varepsilon(1+4\xi) \frac{\eta_1^2\left(\eta_1^2 - \theta_C\right)}{-3\eta_1^4 + \left[2\theta_C + (\theta_A - 1)(\theta_C - 1)\right]\eta_1^2 + \theta_C^2},\\
\nu_3 &= i\eta_3 + \frac{1}{2}\varepsilon(1+4\xi) \frac{\eta_3^2\left(\eta_3^2 - \theta_C\right)}{-3\eta_3^4 + \left[2\theta_C + (\theta_A - 1)(\theta_C - 1)\right]\eta_3^2 + \theta_C^2},\\
\nu_4 &= -i\eta_3 + \frac{1}{2}\varepsilon(1+4\xi) \frac{\eta_3^2\left(\eta_3^2 - \theta_C\right)}{-3\eta_3^4 + \left[2\theta_C + (\theta_A - 1)(\theta_C - 1)\right]\eta_3^2 + \theta_C^2}
\end{aligned} \qquad (2.16)$$

where

$$\eta_1 = i\sqrt{\frac{1}{2}\left(3\lambda_A + 1 + \theta_A\theta_C + \sqrt{(3\lambda_A + 1 + \theta_A\theta_C)^2 + 12\lambda_A\theta_C(\theta_A - 1)}\right)},$$

$$\eta_3 = i\sqrt{\frac{1}{2}\left(-3\lambda_A - 1 - \theta_A\theta_C + \sqrt{(3\lambda_A + 1 + \theta_A\theta_C)^2 + 12\lambda_A\theta_C(\theta_A - 1)}\right)}.$$

Equations' (2.5) degree of stability (the rightmost real part of characteristic exponents) equals either $-\varepsilon\sigma_1$ or $-\varepsilon\sigma_3$ depending on satellite and control parameters. Consider example satellite with inertia tensor $\mathbf{J} = \mathrm{diag}(1.5, 1.8, 1.1)$ kg·m^2 (several tens kilograms) orbiting at altitude of 1000 km and implementing damping algorithm with gain $k = 5 \cdot 10^5$ N·m·s/T^2 ($\varepsilon \approx 0.18$). Fig. 1 presents the degree of stability ζ (in this case $\zeta = -\varepsilon\sigma_3$) with respect to the flywheel angular momentum.

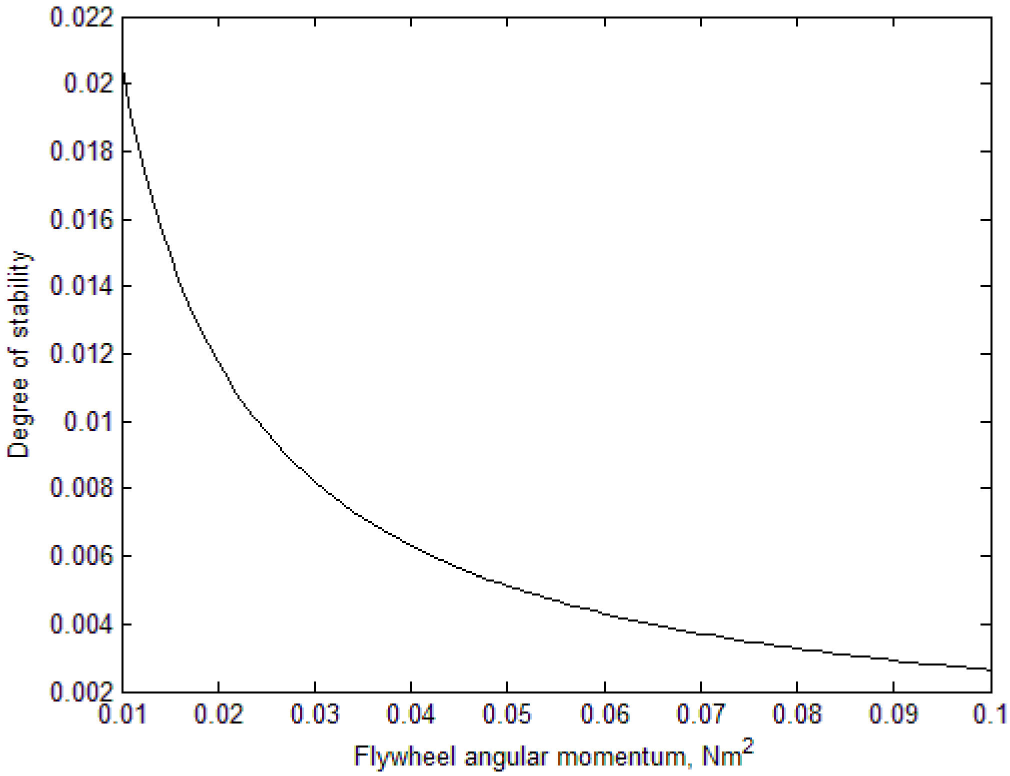

Fig. 1. Degree of stability with respect to flywheel angular momentum

As the flywheel angular momentum rises degree of stability falls, corresponding to increased transient motion period. It is hard to stabilize stronger flywheel. For example flywheel angular momentum 0.01 N·m leads to the degree of stability $\zeta \approx 0.02$. Numerical simulation shows the angle between axes O_aX_2 and O_ax_2 to fall from 9.2° to 1.1° in 10^5 seconds. Theoretical estimate of terminal angle is 0.8°. Angular momentum 0.05 N·m decreases degree of stability to $\zeta \approx 0.005$. The same angle falls to 5.6° in numerical experiment and equals 5.8° according to theoretical estimate. Approximate result (2.16) may be used for quite accurate prediction of control time-response in transient motion.

3. Nominal motion in gravitational attitude

Nominal motion with prevailing gravitational restoring torque corresponds to small relative angular velocity and close coincidence of reference frames $Ox_1x_2x_3$ and $OX_1X_2X_3$. Stability conditions (2.1) are satisfied. Adding energy dissipation through magnetic damping control makes this equilibrium position asymptotically stable. However in-plane stability is ensured only with gravitational restoring torque. Its value is barely adequate for precise pointing for most satellites. In case no disturbances are taken into account and damping control (2.3) is used necessary attitude is achieved with precision. However damping algorithm may be replaced with a simplified one

$$\mathbf{m} = -k\frac{d\mathbf{B}_x}{dt}. \tag{3.1}$$

This control is related to (2.3) through geomagnetic induction vector derivative in bound frame,

$$\frac{d\mathbf{B}_x}{dt} = \mathbf{A}\frac{d\mathbf{B}_X}{dt} - \mathbf{\Omega}\times\mathbf{B}_x. \tag{3.2}$$

Algorithm (3.1) is extensively used for fast rotation damping. In this case the first term on the right side in (3.2) may be neglected. Geomagnetic induction vector rotation rate in orbital frame is of the order of the orbital velocity. Nominal motion corresponds to slow rotation of the satellite, so we cannot omit this term. Our aim is to define this term influence (attitude accuracy) on the satellite motion in nominal regime. Write equations (1.2)-(1.4) in dimensionless form

$$\begin{aligned}
\frac{d\omega_1}{du} &= h_A\omega_3 + \lambda_A\left(\omega_2\omega_3 - 3a_{23}a_{33}\right) + \varepsilon\frac{C}{A}\overline{M}_{1x},\\
\frac{d\omega_2}{du} &= \lambda_B\left(\omega_1\omega_3 - 3a_{13}a_{33}\right) + \frac{C}{B}\varepsilon\overline{M}_{2x},\\
\frac{d\omega_3}{du} &= -h_C\omega_1 - \lambda_C\left(\omega_1\omega_2 - 3a_{13}a_{23}\right) + \varepsilon\overline{M}_{3x},\\
\frac{d\alpha}{du} &= \frac{1}{\cos\beta}\left(\omega_2\cos\gamma - \omega_3\sin\gamma\right) - 1,\\
\frac{d\beta}{du} &= \omega_2\sin\gamma + \omega_3\cos\gamma,\\
\frac{d\gamma}{du} &= \omega_1 - \mathrm{tg}\,\beta\left(\omega_2\cos\gamma - \omega_3\sin\gamma\right)
\end{aligned} \tag{3.3}$$

where $\lambda_A = \dfrac{B-C}{A}$, $\lambda_B = \dfrac{C-A}{B}$, $\lambda_C = \dfrac{B-A}{C}$, $h_A = \dfrac{h}{A\omega_0}$, $h_C = \dfrac{h}{C\omega_0}$, $\varepsilon = \dfrac{kB_0^2}{C\omega_0}$, $\overline{M}_{1x}$ are damping torque dimensionless components. Notations introduced in (2.2) are the same. Note also that angular velocity components are dimensionless here.

Equations (3.3) allow stationary solution $\alpha = \beta = \gamma = 0,\ \omega_1 = \omega_3 = 0,\ \omega_2 = 1$ in case no magnetic control is implemented. This solution corresponds to the necessary attitude. We will find periodic solutions arising from this one due to small magnetic control torque using Poincare method [5]. Equations (3.3) have form

$$\mathbf{x}=\mathbf{f}(\mathbf{x})+\varepsilon\mathbf{g}(\mathbf{x})$$

where $\mathbf{x}=(\omega_1,\omega_2,\omega_3,\alpha,\beta,\gamma)$. Solution can be found as $\mathbf{x}=\mathbf{x}_0+\varepsilon\mathbf{x}_1+O(\varepsilon^2)$ where $\mathbf{x}_0=(0,1,0,0,0,0)$ is stationary generating solution. Substituting solution to the equations of motion leads to

$$\frac{d\mathbf{x}_0}{du}+\varepsilon\frac{d\mathbf{x}_1}{du}=\mathbf{f}(\mathbf{x}_0)+\varepsilon\left(\mathbf{F}(\mathbf{x}_0)\mathbf{x}_1+\mathbf{g}(\mathbf{x}_0)\right)+O(\varepsilon^2)$$

where $F_{ij}=\dfrac{\partial f_i}{\partial x_j}$ and in our case

$$\mathbf{F}(\mathbf{x}_0)=\begin{pmatrix} 0 & 0 & \theta_A & 0 & 0 & -3\lambda_A \\ 0 & 0 & 0 & 3\lambda_B & 0 & 0 \\ -\theta_C & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

Function $\mathbf{g}(\mathbf{x}_0)$ can be easily found from (3.2) taking into account $\mathbf{\Omega}=\mathbf{0}$, $\mathbf{A}=\mathbf{E}$ leading to $\dfrac{d\mathbf{B}_x}{dt}=\dfrac{d\mathbf{B}_X}{dt}$, $\mathbf{B}_x=\mathbf{B}_X$ and finally

$$\mathbf{g}(\mathbf{x}_0)=\left(-2\frac{C}{A}\cos u\sin i\cos i,-2\frac{C}{B}\sin^2 i,\sin u\sin i\cos i,0,0,0\right)^T.$$

Grouping terms with ε we find equations for $\mathbf{x}_1$

$$\begin{aligned} &\frac{dx_1}{du}=\theta_A x_3-3\lambda_A x_6-2\frac{C}{A}\sin i\cos i\cos u, \\ &\frac{dx_2}{du}=3\lambda_B x_4-2\frac{C}{B}\sin^2 i, \\ &\frac{dx_3}{du}=-\theta_C x_1+\sin i\cos i\sin u, \\ &\frac{dx_4}{du}=x_2,\ \frac{dx_5}{du}=x_3+x_6,\ \frac{dx_6}{du}=x_1-x_5. \end{aligned} \tag{3.4}$$

Equations for x_2 and x_4 (in-plane motion) are separated and represented as

$$\ddot{x}_4-3\lambda_B x_4=-2\frac{C}{B}\sin^2 i.$$

Conditions of stability (2.1) lead to $\lambda_B < 0$. General solution is oscillation in the vicinity of necessary attitude. Attitude accuracy is determined mainly by partial solution

$$x_2 = \frac{2C\sin^2 i}{3(C-A)}. \tag{3.5}$$

Homogeneous equations for x_1, x_3, x_5, x_6 has matrix

$$\mathbf{Q} = \begin{pmatrix} 0 & \theta_A & 0 & -3\lambda_A \\ -\theta_C & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

whose characteristic exponents are found from equation

$$\det(\mathbf{Q} - \lambda\mathbf{E}) = \lambda^4 + a\lambda^2 + b = 0$$

where $a = 1 + 3\lambda_A + \theta_A\theta_C$, $b = \theta_C(3\lambda_A + \theta_A)$.

Flywheel with sufficient angular momentum allows to satisfy $a^2 - 4b = \theta_A^2\theta_C^2 + (1 + 3\lambda_A^2) - 2\theta_A\theta_C + 6\lambda_A\theta_A(\theta_C - 2) > 0$ and $a > 0$. All four characteristic exponents are imaginary. General solution is oscillation again. Partial solution may be found in a form

$$x_5 = A_1\sin u + A_2\cos u,\ x_6 = B_1\sin u + B_2\cos u.$$

Substituting to (3.4) and grouping terms with $\sin u$ and $\cos u$ leads to

$$(1-\theta_A)A_1 + (-1 + \theta_A + 3\lambda_A)B_2 = -2\frac{C}{A}\sin i\cos i,$$

$$(-1 + \theta_A + 3\lambda_A)A_2 + (-1 + \theta_A)B_1 = 0,$$

$$(-1 + \theta_C)A_2 + (-1 + \theta_C)B_1 = 0,$$

$$(-1 + \theta_C)A_1 + (1 - \theta_C)B_2 = \sin i\cos i.$$

Solving this equation we finally find partial solution for (3.4)

$$x_1 = \frac{1}{\theta_C - 1}\sin i\cos i\sin u,$$

$$x_2 = \frac{2C\sin^2 i}{3(C-A)},$$

$$x_3 = \frac{1}{\theta_C - 1}\sin i\cos i\cos u,$$

$$x_4 = 0,$$

$$x_5 = \frac{1}{3\lambda_A}\left(\frac{\theta_A - 1}{\theta_C - 1} - 2\frac{C}{A} + 3\lambda_A\right)\sin i \cos i \sin u,$$

$$x_6 = \frac{1}{3\lambda_A}\left(\frac{\theta_A - 1}{\theta_C - 1} - 2\frac{C}{A}\right)\sin i \cos i \cos u.$$

Magnetic control influence is now found. The main part is constant deviation from necessary attitude in orbital plane. This deviation rises as control torque magnitude rises (parameter ε), inclination rises (inclination governs general magnetic control system effectiveness [6]) and falls as gravitational torque increases (difference between inertia moments). Apart from this constant deviation small oscillations with orbital frequency occur. These oscillations may cause resonance effect if matrix **Q** characteristic exponent equals $\pm i$. This is valid if $\lambda^2 = \frac{1}{2}\left(-a \pm \sqrt{a^2 - 4b}\right) = -1$ and therefore $\theta_C - 1 = 0$. This is not the case for the appropriate flywheel.

Fig. 2 brings numerical simulation for the satellite with inertia tensor $\mathbf{J} = \mathrm{diag}(1.5, 1.8, 1.1)$ kg·m^2 on circular orbit with 1000 km altitude and 60° inclination, control gain is $k = 5 \cdot 10^5$ N·m·s/T^2.

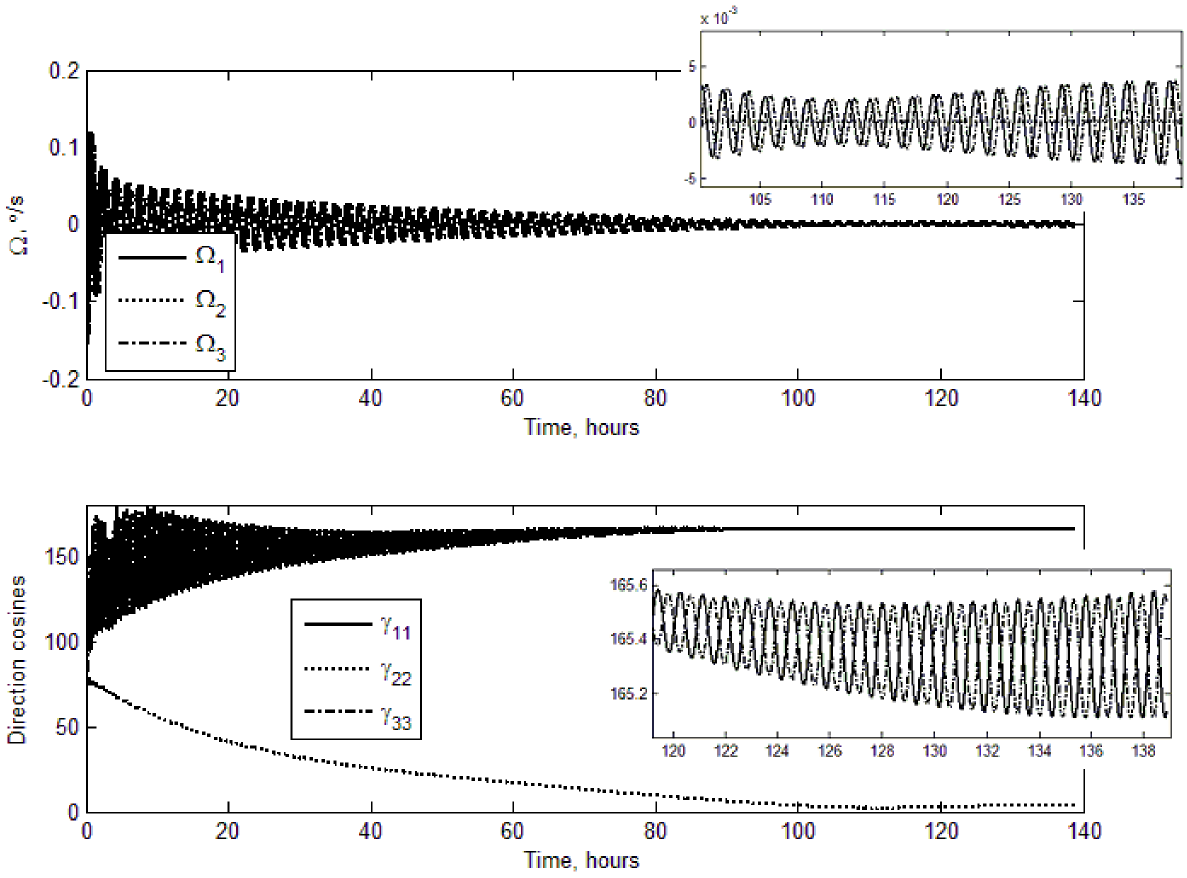

Fig. 2. Attitude accuracy

γ_{jj} are angles between axes of orbital and bound reference frames. In-plane attitude accuracy is about 14.5°, while theoretical result (3.5) predicts accuracy of 13.6°. Relation (3.5) may be quite useful instrument for attitude accuracy estimation. Fig. 2 reveals time-response problem. Transient motion takes more than 80 hours. This can be mitigated by increasing control gain. However tenfold increase in gain to $k = 5 \cdot 10^6$ N·m·s/T^2 leads to satellite alignment with geomagnetic field. According to (3.5) accuracy is about 130°. Increased gain should be accompanied either with better damping control (2.3) without term $d\mathbf{B}_X / dt$ or with gravity-gradient boom (Fig. 3).

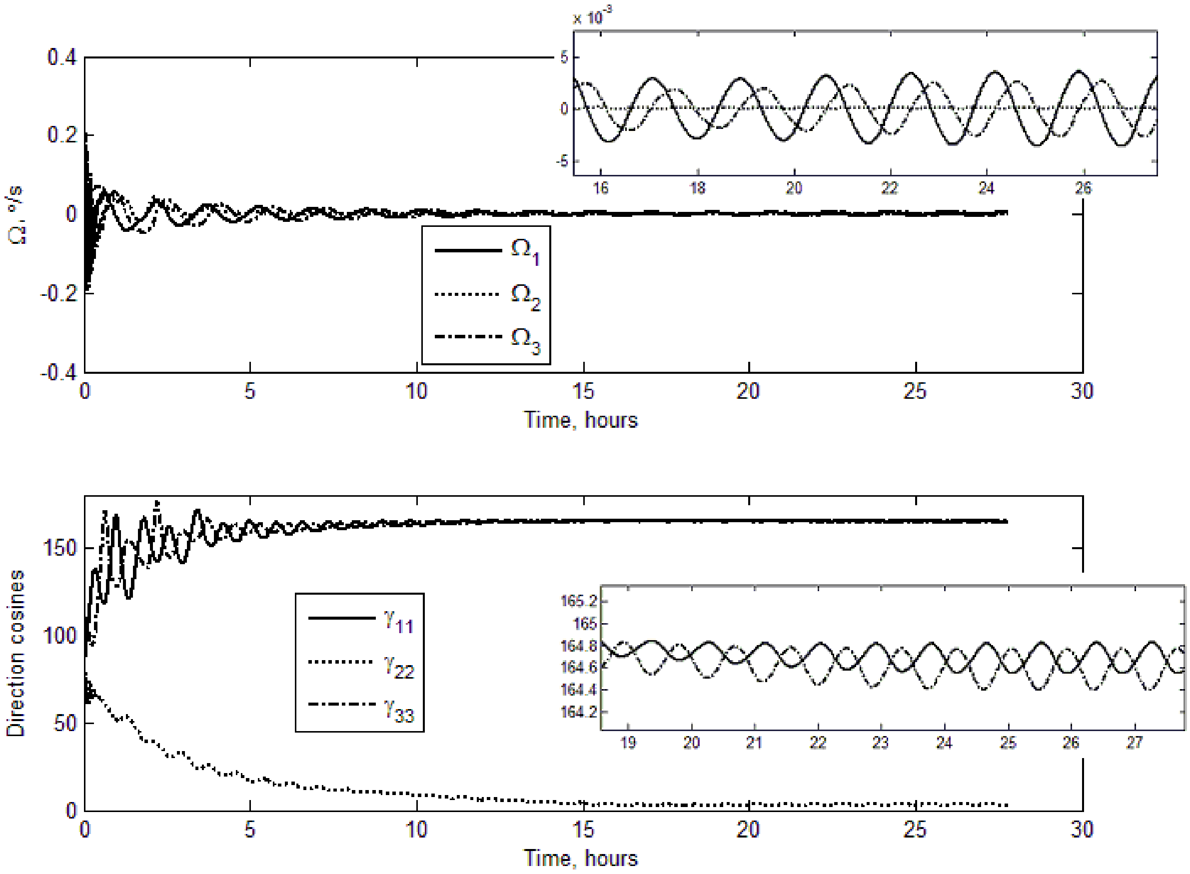

Fig. 3. Attitude accuracy with gravity-gradient boom

Gravity-gradient boom with 1.5 m length and 1.5 kg tip mass allows increased control gain with terminal accuracy of about 15.5° (theoretical prediction 14.5°). Transient motion takes a couple of hours.

4. Arbitrary in-plane attitude

Magnetic control system may be used to provide any equilibrium position $\alpha=\alpha_0$ in orbital plane and its asymptotical stability. Gravitational torque becomes a disturbing one in this case. Assume that after transient motion $\beta,\gamma\sim 0$, $\Omega_1,\Omega_2\sim 0$, $\Omega_2\sim 1$. Equations (1.3)-(1.4) are

$$\ddot{\gamma}+(1-\theta_A)\dot{\beta}+\theta_A\gamma=-3\lambda_A\left(\beta\sin\alpha\cos\alpha+\gamma\cos^2\alpha\right)+\varepsilon\frac{C}{A}\overline{M}_1,$$

$$\ddot{\alpha}=3\lambda_B\sin\alpha\cos\alpha+\varepsilon\frac{C}{B}\overline{M}_2, \qquad (3.6)$$

$$\ddot{\beta}+(\theta_C-1)\dot{\gamma}+\theta_C\beta=-3\lambda_C\sin\alpha\cos\alpha+\varepsilon\overline{M}_3$$

where small parameter ε is not yet defined since control is not defined.

Second equation in (3.6) representing in-plane motion is separated if second control torque component is independent on β,γ. Suppose this control torque component to be

$$\overline{M}_2=k_r\sin(\alpha_0-\alpha). \qquad (3.7)$$

Introducing notation $\rho=\alpha-\alpha_0$ in-plane motion becomes

$$\ddot{\rho}+k_r\varepsilon\frac{C}{B}\sin\rho=3\lambda_B\sin(\rho+\alpha_0)\cos(\rho+\alpha_0). \qquad (3.8)$$

In case gravitational torque is not acting on the satellite ($\lambda_B=0$) equation (3.8) represents oscillations in the vicinity of necessary attitude $\rho=0$. Asymptotic stability of this position (tendency exactly to zero) is provided by energy dissipation according to (2.3). Gravitational torque changes equilibrium position to (linear approximation)

$$\rho_0=\frac{\sin\alpha_0\cos\alpha_0}{\sin^2\alpha_0-\cos^2\alpha_0+k_r\varepsilon C/3(C-A)}. \qquad (3.9)$$

Control (3.7) implementation encounters a number of problems. Magnetic control system induces disturbing torque out of orbital frame since $M_2=m_3B_1-m_1B_3$. Assume that only the first magnetorquer is used and $m_2=m_3=0$. Control restoring torque in this case is

$$\mathbf{M}_r=\left(0,k_r\varepsilon\sin(\alpha_0-\alpha),-k_r\varepsilon\sin(\alpha_0-\alpha)\frac{B_{2x}}{B_{3x}}\right)^T. \qquad (3.10)$$

Another magnetorquer may be used if B_{3x} is close to zero. However the satellite moves almost in orbital plane and $B_{3x}\approx B_0\sin\alpha\cos u\sin i+B_0\cos\alpha\cos i$. Angle α is far from both 0 and π/2, B_{3x} is close to zero only for polar orbit and $u=\pi/2$.

Control (3.10) may be used without additional conditions. Control (3.10) involves disturbing third component. Its influence in equations (3.6) is assessed numerically. Another problem is utilization of Euler angle in control (3.10). Numerical and onboard control implementation requires quaternion or direction cosines matrix elements. Taking into account direction cosines matrix in case $\beta,\gamma \sim 0$ and omitting small terms control (3.10) may be rewritten as

$$\mathbf{M}_r = \left(0, k_r\varepsilon\left(\sin\alpha_0 a_{11} + \cos\alpha_0 a_{13}\right), -k_r\varepsilon\left(\sin\alpha_0 a_{11} + \cos\alpha_0 a_{13}\right)\frac{B_{2x}}{B_{3x}}\right)^T. \qquad (3.11)$$

This control is used for numerical simulation even in transient motion with angles β,γ being not close to zero. Control construction requires one more operation. Since restoring and damping components are implemented simultaneously they should differ by magnitude. Both restoring and damping dipole moments are scaled in such a way that each has its maximum component at some defined value. This one is greater for the restoring part. Control (3.11) has clear disturbing effect in transient motion in this case. However flywheel and energy dissipation ensure proper transient motion result regardless of disturbing torques. This allows straightforward control cycle without additional switching conditions.

Fig. 4 brings numerical simulation result for necessary attitude angle $\alpha_0 = 40^\circ$ (far from gravitationally stable equilibrium), restoring control dipole moment is 3.2 A·m^2 (maximum component value), damping moment is 1.2 A·м^2, other parameters hold. Relation (3.9) should be adapted to link ideal control parameter k_r with magnetorquers dipole moments used in numerical simulation. After this adaptation accuracy of about 0.45° is anticipated while numerical simulation results show accuracy of 0.6°. Out of plane accuracy is about 1.5° due to disturbing control component.

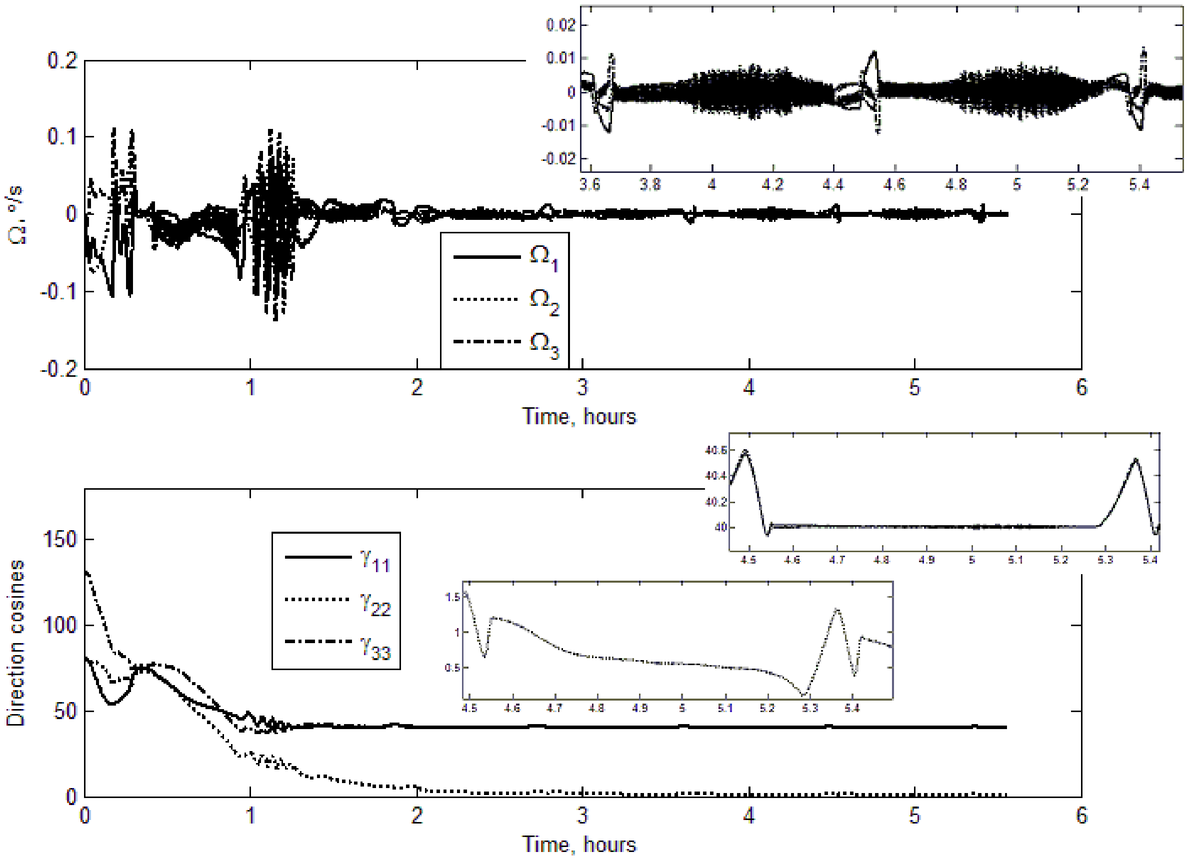

Fig. 4. Arbitrary in-plane attitude

Conclusion

Transient and nominal motions are considered for a dual-spin satellite. Active control is provided by magnetorquers. Approximate formulae for characteristic exponents are found for transient motion of polar satellite. This result may be used to estimate control system time-response during satellite and its attitude control system design. Attitude accuracy estimate is found for nominal gravitationally stable motion in case coarse angular velocity damping algorithm is used. Control algorithm is proposed to provide arbitrary attitude in orbital plane. Attitude accuracy due to disturbing gravitational torque is found. Numerical simulation results are presented.

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