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Asymptotics of L_p-norms of Laguerre polynomials and entropic moments of D-dimensional oscillator

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Асимптотика L_p-норм многочленов Лагерра и энтропийные моменты D-мерного осциллятора. Препринт Института прикладной математики им. М.В. Келдыша РАН, Москва, 2015

Асимптотика L_p-нормы многочленов Лагерра $L_n^{(\alpha)}$ с весом получена для $n \to \infty$ и $p > 0$. Этот результат мотивирован вычислениями энтропийных моментов квантово-механических плотностей вероятностей высокоэнергетических (ридбергских) состояний многомерного осциллятора.

Ключевые слова: Асимптотический анализ; ортогональные многочлены; информационная энтропия.

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The asymptotics of the weighted L_p-norms of the Laguerre polynomials is determined for $n \to \infty$ and $p > 0$. The result is motivated by calculations the entropic moments of the quantum-mechanical probability density of the highly-excited (Rydberg) states of D-dimensional oscillator.

Key words: Asymptotical analysis; orthogonal polynomials; information entropy.

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1. Statement of problem.

The radial components of the wave functions of the D-dimensional isotropic oscillator (whose potential is $V_D(r) = \frac{\lambda^2 r^2}{2}$) in the position space $\vec{r} \in \mathbb{R}^D$, $r := |\vec{r}|$ are given by

$$\Psi_{n,l}(r) = \mathrm{Const}(n, b, \lambda, D)\ r^l e^{-\lambda r^2/2}\, \widehat{L}_n^{(l+D/2-1)}(\lambda r^2)\ ,$$

where $\widehat{L}_n^{(\alpha)}(x)$ are the Laguerre polynomials which are orthonormal with respect to the weight function

$$w_\alpha(x) = x^\alpha e^{-x}\ . \tag{1.1}$$

The wave functions $\Psi_{n,l}$ with quantum numbers n, l correspond to the energy levels

$$E_{n,l} = \lambda \left(2n + l + \frac{D}{2} \right)\ , \quad n = 0,1,2,\ldots\ ,\ l = 0,1,2,\ldots$$

The squared modulus of this wave functions describes the position probability distribution density $\rho_{n,l} = |\Psi_{n,l}|^2$.

J. S. Dehesa has posed a problem to obtain the asymptotics of the entropic moments

$$\int_0^\infty \rho_{n,l}^p(r)\, r^{D-1} dr\ , \quad n \to \infty\ ,$$

i.e. the entropic moments for the Rydeberg (high energy) states. Thus, we need to study the asymptotics of the L_p-norm of the Laguerre polynomials

$$N_n(D, p) = \int_0^\infty \left(\left[\widehat{L}_n^{(\alpha)}(x) \right]^2 w_\alpha(x) \right)^p x^\beta\, dx\ , \quad p > 0\,, \tag{1.2}$$

where

$$\alpha = l + \frac{D}{2} - 1\,,\ l = 0,1,2,\ldots, \quad \text{and} \quad \beta = (p-1)(1 - D/2)\ . \tag{1.3}$$

We note that (1.3) and (1.1) guarantee the convergence of integral (1.2) at zero, i.e. the condition

$$\beta + p\alpha = pl + \frac{D}{2} - 1 > -1\ ,$$

is always satisfied for physically meanfull parameters (1.3).

2. Statements and discussions of the results.

The asymptotic behavior of $N_n(D,p)$ as $n \to \infty$ essentially depends on the values of the parameters D and p (i.e. α, β and p). In fact different regions of integration in (1.2) for different values of the parameters give the dominant contribution in the magnitude of the integral $N_n(D,p)$. Thus we have to use various asymptotical representation for the Laguerre polynomials for different scales.

Roughly speaking in the neighborhood of zero (i.e. the left end point of the interval of orthogonality) the Laguerre polynomials can asymptotically be presented by means of Bessel functions (taken for expanding scale of the variable). Then (to the right) oscillatory behavior of the polynomials (in the bulk region of zeros location) is modeled asymptotically by means of the trigonometric functions and at the neighborhood of the extreme right zeros asymptotics is given by Airy functions. Finally, in the neighborhood of the infinity point the polynomials have growing asymptotics. Moreover, there are regions where these asymptotics match each other. Namely, asymptotics of the Bessel functions for big arguments match the trigonometric function, as well as the asymptotics of the Airy functions do the same. Altogether, there are five asymptotical regimes which can give (depending on D and p) the dominant contribution in the asymptotics of $N_n(D,p)$. Three of them exhibit the growth of $N_n(D,p)$ as some degree of n with an exponent which depends on D and p. We call these regimes Bessel, Airy and cosine (or oscillatory) regimes.

We define the constants which stay in front of the degree of n in the asymptotics of $N(D,p)$ for these regimes. For the Bessel regime we denote

$$C_B(\alpha,\beta,p) := 2\int_0^\infty t^{2\beta+1}|J_\alpha|^{2p}(2t)\,dt\,. \tag{2.1}$$

For the Airy regime we denote

$$C_A(p) := \int_{-\infty}^{+\infty}\left[\frac{2\pi}{\sqrt[3]{2}}\,\mathrm{Ai}^2\left(-\frac{t\sqrt[3]{2}}{2}\right)\right]^p dt\,. \tag{2.2}$$

For the cosine regime we denote

$$C(\beta,p) := \frac{2^{\beta+1}}{\pi^{p+1/2}}\,\frac{\Gamma(\beta+1-p/2)\,\Gamma(1-p/2)\,\Gamma(p+1/2)}{\Gamma(\beta+2-p)\,\Gamma(1+p)}\,. \tag{2.3}$$

Definitions for the Bessel and the Airy functions are given below, see (3.5), (3.10) and (3.11).

There are also two transition regimes: cosine-Bessel and cosine-Airy. If these regimes dominate in integral (1.2), then the asymptotics of $N(D,p)$ besides the degree on n have the factor $\ln n$. It is also curious to mention that if these regimes dominate then the gamma factors in constant $C(\beta,p)$ in (2.3) for the oscillatory cosine regime explode. For the cosine-Bessel regime it happens for $\beta+1-p/2=0$, and for the cosine-Airy regime it happens for $1-p/2=0$.

Now we are going to state the asymptotics results. We split them in three theorems.

Theorem 1. *Let $D \in (2,\infty)$. Denoting*

$$p^* := \frac{D}{D-1}\,, \tag{2.4}$$

we have for (1.2), as $n \to \infty$

$$N_n(D,p) = \begin{cases} C(\beta,p)\,(2n)^{(1-p)\,D/2}\,(1+\bar{\bar{o}}(1)), & p \in (0,p^*) \\[2ex] \dfrac{2}{\pi^{p+1/2}n^{p/2}}\,\dfrac{\Gamma(p+1/2)}{\Gamma(p+1)}\,(\ln n + \underline{\underline{O}}(1))\,, & p = p^* \\[2ex] C_B(\alpha,\beta,p)\,n^{(p-1)D/2-p}\,(1+\bar{\bar{o}}(1)), & p > p^* \end{cases}, \tag{2.5}$$

where the constants C, C_B are defined in (2.3), (2.1) respectively and dependence of the parameters $\alpha(l,D), \beta(p,D)$ on l,p,D is defined in (1.3).

To comment on this result we note that

$$\beta(p^*,D) - \frac{p^*}{2} = (p^*-1)\left(1-\frac{D}{2}\right) - \frac{p^*}{2} = \frac{1}{D-1}\left(1-\frac{D}{2}-\frac{D}{2}\right) = -1\,,$$

therefore from (2.3) we have $C(\beta,p)=\infty$. Thus, when $D>2$ we have that for $p \in (0,p^*)$ the region of $\mathbb{R}_+$ where the Laguerre polynomials exhibit the cosine asymptotics contributes the dominant part in the integral (1.2). For $p=p^*$ the transition cosine-Bessel regime determines the asymptotics of $N_n(D,p^*)$, and for $p>p^*$ the Bessel regime plays the main role.

The next result is

Theorem 2. *Let $D=2$. We have for (1.2), as $n \to \infty$*

$$N_n(2,p) = \begin{cases} C(0,p)\,(2n)^{(1-p)}\,(1+\bar{\bar{o}}(1))\,, & p \in (0,\,2) \\[2ex] \dfrac{\ln n + \underline{\underline{O}}(1)}{\pi^2 n}\,, & p = 2 \\[2ex] \dfrac{C_B(\alpha,0,p)}{n}\,(1+\bar{\bar{o}}(1))\,, & p > 2 \end{cases}. \tag{2.6}$$

A peculiarity of the case of the dimension $D = 2$ is in the following. We have from the Theorems 1 and 2

$$\lim_{D\to 2+} N(D,p) = N(2,p)\ , \quad p \in (0,2) \cup (2,\infty)\ .$$

However, from the Theorem 1 we have

$$\lim_{D\to 2+} N(D,2) = \frac{3(\ln n + \underline{\underline{O}}(1))}{4\pi^2 n}\ . \tag{2.7}$$

At the same time the Theorem 2 states:

$$N(2,2) = \frac{\ln n + \underline{\underline{O}}(1)}{\pi^2 n}\ .$$

Indeed, as we shall prove below, the magnitude of the integral $N(2,2)$ is performed mainly by two regions of $\mathbb{R}_+$ (with the same order of contribution). The first one is at the origin (Bessel-cosine regime), and the second one is around the right-extreme zeros of the Laguerre polynomials (Airy-cosine regime). The first region gives the contribution in $N(2,2)$ as in (2.7). The second one gives the rest of the contribution

$$\frac{\ln n + \underline{\underline{O}}(1)}{4\pi^2 n}\ . \tag{2.8}$$

Thus, for $D = 2$ and $p = 2$ we have the competition of two transition regimes, namely the Bessel-cosine and Airy-cosine regimes.

The concluding result on the asymptotics of $N(D,p)$ (we recall β is defined in (1.3)) is the following

Theorem 3. *Let $D \in [0,2)$. We have for (1.2), as $n \to \infty$ and $p \in (0,2]$*

$$N(D,p) = \begin{cases} C(\beta,p)\,(2n)^{(1-p)\frac{D}{2}}\,(1+\bar{\bar{o}}(1))\ , & p \in (0,2) \\ \dfrac{\ln n + \underline{\underline{O}}(1)}{\pi^2 (4n)^{1-\beta}}\ , & p = 2 \end{cases}\ . \tag{2.9}$$

Denoting $\widetilde{p} := \dfrac{-2+3D}{-4+3D}$, *we have for $p > 2$ and $4/3 < D < 2$*

$$N_n(D,p) = \begin{cases} \dfrac{C_A(p)}{\pi^p}\,(4n)^{(\frac{1-2p}{3}+\beta)}(1+\bar{\bar{o}}(1))\,, & p \in (2,\widetilde{p}) \\ \left(\dfrac{C_A(p)}{\pi^p}\,4^{(\frac{1-2p}{3}+\beta)} + C_B(\alpha,\beta,p)\right) n^{-\beta-1}\ , & p = \widetilde{p} \\ C_B(\alpha,\beta,p)\,n^{-\beta-1}\ , & p \in (\widetilde{p},\infty) \end{cases}\ , \tag{2.10}$$

and we conclude the case $p > 2$ for $D \leqslant 4/3$

$$N(D,p) = \frac{C_A(p)}{\pi^p}\,(4n)^{\left(\frac{1-2p}{3}+\beta\right)}(1+\bar{\bar{o}}(1))\,, \qquad p \in (2,\infty)\,. \tag{2.11}$$

Here we see that the oscillatory regime in (2.9) for $p \in (0{,}2)$ matches the same regime in (2.5) and (2.6) for $p < p^*$. But for $p = 2$ the Airy-cosine regime wins vs Bessel-cosine regime and we have only the contribution of (2.6) in $N(D,p)$. For $p \geq 2$ we get a new phenomena – the role of the oscillatory regime disappears and for the first time the Airy and Bessel regimes becomes competitive.

3. Asymptotics of the Laguerre polynomials

In the proofs of the stated theorems we use the asymptotical representation for the Laguerre polynomials $L_n^{(\alpha)}(x)$ defined by [7, 9]

$$L_n^{(\alpha)}(x) = \sum_{\nu=0}^{n} \binom{n+\alpha}{n-\nu} \frac{(-x)^\nu}{\nu!} \tag{3.1}$$

with norm

$$\|L_n^{(\alpha)}\|^2 = \Gamma(\alpha+1)\binom{n+\alpha}{n}\,. \tag{3.2}$$

For the distinct scales of the variable x with respect to n the Laguerre polynomials have different asymptotics.

For the Bessel regime (i.e. when x is small with respect to n there is Hilb asymptotics (see [9] , eq.(8.22.4))

$$e^{-\frac{x}{2}}x^{\alpha/2}L_n^{(\alpha)}(x) = \frac{(n+\alpha)!}{n!}\,(N\,x)^{-\alpha/2}J_\alpha(2\sqrt{N\,x}) + \varepsilon(x,n)\,, \tag{3.3}$$

where

$$N = n + \frac{\alpha+1}{2}\,, \qquad \varepsilon(x,n) = \begin{cases} x^{\alpha/2+2}\,\underline{\underline{O}}(n^\alpha)\,, & 0 < x < \frac{c}{n} \\ x^{5/4}\,\underline{\underline{O}}(n^{\alpha/2-3/4})\,, & \frac{c}{n} < x < C \end{cases}\,, \tag{3.4}$$

and the Bessel function is defined by

$$J_\alpha(z) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!\,\Gamma(\nu+\alpha+1)}\left(\frac{z}{2}\right)^{\alpha+2\nu}\,. \tag{3.5}$$

For the transition region between Bessel regime and oscillatory regime we use the asymptotics of the Bessel function [7]

$$J_\alpha(z) = \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + e^{|\mathrm{Im}\,z|}\underline{\underline{O}}\left(\frac{1}{z}\right)\,, \quad |\arg z| < \pi\,. \tag{3.6}$$

The following regimes: oscillatory, growing and Airy are described by the Plancherel-Rotach asymptotics (see [9]):

for $x = (4n + 2\alpha + 2)\cos^2\varphi,\ \varepsilon \leqslant \varphi \leqslant \dfrac{\pi}{2} - \varepsilon n^{-1/2}$

$$e^{-x/2}\, L_n^{(\alpha)}(x) = (-1)^n\, (\pi\sin\varphi)^{-1/2}\, x^{-\alpha/2-1/4}\, n^{\alpha/2-1/4} \times \\ \times \left\{\sin\left[\left(n + \tfrac{\alpha+1}{2}\right)\, (\sin 2\varphi - 2\varphi) + \tfrac{3\pi}{4}\right] + (nx)^{-1/2}\, O(1)\right\}\ ; \tag{3.7}$$

for $x = (4n + 2\alpha + 2)\ \mathrm{ch}^2\,\varphi,\ \varepsilon \leqslant \varphi \leqslant \omega$

$$e^{-x/2}\, L_n^{(\alpha)}(x) = \tfrac{1}{2}(-1)^n\, (\pi\,\mathrm{sh}\,\varphi)^{-1/2}\, x^{-\alpha/2-1/4}\, n^{\alpha/2-1/4} \times \\ \times \exp\left[\left(n + \tfrac{\alpha+1}{2}\right)\, (2\varphi - \mathrm{sh}\, 2\varphi)\right]\, \left[1 + O(n^{-1})\right]\ ; \tag{3.8}$$

and for $x = 4n + 2\alpha + 2 - 2\left(\tfrac{2n}{3}\right)^{1/3} t\ ,\ |t| < Const$

$$e^{-x/2}\, L_n^{(\alpha)}(x) = (-1)^n\, \pi^{-1}\, 2^{-\alpha-1/3}\, 3^{1/3}\, n^{-1/3}\{A(t) + O(n^{-2/3})\} \tag{3.9}$$

where $A(t)$ is Airy function (see [9])

$$A(t) = \frac{\pi}{3}\, \left(\frac{t}{3}\right)^{1/2}\, \left(J_{-1/3}(2(\frac{t}{3})^{3/2}) + J_{1/3}(2(\frac{t}{3})^{3/2})\right)\ , \tag{3.10}$$

the solution of the equation

$$\frac{d^2}{dt^2}y + \frac{1}{3}\, t\, y = 0\ ,$$

bounded as $t \to \infty$. In (2.2) we use the following normalization for the Airy function as

$$A(t) = \frac{\pi}{\sqrt[3]{3}}\, \mathrm{Ai}\left(-t/3\sqrt{3}\right)\ . \tag{3.11}$$

During the last two decades there was a substantional progress in proving global asymptotical representations for orthogonal polynomials (see papers of Percy Deift with coauthors[10], [11], [12], Roderic Wang with coauthors [13], [14] and papers [3], [2]). In practice it means that the classical asymptotics formulas (like Hilb and Plancherel-Rotach) hold true in wider domains providing matching of the asymptotics in the transition zones (for example, see in [1], [3], [11], [12] for Hermite polynomials). In our paper we assume that matching of the classical asymptotics holds true for the Laguerre polynomials as well.

4. Proofs

For all three theorems we use the unified approach. We split in (1.2) the domain of integration $\mathbb{R}_+$ into nine intervals:

$$N_n(D,p) = \frac{\int\limits_0^\infty ((L_n^{(\alpha)}(x))^2\, w(x))^p\, x^\beta\, dx}{\|L_n^{(\alpha)}\|^{2p}} = n^{-p\alpha}\left(\sum_{j=1}^{9} I_j\right)\,,$$

where

$$I_j := \int\limits_{\triangle_j} ((L_n^{(\alpha)}(x))^2\, w(x)^p\, x^\beta\, dx\,, \tag{4.1}$$

and

$$\begin{aligned}
&\Delta_1 = [0, M/n]\,; \qquad \Delta_2 = [M/n, 1]\,; \qquad \Delta_3 = [1, (4-\varepsilon)n]\,; \\
&\Delta_4 = [(4-\varepsilon)n, 4n - n^{\frac{1}{3}+\theta}]\,; \qquad \Delta_5 = [4n - n^{\frac{1}{3}+\theta}, 4n - Mn^{\frac{1}{3}}]; \\
&\Delta_6 = [4n - Mn^{\frac{1}{3}}, 4n]\,; \qquad \Delta_7 = [4n, 4n + Mn^{\frac{1}{3}}]\,; \\
&\Delta_8 = [4n + Mn^{\frac{1}{3}}, 4n + n^{\frac{1}{3}+\theta}]\,; \qquad \Delta_9 = [4n + n^{\frac{1}{3}+\theta}, \infty]\,,
\end{aligned} \tag{4.2}$$

for some big $M > 0$, small $\varepsilon > 0$ and $\theta > 0$. Then we replace $L_n^{(\alpha)}\, w$ in (4.1) by their asymptotics. For $j = 1$ we use Hilb asymptotics (3.3)-(3.4); for $j = 2$ we use Hilb asymptotics (3.3)-(3.4) and Bessel function asymptotics (3.6); for $j = 3,4$ we use oscillatory asymptotics of Plancherel-Rotach (3.7); for $j = 5,6,7,8$ we use Airy asymptotics of Plancherel-Rotach (3.9); for $j = 9$ we use growing asymptotics of Plancherel-Rotach (3.8).

Eventually we estimate the contribution of each integral from $\{I_j\}_{j=1}^9$ finding the dominating terms.

4.1. Proof of Theorem 1.

We have $D > 2$ and $p^* = \frac{D}{D-1}$.

We start with $p > p^*$. For this case in the representation (4.1) for $N_n(D,p)$ by the sum of integrals $\sum\limits_{j=1}^{9} I_j$ (see (4.1), (4.2)) the main contribution is given by I_1. We have

$$I_1 = \int\limits_0^{M/n} (w^{1/2}(x)\, \widehat{L}_n^{(\alpha)}(x))^{2p}\, x^\beta\, dx =$$

$$= \int_0^{M/n} \left[\left(\frac{(n+\alpha)!}{n!} \right)^2 (Nx)^{-\alpha} J_\alpha^2(2\sqrt{Nx}) + O\left(x^{\alpha/2+2} n^\alpha \right) \right]^p x^{p\,\alpha+\beta}\, dx \; .$$

Making the change of the variable $t := \sqrt{Nx}$, we continue

$$I_1 \simeq n^{2p\,\alpha} \cdot N^{-p\,\alpha-\beta-1} \int_0^{\sqrt{\frac{MN}{n}}} 2t^{2p\,\alpha+2\beta+1} t^{-2p\,\alpha} |J_\alpha^{2p}|\,(2t)\, dt \simeq$$

$$\simeq n^{p\,\alpha-\beta-1} \int_0^{\sqrt{M}} 2t^{2\beta+1} |J_\alpha^{2p}|\,(2t)\, dt \; . \tag{4.3}$$

The last integral converges at zero. Indeed the integrand has there the order of singularity $2p\,\alpha+2\beta+1 > -1$ due to (3). The order of singularity of the integrand at infinity is $2\beta+1-p < -1$ due to $p > p^*$. Since the parameter M is arbitrary in our partition of $\mathbb{R}_+$ in (4.2)), we take $M \to \infty$ and obtain

$$n^{-p\,\alpha} I_1 \simeq n^{-\beta-1} \int_0^\infty 2t^{2\beta+1} |J_\alpha|^{2p}(2t)\, dt \; . \tag{4.4}$$

In fact, the contribution in N_n of the remaining integrals I_j, $j = 2, \ldots, 9$ for $D > 2$, $p > p^*$ is less (we will see it latter). Thus (due to (3), (4)) asymptotics (4.4) is the same as in (2.5) for $p > p^*$.

Now $p = p^*$. For this case the dominant behavior have two integrals I_2 and I_3. Indeed, we have from (4.3)

$$n^{-p\,\alpha} I_1 = O\left(\frac{M^{p\,\alpha+\beta+1}}{n^{\beta+1}} \right) + \delta_n \; , \quad \delta_n = \frac{M^{p\,\alpha+\beta+3}}{n^{\beta+3}} \; . \tag{4.5}$$

We note, that from (3) we have

$$\beta - \frac{p^*}{2} = (p^*-1)\,(1 - \frac{D}{2}) - \frac{p^*}{2} = -1 \; . \tag{4.6}$$

Estimating I_2 we use the asymptotics of the Bessel function (3.6)

$$n^{-p\,\alpha} I_2 = \int_{M/n}^1 J_\alpha^{2p}(2\sqrt{Nx})\, x^\beta\, dx + \tilde{\delta}_n =$$

$$= \int_{M/n}^1 \frac{1}{\pi^p (Nx)^{p/2}} \left\{ \cos\left(2\sqrt{Nx} - (2\alpha+1)\cdot\frac{\pi}{4} \right) + \underline{\underline{O}}\left(\frac{1}{\sqrt{N}} \right) \right\}^{2p} x^\beta\, dx + \tilde{\delta}_n \; .$$

Using ([4], Lemma 2.1) we continue for $n \to \infty$

$$n^{-p\,\alpha} I_2 = \frac{1}{\pi} \int\limits_0^{\pi} |\cos\theta|^{2p}\, d\theta \int\limits_{M/n}^{1} \frac{x^{-p/2+\beta}\, dx}{\pi^p\, N^{p/2}}\, (1 + \bar{\bar{o}}(1)) \ .$$

The first integral is

$$\int\limits_0^{\pi} |\cos\theta|^{2p}\, d\theta = \frac{\sqrt{\pi}\,\Gamma(p+1/2)}{\Gamma(p+1)} \ .$$

Computing the second integral for $p = p^*$ (see (4.6)) we obtain

$$n^{-p^*\alpha} I_2 = \frac{\Gamma(p^*+1/2)\,(\ln n + \underline{\underline{O}}(1))}{\pi^{p^*+1/2}\,\Gamma(p^*+1)\, N^{p/2}} \ . \tag{4.7}$$

The Plancherel-Rotach asymptotics (3.7) for $\varphi = \arccos\sqrt{\frac{x}{4N}}$ can be transformed to

$$\frac{x^\alpha}{n^\alpha}\left(e^{x/2} L_n^\alpha(x)\right)^2 = \frac{2\sin^2\left[\frac{1}{2}\sqrt{x(4N-x)} - 2N\arccos\sqrt{\frac{x}{4N}} + \frac{3\pi}{4}\right] + O\left(\frac{1}{\sqrt{nx}}\right)}{\pi\sqrt{x(4N-x)}} \ . \tag{4.8}$$

Substituting it in I_3 and using ([4], Lemma 2.1) we have for I_3, as $n \to \infty$

$$n^{-p^*\alpha} I_3 = \int\limits_1^{(4-\varepsilon)n} \frac{x^{\alpha p^*}}{n^{\alpha p^*}} \left(e^{x/2} L_n^{(\alpha)}(x)\right)^{2p^*} x^\beta dx =$$

$$= \left(\frac{2}{\pi\sqrt{4n}}\right)^{p^*} \frac{1}{\pi} \int\limits_0^{\pi} |\sin\theta|^{2p^*} d\theta \cdot \int\limits_1^{(4-\varepsilon)n} x^{\beta - p^*/2} dx \ .$$

Thus I_3 gives the same contribution in $N_n(D, p^*$ as I_2 in (4.7)

$$n^{-p^*\alpha} I_3 = \frac{\Gamma(p^*+1/2)\,(\ln n + \underline{\underline{O}}(1))}{\pi^{p^*+1/2}\,\Gamma(p^*+1)\, N^{p/2}} \ . \tag{4.9}$$

We see from (4.5) that for $p = p^*$ the contribution from I_1 in $N_n(D, p^*)$ is less than that from I_2 and I_3. The same can be shown for the contribution of other integrals. Thus summing up (4.7) and (4.9) we arrive at (2.5) for $p = p^*$.

It remains to consider the case $p \in (0, p^*)$. The dominant contribution here is given by I_3. Substituting in I_3 asymptotics (4.8), making change of variable $t := \sqrt{\frac{x}{4n}}$ and using ([4], Lemma 2.1) we arrive to

$$N^{-p\,\alpha} I_3 = \left(\frac{2}{\pi 4n}\right)^p (2\sqrt{n})^{2\beta+2}\,\frac{1}{\pi}\int\limits_0^\pi |\sin\theta|^{2p} d\theta \cdot \int\limits_0^1 \frac{t^{2\beta+1}\,dt}{t^p(1-t^2)^{p/2}}\,(1+\bar{\bar{o}}(1))\,.$$

The last integral can be evaluated explicitly

$$\int\limits_0^1 \frac{t^{2\beta+1}\,dt}{t^p(1-t^2)^{p/2}} = \frac{1}{2}\,\frac{\Gamma(\beta+1-p/2)\,\Gamma(1-p/2)}{\Gamma(\beta+2-p)}\,.$$

Thus we obtain

$$n^{-p^*\alpha} I_3 = \frac{2^{\beta+1}}{\pi^{p+1}}\,\frac{\Gamma(\beta+1-p/2)\,\Gamma(1-p/2)\,\Gamma(1+p/2)}{\Gamma(\beta+2-p)\,\Gamma(1+p)}\,(2n)^{1-p+\beta}\,(1+\bar{\bar{o}}(1))\,. \tag{4.10}$$

It is clear, that the contributions of I_1 and I_2 is less than I_3. The same can be shown for the contribution of other integrals. Theorem is proved.

4.2. Proof of Theorem 2. We have $D = 2$. Then $\beta \equiv 0$ and $p^* = 2$.

We start with $p > 2$. Like for the case $p > p^*$ for $D > 2$, we see that dominant contribution in $N_n(D, p)$ is given by I_1, see (4.1) – (4.2). Indeed, we have

$$\int\limits_0^{M/n} \left(w^{1/2}(x)\,\widehat{L}_n^{(\alpha)}(x)\right)^{2p} dx =$$

$$= \int\limits_0^{M/n} \left[\frac{n!}{(n+\alpha)!}\,\left(\frac{(n+\alpha)!}{n!}\right)^2 (Nx)^{-\alpha} J_\alpha^2(2\sqrt{Nx}) + x^{\alpha+4} O(n^\alpha)\right]^p x^{p\,\alpha}\,dx \simeq$$

$$\simeq \frac{1}{n}\left(\int\limits_0^{\sqrt{M}} 2t\,|J_\alpha|^{2p}(2t)\,dt + \bar{\bar{o}}(1)\right)\,.$$

Since M is an arbitrary constant, we let $M \to \infty$. At the same time, we see that the sum $J_6 + J_7$ also gives a perceptible contribution

$$\int\limits_{4N-Mn^{1/3}}^{4N+Mn^{1/3}} \left(w^{1/2}(x)\,\widehat{L}_n^{(\alpha)}(x)\right)^{2p} dx = \int\limits_{-M}^{M} \left[(2n)^{-2/3} A_i^2\left(-\frac{t}{2^{4/3}}\right)\right]^p n^{1/3}\,dt\,(1+\bar{\bar{o}}(1)\,. \tag{4.11}$$

However, for $p > 2$

$$1/3 - p\,2/3 < -1\,. \tag{4.12}$$

Thus the only contribution of I_1 plays the role, and we obtain (2.6) for $p > 2$.

Now $p = 2$. In comparison with the case $p = p^*$ for $D > 2$, not only the transition zone for the Bessel-cosine regimes (i.e. integrals I_2 and I_3) plays the role, but the transition zone for the cosine-Airy regimes (i.e. integrals I_4 and I_5) plays the role too.

For I_2 and I_3, substituting $p^* = 2$ in (4.7) and (4.9), we get

$$n^{-2\alpha}(I_2 + I_3) = \frac{3\ln n + \underline{\underline{O}}(1)}{4\pi^2 n}\,. \tag{4.13}$$

The second transition zone is $\big[(4-\varepsilon)n, 4n - n^{1/3+\theta}\big] \cup \big[4n - n^{1/3+\theta}, 4n - M\cdot n^{1/3}\big]$. For the oscillatory Plancherel-Rotach asymptotics (3.7) we have

$$\begin{gathered}\int\limits_{(4-\varepsilon)N}^{4N-n^{1/3+\theta}} \left[\frac{2\sin^2\left(\frac{1}{2}\sqrt{x(4N-x)} - 2N\arccos\sqrt{\frac{x}{4N}} + \frac{3\pi}{4}\right) + O\left(\frac{1}{\sqrt{Nx}}\right)}{\pi\sqrt{x(4N-x)}}\right]^2 dx = \\ = \frac{1}{\pi}\int\limits_0^{\pi} \sin^4\varphi\,d\varphi \cdot \int\limits_{4N}^{4N-n^{1/3+\theta}} \frac{4\,dx}{\pi x(4N-x)} = \frac{3}{8\pi^2 n}\left(\left(\tfrac{2}{3}-\theta\right)\ln n + \underline{\underline{O}}(1)\right)\,.\end{gathered} \tag{4.14}$$

For I_5 using (3.9) and asymptotics for the Airy function (see in [12])

$$\mathrm{Ai}^4\left(-\frac{t}{2^{4/3}}\right) \simeq \frac{(1+\sin(t^{3/2}/3))^2}{4\pi^2(t/2^{4/3})}\,, \quad t\to\infty\,,$$

we obtain

$$\begin{gathered}\int\limits_{4N-n^{(1/3+\theta)}}^{4n-Mn^{1/3}} (w^{1/2}(x)\widehat{L}_n^{(\alpha)}(x))^2 dx \simeq \int\limits_M^{n^\theta} \left[(2n)^{-2/3}\mathrm{Ai}^2\left(-\tfrac{t}{2^{4/3}}\right)\right]^2 n^{1/3} dt \simeq \\ \simeq \frac{1}{4\pi^2 n}\int\limits_0^{\pi}(1+\sin\varphi)^2 d\varphi \int\limits_M^{n^\theta}\frac{dt}{t} = \frac{3(\theta\ln n + \underline{\underline{O}}(1))}{8\pi^2 n}\,.\end{gathered} \tag{4.15}$$

Summing (4.14), (4.15) and (4.13) we get (2.6) for $p = 2$.

The remaining case is $p < 2$. Here we proceed in the same manner as for the case $p < p^*, D > 2$, and we get (4.10) for $\beta = 0$. Theorem is proved.

4.3. Proof of Theorem 3. We have $D \in [0,2)$, $\beta > 0$ for $p > 1$, therefore $p^* = 2$, as in the previous case.

We start with $p > 2$. Now the competition between I_1 and $I_6 + I_7$ becomes crucial. We already know for I_1 from (4.4) that

$$n^{-p\,\alpha} I_1 = C_B n^{-\beta-1} \; .$$

To get the asymptotics for $n^{-p\,\alpha}(I_6 + I_7)$ we substitute x^β in the left-hand side of (4.11)

$$\int\limits_{4n-Mn^{1/3}}^{4n+Mn^{1/3}} (w^{1/2}(x)\widehat{L}_n^{(\alpha)}(x))^{2p} x^\beta dx \simeq 2^{2\beta} n^{\frac{1-2p}{3}+\beta} C_A \; .$$

Now instead of inequality (4.12), we have for $D > 4/3$ the solution $p = \tilde{p}$ of the equation (where β is from (3))

$$-\beta - 1 = 1 - \frac{2p}{3} + \beta \Rightarrow \tilde{p} = \frac{-2+3D}{-4+3D} \; .$$

Thus we have obtained (2.11) and (2.10).

Now $p = 2$. In comparison with the previous cases, we have that the only the transition zone for the cosine-Airy regimes plays the role. Substituting x^β in the left-hand sides of (4.14) and (4.15) we arrive at (2.9), $p = 2$.

Finally for $p \in (0,2)$, we have

$$1 + \beta - p > -\beta - 1 \; ,$$

and

$$1 + \beta - p > \frac{1-2p}{3} + \beta \; .$$

Thus only the oscillatory integral I_3 gives the contribution to the asymptotics of $N_n(D, p)$, and from (4.10) we complete proof of (2.9).

Theorem is proved.

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