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РОССИЙСКАЯ АКАДЕМИЯ НАУК
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ИМЕНИ М.В. КЕЛДЫША

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SPECIAL SOLUTIONS TO CHAZY EQUATION

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We consider the classical Chazy equation, which is known to be integrable in hypergeometric functions. But this solution has remained purely existential and was never used numerically. We give explicit formulas for hypergeometric solutions in terms of initial data. We have found a special solution of this type in the upper half plane H with the same tessellation of H as that of the modular group. This allowed us to derive some new identities for the Eisenstein series. We constructed a special solution in the unit disk and gave an explicit description of singularities on its natural boundary. Finally, we found an explicit global solution to Chazy equation in elliptic and theta functions. The results have some applications to analytic number theory.

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Рассматривается классическое уравнение Чейзи, которое, как известно, интегрируемо в гипергеометрических функциях. Однако это решение оставалось чисто экзистенциальным и никогда численно не использовалось. Мы даем явные формулы для гипергеометрических решений в терминах начальных данных. Найдено специальное решение этого вида в верхней полуплоскости H, которое порождает такое же разбиение H как и модулярная группа. Это позволило вывести некоторые новые тождества для рядов Эйзенштейна. Построено специальное решение в единичном круге и дано явное описание особенностей на его естественной границе. Наконец, мы нашли явное глобальное решение уравнения Чейзи в эллиптических и тета функциях. Результаты имеют приложение к аналитической теории чисел.

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§ 1. Introduction

In recent years there is a marked increase in popularity of the so called Painlevé analysis. In this connection the classical Chazy equation is mentioned regularly. This equation takes the form

$$y''' - 2\,y\,y'' + 3\,(y')^2 = 0, \tag{1}$$

where $y = y(x)$, and prime denotes (here and below) the differentiation with respect to the appropriate variable.

Chazy introduced this equation in the framework of Painlevé theory as an attempt to generalize this theory to ODEs of higher order. Instead, a counterexample was produced: the equation (1) is integrable, but its movable singularities are not isolated poles but chains of essential singularities that form natural boundaries for all its solutions.

Ironically, the paper by Chazy [1] (actually, a short communication) was submitted for publication in 1909 by Painlevé himself.

Chazy equation had a long prehistory that dates back to works by Darboux and Halphen [2]. Far from being a mathematical curiosity, Eq (1) have found numerous applications to problems of mathematical physics (see a historical review and a list of references in [3, 4]).

Eq (1) bears a striking similarity to the Blasius equation (see [5]), which is not coincidental. They both can be derived from Navier-Stokes equations in the Prandtl boundary layer theory (see [4]). But unlike Blasius equation, which is not integrable, since it possesses rather complicated singularities (see [6, 7]), Chazy equation is known to be integrable in hypergeometric functions with the help of Schwarz triangle function (Sect. 2).

The hypergeometric solution given by Chazy, however, was never used numerically as far as we know, and thus remained, to all intents and purposes a purely existential one. We will demonstrate in Sect. 2 that this fact has a simple explanation: the computations are extremely cumbersome.

In fact, we believe that explicit formulas for hypergeometric solutions given in terms of initial data for Eq (1) were never produced before, although some claims to this effect can be found in [8, p. 339-340].

The solution given by Chazy is not unique. In Sect. 2 we find four types of hypergeometric solutions to Eq (1) as well as one such solution to Blasius equation.

In Sect. 3, we give explicit formulas for coefficients of hypergeometric parametrizations to Eq (1) in terms of initial data.

One of our solutions coincides with the Chazy solution. Another is equivalent to that with a simple change of variables. Yet another turns out to be a fancy parametrization of a special solution to Eq (1). But the fourth solution has a symmetry that we use for construction of a rather special solution to Eq (1) in the upper half-plane H.

The solution to Eq (1) given by Chazy is related to the Schwarz triangle function $s = s(x, \pi/3, \pi/2, 0)$ (see [9]). Our solution is related to the function $s = s(x, \pi/3, \pi/3, 0)$, which gives a tessellation of the upper half plane identical with the tessellation given by the classical modular group $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ (see [10]).

In Sect. 4, we construct a special solution to Eq (1) in H from purely symmetrical considerations (the H-solution). It turns out (Sect. 5) that the H-solution coincides with Eisenstein series solution to Chazy equation that was found fairly recently (see [3]). This allowed us to find some new identities for Eisenstein series and the sum of divisors function $\sigma(n)$.

In Sect. 6, we construct a special D-solution in the unit circle and describe asymptotic behavior of this solution at its natural boundary. It turns out that the singularities at the boundary mimic simple poles if we move to the boundary in a suitable direction. This information has numerical applications.

In Sect. 7, we construct a special solution to Chazy equation in elliptic integrals (L-solution). The L-solution is valid in the left half-plane, and thus, technically, is the same as the H-solution. However, unlike hypergeometric parametrizations, the elliptic one gives a global parametrization that allows to solve Eq (1) explicitly. In particular, in Sect. 8 we find explicit formulas for the natural boundary and its center.

Finally, in Sect. 9 we express our L-solution in theta functions and compare this representation with the solution found by Halphen (see [2, 3]). It turns out, that this solution gives a power generating function for the sum of divisors function $\sigma(n)$. The Sloane online encyclopedia of integer sequences (see [13]) reports this generating function as missing.

§ 2. Hypergeometric solutions to Chazy equation

In his short report [1], Chazy related the solution to Eq (1) to the solution of Darboux-Halphen system (see details in [8, pp. 335-337]).

The solution given by Chazy takes the form

$$x = \frac{p(s)}{q(s)}, \quad y = \frac{6}{q}\,\frac{dq}{dx}, \tag{2}$$

where p and q are two independent solutions to the Gauss' hypergeometric equation

$$s\,(1-s)\,f''(s) + (c - (a+b+1)\,s)\;f'(s) - a\,b\,f(s) = 0, \tag{3}$$

with specific values a, b, c that we give below.

The classical result by Schwarz (see [9, p. 206], [14, p. 129]) states that the function $x = p(s)/q(s)$ maps the upper half-plane in the variable s onto the curvilinear triangle with the (normalized) angles (i.e., divided by π)

$$\alpha = 1 - c, \quad \beta = c - a - b, \quad \gamma = a - b.$$

Since the inverse function $s = s(x, \alpha, \beta, \gamma)$ is a Schwarz triangle function defined only in the interior of a circle (obtained by analytical continuation from the initial triangle, i.e., the *master tile*), it follows that the general solution to Chazy equation has a (movable) natural boundary. We will dispute this statement in the part "it follows".

The necessity to demonstrate the natural boundary explicitly for Eq (1) was pointed out in [11], where a method was proposed to do so with the help of an asymptotic series that "breaks down" at the boundary (see Sect. 4).

Rather than use already known results, let us pretend that we do not know the solution (2) other than it should be given in a specific hypergeometric form.

Let us have two copies of Eq (3): one for $f(s) = p(s)$, and another for $f(s) = q(s)$. Then

$$p(s) = \ {}_2F_1(\frac{a, b}{c}; s) := F(a, b, c, s), \ q(s) = s^{1-c}\, F(a{-}c{+}1, b{-}c{+}1, 2{-}c, s) \quad (4)$$

are two linearly independent solutions to Eq (3). Here we use a shorthand $F(a, b, c, s)$ for the classical hypergeometric function.

We remark that a different second independent hypergeometric solution can be obtained with the help of $F(a, b, c, 1 - s)$, as it was done in [15, formula (29)], but it involves infinite series, while we need a closed form solution.

We seek solutions to Eq (1) in the form

$$x(s) = p(s)/q(s), \quad y(s) = d\, q(s)\, q'(s)\ \left(p'(s)\, q(s) - q'(s)\, p(s)\right)^{-1}. \qquad (5)$$

with the parameters $\{a, b, c, d\}$ unspecified and to be found.

The function $y(s)$ in (5) is obtained by the chain differentiation rule in (2) with the coefficient 6 replaced by unknown d. We note that the denominator in (5) is a Wronskian of the two independent solutions to Eq (3), and thus does not vanish.

We cannot reproduce intermediate formulas, which are very cumbersome, so we only describe here the necessary steps to be taken. Needless to say, this can only be done on a computer with a symbolic manipulation package (CAS). We used Maple.

First, we make the change of independent variable $x = p(s)/q(s)$ in Eq (1). Here p and q are purely symbolic, i.e., the formulas (4) are not used. We obtain an equation for $y = y(s)$ dependent on p and q.

Then we substitute $y(s)$ there as in (5), factor this expression, and cancel nonvanishing factors. We obtain an ODE dependent on p, q, and the parameters $\{a, b, c, d\}$.

Next, we differentiate twice both copies of hypergeometric equations for $p(s)$ and $q(s)$. Using these 2 systems of 3 linear equations, we express higher derivatives of p and q in terms of their first derivatives. (Of course, it needs to be done only for p, then we put q for p.)

Then we use these expressions to simplify and factor the previous ODE.

One of the factors would be (a power of) the Wronskian for p and q, and thus can be omitted. The other factor is a polynomial in s, $q(s)$, $q'(s)$, and $\{a, b, c, d\}$.

This polynomial should be identical zero. It can only happen if all coefficients at s, $q(s)$, and $q'(s)$, dependent on $\{a, b, c, d\}$, vanish.

This can be, as it turns out, if and only if $d = 6$, and $\{a, b, c\}$ satisfy the following equation

$$(2\,c-1)\,(3\,c-2)-(12\,c\,b+12\,a\,c-5\,b+1-5\,a-2\,c-24\,a\,b)\,s+6\,(a-b)^2\,s^2 = 0.$$

This equation has exactly 4 different solutions:

$$\begin{array}{llllll}
(s_0)\colon\ a = 0, & b = 0, & c = 1/2; & (\alpha = 1/2, & \beta = 1/2, & \gamma = 0); \\
(s_1)\colon\ a = 1/12, & b = 1/12, & c = 1/2; & (\alpha = 1/2, & \beta = 1/3, & \gamma = 0); \\
(s_2)\colon\ a = 1/12, & b = 1/12, & c = 2/3; & (\alpha = 1/3, & \beta = 1/2, & \gamma = 0); \\
(s_3)\colon\ a = 1/6, & b = 1/6, & c = 2/3; & (\alpha = 1/3, & \beta = 1/3, & \gamma = 0).
\end{array}$$

The solution (s_0) has the sum of angles equal to π, and thus cannot provide a tessellation (or tiling) of a hyperbolic plane (i.e., a circle). It can also be seen from the fact that, for this solution, the ratio $(y''(x))^3/(y'(x))^2 \equiv 2/3$. Thus we simply obtained a fancy parametrization of a special solution $y(x) = -6/(x + \text{const})$ to Eq (1).

The same situation occurs if we try to solve the Blasius equation

$$y''' + 2y\,y'' = 0 \tag{6}$$

in the same manner. Then we would obtain $d = -3/2$, $a = 0$, and b, c arbitrary. Thus we obtain Schwarz triangle functions with angles

$$\alpha = 1 - c, \quad \beta = c - b, \quad \gamma = -b; \qquad \alpha + \beta + \gamma = 1 - 2\,b.$$

The hymergeometric solution of (6) takes the form

$$
\begin{aligned}
&f_1(s) = 1, \quad f_2(s) = s^{1-c}\,F(1-c, 1-c+b, 2-c, s),\\
&p(s) = c_0\, f_1(s) + d_0\, f_2(s); \quad q(s) = a_0\, f_1(s) + b_0\, f_2(s);\\
&x(s) = p(s)/q(s), \quad y(s) = -\tfrac{2}{3}\,\frac{q(s)\,q'(s)}{x'(s)}, \quad a_0\, d_0 - b_0\, c_0 = 1,
\end{aligned}
\tag{7}
$$

where we used notation similar to that introduced in [8, p. 339].

It is not a trivial matter to verify that the "general" solution (7) is, in fact, a special solution $y(x) = 3/2/(x + \mathrm{const})$ to Blasius equation, whatever arbitrary constants $\{a_0,\, b_0,\, c_0,\, d_0\}$, and $\{b,\, c\}$ are taken. This is easily confirmed numerically. However, the sum of angles cannot be less than π unless $\gamma < 0$, and thus no natural boundaries for Blasius equation.

We also note that the similarity between Eq (1) and the Blasius equation (6) is not coincidental. They both can be derived as a reduction from Navier-Stokes equations in the Prandtl boundary layer theory (see [4, 5])).

We return to Chazy equation.

The solution (s_1) is the solution given by Chazy. The sum of angles here is less than π, and so the solution (s_1) gives a tiling of a circle corresponding to the action of the $(2, 3, \infty)$-triangle group (see Fig. 6.5.1 in [8, p. 338]).

However, it is by no means obvious or proven that the hypergeometric solution (s_1) gives a general solution to Chazy equation. It seems that this objection has completely escaped notice until it was raised implicitly in [8, p. 339]. We will deal with this matter in Sect. 3.

The solution (s_2) looks different from (s_1), but it is, in fact, the same one, which is obvious from the angles of the Schwarz triangle. These two solutions transform into each other under the substitution $s \to 1 - s$.

Finally, the solution (s_3) is a new one (in the form that we give in Sect. 3). It has 2 equal angles, and thus possesses a symmetry that the solution (s_1) is lacking. As we will see in Sect. 4, this simple fact has far reaching consequences. In particular, it relates the Schwarz triangle function $s = s(x, 1/3, 1/3, 0)$ to the classical modular group Γ in the upper half-plane, with the master tile being the fundamental region.

The Schwarz triangle function $s = s(x, 1/3, 1/3, 0)$ was already used in connection with Chazy equation in (see [15]), but not in the form suitable for computations.

§ 3. Explicit formulas for hypergeometric solutions

As it was stated in Sect. 2, the general hypergeometric solution to Eq (1) locally takes the form

$$x(s) = \frac{p(s)}{q(s)}, \quad y(s) = \frac{6\, q'(s)}{q(s)\, x'(s)}, \tag{8}$$

where

$$\begin{aligned} &p(s) = c_0\, f_1(s) + d_0\, f_2(s); \quad q(s) = a_0\, f_1(s) + b_0\, f_2(s); \quad a_0\, d_0 - b_0\, c_0 = 1, \\ &f_1(s) = F(a,b,c,s), \quad f_2(s) = s^{1-c}\, F(a-c+1, b-c+1, 2-c, s), \end{aligned} \tag{9}$$

and $\{a, b, c\}$ are taken either as in (s_1) (Chazy solution), or as in (s_3).

We note that a global solution to Chazy equation is completely determined by four parameters (initial values)

$$x = x_0, \quad y_0 = y(x_0), \quad y_1 = y'(x_0), \quad y_2 = y''(x_0),$$

where x_0, y_0, y_1, and y_2 are arbitrary complex numbers.

The normalizing condition for a_0, b_0, c_0, and d_0 means that, with an unknown initial value $s = s_0$, we have in total 4 values to be determined. However, the example with the Blasius equation makes it clear that we cannot simply assume that a highly nonlinear system of 4 equations (see below) would necessarily and always have a solution.

The values a_0, b_0, c_0, and d_0 were never computed explicitly as far as we know.

The first (and may be unique) attempt to compute these values was made in [8, pp. 339-340]. However, this effort proved to be only partially successful. Namely, the formulas in [8] are correct only up to the formula (6.5.73b), i.e., the initial value s_0 was found. Then the formulas (6.5.74) simplify to $a_0 = a_0$ and $b_0 = b_0$ (in our notation), and so cannot be used to compute these values.

The system of 4 (eventually algebraic) equations for the unknown values a_0, b_0, c_0, d_0, and s_0 is obtained by differentiation of Eq (8) for $y(s)$ twice with the use of Eq (9). We follow the same path in solving this system as outlined in [8, pp. 339-340].

First, we consider the case (s_1), i.e., the Chazy solution.

The two independent solutions to Eq (3) take the form

$$f_1(s) = F(1/12, 1/12, 1/2, s), \quad f_2(s) = s^{1/2}\, F(7/12, 7/12, 3/2, s).$$

We also need the Wronskian of these 2 solutions:

$$W = f_2'(s)\, f_1(s) - f_2(s)\, f_1'(s) = (1-s)^{-2/3}\, s^{-1/2}/2.$$

Using these formulas as well as normalizing condition $a_0\, d_0 - b_0\, c_0 = 1$, we obtain the (local) hypergeometric parametrization of the solution (s_1) to Eq (1)

$$\begin{aligned}
&p(s) = c_0\, f_1(s) + d_0\, f_2(s); \quad q(s) = a_0\, f_1(s) + b_0\, f_2(s);\\
&x(s) = p(s)/q(s),\\
&y(s) = 12\, s^{1/2}\, (1-s)^{2/3}\, q(s)\, q'(s),\\
&\tfrac{dy}{dx}(s) = \tfrac{1}{6}\, (1-s)^{1/3}\, (144\, s\, (1-s)\, q'(s)^2 + q(s)^2)\, q(s)^2,\\
&\tfrac{d^2y}{dx^2}(s) = \tfrac{1}{9}\, s^{1/2}\, (18\, (1-s)\, q'(s)\, (q(s)^2 + 48\, s\, q'(s)^2\, (1-s)) - q(s)^3)\, q(s)^3,
\end{aligned} \tag{10}$$

where the initial value s_0 is found by the formula

$$s_0 = \frac{1}{9}\, \frac{\left(y_0^3 + 9\, y_2 - 9\, y_0\, y_1\right)^2}{9\, y_2^2 + 2\, y_0^3\, y_2 - 18\, y_0\, y_1\, y_2 + 24\, y_1^3 - 3\, y_0^2\, y_1^2}, \tag{11}$$

which coincides with the formula (6.5.73b) in [8, p. 340].

However, a_0, b_0, c_0, and d_0 are found by these formulas

$$\begin{aligned}
&r = \epsilon\, (y_0^2 - 6\, y_1)^{1/4}/(1-s_0)^{1/12},\\
&a_0 = (-y_0\, f_2(s_0) + 12\, r^2\, s_0^{1/2}\, (1-s_0)^{2/3}\, f_2'(s_0))/r/6,\\
&b_0 = (\ \ y_0\, f_1(s_0) - 12\, r^2\, s_0^{1/2}\, (1-s_0)^{2/3}\, f_1'(s_0))/r/6,\\
&c_0 = a_0\, x_0 - f_2(s_0)/q(s_0), \quad d_0 = (1 + b_0\, c_0)/a_0.
\end{aligned} \tag{12}$$

where $\epsilon^4 = -1$, and the value of the root is taken such that $\frac{d^2y}{dx^2}(s_0) = y_2$. This last condition is satisfied for two roots of the equation $\epsilon^4 = -1$, which give a_0, b_0, c_0, and d_0 that differ in sign (i.e., equivalent to each other).

We note that some root of unity is necessary to determine by initial values. This situation is typical in the theory of modular forms, and we will encounter it in the following sections.

The solution given above is valid only in the master tile, i.e., it is, in a sense, a local one (since we cannot accept analytical continuation as a closed form solution). However, hypergeometric functions can be computed outside their radius of convergence in the upper/lower s-half-plane. Thus the above formulas may be used numerically in the adjacent tile, i.e., the triangle symmetrical to the master tile with respect to the arc $s \in [0, 1]$.

We give a numerical example that will help to verify these formulas.

Let $x_0 = 0$, $y_0 = 0.1$, $y_1 = 0.2$, and $y_2 = 0.3$. Then we find

$$\begin{aligned}
&s_0 = 0.790418749844538,\\
&a_0 = i\, 1.224789099142053, \quad b_0 = -i\, 0.048042121436388,\\
&c_0 = i\, 1.002993933669566, \quad d_0 = -i\, 0.855809344723555.
\end{aligned} \tag{13}$$

We take $s = 0.2$ and find by the formulas (10):

$$\begin{aligned}
&x(s) = 0.500806544488046, \quad y(s) = 0.235767617269862,\\
&\tfrac{dy}{dx}(s) = 0.336470151854129, \quad \tfrac{d^2y}{dx^2}(s) = 0.230694566409036.
\end{aligned} \tag{14}$$

These values can also be obtained by numerical integration of Eq (1) from $x = 0$ to $x = x(s = 0.2)$. An additional test is to compute a_0, b_0, c_0, and d_0 by (14) as initial values. They are the same, since we are in the same master tile.

As it was noted in [8, p. 340], the equation for $x(s)$ can be written as

$$x(s) = \frac{c_0 + d_0\,\tau}{a_0 + b_0\,\tau}, \qquad \tau = \frac{f_2(s)}{f_1(s)},$$

i.e., the maps $x(s)$ and $\tau(s)$ are related by Möbius transformation. Since the center of the orthogonal circle (alias natural boundary) for $\tau(s)$ is located at $s = 0$, i.e., at $\tau = 0$, it was concluded in [8] that the center of the orthogonal circle for the solution to Chazy equation is at $x(0) = c_0/a_0$. However, this is not true. The point $x(0) = c_0/a_0$ is only the vertex of the master tile with the angle $\pi/2$. It can be located anywhere within the natural boundary where these points go under the tessellation (i.e., under the action of the $(2, 3, \infty)$-triangle group). Thus the assertion about the radius of the orthogonal circle (formula (6.5.75) in [8]) is also not true.

On the other hand, we can always obtain a point on the natural boundary taking the limit $s \to \infty$, i.e., moving to the vertex with zero angle (since all such vertices are on the orthogonal circle, see [9]). And we can obtain a second point on the natural boundary taking the limit $s \to \infty$ in some other direction. Unfortunately, we cannot produce a third point in this manner.

In Sect. 6 we give explicit formulas that will illustrate these two observations.

Now we consider the case (s_3), that we believe is more useful, since it will give an access to the classical modular group Γ (see [10]).

The two independent solutions to Eq (3) take the form

$$f_1(s) = F(1/6, 1/6, 2/3, s), \quad f_2(s) = s^{1/3}\,F(1/2, 1/2, 4/3, s).$$

The Wronskian of these 2 solutions:

$$W = f_2'(s)\,f_1(s) - f_2(s)\,f_1'(s) = (1-s)^{-2/3}\,s^{-2/3}/3.$$

Using these formulas as well as normalizing condition $a_0\,d_0 - b_0\,c_0 = 1$, we obtain the (local) hypergeometric parametrization of the solution (s_3) to Eq (1)

$$\begin{aligned}
&p(s) = c_0\,f_1(s) + d_0\,f_2(s); \quad q(s) = a_0\,f_1(s) + b_0\,f_2(s);\\
&x(s) = p(s)/q(s),\\
&y(s) = 18\,s^{2/3}\,(1-s)^{2/3}\,q(s)\,q'(s),\\
&\tfrac{dy}{dx}(s) = \tfrac{3}{2}\,s^{1/3}\,(1-s)^{1/3}\,(36\,s\,(1-s)\,q'(s)^2 + q(s)^2)\,q(s)^2,\\
&\tfrac{d^2y}{dx^2}(s) = \tfrac{3}{2}\,(18\,s\,(1-s)\,q'(s)\,(12\,s\,(1-s)\,q'(s)^2 + q(s)^2) + q(s)^3\,(1-2\,s))\,q(s)^3,
\end{aligned} \tag{15}$$

where the initial value s_0 is found by the formula

$$\begin{aligned} s_0 = &\tfrac{1}{6}\,(6\,y_2\,y_0^3 - 9\,y_1^2\,y_0^2 - 54\,y_1\,y_2\,y_0 + 27\,y_2^2 + 72\,y_1^3 \\ &+ \lambda\sqrt{G}\,(9\,y_2 - 9\,y_0\,y_1 + y_0^3))/G, \quad \lambda^2 = 1, \\ G = &\;2\,y_2\,y_0^3 + 24\,y_1^3 - 3\,y_1^2\,y_0^2 + 9\,y_2^2 - 18\,y_0\,y_1\,y_2. \end{aligned} \tag{16}$$

But the sum of these two values s_0 in (16) is 1, so they give the same master tile, only flipped over with respect to the midsection passing through $s_0 = 1/2$. Thus, incidentally, this parametrization composes one master tile from the two of the (s_1)-type connecting them in such a way that two adjacent vertices with the angles $\pi/2$ merge together at the middle point $s_0 = 1/2$.

However, a_0, b_0, c_0, and d_0 are different for these two values of s_0, although found by the same formulas:

$$\begin{aligned} r &= \epsilon\,(1/3)^{1/2}\,(y_0^2 - 6\,y_1)^{1/4}/(1-s_0)^{1/12}/(s_0)^{1/12}, \\ a_0 &= (-y_0\,f_2(s_0) + 18\,r^2\,s_0^{2/3}\,(1-s_0)^{2/3}\,f_2'(s_0))/r/6, \\ b_0 &= (\;\;y_0\,f_1(s_0) - 18\,r^2\,s_0^{2/3}\,(1-s_0)^{2/3}\,f_1'(s_0))/r/6, \\ c_0 &= a_0\,x_0 - f_2(s_0)/q(s_0), \quad d_0 = (1 + b_0\,c_0)/a_0. \end{aligned} \tag{17}$$

where $\epsilon^4 = -1$, and the value of the root is taken such that $\frac{d^2y}{dx^2}(s_0) = y_2$. This last condition is satisfied (as before) for 2 roots of the equation $\epsilon^4 = -1$, which give a_0, b_0, c_0, and d_0 that differ in sign (i.e., equivalent to each other).

For numerical test, we take as before $x_0 = 0$, $y_0 = 0.1$, $y_1 = 0.2$, and $y_2 = 0.3$. Then

$$\begin{aligned} &s_0 = 0.944527487857764, \quad (\lambda = 1), \\ &a_0 = i\,0.898963093828840, \quad b_0 = -i\,0.144742025729118, \\ &c_0 = i\,1.863635097940293, \quad d_0 = -i\,1.412456560243969, \end{aligned} \tag{18}$$

and the second bunch of constants is

$$\begin{aligned} &s_0 = 0.055472512142235, \quad (\lambda = -1), \\ &a_0 = -0.766279841777836, \quad b_0 = -0.007530939525835, \\ &c_0 = \;\;0.499948432481262, \quad d_0 = -1.300092817628603. \end{aligned} \tag{19}$$

We will use these values in Sect. 6.

§ 4. A special solution to Chazy equation

As it was implied in [11], the natural boundary for solutions to Chazy equation (and possibly for some other equations with the same phenomenon) needs to be seen or demonstrated in some form other than relying on hypergeometric representation. To this effect, a local asymptotic method of seeing the natural boundary was suggested in [11] (see also [12, pp. 196-198]). It was stated that the natural boundary is located where a certain asymptotic expansion brakes down, i.e., at the boundary of its convergence.

However, this statement cannot be true, since "braking down" of the expansion means just that, i.e., the boundary of its convergence. This fact by no means implies impossibility of analytical continuation of the expansion beyond the boundary of its convergence. The fact that in this case analytical continuation is impossible is due to the existence of the natural boundary, which still remains to be seen.

In this section we produce the natural boundary quite naturally, by producing a special solution to Chazy equation.

We have seen in Sect. 3 that computation of parameters $\{a_0, b_0, c_0, d_0\}$ for the hypergeometric solution from the initial values of a solution to Chazy equation is a laborious task. It turns out that the reverse process is much easier.

The type of symmetry that the solution (s_3) possesses suggests that there is a connection with the modular group Γ in the upper half-plane H.

We use standard notation and assume the properties of the modular group Γ to be known. They are studied in many books on modular forms and number theory (see [10]). Here and below, we reserve t as an independent variable in $H = \{t : \mathrm{Im}\,(t) > 0\}$; and we use a copy of Chazy equation (1) for the variable $w(t)$ in H.

We recall that the fundamental region of Γ is defined as

$$W = \{t \in H : \ |t| > 1, \ |\mathrm{Re}(t)| < 1/2\} \tag{20}$$

with the left half of the boundary of W attached (see Fig. 1 in [16, p. 6]).

It is a simple observation that the region W is, in fact, a hyperbolic triangle with the angles $\pi/3$, $\pi/3$, and 0; and the action of the modular group in H is identical with the action of the $(3, 3, \infty)$-triangle group in a special circle, namely, the upper half-plane H. Thus the solution (s_3) generates the same tessellation in H as the modular group Γ (see Fig. 1).

Let us construct a special solution of (s_3)-type that fits the fundamental region W, and that we call the *H-solution*.

For this, we use only three properties of the H-solution that follow directly from the symmetry that we assign to it. Namely, the point $s = 0$ corresponds to the left corner of W, i.e., $t = -1/2 + i\sqrt{3}/2$; the point $s = 1$ goes to the right corner of W, i.e., $t = 1/2 + i\sqrt{3}/2$; and the point $s = 1/2$ is placed at the middle, i.e., corresponds to the point $t = i$.

We could have placed the point $s = 0$ instead of $s = 1$ and vice versa, but otherwise there is no choice, since the solution must fit the triangle W.

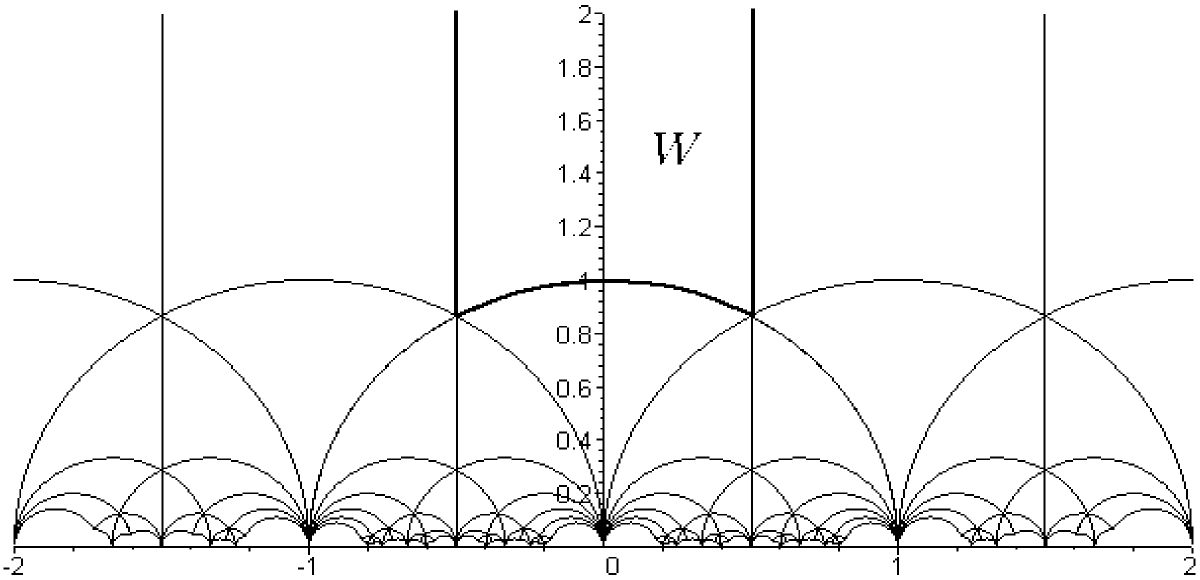

Fig 1. Tessellation of H by the modular group.

Thus we have 3 linear equations plus normalizing condition for the parameters a_0, b_0, c_0, d_0 of the (s_3)-type hypergeometric solution. Here we use only the expression for $x = x(s)$ in (15) for these equations.

This system has exactly 2 solutions which are equivalent, i.e., they differ in sign. The system is solved algebraically. However, the solution is expressed in values of hypergeometric functions at the arguments $s = 1/2$ and $s = 1$. The solution simplifies with some nontrivial hypergeometric and Γ function identities. They can be found in standard reference books and online databases. We omit these technical details and give the final result

$$\begin{aligned} a_0 &= \tfrac{1}{9}\,\frac{2^{1/6}\,\pi^{3/2}\,(3^{1/2}+3+(3^{1/2}-3)\,i)}{\Gamma(2/3)^3}, \quad c_0 = i\,\bar{a}_0, \\ b_0 &= \tfrac{3}{16}\,\frac{2^{5/6}\,\Gamma(2/3)^3\,(1-3^{1/2}+(1+3^{1/2})\,i)}{\pi^{3/2}}, \quad d_0 = i\,\bar{b}_0, \\ a_0 &\approx 1.323530303838272 - 0.354638876071583\,i, \\ b_0 &\approx -0.109054977342659 + 0.406998716261079\,i. \end{aligned} \tag{21}$$

Knowing parameters (21), and using formulas (15), we find initial values for the H-solution in original variable t:

$$\begin{aligned} & t = i, \quad w(t) = 3\,i, \\ & w'(t) = \tfrac{1}{128}\,\frac{\Gamma(1/4)^8}{\pi^4} - \tfrac{3}{2} \approx 0.894633974746304, \\ & w''(t) = i\left(\tfrac{3}{128}\,\frac{\Gamma(1/4)^8}{\pi^4} - \tfrac{3}{2}\right) \approx i\,5.683901924238913. \end{aligned} \tag{22}$$

The point $s = 1/2$ (i.e., $t = i$) is the (appointed) center of the orthogonal circle H, that will be better seen in Sect. 5. The point $s = 0$ in the left corner of W, i.e., one of the vertices of the master tile W, is

$$
\begin{aligned}
&t = -1/2 + i\sqrt{3}/2, \quad w(t) = 2\,i\sqrt{3}, \quad w'(t) = -2, \\
&w''(t) = -i\left(\tfrac{4}{\sqrt{3}} + \tfrac{3}{512}\,\frac{\Gamma(1/3)^{18}}{\pi^9}\right) \approx -12.236718925289588\,i.
\end{aligned} \tag{23}
$$

The initial values for the point $s = 1$ in the right corner of W are identical with (23) except $t = 1/2 + i\sqrt{3}/2$.

Now we can verify directly that (22) and (23) are initial values of the same H-solution to (a w-copy of) Chazy equation using analytical continuation in variable t. This can be done as follows

$$
\begin{aligned}
&w(t) = \sum_{n=0}^{\infty} g_n\,(t-t_0)^n, \quad g_0 = w(t_0),\ g_1 = w'(t_0),\ g_2 = w''(t_0)/2, \\
&n\,(n-1)\,(n-2)\,g_n = \left(\sum_{m=1}^{n-1} m\,(5\,m - 3\,n + 1)\,g_{n-m-1}\,g_m\right), \quad n > 2.
\end{aligned} \tag{24}
$$

Analytical continuation of the type (24) can be made very accurate if sufficiently small steps $|t - t_0|$ are taken. For example, taking $N + 1$ Taylor coefficients is equivalent to numerical integration with a Runge-Kutta algorithm of the N-th order. We remark that there are no standard routines for numerical integration of ODEs in complex variable (as far as we know).

Let us consider some properties of the H-solution.

The upper half-plane in hypergeometric variable s is mapped by the map $x(s) = p(s)/q(s)$ in (15) onto the master tile W. It follows that the lower s-half-plane is mapped onto the image of W in H under one of the generators of the modular group, namely, $S : t \to -1/t$. (The second generator is $T : t \to t + 1$, see [16, p. 6].) The image $S(W)$ is the hyperbolic triangle below W (see Fig. 1).

One can verify that the line $1/2 + i\,s$, $s \to +\infty$ in the upper s-half-plane corresponds to the line $t \to +\infty\,i$ in H. In addition, $w(t) \to i\,\pi$, $w'(t) \to 0$, and $w''(t) \to 0$.

Similarly, the line $1/2 + i\,s$, $s \to -\infty$ in the lower s-half-plane corresponds to the line $t \to +i\,0$ in H. In addition, $w(t) \to -\infty\,i$, $w'(t) \to +\infty$, and $w''(t) \to +\infty\,i$.

Both limits above belong to the natural boundary of the H-solution. Thus a solution to Chazy equation can take finite values on its natural boundary. The point at the origin in H is usually called an essential singularity. However one should keep in mind that it is a figure of speech, i.e., this point is not an isolated essential singularity in a usual sense.

Since the upper half-plane H is covered by images of W obtained by the action of the modular group Γ, which is identical with the action of $(3, 3, \infty)$-triangle group in H, we see (Fig. 1) that the natural boundary here is the real axis.

Thus the power series (24) developed at $t_0 = i$ converges at the corners of the master tile W, since the distance from (22) to (23) is $(3^{1/2} - 1)/2^{1/2} \approx 0.517638 < 1$.

In Sect. 8 we will demonstrate that the H-solution is, in a sense, a general solution to Chazy equation.

Now we construct the same H-solution in another manner.

Let a solution $w(t)$ to Chazy equation be defined in H and take 2 values at the two corners of its master tile W:

$$w(\mp 1/2 + i\sqrt{3}/2) = 2\,i\sqrt{3}.$$

Then $w(t)$ coincides with the H-solution.

The proof is to demonstrate that an (s_3)-type hypergeometric solution is uniquely restored with the given data.

For this, we use $y(s) = w(s)$ in (15) first. For $s = 0$, we obtain the equation

$$6\,a_0\,b_0 = 2\,i\sqrt{3}.$$

For $s = 1$, we obtain the equation

$$10\,a_0\,b_0 + \frac{9}{4}\,a_0^2\,\frac{\Gamma(2/3)^6\,3^{1/2}\,2^{2/3}}{\pi^3} + \frac{32}{27}\,b_0^2\,\frac{\pi^3\,2^{1/3}\,3^{1/2}}{\Gamma(2/3)^6} = 2\,i\sqrt{3}.$$

Note that we cannot just substitute $s = 1$ in (15) but need to use hypergeometric transformation formulas to resolve indeterminate forms $0\,\infty$ (see [17]).

Next we write 2 equations for $x(0)$ and $x(1)$, which take the form

$$\frac{c_0}{a_0} = -1/2 + i\,\sqrt{3}/2, \qquad \frac{9\,c_0\,3^{1/2}\,\Gamma(2/3)^6 + 4\,d_0\,\pi^3\,2^{1/3}}{9\,a_0\,3^{1/2}\,\Gamma(2/3)^6 + 4\,b_0\,\pi^3\,2^{1/3}} = 1/2 + i\,\sqrt{3}/2.$$

The above system of 4 equations has 4 solutions that split into 2 pairs of equivalent ones. Only one pair satisfies the normalizing condition and gives (21).

These computations can be generalized as follows.

Theorem 1. *An (s_3)-type solution to Chazy equation is completely determined by the two values that the solution takes at the two corners of its master tile with the angles $\pi/3$. One of the corners can be placed anywhere in the complex plane, then the other corner is determined uniquely.*

Proof is a repetition of the above computation. Namely, two equations for the coefficients a_0 and b_0 are $y(s = 0) = y_0$ and $y(s = 1) = y_1$. They give a biquadratic algebraic equation. Then two equations for the coefficients c_0 and d_0 are the normalizing condition and $x(s = 0) = x_0$ (or $x(s = 1) = x_1$). These equations always have 2 equivalent solutions.

An analogous theorem is true for an (s_1)-type solution, i.e., for the classical Chazy solution.

§ 5. Eisenstein series solution to Chazy equation

For the past couple of decades or so the works of Ramanujan enjoy ever increasing popularity. This is, no doubt, in no small part due to the publication of his "Lost notebooks" and appearance of new editions of his other publications.

Thus it was found fairly recently that Eq (1) has a very special solution in the form of an Eisenstein series.

Namely, consider a (normalized) Eisenstein series

$$P(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n)\, q^{2\,n}, \tag{25}$$

where we use standard notation for the *nome* $q = \exp(i\,\pi\,t)$ and the sum of divisors function $\sigma(n)$ (see [10, 16]).

In addition to $P(q)$, Ramanujan introduced two other Eisenstein series $Q(q)$ and $R(q)$, and gave a system of 3 differential equations of the first order that the Eisenstein series P, Q, R satisfy (see [18, 19]) (actually, Ramanujan used the nome $q = \exp(2\,i\,\pi\,t)$).

If we eliminate Q, R from this system, we obtain an equation of the 3-rd order for $P(q)$:

$$\delta^{(3)} P(q) - 2\,P(q)\,\delta^{(2)} P(q) + 3\,(\delta P(q))^2 = 0, \quad \delta := q\,\frac{d}{dq}, \tag{26}$$

which is remarkably similar to Chazy equation. We call it the *mock-Chazy equation.*

Now, if we make the change of variables in w-copy of Eq (1):

$$q = \exp(i\,\pi\,t), \quad w(q) = i\,\pi\,P(q),$$

then we obtain Eq (26), as it is easy to verify.

Thus a special solution to Chazy equation in the upper half-plane H takes the form

$$w(t) = i\,\pi\left(1 - 24 \sum_{n=1}^{\infty} \sigma(n)\,\exp(2\,n\,i\,\pi\,t)\right), \tag{27}$$

which is usually called a Fourier expansion of the Eisenstein series P.

We remark that Ramanujan derived his equations for the Eisenstein series P, Q, R in [18] with highly non-trivial but algebraic means. Up to date, this is the only proof that (27) is a solution to Chazy equation.

Here we give another proof of this fact based on rather advanced but elementary properties of the function $\sigma(n)$.

We need to prove that the series (25) is a formal solution to Eq (26).

Let us pretend that we do not know the coefficients $\sigma(n)$, i.e., that they are simply placeholders. We substitute a formal power series (25) into Eq (26) and collect similar terms. We obtain $\sigma(1) = 1$ and the following recurrence relation for the unknown coefficients $\sigma(n)$:

$$n^2\,(n-1)\,\sigma(n) = 12\left(\sum_{m=1}^{n-1} m\,(3\,n - 5\,m)\,\sigma(n-m)\,\sigma(m)\right),\ n > 1. \tag{28}$$

Thus we need to prove that the above convolution sum gives the sum of divisors function.

The relation (28) is, in fact, already known (see [15]). But it was derived in [15] from the fact that (27) is a solution to Chazy equation. We, on the other hand, use it to prove the fact. The relation (28) itself can be derived from the formulas (3.15) and (3.16) in [20]. We take $3 \times$ (3.15) minus $2 \times$ (3.16) and obtain the identity (28) by eliminating the function $\sigma_3(n)$ from both identities (3.15) and (3.16) in [20].

In Sect. 7 we give yet another proof that (27) is a solution to Chazy equation.

An immediate consequence is the fact that the series in Eq (27) converges for $\mathrm{Im}\,(t) > 0$, i.e., in the upper half plane H, which follows from the well known estimates of the growth of the function $\sigma(n)$ (see [21] and references there).

Thus (27) is a global solution to Eq (1) defined in a special circle H. Since the axis $\mathbb{R}$ is a natural boundary for such solutions, we deduce that the analytical function $P(q)$ has the unit circle $|q| = 1$ as a natural boundary.

Theorem 2. *The solution* (27) *is the* (s_3)*-type* H*-solution with the constants* (21).

Proof. Let us consider how a solution to Eq (1) changes under the Möbius transformation, i.e., with the change of independent variable

$$x = \frac{a\,t + b}{c\,t + d}, \quad a\,d - b\,c \neq 0, \tag{29}$$

where $a\,d - b\,c = 1$ usually but not necessarily. This was already done in the paper [1] by Chazy himself. We have

$$y(x) = (c\,t + d)\,\frac{(c\,t + d)\,w(t) + 6\,c}{a\,d - b\,c}. \tag{30}$$

We will also need formulas for $y'(x)$ and $y''(x)$ under the transformation (29). But since they can be obtained by the chain differentiation rule, we omit them.

Now we consider a special transformation (29) with $a = 0$, $b = -1$, $c = 1$, and $d = 0$, i.e., the generator $S : t \to -1/t$ of the modular group. Application of (30) to (27) gives another solution valid in H:

$$w(t) = -\frac{6}{t} + \frac{i\,\pi}{t^2}\left(1 - 24\sum_{n=1}^{\infty}\sigma(n)\,\exp\left(\frac{-2\,n\,i\,\pi}{t}\right)\right). \tag{31}$$

Note that we use the same notation $w(t)$ in the lhs(31) as in lhs(27) since we intend to prove that they are identical.

The arc $t = [-1/2 + i\sqrt{3}/2, \ldots, 1/2 + i\sqrt{3}/2]$, $|t| = 1$ is invariant under S but flipped over with respect to the middle point $t = i$. Since (27) is invariant under the action of the second generator of the modular group, namely, under $T : t \to t + 1$, i.e., the function $w(t)$ in (27) is 1-periodic, it follows that the function $w(t)$ in (31) takes the same values at both ends of the arc, which are the corners of the master tile W.

If we substitute the ends of the arc $t = \mp 1/2 + i\sqrt{3}/2$ in (27), we obtain the same formula (as expected); but if we substitute them into (31), we obtain two different expressions that must be equal. Hence we obtain an equation that allows to evaluate the following sum

$$\sum_{n=1}^{\infty}(-1)^n\,\sigma(n)\,e^{-n\,\pi\,\sqrt{3}} = \frac{(\pi\,\sqrt{3} - 6)\,\sqrt{3}}{72\pi} \approx -0.004277407951816. \tag{32}$$

The same sum is involved in the expression for $w(t)$ in (27) at $t = \mp 1/2 + i\sqrt{3}/2$. Thus we obtain (almost gratis) $w(\mp 1/2 + i\sqrt{3}/2) = 2\,i\,\sqrt{3}$ in (27), and hence in (31).

Now we observe that a global solution (27) in H is completely determined by its values in the master tile W, since transformations (29) of the modular group will cover the whole half-plane H by images of W (see Fig 1). Here we implicitly use the fact that transformations of the modular group Γ are the actions of the $(3, 3, \infty)$-triangle group in H.

Since we have a second representation of such a solution in H, namely, a (s_3)-type hypergeometric one, we have the statement of Theorem 2 by Theorem 1. **End of proof**.

As an immediate corollary, we see that solutions (27) and (31) are two representations of the same solution, since they are both defined in H and have the same values at the corners of the master tile W. Thus we obtain the famous identity for Eisenstein series, i.e., rhs(27)=rhs(31). Using (30), this identity can be written in the following form

$$w(S(t)) = t^2\,w(t) + 6\,t, \tag{33}$$

where $G_2(t) = -i\,\frac{\pi}{3}\,w(t)$ (see formula (50) in [10, p. 69], and [16, p. 19]). Of course, t in (33) can be replaced by any $g(t)$, where $g \in \Gamma$.

Collecting previously stated facts, we have given a purely algebraic proof of this identity (with the help of some well know hypergeometric ones). A direct proof of (33) outlined in [10] as a series of exercises requires rather advanced complex analysis.

Let us exploit the trick we used in the proof of Theorem 2 further.

If we substitute the ends of the arc $t = \mp 1/2 + i\sqrt{3}/2$ into the first derivative of (31), we obtain two different expressions that must be equal. Thus we can evaluate the following sum

$$\sum_{n=1}^{\infty} (-1)^n\, n\, \sigma(n)\, e^{-n\pi\sqrt{3}} = -\frac{1}{24\pi^2} \approx -0.004221715985097. \tag{34}$$

This sum is involved in the expression for $w'(t)$ in (27) at $t = \mp 1/2 + i\sqrt{3}/2$. Thus we obtain $w'(\mp 1/2 + i\sqrt{3}/2) = -2$ in (27) as well as in (31). This was quite expected by Theorem 2 and (23). But if we use this trick yet again for the second derivative $w''(t)$ in (31), we gain nothing new, since the sum we need is cancelled.

So we need hypergeometric representation of (27) to evaluate the following sum

$$\sum_{n=1}^{\infty} (-1)^n\, n^2\, \sigma(n)\, e^{-n\pi\sqrt{3}} = -\frac{1}{96\,\pi^3}\left(\frac{4}{\sqrt{3}} + \frac{3}{512}\,\frac{\Gamma(1/3)^{18}}{\pi^9}\right), \tag{35}$$

which is $\approx -0.004110968351753$.

Since we can differentiate Eq (1) indefinitely, we obtain the following

Theorem 3. *The sums*

$$\sum_{n=1}^{\infty} (-1)^n\, n^k\, \sigma(n)\, e^{-n\pi\sqrt{3}}, \quad k \in \mathbb{N}$$

are rational functions of $\sqrt{3}$*,* π*, and* $\Gamma(1/3)$ *with integer coefficients.*

Now we explore the values of (27) and its derivatives at the middle point $t = i$.

If we substitute $t = i$ in (33), we obtain a rather famous identity

$$P(e^{-\pi}) = 1 - 24\sum_{n=1}^{\infty} \sigma(n)\, e^{-2\pi n} = \frac{3}{\pi}. \tag{36}$$

However, if we differentiate (31) or, which is the same, (33), we gain nothing new, since the sum we need is cancelled (as was before, only then we were able to move a step further). So we use (22) to obtain the following evaluations:

$$\sum_{n=1}^{\infty} n\,\sigma(n)\, e^{-2\pi n} = \frac{1}{48\,\pi^2}\left(\frac{1}{128}\,\frac{\Gamma(1/4)^8}{\pi^4} - \frac{3}{2}\right) \approx 0.001888445275998. \tag{37}$$

and

$$\sum_{n=1}^{\infty} n^2\,\sigma(n)\,e^{-2\pi n} = \frac{1}{96\,\pi^3}\left(\frac{3}{128}\frac{\Gamma(1/4)^8}{\pi^4} - \frac{3}{2}\right) \approx 0.001909526652338. \qquad (38)$$

Thus we obtain

Theorem 4. *The sums*

$$\sum_{n=1}^{\infty} n^k\,\sigma(n)\,e^{-2\pi n}, \quad k \in \mathbb{N}$$

are rational functions of π *and* $\Gamma(1/4)$ *with integer coefficients.*

Theorems 3 and 4 are clearly in connection with algebraic independence of the constants π, $\exp(\pi)$, and $\Gamma(1/3)$ (respectively, $\Gamma(1/4)$) (see [23, p. 27]).

Using (33) and (22) (or (23)), we can obtain a plethora of similar identities that will appear under the action of the modular group.

§ 6. A solution to Chazy equation in a circle

The H-solution constructed in Sect. 4 from purely symmetrical considerations turned out to be the Eisenstein series solution (27). But the upper half-plane is a special circle which is unbounded. Let us construct a similar solution to Eq (1) in the unit circle D. The simplest way to do this is to apply a suitable Möbius transformation (29) to the H-solution.

We again use symmetry as a primary consideration. Let $t = i$ be mapped into the center of D, i.e., into the origin $x = 0$. Let the point $t = \infty$ be mapped into $x = +i$, and the point $t = 0$ into $x = -i$. Then the Möbius transformation is determined uniquely. We have

$$x = \frac{t - i}{-i\,t + 1}. \qquad (39)$$

Applying (39) to the H-solution, we obtain a solution to Eq (1) in the unit circle. We call it the *D-solution.*

The corners of the master tile W (alias fundamental region in H) are mapped into $x = \mp(2 - \sqrt{3})$, and $x = i$. It is a simple and straightforward calculation to transfer the hypergeometric data (21) to the master tile in D (see Fig. 2).

Let us consider all possible forms of the D-solution as a (transformed) Eisenstein series.

As is well known, any member of the modular group Γ is given by the Möbius transformation

$$M : t \to \frac{l\,t + m}{k\,t + j}, \quad l\,j - m\,k = 1, \quad l, j, m, k \in \mathbb{Z}. \qquad (40)$$

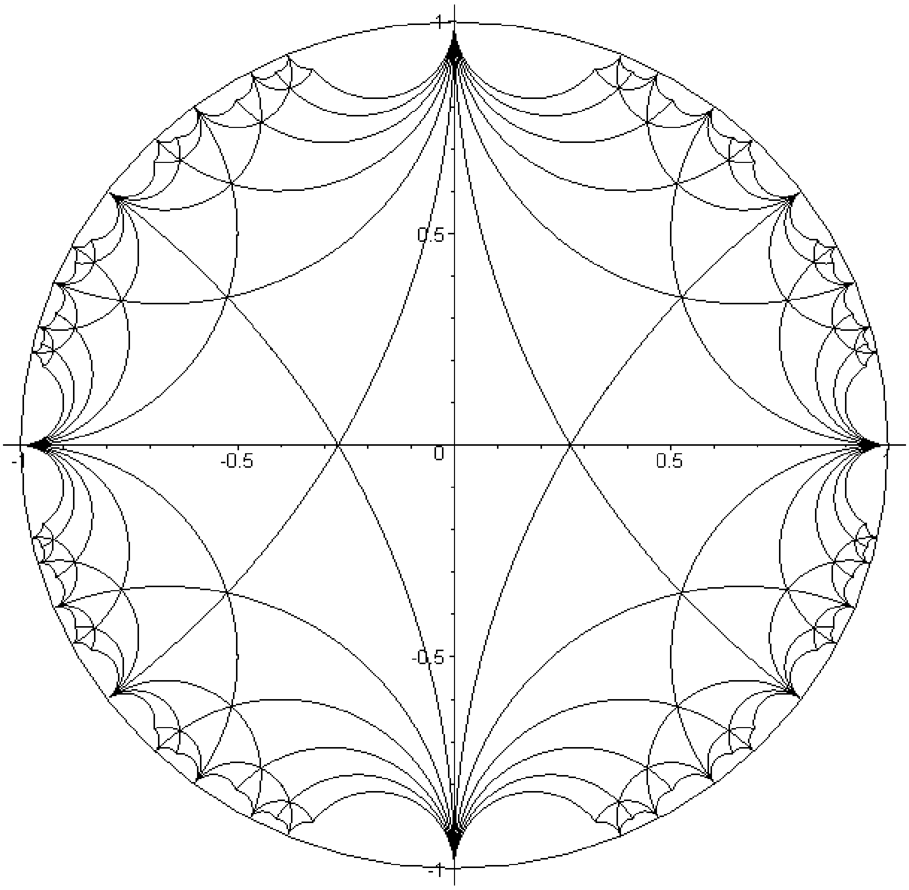

Fig 2. Tessellation of D by $(3,3,\infty)$-triangle group.

Thus the most general form of the D-solution is given by (39) applied to the H-solution, but where t is replaced by $M(t)$ as in (40). After some simplifications, we obtain

$$y(x) = -\frac{6}{x-\beta} - \frac{\delta\left(1 - 24\sum\limits_{n=1}^{\infty}\sigma(n)\,\exp\left(2\,n\,\gamma\,\dfrac{x-\alpha}{x-\beta}\right)\right)}{(x-\beta)^2}, \tag{41}$$

where

$$\alpha = \frac{i\,m + j}{m + i\,j}, \quad \beta = \frac{i\,l + k}{l + i\,k}, \quad \gamma = \pi\,\frac{j - i\,m}{l + i\,k}, \quad \delta = \frac{2\,\pi\,i}{(l + i\,k)^2}. \tag{42}$$

We stress that (41) is one and the same solution valid in the whole unit circle D. This follows from the fact that the series in (41) converge in D by construction, and perforce represent one and the same analytical function in D with the natural boundary $|x| = 1$.

Initial values of the D-solution at the origin are obtained from (22) by (39):

$$y(0) = 0, \quad \frac{dy}{dx}(0) = \frac{1}{32}\,\frac{\Gamma(1/4)^8}{\pi^4}, \quad \frac{d^2y}{dx^2}(0) = 0.$$

These values together with the recurrence relation (24) determine the D-solution uniquely as a power series in x convergent in D. However, the series (41) present a rather unusual phenomenon, i.e., asymptotic series developed at boundary (and singular) points of D and convergent in the whole D (some of them rapidly). Numerical experiments reveal that they are more useful from the computational viewpoint than the power series.

Substitution of various corners of the tessellation (see Fig. 2) into (41) and its derivatives gives infinitely many identities of the type given in Sect. 5.

Let us consider the natural boundary $|x| = 1$ of the D-solution in more detail.

Obviously, $|\alpha| = |\beta| = 1$. If $x \to \beta$ in radial direction, i.e., $x = \tau\,\beta$, $0 < \tau \to 1$, then all terms in the sum in (41) vanish, since

$$\mathrm{Re}\,(\gamma\,(\tau\,\beta - \alpha)/(\tau\,\beta - \beta)) = -\frac{\pi\,(1+\tau)}{(1-\tau)\,(l^2+k^2)} \to -\infty.$$

Thus the point $x = \beta$ is similar to a double pole singularity (in this direction). We call it a β-*singularity*. But if $x \to \alpha$, then all exponents in the sum in (41) tend to 1. We call such a point an α-*singularity*.

Both α and β give Pythagorean triples, as it is easy to see that

$$\alpha = \frac{2\,u}{1+u^2} + i\,\frac{1-u^2}{1+u^2},\ u = j/m; \quad \beta = \frac{2\,v}{1+v^2} + i\,\frac{1-v^2}{1+v^2},\ v = k/l, \tag{43}$$

and $(2\,u)^2 + (1-u^2)^2 = (1+u^2)^2$.

Both α- and β-singularities are everywhere dense on the unit circle. In fact, they are one and the same thing, that follows from this

Proposition. *In* (43), *u or v can be an arbitrary rational number.*

Proof. We need to prove that whatever $u \in \mathbb{Q}$ (or $v \in \mathbb{Q}$) is taken, there exists $v \in \mathbb{Q}$ (or $u \in \mathbb{Q}$) such that the normalizing condition $l\,j - m\,k = 1$ is satisfied.

Without loss of generality, we take $0 < v = k/l$, where k, l are coprime. We need to find an integer $j = (1 + m\,k)/l$ for an integer m. Consider the sequence

$$(1 + m\,k) \bmod l, \quad m = 1, \ldots, l.$$

All terms of this sequence must be different, otherwise

$$(1 + m_1\,k) = (1 + m_2\,k) \bmod l$$

for some m_1 and m_2, i.e., $(m_2 - m_1)\,k = 0 \bmod l$, which is impossible, since $|m_2 - m_1| < l$, and l, k are coprime. By pigeon hole argument, there is an m such that $(1 + m\,k) = 0 \bmod l$. **End of proof**.

We could not find in literature a rigorous definition of the term "natural boundary" for an analytical function. Usually, it is understood to be some Jourdan curve where singularities are everywhere dense. However, a natural boundary of an analytical function must not necessarily be a Jourdan curve. It is an easy exercise to construct a function analytical inside the Koch star. Thus natural boundaries can have a fractal structure.

What is remarkable about the natural boundary given here is that it is generated by an ODE.

A general solution to Eq (1) in an arbitrary circle on the Riemann sphere can be obtained by a suitable Möbius transformation of the D-solution. But there is a simpler way.

Theorem 5. *Let the constants α, β, γ in* (41) *be arbitrary complex numbers, and $\delta = \gamma\,(\beta-\alpha)$. Then the formula* (41) *gives the general representation of a solution to Eq* (1) *in the form of an Eisenstein series.*

Proof. The formula (41) gives a formal solution to Eq (1), which is verified as follows. We substitute

$$y(x) = -6/(x-\beta) - \gamma\,(\beta-\alpha)\,P(\gamma\,(x-\alpha)/(x-\beta))/(x-\beta)^2,$$

into Eq (1), where P is symbolic. Then we make the change of variable $t = \gamma\,(x-\alpha)/(x-\beta)$ in the equation for P, and recover the Chazy equation. Hence we can take $P(\exp(t))$ as the Eisenstein series (25), which is a solution as we already know. The expansion (41) depends on 3 arbitrary parameters, and it is convergent in some circle provided

$$\left|\exp\left(\gamma\,\frac{x-\alpha}{x-\beta}\right)\right| < 1,$$

which follows from the growth estimates of the function $\sigma(n)$ (see [21]). **End of proof**.

The Eisenstein series solution in Theorem 5 can be considered as a special case of flat expansions (see [22]).

A typical "Painlevé analysis" of singularities of solutions to Chazy equation can be found in [12, pp. 196-198].

The natural boundary (alias orthogonal circle) is determined uniquely from the given constants α, β, γ.

First, let $\mathrm{Re}\,(\gamma) \neq 0$. Then the center x_0 and the radius R are

$$x_0 = (\alpha+\beta+i\,(\alpha-\beta)\,\mathrm{Im}\,(\gamma)/\mathrm{Re}\,(\gamma))/2, \quad R = |\gamma|\,|\beta-\alpha|/\mathrm{Re}\,(\gamma)/2. \quad (44)$$

We can verify with (42) that $x_0 = 0$ and $R = 1$ for D-solutions (41).

If $\mathrm{Re}(\gamma) = 0$, then the orthogonal circle is the half-plane

$$0 < \mathrm{Im}(\gamma)\,\mathrm{Im}((x-\alpha)/(x-\beta)).$$

The problem remains how to find the constants α, β, γ from the given initial values for a solution to Eq (1). It is clear that there should be infinitely many such triples for one and the same solution.

It turns out that this task can be done numerically due to the observation that β-singularities mimic double poles if we move to the boundary in a suitable direction. Thus we need some points on the boundary.

As it was promised in Sect. 3, here we give explicit formulas for all 4 corners of the master tile (and its adjacent copy) that are determined uniquely by the given initial values.

First, we consider the (s_1)-type hypergeometric solution (i.e., given by Chazy).

We assume the constants a_0, b_0, c_0, d_0, and s_0 be already found by the formulas (12) in Sect. 3.

Here and below we skip some routine but very cumbersome hypergeometric transformations (see [24]).

The two points at $s=\infty$ are the vertices with zero angle of the two triangles, one of which being the master tile and the other adjacent to it with the common arc $s=[0,1]$ (see Fig. 2). These points are

$$z_{1,2} = \frac{4\,c_0\,\pi^2 \pm i\,\sqrt{3}\,d_0\,\Gamma(1/4)^4}{4\,a_0\,\pi^2 \pm i\,\sqrt{3}\,b_0\,\Gamma(1/4)^4}. \tag{45}$$

The point $s=0$ is the vertex with the angle $\pi/2$. Here we only need to expand the formulas (10) in power series. Initial values in original coordinates are

$$x = c_0/a_0, \quad y = 6\,a_0\,b_0, \quad y' = a_0^2\,(a_0^2 + 36\,b_0^2)/6, \quad y'' = a_0^3\,b_0\,(a_0^2 + 12\,b_0^2). \tag{46}$$

This point coincides with the point $s=1/2$ for the (s_3)-type hypergeometric solution (see below).

The point $s=1$ is the vertex with the angle $\pi/3$. Initial values in original coordinates are

$$x = \frac{4\,\pi^2\,(2+\sqrt{3})\,c_0 + \sqrt{3}\,d_0\,\Gamma(1/4)^4}{4\,\pi^2\,(2+\sqrt{3})\,a_0 + \sqrt{3}\,b_0\,\Gamma(1/4)^4}, \quad y = 4\,\sqrt{3}\,a_0\,b_0 + \frac{4\,\pi^2\,a_0^2}{\Gamma(1/4)^4} + \frac{3}{4}\,\frac{b_0^2\,\Gamma(1/4)^4}{\pi^2},$$

$$\begin{aligned} y' &= \frac{\left(16\,a_0^2\,\pi^4 + 16\,a_0\,b_0\,\pi^2\,\sqrt{3}\,\Gamma(1/4)^4 + 3\,b_0^2\,\Gamma(1/4)^8\right)^2}{96\,\pi^4\,\Gamma(1/4)^8}, \\ y'' &= (108\,G - H^3)\,H^3/9, \end{aligned} \tag{47}$$

where

$$G = \frac{1}{27}\,\frac{\pi^{3/2}\,\Gamma(2/3)^3\,\left((8\sqrt{3}-12)\,a_0\,\pi^2 + 3\,b_0\,\Gamma(1/4)^4\right)^3}{\Gamma(1/4)^{12}\,\Gamma(7/12)^6},$$

$$H = \frac{1}{9}\,\frac{\sqrt{3}\,\pi^{3/2}\,\left(4\,\Gamma(11/12)^4\,a_0 + 3\,b_0\,\Gamma(2/3)^4\right)}{\Gamma(2/3)^5\,\Gamma(11/12)^2}.$$

The computations for these formulas are extremely cumbersome, and we only give them for the sake of completeness. They are, actually, not needed if we use the (s_3)-type solution.

Now we turn to the (s_3)-type solution and assume the (two sets of) constants a_0, b_0, c_0, d_0, and (both) s_0 be found by the formulas (17) in Sect. 3. We recall that there are two sets of constants: one for $s = s_0$, and another for $s = 1 - s_0$ corresponding to the sign $\pm$ in the formula (16).

As before, the points at $s = \infty$ are the vertices with zero angle of the two triangles adjacent to each other by the common arc $s = [0, 1]$. These points are

$$z_{1,2} = \frac{9\,c_0\,\Gamma(2/3)^6\,3^{1/2} + 4\,\pi^3\,2^{1/3}\,(1 \pm i\,3^{1/2})\,d_0}{9\,a_0\,\Gamma(2/3)^6\,3^{1/2} + 4\,\pi^3\,2^{1/3}\,(1 \pm i\,3^{1/2})\,b_0}. \tag{48}$$

These are the same points as (45) except z_1 may be z_2 and vice versa. The same is true if we take the second bunch of constants a_0, b_0, c_0, d_0. This follows from the fact that the master tile of the (s_1)-type solution together with an adjacent triangle with the common arc $s = [-\infty, 0]$ form the master tile of the (s_3)-type solution (see Fig. 2).

Thus we cannot find a third point on the boundary by employing different types of hypergeometric parametrization.

For the (s_3)-type solution, both points $s = 0$ and $s = 1$ are the vertices with the angle $\pi/3$. Thus we only need to expand the formulas (15) in power series in order to obtain initial values in original coordinates at both points. For $s = 0$, we have

$$x = c_0/a_0, \quad y = 6\,a_0\,b_0, \quad y' = 6\,a_0^2\,b_0^2, \quad y'' = \frac{3}{2}\,a_0^3\,(a_0^3 + 8\,b_0^3). \tag{49}$$

This point coincides with the point $s = 1$ in (47) for one set of constants a_0, b_0, c_0, d_0.

Now, instead of doing advanced hypergeometric transformations for the point $s = 1$, as we had to do for the (s_1)-type solution, here we only need to take the second bunch of constants a_0, b_0, c_0, d_0, and use the formulas (49).

For numerical example, we take $x = 0$, $y(0) = 0.1$, $y'(0) = 0.2$, and $y''(0) = 0.3$, that were already used in Sect. 3.

We use (s_1)-type solution, i.e., the formulas (45), and obtain the two points on the natural boundary

$$z_{1,2} = 2.199583022030692 \pm i\,4.643060020871425. \tag{50}$$

We use them as α- and β-singularities in the Eisenstein series (41). The formulas (48) give the same $\alpha = z_2$ and $\beta = z_1$.

Analytical continuation towards one of these points gives an approximation to the value of γ. We need only to omit the sum in (41) and use rational approximation. Then we use full expansion (41) at the initial value $x = 0$ and find γ by Newton iterations. The expansion (41) is truncated with sufficient number of terms (we took 100, which is superfluous). Here we use the fact (verified empirically) that the series (41) (at least some of them) converge rapidly inside natural boundary. Thus we obtain

$$\gamma = 2.631142063353375 - i\,1.716594257109669.$$

Then we find the center and the radius of the orthogonal circle by the formula (44):

$$x_0 = -0.829614936625082, \quad R = 5.543829599666541. \tag{51}$$

Let us now compute the corners of both master tiles, i.e., the (s_1)- and (s_3)-types.

For the (s_1)-type solution, we compute by the formulas (46) and (47) with the constants (13) found in Sect. 3.

$$\begin{aligned} p(s=0) = \{x = {} & 0.818911545156752, y = 0.353048799809683, \\ & y' = 0.395828079819275, y'' = 0.134857159376889\}, \\ p(s=1) = \{x = {} & -0.652435840307814, y = 0.034624842889771, \\ & y' = 0.000199813290856, y'' = 0.303681938129847\}. \end{aligned} \tag{52}$$

For the (s_3)-type solution, we compute by the formula (48) with the constants (18) and (19). We find that $p(s = 0)$ above corresponds to the middle point $s = 1/2$ for this parametrization. The point $p(s = 1)$ above corresponds to the point $s = 0$ for the constants (19). And finally, the point $s = 0$ for the constants (18) is

$$\begin{aligned} p_2(s=0) = \{ & x = 2.073094113355361, y = 0.780706435539009, \\ & y' = 0.101583756415337, y'' = -0.765231105647615\}, \end{aligned}$$

We observe that the center of the orthogonal circle lies outside of both master tiles.

Now we need to justify the trouble we took to compute the corners of master tiles.

The corner of the master tile of the (s_1)-type solution with the angle $\pi/2$, i.e., at $s = 0$, is fairly useless, since it lies inside the two adjacent triangles of the (s_3)-type solution (see Fig. 2). However, any corner with the angle $\pi/3$ is close to a triangle adjacent to the master tile by the segment $s = [-\infty, 0]$ or $s = [1, \infty]$ (for the (s_3)-type solution). This fact allows to use analytical continuation in its simplest form, i.e., the power series (24), to obtain initial values for a solution to Eq (1) inside this adjacent triangle. This second tile with its mirror image (i.e., adjacent by the segment $s = [0, 1]$) has exactly one point on the natural boundary different from the two already found (see Fig. 2).

Thus we need to recompute the constants a_0, b_0, c_0, d_0 of hypergeometric solution (of any type) and use the formulas (45) or (48) to obtain a third point on the boundary.

Any three points on the circle determine it uniquely.

We performed these computations for the left corner of the master tile $p(s = 1)$ in (52) and obtained the third point on the boundary

$$z_3 = -6.373444536291623. \tag{53}$$

One can easily verify that the center and the radius of the orthogonal circle given above (and found numerically) agree with this additional point.

§ 7. A solutions to Chazy equation in elliptic integrals

As we have seen, the existence of Eisenstein series solutions to Eq (1) was derived either from Ramanujan's result on Eisenstein series P, Q, R (see [18]) or from the existence of a special convolution sum (28) for the sum of divisors function $\sigma(n)$ (see [20]).

Here we give another proof of the fact that the series P (25) satisfies Eq (1) after a suitable change of variables. This proof is based on the famous Jacobi's identity for the elliptic function ns (see formula (27) in [24, p. 869])

$$\frac{1}{\mathrm{sn}^2\,(u)} = \frac{\pi^2}{4\mathbf{K}^2}\,\mathrm{cosec}^2\left(\frac{\pi\,u}{2\mathbf{K}}\right) + \frac{\mathbf{K}-\mathbf{E}}{\mathbf{K}} - \frac{2\,\pi^2}{\mathbf{K}^2}\sum_{n=1}^{\infty}\frac{n\,q^{2\,n}\,\cos\left(\dfrac{n\,\pi\,u}{\mathbf{K}}\right)}{1-q^{2\,n}}, \tag{54}$$

where

$$q = \exp(-\pi\mathbf{K}'/\mathbf{K}) = \exp(i\,\pi\,t), \qquad \left|\mathrm{Im}\,\left(\frac{u}{\mathbf{K}}\right)\right| < 2\,\mathrm{Im}\,(t), \tag{55}$$

(see also [25, p. 535]). Note that the convergency condition given for (54) in [24, p. 869] is not correct.

Here and below we use standard notation for elliptic and theta functions (see [24]), so prime does not necessarily denote a derivative.

Among 27 formulas for trigonometric series of elliptic functions given in [24], the formula (54) is the only one suitable for our purpose. We need to somehow get rid of the cosine in the sum in (54).

We substitute $u = 2\mathbf{K}(k) + z$ in (54), and, using the identity $\mathrm{sn}\,(2\mathbf{K}(k), k) \equiv 0$, we resolve an indeterminate form $\infty - \infty$ in (54) as $z \to 0$. After some simplifications, we obtain the identity

$$P(q) = 1 - 24 \sum_{n=1}^{\infty} \frac{n\, q^{2\,n}}{1 - q^{2\,n}} = \frac{4\mathbf{K}(k)\,((k^2-2)\mathbf{K}(k) + 3\mathbf{E}(k))}{\pi^2}. \qquad (56)$$

The lhs(56) is the same Ramanujan's Eisenstein series P (25) written in the form of Lambert series. The identity (56) is, in a sense, known. It can be found in a footnote of Ramanujan's paper in [18, p. 140]. We could not find this identity anywhere else. As far as we know, Ramanujan did not use this identity or gave any proof.

Thus we need to prove that rhs(56) is a solution to the mock-Chazy equation (26) after the change of independent variable $q(k) = \exp(-\pi\mathbf{K}'/\mathbf{K})$. This substitution gives (after a rather cumbersome calculation) the following equation

$$\begin{aligned} &3\,\mathbf{K}\,\pi^2\,(1-k^2)\,(\delta\,P(k))^2 + 4\,(\mathbf{K}^3\,(3-k^2) + 3\,\mathbf{K}\,\mathbf{E}\,(\mathbf{E} - 2\,\mathbf{K}) \\ &+\pi^2\,P(k)\,(\mathbf{K}-\mathbf{E}))\,\delta\,P(k) + 2\,\mathbf{K}\,(1-k^2)\,(6\,\mathbf{K}\,(\mathbf{E}-\mathbf{K}) - \\ &\pi^2\,P(k))\,\delta^{(2)}\,P(k) + 2\,\mathbf{K}^3\,(1-k^2)^2\,\delta^{(3)}\,P(k) = 0, \quad \delta := k\,\frac{d}{dk}, \end{aligned} \qquad (57)$$

which is obtained with the use of the Legendre identity for elliptic integrals (see [24])

$$\mathbf{K}\mathbf{E}' + \mathbf{K}'\mathbf{E} - \mathbf{K}\mathbf{K}' = \pi/2. \qquad (58)$$

Then it is relatively easy to verify that $P(k) =$ rhs(56) is indeed a solution to Eq (57).

Yet another approach seems more illuminating. We use the fact that Eq (1) is invariant under the change of variables

$$x \to x/A, \quad y(x) \to A\,y(x),$$

where A is an arbitrary constant. We need to prove that $y(q) = i\,\pi\,P(q)$ is a solution to Eq (1).

We make the change of independent variable in Eq (1)

$$x = \mathbf{K}'/\mathbf{K}/A = F(\tfrac{1}{2}, \tfrac{1}{2}, 1, 1-k^2)/F(\tfrac{1}{2}, \tfrac{1}{2}, 1, k^2)/A, \qquad (59)$$

Then we verify that

$$\begin{aligned} &y(k) = \frac{4\,A\mathbf{K}\,((2-k^2)\mathbf{K} - 3\mathbf{E})}{\pi} = \\ &A\,\pi\,F(\tfrac{1}{2}, \tfrac{1}{2}, 1, k^2)\,((2-k^2)\,F(\tfrac{1}{2}, \tfrac{1}{2}, 1, k^2) - 3\,F(-\tfrac{1}{2}, \tfrac{1}{2}, 1, k^2)). \end{aligned} \qquad (60)$$

is a solution to the transformed Chazy equation. This is done again with the application of Legendre identity. Taking $A = -i$, we recover the solution $y(q) = i\,\pi\, P(q)$.

The substitution (59) is similar to the hypergeometric one given by Chazy (see Sect. 2). The hypergeometric parameters here are $a = 1/2$, $b = 1/2$, and $c = 1$. This would have given the tessellation of the type (∞, ∞, ∞), i.e., the Poincaré model of the hyperbolic plane, but $k = k(x)$ in (59) is not a Schwarz triangle function.

We recall that the variable t was reserved for a copy of Chazy equation in the upper half-plane. Here it is more convenient to take $A = -1/\pi$ in (59) and (60) and consider a standard solution to Eq (1) in the left half-plane, that we denote iH.

Thus we have $x = -\pi\mathbf{K}'/\mathbf{K} = i\,\pi\, t$, and a solution to Eq (1): $y(x) =$rhs(56) valid in $iH = \mathrm{Re}(x) < 0$. We call it the L-solution.

Theorem 6. *The L-solution* (56) *is obtained from the H-solution with the Möbius transformation $x = i\,\pi\, t$.*

Proof is obvious. The Eisenstein series solution $w(t)$ (27) transforms under this substitution to $i\,\pi\, y(x)$, and we recover the solution (56), since the Lambert series lhs(56) and the series (25) are one and the same thing (see [17]). **End of proof**.

Now let us pretend that we do not know the Eisenstein series identity (33). By definition, $x = x(k) = -\pi\mathbf{K}'/\mathbf{K}$. We denote $x' = \pi^2/x = -\pi\mathbf{K}/\mathbf{K}'$ and obtain a second L'-solution (56) where k is replaced by k'. We combine the two L-solutions in this way

$$\pi^2\, y(x') + x^2\, y(x) + 6\, x = 0, \tag{61}$$

and after a simple calculation recover the Legendre identity for elliptic integrals (58). Thus the identity for Eisenstein series (33) and Legendre identity (58) are one and the same thing.

This can be considered as yet another algebraic proof of (33), since Jacobi derived the identity (54) in his “Fundamenta nova” with purely algebraic manipulations (see [26]).

The identities for the sum of divisors function found in Sect. 5 can be duplicated with the use of elliptic integrals. For example, if we put $x = -\pi$ in (56), i.e., $q = \exp(-\pi)$, we obtain the identity (36), since $K = K'$ for $k = 1/\sqrt{2}$.

§ 8. Möbius transformations of the L-solution

In Sect. 4 we promised to prove that the H-solution gives a general solution to Chazy equation with a suitable Möbius transformation. This statement seems fairly obvious given the general nature of such transformations, but it remains existential unless we find explicit formulas for such a transformation for any given initial data in original coordinates.

We cannot follow this plan for any hypergeometric representation of the H-solution, since hypergeometric solutions are, in a sense, local ones. They are valid only in the two adjacent triangles that are images of the upper and lower half-planes (in variable s). Then we must use analytical continuation to move to other triangles, and this is not a closed form operation.

The Eisenstein series representations are global and very effective numerically, but we do not know how to evaluate infinite series at arbitrary arguments exactly.

Thus both approaches that we already used in our computations are not suitable for our purpose.

It turns out that the L-solution, which is basically the same as the H- and D-solutions, solves Chazy equation explicitly, globally, and in a closed form.

Here we will need a third copy of Chazy equation for the function $u(z)$.

Let four parameters (initial values) be given

$$z = z_0, \quad u_0 = u(z), \quad u_1 = u'(z), \quad u_2 = u''(z), \tag{62}$$

where z_0, u_0, u_1, and u_2 are arbitrary complex numbers. These parameters determine the solution $u(z)$ to the u-copy of Eq (1) uniquely.

We need to find 4 constants a, b, c, and d (actually, 3, since $a\,d - b\,c = 1$) and the value of the modulus k such that initial values of the L-solution at k transform into initial values (62).

For the L-solution, we have by the chain differentiation rule

$$\begin{aligned} &x_0 = x = -\pi\,\mathbf{K}'/\mathbf{K}, \quad y_0 = y(x) = 4\,\mathbf{K}\,(3\,\mathbf{E} - \mathbf{K}\,(2-k^2))/\pi^2, \\ &y_1 = \tfrac{dy}{dx}(x) = 8\,\mathbf{K}^2\,(3\,\mathbf{E}^2 - 2\,(2-k^2)\,\mathbf{K}\,\mathbf{E} + (1-k^2)\,\mathbf{K}^2)/\pi^4, \\ &y_2 = \tfrac{d^2y}{dx^2}(x) = 96\,\mathbf{K}^3\,\mathbf{E}\,(\mathbf{K}-\mathbf{E})\,(\mathbf{K}\,(1-k^2) - \mathbf{E})/\pi^6. \end{aligned} \tag{63}$$

A general Möbius transformation of a solution $y(x)$ to Eq (1) takes the form

$$\begin{aligned} &z_0 = z = (a\,x + b)/h, \quad u_0 = u(z) = h\,(6\,c + h\,y_0), \\ &u_1 = \tfrac{du}{dz}(z) = h^2\,(6\,c^2 + 2\,h\,c\,y_0 + h^2\,y_1), \\ &u_2 = \tfrac{d^2u}{dz^2}(z) = h^3\,(12\,c^3 + 6\,h\,c^2\,y_0 + 6\,h^2\,c\,y_1 + h^3\,y_2), \end{aligned} \tag{64}$$

where $h = c\,x + d$, and $a\,d - b\,c = 1$.

Thus we need to solve the system of equations (64) explicitly with respect to k, and $\{a, b, c, d\}$, where the values on the left are arbitrary complex numbers (62), and the values on the right are computed by formulas (63). This can be done as follows.

First, we take the 2nd Eq (64) and find

$$c = \frac{1}{6}\,\frac{u_0 - h^2\,y_0}{h}, \tag{65}$$

where y_0 and h remain symbolic. Then we substitute this into the 3rd Eq (64) and find

$$h = \eta\left(\frac{u_0^2 - 6\,u_1}{y_0^2 - 6\,y_1}\right)^{1/4} = \frac{\eta\,\pi}{2}\,\frac{(u_0^2 - 6\,u_1)^{1/4}}{\mathbf{K}\,(1 - k^2 + k^4)^{1/4}}, \quad \eta^4 = 1. \tag{66}$$

Now we observe that h in (66) depends only on one undetermined value, i.e., k. Here we assume $u_0^2 - 6\,u_1 \neq 0$.

Now we use (65), (66), and explicit formulas (63) in the 4th Eq (64) and find (after rather cumbersome but straightforward calculation)

$$\alpha := \frac{1}{4}\,\frac{(u_0^2 - 6\,u_1)^3}{(u_0^3 - 9\,u_0\,u_1 + 9\,u_2)^2} = \frac{(1 - k^2 + k^4)^3}{(k^2 - 2)^2\,(2\,k^2 - 1)^2\,(k^2 + 1)^2}. \tag{67}$$

The L-solution (63) depends on the parameter $m = k^2$ rather than the modulus k. Thus Eq (67) takes the form

$$4\,\alpha - 1 - 3\,(4\,\alpha - 1)\,m - 3\,(\alpha + 2)\,m^2 + (26\,\alpha + 7)\,m^3 -$$
$$3\,(\alpha + 2)\,m^4 - 3\,(4\,\alpha - 1)\,m^5 + (4\,\alpha - 1)\,m^6 = 0.$$

This equation is recurrent, and thus can always be solved explicitly. We have

$$m^3 - \frac{3}{2}\,\frac{(12\,\alpha - 3)^{1/2} - 3)}{(12\,\alpha - 3)^{1/2}}\,m^2 - \frac{3}{2}\,\frac{(12\,\alpha - 3)^{1/2} + 3)}{(12\,\alpha - 3)^{1/2}}\,m + 1 = 0, \tag{68}$$

and another copy of this equation where m is replaced by $1/m$.

Thus, unless $\alpha = 1/4$, we always have 6 solutions $m = m_i$ and $m = 1/m_i$, $i = 1, 2, 3$ to Eq (68). If $\alpha = 1/4$, then Eq (67) has 4 solutions: $k = 0$, $k = \pm 1$, and $k = \infty$. This case is treated separately.

Since we know k, we know h, and c. Then, with the use of the 1st Eq (64), we have

$$d = h - c\,x, \quad b = d\,z - x/h, \quad a = (1 + b\,c)/d. \tag{69}$$

Thus all the values are determined uniquely but in 6 different packs dependent on $m = k^2$. In addition, the value η remains undetermined. It turns out that all 6 packs of the values are good if we take either $\eta = 1$ or $\eta = i$ in each case. This decision on η is made after the substitution of the found values into (64). The equation for u_2 is satisfied only for 2 primitive roots of $\eta^4 = 1$, and the sign $\pm\eta$ is irrelevant, since $\{a, b, c, d\}$ also change sign.

A similar situation appeared in Sect. 3 for hypergeometric solutions. Only there we had $\epsilon^4 = -1$.

Before we move further, we dispatch the special cases (a): $\alpha = 1/4$, and $u_0^2 - 6\,u_1 = 0$ in (66).

The case (a) means that

$$24\,u_1^3 - 3\,u_0^2\,u_1^2 + 2\,u_0^3\,u_2 - 18\,u_0\,u_1\,u_2 + 9\,u_2^2 = 0.$$

This is possible if and only if the initial values belong to a special solution to Chazy equation

$$u(z) = -6/(z-A) + B/(z-A)^2$$

for some constants A and B, as it is easy to verify. This special case has no natural boundary, which explains the values for k.

The case (b) is more tricky. Here we cannot find h by the formula (66). But the formula (67) is still valid, and we have 4 different values of k given by the biquadratic equation $1 - k^2 + k^4 = 0$.

The last statement is better seen as follows.

We use the value $u_1 = u_0^2/6$ in the 2nd and 3rd Eqs (64) and find $y_1 = y_0^2/6$, which, after application of (63), yields

$$\mathbf{K}^4\,(1 - k^2 + k^4) = 0.$$

Thus $k_j = \pm\sqrt{3}/2 \pm i/2$, $k_j^6 = -1$, $j = 1, \ldots, 4$.

Now we use $u_1 = u_0^2/6$, and $y_1 = y_0^2/6$, in the 4th Eq (64) and find

$$h = \kappa \left(\frac{u_0^3 - 18\,u_2}{y_0^3 - 18\,y_2} \right)^{1/6} = \frac{\kappa\,\pi}{2}\, \frac{(\pm i\,\sqrt{3}/9)^{1/6}\,(u_0^3 - 18\,u_2)^{1/6}}{\mathbf{K}(\sqrt{3}/2 \mp i/2)}, \qquad \kappa^6 = 1. \tag{70}$$

Since $u_0^3 - 18\,u_2 \neq 0$, otherwise we get the case (a), then $h \neq 0$ in (70), and we can use (65) and (69) to find $\{a, b, c, d\}$.

Similar computations are needed for the Möbius transformation of the general solution to the L-solution, i.e., the inverse to that constructed above.

The above computations are more simple than what was required to obtain hypergeometric parametrizations of (s_1)- and (s_3)-types. In addition, we need not recompute the coefficients $\{a, b, c, d\}$ as we move from one triangle to another.

Both the L-solution and the solution given by initial values (62) are determined uniquely. Thus the Möbius transformation in (64) is now applied globally after the auxiliary variable h is restored in its original form $h = c\,x + d$.

Thus, we have a parametrization of the natural boundary, which is the image of the imaginary axis. We can always compute the orthogonal circle and its center by just 3 points

$$z_1 = z(0) = b/d, \quad z_2 = z(\infty) = a/c, \quad z_3 = z(i) = (a\,i + b)/(c\,i + d).$$

We skip these elementary but cumbersome calculations.

For example, we can map the L-solution to the D-solution and verify that the center is indeed the origin, and the radius is 1. Here we have

$$z = 0, \quad u_0 = 0, \quad u_1 = \frac{8}{\pi^2}\,\mathbf{K}\left(\frac{1}{\sqrt{2}}\right)^4, \quad u_2 = 0,$$

and, by (67), $k \in \{\pm\sqrt{2}, \pm 1/\sqrt{2}, \pm i\}$.

Taking $k = 1/\sqrt{2}$ and $\eta = 1$, we obtain $x = -\pi$, $h = (1+i)\sqrt{\pi}$, and

$$a = \frac{1-i}{2\sqrt{\pi}}, \quad b = (1-i)\sqrt{\pi}/2, \quad c = \frac{-1-i}{2\sqrt{\pi}}, \quad d = (1+i)\sqrt{\pi}/2.$$

Now we restore $h = c\,x + d$ in (64). Then we have a k-parametrization of the D-solution, where k is any complex number.

If we are given an arbitrary z within the unit circle, we need to find x by the formula

$$x = -\frac{z\,d - b}{z\,c - a}, \tag{71}$$

then solve the equation $x = -\pi\,\mathbf{K}'/\mathbf{K}$ for $k = k(x)$, which can always be solved explicitly (see Sect. 9). Thus we have $u(z)$, $u'(z)$, and $u''(z)$ by (64).

Let us consider where the special values $k = \pm\sqrt{3}/2 \pm i/2$ are mapped in the unit disk.

We skip intermediate calculations and find that these values are mapped into the corners of the master tile of the (s_3)-type solution with the angles $\pi/3$ (and in some neighboring corners with this angle, see Sect. 6). Since we already have initial values at these points obtained from the H-solution, we can compare two sets of formulas and arrive at these evaluations

$$\mathbf{K}(\tfrac{\sqrt{3}}{2} \pm \tfrac{i}{2}) = \tfrac{1}{16\,\pi}\,2^{1/6}\,3^{1/4}\,(1+\sqrt{3} \mp (1-\sqrt{3})\,i)\,\Gamma^3(\tfrac{1}{3}),$$

$$\mathbf{E}(\tfrac{\sqrt{3}}{2} \pm \tfrac{i}{2}) = \tfrac{1}{96\,\pi}\,2^{5/6}\,3^{1/4}\,(1+\sqrt{3} \pm (1-\sqrt{3})\,i)\,(2^{1/3}\sqrt{3}\,\Gamma^6(\tfrac{1}{3}) + 16\,\pi^3)/\Gamma^3(\tfrac{1}{3}).$$

We could not find these formulas in reference books.

To conclude this Section, we apply obtained formulas to the numerical example we used in Sect. 3, 6, i.e., we take initial values $z_0 = 0$, $u_0 = 0.1$, $u_1 = 0.2$, and $u_2 = 0.3$.

By (68), we find 12 values of k and take the first:

$$k = -1.194830742557925 - 0.403533522744867\,i,$$

and $\eta = 1$. Then we find

$$\begin{aligned} a &= 1.123904414385994, & b &= 2.411094984012909 - 1.765424925786088\, i, \\ c &= -0.176341758053477, & d &= 0.511452098886254 + 0.276996985810956\, i. \end{aligned}$$

We observe that one of the points on the orthogonal circle is $z_2 = b/d$ (50), that we have found in Sect. 6, and another is $z_3 = a/c$ (53). For the third point we take $x = i$ in (64) and find

$$z_4 = 4.300787157864557 - 2.100719157573232\, i.$$

It is easy to verify that the center and the radius are exactly the same as we found in (51).

Let us compute initial values at the center of the orthogonal circle.

We take $z = -0.829614936625082$. Then we find $x = -2.900345922212885 + i\,\pi/2$ by (71). Here we can either use Eisenstein series (27) and its derivatives (with y and x instead of w and t, and $q = \exp(x)$), or we can solve the equation $x = -\pi\,\mathbf{K}'/\mathbf{K}$ for k, i.e., we find

$$k = -0.777467228338905 - 0.494246624935179\, i.$$

Then we use (63) and obtain initial values of the L-solution. Then by (64), we have

$$\begin{aligned} &u(z) = 0.039337185273310, \quad u'(z) = -0.053292331179842, \\ &u''(z) = 0.30031323779999. \end{aligned}$$

All decimal places given in this paper are correct (without rounding).

§ 9. A solution of Chazy equation in theta functions

As it is well known, the four classical theta functions are the universal building blocks for anything with the word “elliptic” attached to it. In addition, they are very effective computationally and are among the most well studied special functions of mathematical physics.

Thus it is only natural to express the L-solution to Chazy equation, that we obtained in Sect. 7, in theta functions. Apart from the sense of completeness, it will remove the last numerical aspect of our general solution, namely, the necessity to solve the equation $x = -\pi\,\mathbf{K}'/\mathbf{K}$ for k. Thus the solution we have given in Sect. 8 becomes completely analytical.

We use standard (modern) notation for theta functions (see [17]). For convenience of the reader, we give explicit formulas for the three functions we use.

$$\theta_2(z, q) = 2\, q^{1/4} \sum_{n=0}^{\infty} q^{n\,(n+1)} \cos((2\,n+1)\,z).$$

$$\theta_3(z,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos(2\,n\,z).$$

$$\theta_4(z,q) = 1 + 2\sum_{n=1}^{\infty} (-1)^n\, q^{n^2}\cos(2\,n\,z),$$

where z is an arbitrary complex number, and $q = \exp(i\,\pi\,t)$ is the nome, while t stands for τ as a parameter ($t \in H$). Note that in some books and papers the nome is taken as $q = \exp(2\,i\,\pi\,t)$. Since we are interested in the left-half plane iH, the nome we use is $q = \exp(x) = \exp(-\pi\,\mathbf{K}'/\mathbf{K})$.

As it is customary, we denote the *null theta functions* by the same symbols $\theta_k(0,q) = \theta_k(q)$, $k = 2,3,4$.

We recall that the constants $\{a,b,c,d\}$ for the Möbius transformation from the L-solution to an arbitrary solution $u(z)$ to Chazy equation are determined explicitly by the initial values of $u(z)$ given at just one point.

For any value of z within natural boundary (the image of the imaginary axis), we find $x = x(z)$ by the inverse Möbius transformation (71). Hence we know the nome $q = \exp(x)$. It remains to find $k = k(x) = k(q)$ and then use parametrization (63) and formulas (64).

We have (see [24]),

$$\mathbf{K}(k) = \frac{\pi\theta_2^2(q)}{2\,k} = \frac{\pi}{2}\,\theta_3^2(q), \quad k = \frac{\theta_2^2(q)}{\theta_3^2(q)}.$$

These are classical formulas found by Jacobi.

Thus we completed the solution of Chazy equation.

Although this paper is not about analytical number theory, we have used profitably some of its achievements, in particular, the properties of the sum of divisors function $\sigma(n)$. Thus it seems appropriate to draw some obvious conclusions based on the solutions we have found that reflect back on this function.

It seems to have escaped notice (although we cannot be completely sure) that the Eisenstein series solution to Chazy equation gives a generating function for the sequence $\{\sigma(n),\ n \in \mathbb{N}\}$. The Dirichlet generating function for this sequence is well known. However, the reference books (see [17]) usually give the Lambert series (56) as a generating function, which seems absurd, since both series are different forms of each other.

The Sloane online encyclopedia of integer sequences, which is a very authoritative source of information on the subject, gives several generating functions for this sequence in the form of Lambert series and some infinite products, but frankly admits that the power generating function for this sequence is unknown (see [13]). This is strange, since, as we will see shortly, this function was published (although in bits and pieces) for more than a 100 years.

First, we can plug the $k = k(q)$ that we found in the formula for $y(x)$ in (63) and use Eisenstein series solution on the left. This works, but the formula is not very nice. We can also use hypergeometric representation (60).

We use transformation formulas for elliptic integrals in (63) (see [24]) and find

$$1 - 24 \sum_{n=1}^{\infty} \sigma(n)\, q^{2\,n} = \theta_2^4(q) + \theta_3^4(q) + 12\, q\, \frac{d}{dq} \log \theta_4(q), \tag{72}$$

which amounts to an explicit generating function for the sequence $\{\sigma(n),\ n \in \mathbb{N}\}$.

The series lhs(72) is a formal power series solution to the mock-Chazy equation (26); thus we can verify (72) formally by substitution of rhs(72) into (26).

We can also use various identities for theta functions in order to simplify or modify (72). For example, the famous identity $\theta_2^4(q) + \theta_4^4(q) = \theta_3^4(q)$ allows to eliminate θ_2 function from (72).

However, the most symmetrical form of solution to Eq (26) was found by Halphen (see [2]) in the course of solution of the Darboux-Halphen system (see [3] for historical details). This system takes the form

$$\frac{d}{dz}\,(u_1(z) + u_2(z)) = u_1(z)\, u_2(z),$$

plus two similar equations for indices $(1, 2) \to (2, 3)$, and $(1, 2) \to (1, 3)$.

Chazy derived his equation for $u(z) = 2\,(u_1(z) + u_2(z) + u_3(z))$ in [1]. The solution by Halphen gives

$$1 - 24 \sum_{n=1}^{\infty} \sigma(n)\, q^{2\,n} = 4\,\delta\,(\log \theta_2(q) + \log \theta_3(q) + \log \theta_4(q))\,, \tag{73}$$

where $\delta = q\, d/dq$ as in (26).

Thus we also obtained a (possibly) new identity for theta functions: rhs(72)=rhs(73), which takes the form

$$\theta_2^4(q) + \theta_3^4(q) = 4\,\delta\,(\log \theta_2(q) + \log \theta_3(q) - 2\, \log \theta_4(q))\,.$$

The importance of the formulas (72) and (73) lies with the fact that the growth rate of the function $\sigma(n)$ is intimately related to the Riemann hypothesis. It was established in [21] that Riemann hypothesis is true if and only if

$$\sigma(n) < H_n + \log H_n \exp H_n, \quad 1 < n,$$

where H_n is the harmonic number.

The result of the paper [21] is based on the work [27], etc., which goes far beyond the scope of the present article.

Thus all one needs in order to prove or disprove the Riemann hypothesis is to give an accurate evaluation of a Cauchy integral for the generating function (72) or (73).

As a last minute remark, we notice that the Eisenstein series solution (27) (alias the H-solution found in Sect. 4) is identical with the solution given in [8, p. 342], which is expressed with the help of the discriminant modular form, i.e.,

$$w(t) \equiv \frac{1}{2} \frac{d}{dt} \log \Delta(t), \tag{74}$$

where

$$\Delta(t) = \sum_{n=1}^{\infty} \tau(n)\, q^{2\,n} = q^2 \prod_{n=1}^{\infty} (1 - q^{2\,n})^{24}, \quad q = \exp(i\,\pi\,t),$$

and $\tau(n)$ is Ramanujan's tau function. Here we omitted the coefficient $(2\,\pi)^{12}$, which is cancelled in (74) (see [10, p. 51]).

The proof of the above statement can be found in the Exercise 5 in [10, p. 71]. Namely,

$$G_2(t) = -4\,\pi\,i\,\frac{d}{dt} \log \eta(t),$$

where $G_2(t)$ is the Eisenstein series (of level 2) (see [10, p. 69]), and $\eta(t)$ is the Dedekind eta function.

Since $G_2(t) = -i\,\frac{\pi}{3}\,w(t)$ (see (33)), we integrate the above formula, raise it to 24-th power, and use Theorem 3.3 in [10, p. 51]. Thus we obtain

$$\Delta(t) = \text{const} \, \exp(2 \int (w(t)\,dt)),$$

which is identical with (74).

As a corollary we derive the convolution sum that links Ramanujan's tau function with the sum of divisors function $\sigma(n)$:

$$\tau(1) = 1, \quad (n-1)\,\tau(n) = -24 \left(\sum_{m=1}^{n-1} \sigma(n-m)\,\tau(m) \right), \; n > 1. \tag{75}$$

We could not find this formula in [10], so it is, probably, new.

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