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Yashunsky A.D.

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ИНСТИТУТ ПРИКЛАДНОЙ МАТЕМАТИКИ
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A. D. Yashunsky

Bernoulli distribution transformations by Boolean functions from closed classes

Москва — 2016

Яшунский А. Д.

Преобразования бернуллиевских распределений булевыми функциями из замкнутых классов

Рассматривается задача о приближенном выражении распределений бернуллиевских случайных величин путем применения произвольных булевых функций из замкнутого класса к независимым одинаково распределенным случайным величинам, имеющим заданное распределение. Для всех замкнутых классов булевых функций и всевозможных начальных распределений описаны множества аппроксимируемых распределений.

Ключевые слова: бернуллиевская случайная величина, преобразование, булева функция, замкнутый класс

Alexey Dmitrievich Yashunsky

Bernoulli distribution transformations by Boolean functions from closed classes

We consider the problem of approximating distributions of Bernoulli random variables by applying arbitrary Boolean functions from a closed class to independent identically distributed random variables with a given dsitribution. For every closed class of Boolean functions and any given initial distribution we provide a description of the approximable distribution set.

Key words: Bernoulli random variable, transformation, Boolean function, closed class

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Contents

Introduction

The problems of discrete probability distribution transformations usually consider some operations applied to random variables having values in some finite set with given distributions. The result is a new random variable, whose distribution is easily computed. The goal of research is to describe the set of distributions that are expressible in such a way.

Within this general setting the following details may vary: the set of random variable values (usually the set $E_k = \{0, 1, \ldots, k-1\}$), the applied operations, the set of initial distributions and, finally, one may consider exact or approximate expression of the distributions.

The most-well studied problems are the ones that deal with transformations of Bernoulli distributions, i. e. random variables taking values in the set $\{0, 1\}$. The main focus of research has been on exact expression of distributions with rational components. The results obtained in this area are outlined in R. M. Kolpakov's review [1].

A wide-spread approach to defining the set of applied operations is the following. A set of operations $\mathcal{B}$ on random variables is considered; each operation $f(x_1, \ldots, x_n) \in \mathcal{B}$ has a corresponding operation $\hat{f}$, acting on distributions, transforming the distributions of random variables $X_1, \ldots, X_n$ into the distribution of the random variable $f(X_1, \ldots, X_n)$. Then the algebra of distributions is considered with the set of operations being $\hat{f}, f \in \mathcal{B}$. This approach in fact is equivalent to considering the transformations of random variables by all functions that are expressible as a read-once formula over $\mathcal{B}$.

Early works of R. L. Skhirtladze in this area have demonstrated that even very simple systems present wide possibilities for transforming distributions. In [2] the initial distribution $(1/2, 1/2)$ is shown to generate any distribution with binary rational components when transformed by series-parallel networks (which are equivalent to read-once AND/OR formulas). In [3] read-once formulas containing AND, OR, NOT operations are shown to approximate arbitrary distributions for any given initial distribution.

Later the research focused either on read-once superpositions of functions from “simple” sets $\mathcal{B}$, or the set $\mathcal{B}$ would be considered to contain all Boolean functions, hence superpositions become irrelevant.

The present work investigates a slightly different problem. We suppose that for a given set of Boolean operations $\mathcal{B}$ not only read-once, but arbitrary formulas may be constructed and then applied to independent random Bernoulli variables with given distributions to generate new distributions. The set of initial distributions is supposed to contain just one distribution $(1-p, p)$ and we study the approximation of distributions with arbitrary precision.

This approach naturally leads to considering the distributions of transformations by Boolean functions from closed classes[1], since all possible superpositions of functions from a given set $\mathcal{B}$ are exactly a closed class.

Earlier results on read-once superpositions allow the description of only some of the classes' properties. Yet, these results are essentially used in the present work. In particular, many statements from the author's paper [4] on read-once transformations of Bernoulli distributions have been reformulated in the present work in terms of functions from closed classes.

Note that the paper of F. I. Salimov [5] states some results in terms of Boolean function closed classes, different from the class of all Boolean functions, yet in fact these results are related to read-once superpositions and their formulation in terms of closed classes is a weakening.

The results of the present work allow to describe for any finite system of Boolean functions $\mathcal{B}$ the set of distributions that may be approximated by superpositions of functions from $\mathcal{B}$, applied to independent random variables with Bernoulli distributions from a given finite set.

Definitions and basic properties

Let $x_1 x_2$ denote the *conjunction* (logical AND) of variables x_1 and x_2, $x_1 \vee x_2$ denote the *disjunction* (logical OR) of variables x_1 and x_2, $\overline{x}$ denote the *negation* (logical NOT) of variable x.

Consider a Boolean function $f(x_1, \ldots, x_n)$ and let $X_1, \ldots, X_n$ be mutually independent Bernoulli random variables, equal to 1 with probability p and equal to 0 with probability $1 - p$.

Let
$$h_f(p) = \mathsf{P}(f(X_1, \ldots, X_n) = 1),$$
i. e. $h_f(p)$ expresses the probability of obtaining the value 1 when substituting random variables $X_1, \ldots, X_n$ for the variables of the function f.

For a given function f the expression for $h_f(p)$ is easy to write out. Recall that the *weight* of a tuple $\alpha \in \{0,1\}^n$ is the number of 1's in α. For a Boolean function $f(x_1, \ldots, x_n)$ denote by A_i the number of tuples of weight i on which the function has value 1. Then

$$h_f(p) = \sum_{i=0}^{n} A_i p^i (1-p)^{n-i}.$$

Note that $0 \leqslant A_i \leqslant \binom{n}{i}$. We shall further refer to $h_f(p)$ as the *characteristic polynomial* of the function $f(x_1, \ldots, x_n)$.

[1]Closed classes are, except for some special cases, Boolean function clones that form Post's lattice.

The characteristic polynomials have a decomposition property similar to Boolean functions. For $f(x_1, \dots, x_n) = \overline{x_1} f_0(x_2, \dots, x_n) \vee x_1 f_1(x_2, \dots, x_n)$ we have $h_f(p) = (1-p)h_{f_0}(p) + p h_{f_1}(p)$. This and the definition of the characteristic polynomial easily implies

Property 1. *If functions f and g differ only by insertion or deletion of dummy variables or renaming of variables (without identification) then $h_f(p) = h_g(p)$.*

Recall that the function $f^*(x_1, \dots, x_n) = \overline{f(\overline{x_1}, \dots, \overline{x_n})}$ is *dual* to the function $f(x_1, \dots, x_n)$. One can easily verify

Property 2. $h_{f^*}(p) = 1 - h_f(1-p)$.

For a set $X \subseteq [0,1]$ let $cl(X)$ denote the *closure* of X, i. e. the set X and all its limit points. Let $\mathcal{A}$ be a set of Boolean functions. Let

$$W_{\mathcal{A}}(p) = cl(\{h_f(p) : f \in \mathcal{A}\}).$$

The set $W_{\mathcal{A}}(p)$ is further referred to as the *the set of distributions, approximable by functions*[2] *from $\mathcal{A}$ at the point p.* Essentially, $W_{\mathcal{A}}(p)$ contains all points from the segment $[0,1]$ that may be with arbitrary precision approximated by the set $\mathcal{A}$ functions' characteristic polynomials values at the point p. In particular, if $W_{\mathcal{A}}(p) = [0,1]$ then substituting independent random variables with Bernoulli distributions $(1-p, p)$ for variables of functions from $\mathcal{A}$, one may approximate any Bernoulli distribution with arbitrary precision.

Note that for any set $\mathcal{A}$ we have $W_{\mathcal{A}}(0), W_{\mathcal{A}}(1) \subseteq \{0,1\}$. Further we shall consider $W_{\mathcal{A}}(p)$ for $p \in (0,1)$.

For a set $\mathcal{A}$ let $\mathcal{A}^* = \{f^* : f \in \mathcal{A}\}$ be the set of dual functions. The property 2 implies

Property 3. $W_{\mathcal{A}^*}(p) = \{1 - w : w \in W_{\mathcal{A}}(1-p)\}$.

Besides, one may easily verify

Property 4. *If $\mathcal{A}' \subset \mathcal{A}$, then $W_{\mathcal{A}'}(p) \subseteq W_{\mathcal{A}}(p)$.*

Finally, if the set $\mathcal{A}$ is closed under read-once superposition then the continuity of characteristic polynomials with respect to their variable p implies

Property 5. *If $\mathcal{A}$ is closed and $p' \in W_{\mathcal{A}}(p)$, then $W_{\mathcal{A}}(p) \supseteq W_{\mathcal{A}}(p')$.*

---

[2]Note that the paper [4] uses the symbol $W_{\mathcal{B}}(p)$ for distributions that are approximable by *read-once* superpositions of functions from $\mathcal{B}$.

Further, for the sake of simplicity, unless otherwise stated, we shall consider the formula $f(f_1, \ldots, f_n)$, where $f, f_1, \ldots, f_n$ are symbols of Boolean functions, to represent a read-once superposition of the above functions, i.e. the function $f(f(\tilde{x}^{(1)}), \ldots, f_n(\tilde{x}^{(n)}))$, where $\tilde{x}^{(1)}, \ldots, \tilde{x}^{(n)}$ are pairwise disjoint sets of variables. Note that read-once superposition is, in a sense, in agreement with characteristic polynomial composition, precisely $h_{f(g,\ldots,g)}(p) = h_f(h_g(p))$.

Closed classes

A set of Boolean functions A is called a *closed class* if it is closed under superposition of functions and insertion/deletion of dummy variables (this definition agrees with property 1). For the sake of exposition clarity we shall now list all closed classes of Boolean functions. We shall be using notation conventions[3] from [6].

We introduce some of the necessary concepts, for all the other definitions see [6].

A function $f(x_1, \ldots, x_n)$ preserves $c \in \{0, 1\}$ if $f(c, \ldots, c) = c$.

For tuples $\alpha, \beta \in \{0, 1\}^n$ we shall say that $\alpha \leqslant \beta$ if for every i we have $\alpha_i \leqslant \beta_i$. A function $f(x_1, \ldots, x_n)$ is *monotone* if for every pair of tuples α, β, such that $\alpha \leqslant \beta$, holds $f(\alpha) \leqslant f(\beta)$.

A function f is *self-dual* if $f = f^*$.

A function f *possesses property* $\langle 1^\mu \rangle$ (*property* $\langle 0^\mu \rangle$), $\mu = 2, 3, 4, \ldots$, if any μ tuples on which the function takes value 1 (respectively 0) share a common unit (respectively zero) component. A function f *possesses property* $\langle 1^\infty \rangle$ (*property* $\langle 0^\infty \rangle$) if for some variable x_i: $f \leqslant x_i$ (respectively $f \geqslant x_i$).

We shall denote by $x_1 \oplus x_2$ the sum *modulo* 2 of the variables x_1 and x_2. A function $f(x_1, \ldots, x_n)$ is said to be *affine* if it may be represented as $c_0 \oplus c_1x_1 \oplus c_2x_2 \oplus \ldots \oplus c_nx_n$ for some constants $c_0, c_1, \ldots, c_n \in \{0, 1\}$.

A function $f(x_1, \ldots, x_n)$ is said to be a *conjunction* if it may be represented as $c_0(c_1 \vee x_1)(c_2 \vee x_2) \cdots (c_n \vee x_n)$ for some constants $c_0, c_1, \ldots, c_n \in \{0, 1\}$. A function $f(x_1, \ldots, x_n)$ is said to be a *disjunction* if it may be represented as $c_0 \vee (c_1x_1) \vee (c_2x_2) \vee \ldots \vee (c_nx_n)$ for some constants $c_0, c_1, \ldots, c_n \in \{0, 1\}$.

Closed classes of Boolean functions are exhausted by the list below.

1. The class of all Boolean functions P_2.
2. The class of 0-preserving functions T_0; 1-preserving functions T_1; constant preserving functions T_{01}.

[3] Unfortunately, the closed classes in Russian literature are denoted by symbols that differ from the symbols traditionally used for clones from Post's lattice.

3. The class of monotone functions M; 0-preserving monotone functions M_0; 1-preserving monotone functions M_1; constant preserving monotone functions M_{01}.

4. The class of self-dual functions S; constant preserving self-dual functions S_{01}; monotone self-dual functions SM.

5. Countable class families I^μ, MI^μ, I_1^μ, MI_1^μ ($\mu = 2, 3, \ldots, \infty$) of functions that possess the property $\langle 1^\mu \rangle$, are monotone (MI^μ, MI_1^μ) and 1-preserving (I_1^μ, MI_1^μ).

6. Countable class families O^μ, MO^μ, O_0^μ, MO_0^μ ($\mu = 2, 3, \ldots, \infty$) of functions that possess the property $\langle 0^\mu \rangle$, are monotone (MO^μ, MO_0^μ) and 0-preserving (O_0^μ, MO_0^μ).

7. The class of affine functions L; 0-preserving affine functions L_0; 1-preserving affine functions L_1; self-dual affine functions SL; constant preserving affine functions L_{01}.

8. The class of conjunctions K; 0-preserving conjunctions K_0; 1-preserving conjunctions K_1; constant preserving conjunctions K_{01}.

9. The class of disjunctions D; 0-preserving disjunctions D_0; 1-preserving disjunctions D_1; constant preserving disjunctions D_{01}.

10. The class of essentially unary functions U; self-dual unary functions SU; monotone unary functions MU; constant preserving unary functions U_{01}.

11. The class of constants[4] C; 0-preserving constants C_0; 1-preserving constants C_1.

The Hasse diagram of closed classes is represented in fig. 1.

Classes, containing M_{01}

Statement 1 [4]. $W_{M_{01}}(p) = [0, 1]$ *for all* $p \in (0, 1)$.

Proof. Consider the following Boolean functions

$$f_{n,m} = \bigvee_{i=1}^{m} x_{i1} x_{i2} \cdots x_{in}.$$

[4]These are the classes that are *not* clones.

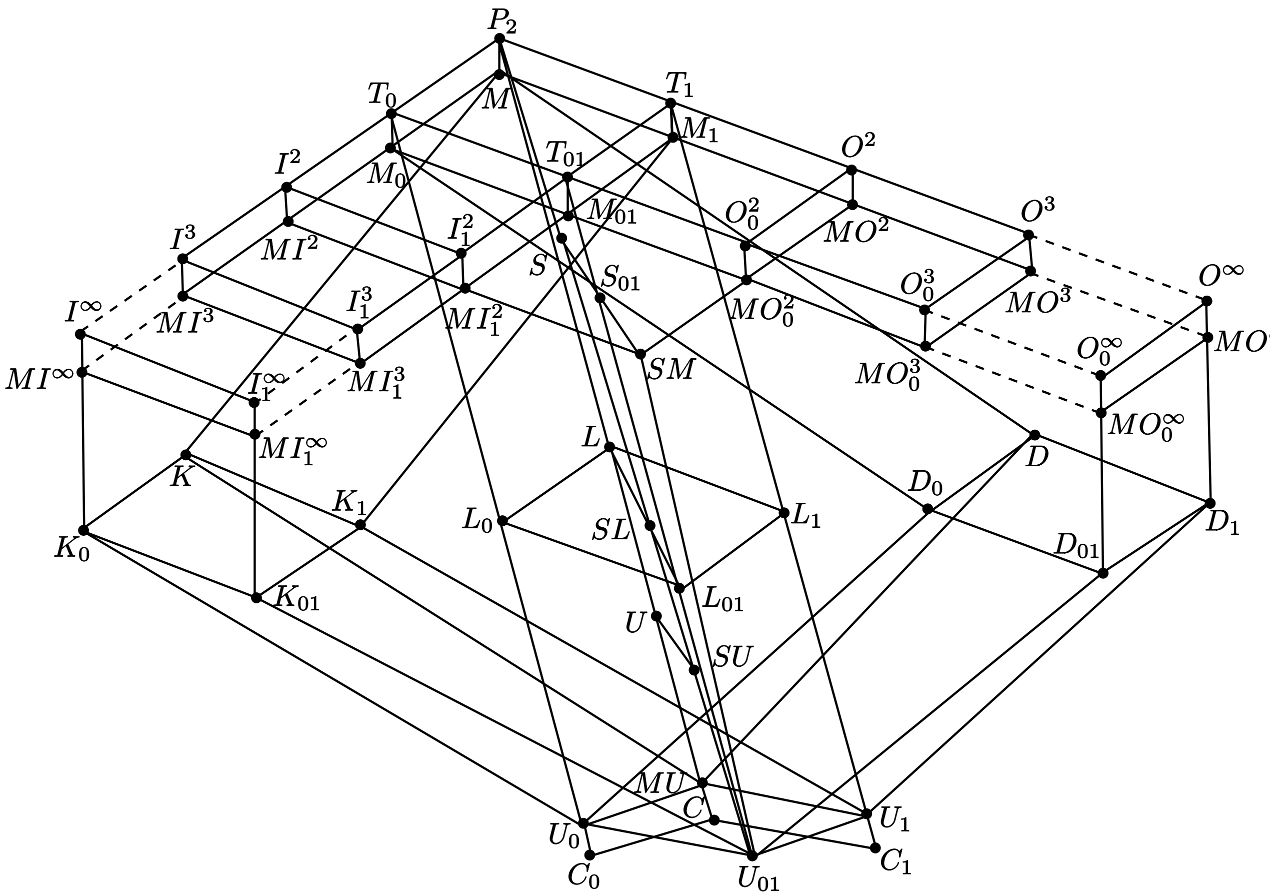

Figure 1

Evidently, $f_{n,m} \in M_{01}$. One can easily verify that $h_{f_{n,m}}(p) = 1 - (1 - p^n)^m$. Let us now demonstrate that for all $p \in (0, 1)$, any $\xi \in [0, 1]$ and arbitrary $\varepsilon > 0$ there exist such N, M, that $h_{f_{N,M}}(p) \in (\xi - \varepsilon, \xi + \varepsilon)$.

If $\xi = 0$, let us consider the functions $f_{n,1}$: $h_{f_{n,1}}(p) = p^n$. One can easily see that for any $\varepsilon > 0$ there exists such an N, that $p^N < \varepsilon$, which implies that $h_{f_{N,1}}(p)$ approximates the point $\xi = 0$ with precision ε. In a similar way, there exists such an M that $h_{f_{1,M}}(p)$ approximates the point $\xi = 1$ with precision ε. We further restrict our proof to $\xi \in (0, 1)$.

Let N be such that $1 - p^N > 1 - \xi$ and $p^N < \varepsilon$. Note that the sequence $(1 - p^N)^m$, $m = 1, 2, \ldots$ is decreasing to zero, since $1 - p^N < 1$ and $\lim\limits_{m\to\infty}(1 - p^N)^m = 0$. Therefore there exists such an M that

$$(1 - p^N)^{M+1} \leqslant 1 - \xi < (1 - p^N)^M.$$

Consider the difference:

$$(1 - p^N)^M - (1 - \xi) \leqslant (1 - p^N)^M - (1 - p^N)^{M+1} = (1 - p^N)^M p^N \leqslant p^N < \varepsilon.$$

Hence, $(1 - p^N)^M \in (1 - \xi, 1 - \xi + \varepsilon)$, which implies $1 - (1 - p^N)^M \in \in (\xi - \varepsilon, \xi) \subset (\xi - \varepsilon, \xi + \varepsilon)$. $\square$

By virtue of property 4 the statement 1 is easily generalized into

Statement 2. *Let $\mathcal{A}$ be one of the closed classes M_{01}, M_0, M_1, T_0, T_1, T_{01}, M, P_2. Then $W_{\mathcal{A}}(p) = [0,1]$ for any $p \in (0,1)$.*

Corollary. *Let $p \in (0,1)$, $\xi \in [0,1]$, $\varepsilon > 0$ be fixed and $\mathcal{A}$ be one of the closed classes M_{01}, M_0, M_1, T_0, T_1, T_{01}, M, P_2. Then there exists such a function $f \in \mathcal{A}$, that $h_f(p) \in (\xi - \varepsilon, \xi + \varepsilon)$.*

Classes S and S_{01}

By virtue of property 2 for any self-dual function f the identity $h_f(p) = 1 - h_f(1-p)$ holds, consequently $h_f(1/2) = 1/2$, which implies

Statement 3. $W_S(1/2) = \{1/2\}$, $W_{S_{01}}(1/2) = \{1/2\}$.

Let us show that with the exception of the "special" point $p = 1/2$ the functions from classes S and S_{01} also allow the approximation of an arbitrary Bernoulli distribution.

We shall say that a Boolean function f at the point $p \in (0,1)$ is a 0_ε-function, if $h_f(p) < \varepsilon$ (respectively, a 1_ε-function if $h_f(p) > 1 - \varepsilon$).

Let us show that substituting 0_ε- and 1_ε-functions for another function's variables differs very little from substituting the constants 0 and 1 respectively.

Lemma 1 [4]. *Let $f(x_1, \ldots, x_n)$ be an arbitrary Boolean function, $p \in (0,1)$, $\varepsilon > 0$ be fixed and g be a Boolean function which is an α_ε-function at the point p. Let $f_\alpha = f(\alpha, x_2, \ldots, x_n)$, $f' = f(g, x_2, \ldots, x_n)$. Then $|h_{f_\alpha}(p) - h_{f'}(p)| < \varepsilon$.*

Proof. Let $f_{\bar{\alpha}} = f(\bar{\alpha}, x_2, \ldots, x_n)$. Using the function f decomposition on its variable x_1 we obtain

$$h_{f'}(p) = (1 - h_g(p))h_{f_0}(p) + h_g(p)h_{f_1}(p). \tag{1}$$

Consider first $\alpha = 0$. Then $f_\alpha = f_0$, and (1) implies:

$$h_{f_\alpha}(p) - h_{f'}(p) = h_{f_0}(p) - h_{f'}(p) = h_g(p)(h_{f_0}(p) - h_{f_1}(p)).$$

Since g is a 0_ε-function we obtain that $|h_{f_\alpha}(p) - h_{f'}(p)| < \varepsilon |h_{f_0}(p) - h_{f_1}(p)|$.

Let now $\alpha = 1$. Then $f_\alpha = f_1$, and (1) implies:

$$h_{f_\alpha}(p) - h_{f'}(p) = (1 - h_g(p))(h_{f_1}(p) - h_{f_0}(p)).$$

Since g is a 1_ε-function, we obtain that $|h_{f_\alpha}(p) - h_{f'}(p)| < \varepsilon |h_{f_1}(p) - h_{f_0}(p)|$.

Note that $|h_{f_1}(p) - h_{f_0}(p)| \leqslant \max\{h_{f_1}(p), h_{f_0}(p)\} \leqslant 1$, which in both cases implies $|h_{f_\alpha}(p) - h_{f'}(p)| < \varepsilon$. The lemma is proved. □

Corollary [4]. *Let* $f(x_1, \dots, x_n)$ *be an arbitrary Boolean function,* $p \in (0,1)$*,* $\varepsilon > 0$ *and constants* $\alpha_1, \dots, \alpha_m \in \{0,1\}$ *be fixed and Boolean functions* $g_1, \dots, g_m$ *be* $(\alpha_i)_\varepsilon$*-functions at the point* p*. Let*

$$f_\alpha = f(\alpha_1, \dots, \alpha_m, x_{m+1}, \dots, x_n), \quad f' = f(g_1, \dots, g_m, x_{m+1}, \dots, x_n).$$

Then $|h_{f_\alpha}(p) - h_{f'}(p)| < m\varepsilon$*.*

The above corollary allows to reduce the verification of every point's approximability to the approximability of just two points 0 and 1 for many closed classes.

Lemma 2 [4]. *Let* $\mathcal{A}$ *be a closed class of Boolean functions,* $\mathcal{A} \not\subseteq L, K, D$*. For a fixed* $p \in (0,1)$ *the set* $W_\mathcal{A}(p)$ *coincides with* $[0,1]$ *iff* $0, 1 \in W_\mathcal{A}(p)$*.*

Proof. The necessity of the condition is evident, let us show its sufficiency. Let $p \in (0,1)$ be fixed, we shall demonstrate that for every $\xi \in [0,1]$ and $\varepsilon > 0$ there exists such a function $f \in \mathcal{A}$, that $h_f(p) \in (\xi - \varepsilon, \xi + \varepsilon)$.

Consider the class[5] $[\mathcal{A} \cup \{0,1\}]$. Since $\mathcal{A} \not\subseteq L, K, D$, the class $[\mathcal{A} \cup \{0,1\}]$ is either M or P_2. In both cases the corollary of statement 2 implies there exists such a function $g \in [\mathcal{A} \cup \{0,1\}]$, that $h_g(p) \in (\xi - \varepsilon/2, \xi + \varepsilon/2)$.

Then there exists such a function $\varphi \in \mathcal{A}$ that

$$g(x_1, \dots, x_n) = \varphi(\alpha_1, \dots, \alpha_m, x_1, \dots, x_n),$$

where $\alpha_1, \dots, \alpha_m \in \{0,1\}$. Since by lemma's conditions $0, 1 \in W_\mathcal{A}(p)$ at the given point p, the class $\mathcal{A}$ contains functions φ_0 and φ_1, which are a $0_{\varepsilon/2m}$- and a $1_{\varepsilon/2m}$-function respectively.

Consider the function

$$f = \varphi(\varphi_{\alpha_1}, \dots, \varphi_{\alpha_m}, x_1, \dots, x_n).$$

By virtue of lemma 1 we obtain that

$$|h_f(p) - h_g(p)| < m \cdot \frac{\varepsilon}{2m} = \frac{\varepsilon}{2}.$$

Together with the condition $h_g(p) \in (\xi - \varepsilon/2, \xi + \varepsilon/2)$ this implies that $h_f(p) \in (\xi - \varepsilon, \xi + \varepsilon)$, q.e.d. □

The above lemma allows a complete description of $W_S(p)$.

Statement 4. $W_S(p) = \begin{cases} [0,1] \text{ for } p \neq 1/2, \\ \{1/2\} \text{ for } p = 1/2. \end{cases}$

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[5]The notation $[\mathcal{B}]$ stands for closure under superposition of the function set $\mathcal{B}$.

Proof. The case $p = 1/2$ is described by statement 3. Since the class S is dual to itself, the property 3 allows to restrict our consideration to values $p \in (0, 1/2)$. Let us show that for such values of p we have $0, 1 \in W_S(p)$.

Let $m(x_1, x_2, x_3) = x_1x_2 \vee x_1x_3 \vee x_2x_3$ be the majority function. Consider a sequence of functions defined as follows: $f_0 = m$, $f_{n+1} = m(f_n, f_n, f_n)$. Evidently, $f_n \in S$.

Let $F(p) = h_m(p) = 3p^2(1-p) + p^3$. One can easily verify that

$$h_{f_{n+1}}(p) = F(h_{f_n}(p)) = \underbrace{F(F(\dots F}_{n+1 \text{ times}}(p))).$$

For $p \in (0, 1/2)$ we have $F(p) < p$, which implies that $h_{f_n}(p)$ decreases monotonously as n increases. Since $h_{f_n}(p) \geqslant 0$, there exists a limit $\xi = \lim\limits_{n\to\infty} h_{f_n}(p)$, for which $F(\xi) = \xi$ holds. It is easy to see that $\xi = 0$, hence $h_{f_n}(p) \to 0$ as $n \to \infty$. Therefore within the sequence f_n there exist 0_ε-functions for any $\varepsilon > 0$ given beforehand, hence $0 \in W_S(p)$.

Besides that, $\overline{f_n} \in S$ and $h_{\overline{f_n}}(p) \to 1$ as $n \to \infty$, which implies that $1 \in W_S(p)$. Thus, $0, 1 \in W_S(p)$ for $p \in (0, 1/2)$ and by virtue of lemma 2 we obtain $W_S(p) = [0, 1]$. □

Unlike the class S, the class S_{01} does not contain negation, yet it still allows the approximation of arbitrary Bernoulli distributions for $p \neq 1/2$.

Statement 5. $W_{S_{01}}(p) = \begin{cases} [0,1] \textit{ for } p \neq 1/2, \\ \{1/2\} \textit{ for } p = 1/2. \end{cases}$

Proof. As in the proof above, we restrict ourselves to the values $p \in (0, 1/2)$. The functions f_n from the proof of statement 4 belong to S_{01} as well, hence $0 \in W_{S_{01}}(p)$.

Let us now show that $1 \in W_{S_{01}}(p)$. Consider a sequence of Boolean functions

$$g_n(x_1, \dots, x_n) = \begin{cases} 0, \text{ if } x_1 = \dots = x_n = 0, \\ 1, \text{ if } x_1 = \dots = x_n = 1, \\ \overline{x_1}, \text{ on all other tuples.} \end{cases}$$

Note that $g_n \in S_{01}$, $h_{g_n}(p) = 1 - p - (1-p)^n + p^n$.

Let $\varepsilon > 0$ be fixed and let $f \in S_{01}$ be some $0_{\varepsilon/2}$-function for the given $p \in (0, 1/2)$. Since $0 \notin S_{01}$, we have $0 < h_f(p) < \varepsilon/2$.

Consider the functions $\varphi_n = g_n(f, \dots, f)$. Then

$$h_{\varphi_n}(p) = h_{g_n}(h_f(p)) = 1 - h_f(p) - (1 - h_f(p))^n + (h_f(p))^n \geqslant 1 - (h_f(p) + (1 - h_f(p))^n).$$

Since $h_f(p) > 0$, we have $1 - h_f(p) < 1$, and, hence, there exists such an N that $(1 - h_f(p))^N < \varepsilon/2$. Therefore

$$h_{\varphi_N}(p) \geqslant 1 - \left(\frac{\varepsilon}{2} + \frac{\varepsilon}{2}\right) = 1 - \varepsilon.$$

Consequently, φ_N at the point p is a 1_ε-function. Then $1 \in W_{S_{01}}(p)$ and by virtue of lemma 2 we have $W_{S_{01}}(p) = [0, 1]$. □

Classes, contained within L, K or D

The following statements are easily verified.

Statement 6 [4]. *Let $\mathcal{A}$ be one of the classes K, K_0, K_1, K_{01}. Then for any $p \in (0, 1)$ we have $W_{\mathcal{A}}(p) \subseteq \bigcup_n \{p^n\} \cup \{0, 1\}$.*

Statement 7 [4]. *Let $\mathcal{A}$ be one of the classes D, D_0, D_1, D_{01}. Then for any $p \in (0, 1)$ we have $W_{\mathcal{A}}(p) \subseteq \bigcup_n \{1 - (1 - p)^n\} \cup \{0, 1\}$.*

The sets $W_{\mathcal{A}}(p)$ for classes of affine functions are also at most countable.

Statement 8 [4]. *Let $\mathcal{A}$ be one of the classes L, L_0, L_1, L_{01}, SL. Then for any $p \in (0, 1)$ we have $W_{\mathcal{A}}(p) \subseteq \bigcup_n \{\frac{1}{2}(1 \pm (1 - 2p)^n)\} \cup \{\frac{1}{2}\}$.*

Proof. Let $f(x_1, \dots, x_n) \in L$. Without loss of generality we may suppose that all variables of the function f are essential. Then $f = x_1 \oplus x_2 \oplus \dots \oplus x_n \oplus c$. Hence, $h_f(p) = \sum_{\text{odd } i} \binom{n}{i} p^i (1 - p)^{n-i}$ for $c = 0$ and $h_f(p) = \sum_{\text{even } i} \binom{n}{i} p^i (1 - p)^{n-i}$ for $c = 1$.

Since $\sum_i \binom{n}{i} p^i (1-p)^{n-i} = 1$ and $\sum_i \binom{n}{i} (-p)^i (1-p)^{n-i} = (1-2p)^n$, we obtain that $2 \sum_{\text{odd } i} \binom{n}{i} p^i (1-p)^{n-i} = 1 - (1-2p)^n$ and $2 \sum_{\text{even } i} \binom{n}{i} p^i (1-p)^{n-i} = 1 + (1-2p)^n$, which implies $h_f(p) = \frac{1}{2}(1 \pm (1 - 2p)^n)$. One can easily see that the only limit point of such values is $1/2$. The statement is proved. □

The following statement is nearly evident.

Statement 9. *Let $\mathcal{A}$ be one of the classes U, SU, U_0, U_1, MU, C, C_0, C_1, U_{01}. Then for any $p \in (0, 1)$ we have $W_{\mathcal{A}}(p) \subseteq \{0, 1, p, 1 - p\}$.*

Classes with properties $\langle 0^\mu \rangle$, $\langle 1^\mu \rangle$, $\langle 0^\infty \rangle$, $\langle 1^\infty \rangle$

We first determine the structure of the sets $W_{\mathcal{A}}(p)$ for classes that possess the properties $\langle 0^\infty \rangle$, $\langle 1^\infty \rangle$.

Statement 10. *Let $\mathcal{A}$ be one of the classes I^∞, MI^∞, I_1^∞, MI_1^∞. Then $W_{\mathcal{A}}(p) = [0, p]$.*

Let $\mathcal{A}$ be one of the classes O^∞, MO^∞, O_0^∞, MO_0^∞. Then $W_{\mathcal{A}}(p) = [p, 1]$.

Proof. Since classes O^∞, MO^∞, O_0^∞, MO_0^∞ are dual to classes I^∞, MI^∞, I_1^∞, MI_1^∞ respectively, it suffices to prove the statement for classes I^∞, MI^∞, I_1^∞, MI_1^∞.

Let $f(x_1, \ldots, x_n) \in I^\infty$. Then without loss of generality we may represent f as $x_1 f'(x_2, \ldots, x_n)$ and consequently $h_f(p) = p h_{f'}(p) \leqslant p$. Hence, $W_{I^\infty}(p) \subseteq [0, p]$.

By virtue of statement's 2 corollary for any $\xi \in [0, 1]$, $\varepsilon > 0$ there exists such a function $f(x_1, \ldots, x_n) \in M_{01}$, that $h_f(p) \in (\xi - \varepsilon, \xi + \varepsilon)$. Consider $g = y f(x_1, \ldots, x_n)$, where y is a variable different from $x_1, \ldots, x_n$. Then $g \in MI_1^\infty$ and $h_g(p) = p h_f(p)$. Since ξ is arbitrary, we obtain that $W_{MI_1^\infty}(p) \supseteq [0, p]$.

Due to the relations between the considered classes and by virtue of property 4 we obtain that

$$W_{I^\infty}(p) = W_{MI^\infty}(p) = W_{I_1^\infty}(p) = W_{MI_1^\infty}(p) = [0, p].$$

The statement is proved. □

To describe the sets of distributions approximable by classes with properties $\langle 0^\mu \rangle$, $\langle 1^\mu \rangle$, we need two auxiliary statements.

Lemma 3 (generalization of the Erdös — Ko — Rado theorem [7]). *Let X be a finite set of cardinality n. Let $\mathcal{F}$ be a family of subsets of X all having cardinality i, such that for any $F_1, \ldots, F_k \in \mathcal{F}$ holds $\bigcap_{j=1}^{k} F_j \neq \varnothing$. If the inequality $ki/(k-1) \leqslant n$ holds then $|\mathcal{F}| \leqslant \binom{n-1}{i-1}$.*

Corollary. *Let a Boolean function $f(x_1, \ldots, x_n)$ possess the property $\langle 1^\mu \rangle$ and let $\mu i/(\mu - 1) \leqslant n$. Then A_i (the number of tuples of weight i, on which f equals 1) does not exceed $\binom{n-1}{i-1}$.*

Lemma 4. *Let $q \in (0, 1)$ and $k < nq$. Then*

$$\sum_{i=0}^{k} \binom{n}{i} q^i (1-q)^{n-i} \leqslant \frac{nq(1-q)}{(qn-k)^2}.$$

Proof. Let X be a random variable having binomial distribution with parameters n and q. Then

$$\sum_{i=0}^{k}\binom{n}{i}q^i(1-q)^{n-i} = \mathsf{P}\{X \leqslant k\} = \mathsf{P}\{\mathsf{E}X-X \geqslant \mathsf{E}X-k\} \leqslant \mathsf{P}\{|X-\mathsf{E}X| \geqslant \mathsf{E}X-k\},$$

where $\mathsf{E}X$ is the mean of the random variable X. For the chosen X its mean equals $\mathsf{E}X = nq$, while its variance equals $\mathsf{D}X = nq(1-q)$. Due to the condition $k < nq$, we have $\mathsf{E}X - k > 0$. Using Chebyshev's inequality [8], we obtain

$$\mathsf{P}\{|X - \mathsf{E}X| \geqslant \mathsf{E}X - k\} \leqslant \frac{\mathsf{D}X}{(\mathsf{E}X-k)^2} = \frac{nq(1-q)}{(qn-k)^2}.$$

The lemma is proved. □

Statement 11. *Let $\mathcal{A}$ be one of the classes I^μ, MI^μ, I_1^μ, MI_1^μ. Then*

$$W_{\mathcal{A}}(p) = \begin{cases} [0,p], & \textit{if } 0 < p \leqslant 1 - \frac{1}{\mu}, \\ [0,1], & \textit{if } 1 - \frac{1}{\mu} < p < 1. \end{cases}$$

Let $\mathcal{A}$ be one of the classes O^μ, MO^μ, O_0^μ, MO_0^μ. Then

$$W_{\mathcal{A}}(p) = \begin{cases} [0,1], & \textit{if } 0 < p < \frac{1}{\mu}, \\ [p,1], & \textit{if } \frac{1}{\mu} \leqslant p < 1. \end{cases}$$

Proof. By virtue of property 3 it suffices to prove the statement for classes I^μ, MI^μ, I_1^μ, MI_1^μ. Taking into account the property 4 and the statement 10, we obtain that $W_{I^\mu}(p), W_{MI^\mu}(p), W_{I_1^\mu}(p), W_{MI_1^\mu}(p) \supseteq [0,p]$.

Let us now show that for $p < 1 - \frac{1}{\mu}$ we have $W_{I^\mu}(p) \subseteq [0,p]$. Let $\xi = \max W_{I^\mu}(p)$. Since the set $W_{I^\mu}(p)$ contains all its limit points ξ is well-defined. If $\xi = 1$, then by virtue of lemma 2 we have $W_{I^\mu}(p) = [0,1]$. In all other cases, by virtue of property 5 we obtain

$$[0,\xi] \supseteq W_{I^\mu}(p) \supseteq W_{I^\mu}(\xi) \supseteq [0,\xi].$$

Hence $W_{I^\mu}(p) = [0,\xi]$. Let $f(x_1,\ldots,x_n) \in I^\mu$. Then $h_f(p) = \sum\limits_{i=0}^{n} A_i p^i (1-p)^{n-i}$. According to lemma 3 corollary we have $A_i \leqslant \binom{n-1}{i-1}$ for $i \leqslant n(1-\frac{1}{\mu})$. For all other values of i we use the trivial upper bound $A_i \leqslant \binom{n}{i}$. Let $k = \lfloor n(1-\frac{1}{\mu})\rfloor$.

Then

$$h_f(p) \leqslant \sum_{i=0}^{k} \binom{n-1}{i-1} p^i (1-p)^{n-i} + \sum_{i=k+1}^{n} \binom{n}{i} p^i (1-p)^{n-i} =$$

$$= \sum_{i=1}^{k} \binom{n-1}{i-1} p^i (1-p)^{n-i} + \sum_{i=k+1}^{n} \left(\binom{n-1}{i} + \binom{n-1}{i-1} \right) p^i (1-p)^{n-i} =$$

$$= \sum_{i=1}^{n} \binom{n-1}{i-1} p^i (1-p)^{n-i} + \sum_{i=k+1}^{n} \binom{n-1}{i} p^i (1-p)^{n-i} =$$

$$= \sum_{i=0}^{n-1} \binom{n-1}{i} p^{i+1} (1-p)^{n-1-i} + \sum_{j=0}^{n-k-1} \binom{n-1}{j-1} (1-p)^j p^{n-j} =$$

$$= p + (1-p) \sum_{j=0}^{n-k-2} \binom{n-1}{j} (1-p)^j p^{n-1-j}.$$

We now apply the inequality from lemma 4 to the resulting sum above, letting $q = 1-p$. Let us verify the inequality $(n-1)(1-p) > n-k-2$. Indeed:

$$(n-1)(1-p) - (n-k-2) = n-1-p(n-1)-n+k+2 = k+1-(n-1)p.$$

By choice of k we have $k \geqslant n\left(1-\frac{1}{\mu}\right)-1$, and besides $p < 1-\frac{1}{\mu}$, which implies

$$k+1-(n-1)p \geqslant n\left(1-\frac{1}{\mu}\right) - (n-1)\left(1-\frac{1}{\mu}\right) = 1-\frac{1}{\mu} > 0.$$

Thus, by virtue of the inequality from lemma 4 we have

$$h_f(p) \leqslant p + (1-p)\frac{(n-1)p(1-p)}{(k+1-(n-1)p)^2}.$$

One can easily verify that considering k as a function of n, $k(n)$, for $p < 1-\frac{1}{\mu}$ we have

$$\lim_{n\to\infty} \frac{(n-1)p(1-p)}{(k(n)+1-(n-1)p)^2} = 0.$$

Otherwise saying, the characteristic polynomials of functions from I^{μ} with sufficiently many variables cannot considerably exceed the value p at the point p. The obtained estimate easily implies that for every N there exists only a finite number of such functions $f \in I^\mu$, that

$$h_f(p) > p + (1-p)\frac{(N-1)p(1-p)}{(k(N)+1-(N-1)p)^2}.$$

Yet, should there be just one such function f', as proved earlier, the entire segment $[0, h_{f'}(p)]$ would be contained in $W_{I^\mu}(p)$, hence there should be infinitely many such functions. This contradiction shows that in fact for all $f \in I^\mu$ and $p < 1 - \frac{1}{\mu}$ the inequality $h_f(p) \leqslant p$ holds.

Let now $p = 1 - \frac{1}{\mu}$. By the reasoning above for any function $f \in I^\mu$ and any $p' < 1 - \frac{1}{\mu}$ we have $h_f(p') \leqslant p' < 1 - \frac{1}{\mu}$. By continuity of $h_f(p)$ we obtain that $h_f\left(1 - \frac{1}{\mu}\right) \leqslant 1 - \frac{1}{\mu}$.

Thus, for $p \leqslant 1 - \frac{1}{\mu}$ the equality $W_{I^\mu}(p) = [0, p]$ holds and by virtue of property 4 and statement 10 the same is true for classes I_1^μ, MI^μ, MI_1^μ.

Let us now consider the values $p > 1 - \frac{1}{\mu}$. We define a sequence of functions f_n having $n\mu + 1$ variables the following way: on tuples of weight $n\mu - n + 1$ or more let f_n equal 1, on all other tuples let f_n equal 0 (such functions have been considered in [9] as well). Then any μ tuples on which the function f_n equals 1 have at most μn zero components, and therefore all share a component that equals 1. Thus f_n possesses the property $\langle 1^\mu \rangle$. Also, one can easily see that f_n are monotone and 1-preserving. Hence, $f_n \in MI_1^\mu$.

One can easily verify that

$$h_{f_n}(p) = \sum_{i=n\mu-n+1}^{n\mu+1} \binom{n\mu+1}{i} p^i (1-p)^{n\mu+1-i} = 1 - \sum_{i=0}^{n\mu-n} \binom{n\mu+1}{i} p^i (1-p)^{n\mu+1-i}.$$

Let us now apply the inequality from lemma 4 to the obtained sum, letting $q = p$. We note that due to the condition $p > 1 - \frac{1}{\mu}$ we have $(n\mu+1)p - (n\mu - n) > (n\mu+1)(1-\frac{1}{\mu}) - n\mu(1-\frac{1}{\mu}) = 1 - \frac{1}{\mu} > 0$. Consequently,

$$h_{f_n}(p) \geqslant 1 - \frac{(n\mu+1)p(1-p)}{((n\mu+1)p - n(\mu-1))^2}.$$

Thus we see that $h_{f_n}(p) \to 1$ as $n \to \infty$, and consequently, by virtue of lemma 2 we have $W_{MI_1^\mu}(p) = [0,1]$. Due to the property 4 the same is true for classes I^μ, I_1^μ, MI^μ. The statement is proved. □

Class SM

Statement 12. $W_{SM}(p) = \begin{cases} [0,p], \textit{ for } 0 < p < 1/2, \\ 1/2, \textit{ for } p = 1/2, \\ [p,1], \textit{ for } 1/2 < p < 1. \end{cases}$

Proof. The inclusion of $W_{SM}(p)$ into the set described in the statement's conditions follows straight from the inclusion $SM \subseteq MI_1^2 \cap MO_0^2$. In order to prove the statement we have to establish that for every $p \in (0,1)$ all the points of the before mentioned set can indeed be approximated.

Since the class SM is dual to itself, it suffices to prove the statement only for $0 < p < 1/2$, i. e. prove that for every $p \in (0, 1/2)$, any $\xi \in (0,p)$ and any $\varepsilon > 0$ there exists such a function $f \in SM$, that $h_f(p) \in (\xi - \varepsilon, \xi + \varepsilon)$.

Let m be the majority function, consider the sequence of functions:

$$f_n(x_0, x_1, \ldots, x_n) = m(x_0, x_1, m(x_0, x_2, m(x_0, x_3, m(\ldots, m(x_0, x_{n-1}, x_n)\ldots)))).$$

One easily sees that

$$h_{f_n}(p) = p(1-(1-p)^n) + (1-p)p^n = p - (p(1-p)^n - (1-p)p^n).$$

Consider $\chi_n(p) = p - h_{f_n}(p) = p(1-p)^n - (1-p)p^n$. For all $p \in (0, 1/2)$ we have $\chi_n(p) \geqslant 0$ and $\chi_n(p) \to 0$ as $n \to \infty$. Let us show that $\chi_n(p)$ converges to zero uniformly on the segment $[0, 1/2]$.

Since $\chi_n(0) = \chi_n(1/2) = 0$ and $\chi_n(p) \geqslant 0$, the function $\chi_n(p)$ has a global maximum on the segment $[0, 1/2]$ at some point p_0, where the equality $\chi_n'(p_0) = 0$ holds. Differentiating $\chi_n(p)$ with variable p we obtain that the condition $\chi_n'(p_0) = 0$ is equivalent to

$$(1-p_0)^n + p_0^n = n(p_0(1-p_0)^{n-1} + p_0^{n-1}(1-p_0)),$$

which implies

$$\left(\frac{1-p_0}{p_0}\right)^{n-1} = \frac{1}{n}\left(\left(\frac{1-p_0}{p_0}\right)^n + 1\right) - \frac{1-p_0}{p_0}. \tag{2}$$

Since the function $\chi_n(p)$ has the global maximum on the segment $[0, 1/2]$ at the point p_0 and due to the relation (2), we obtain

$$\chi_n(p) \leqslant \chi_n(p_0) = p_0(1-p_0)^n - p_0^n(1-p_0) = p_0^n(1-p_0)\left(\left(\frac{1-p_0}{p_0}\right)^{n-1} - 1\right) =$$

$$= p_0^n(1-p_0)\left(\frac{1}{n}\left(\frac{1-p_0}{p_0}\right)^n + \frac{1}{n} - \frac{1-p_0}{p_0} - 1\right) \leqslant \frac{2}{n}.$$

It follows easily that $\chi_n(p)$ converges uniformly to zero on the entire segment $[0, 1/2]$.

Let $\varepsilon > 0$ be fixed. Then there exists such a number N that $\chi_N(p) < \varepsilon$ for all $p \in [0, 1/2]$. Let us construct a sequence of Boolean functions g_i the following way. Let $g_0 = f_N$, $g_{i+1} = f_N(g_i, \dots, g_i)$. Then one can easily verify that

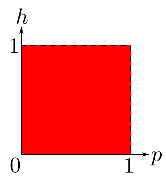
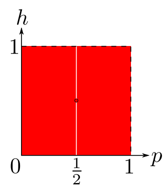
$$h_{g_i}(p) = \underbrace{h_{f_N}(h_{f_N}(\dots h_{f_N}}_{i \text{ times}}(p))). \tag{3}$$

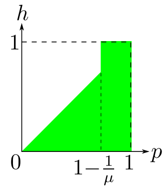
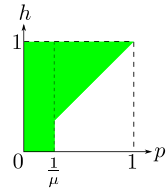
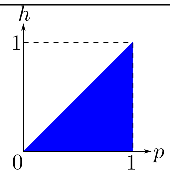
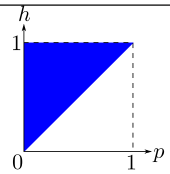
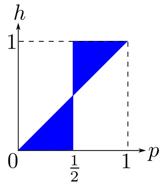
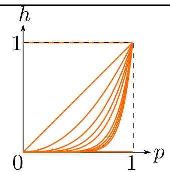
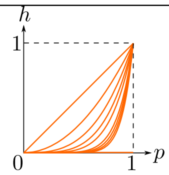
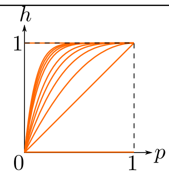
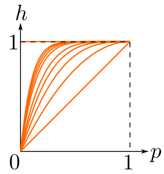
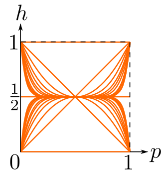
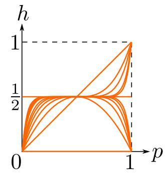
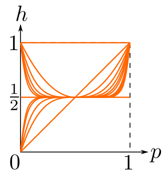
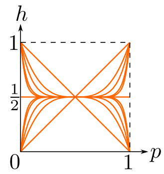
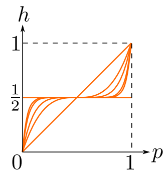
By choice of N we have that $h_{g_i}(p) - h_{g_{i+1}}(p) < \varepsilon$, while $h_{g_i}(p)$ decrease as i grows since $h_{f_N}(p) < p$, and, due to equation (3) and the equality $h_{f_N}(0) = 0$, similarly to the proof of statement 4 we obtain that $\lim\limits_{i\to\infty} h_{g_i}(p) = 0$. This implies that for any point $\xi \in [0, p]$ there exists such a number i that $h_{g_i}(p) \in (\xi - \varepsilon, \xi + \varepsilon)$. The statement is proved. □

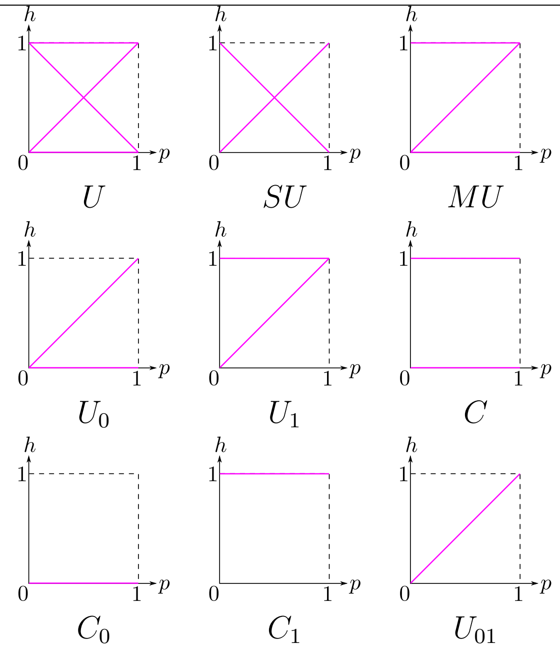
Approximable sets

Joining the statements 1–12, we obtain a grouping of Boolean function closed classes according to the type of the sets of distributions, approximable by functions from these classes. Fig. 2 represents Post's lattice with different types of classes given in different colors. The colors and the characteristics of the approximable sets are outlined in the table below.

The classification thus obtained allows, by the way, to formulate necessary conditions of distribution approximability in the case of distribution transformations by read-once Boolean formulas as well.

Color	Description	Approximable sets
Red	Classes allowing the approximation of arbitrary distributions for all $p \in (0, 1)$, with possible exception of $p = 1/2$	 $P_2, T_0, T_1, T_{01}, M_0, M_1, M_{01}$  S, S_{01}

Color	**Description**	**Approximable sets**
Green	Classes allowing the approximation of arbitrary distributions for some $p \in (0,1)$ and having a continuum set of approximable distributions for other p	 $I^\mu, I_1^\mu, MI^\mu, MI_1^\mu$  $O^\mu, O_0^\mu, MO^\mu, MO_0^\mu$ $\mu = 2, 3, \ldots$
Blue	Classes having a continuum set of approximable distributions for all $p \in (0,1)$, yet not allowing the approximation of arbitrary distributions for any p	 $I^\infty, I_1^\infty, MI^\infty, MI_1^\infty$  $O^\infty, O_0^\infty, MO^\infty, MO_0^\infty$  SM
Orange	Classes having countable sets of approximable distributions for all $p \in (0,1)$	 K, K_1  K_0, K_{01}  D, D_0  D_1, D_{01}  L  L_0  L_1  SL  L_{01}

Color	Description	Approximable sets
Purple	Classes having finite sets of approximable distributions for all $p \in (0,1)$	

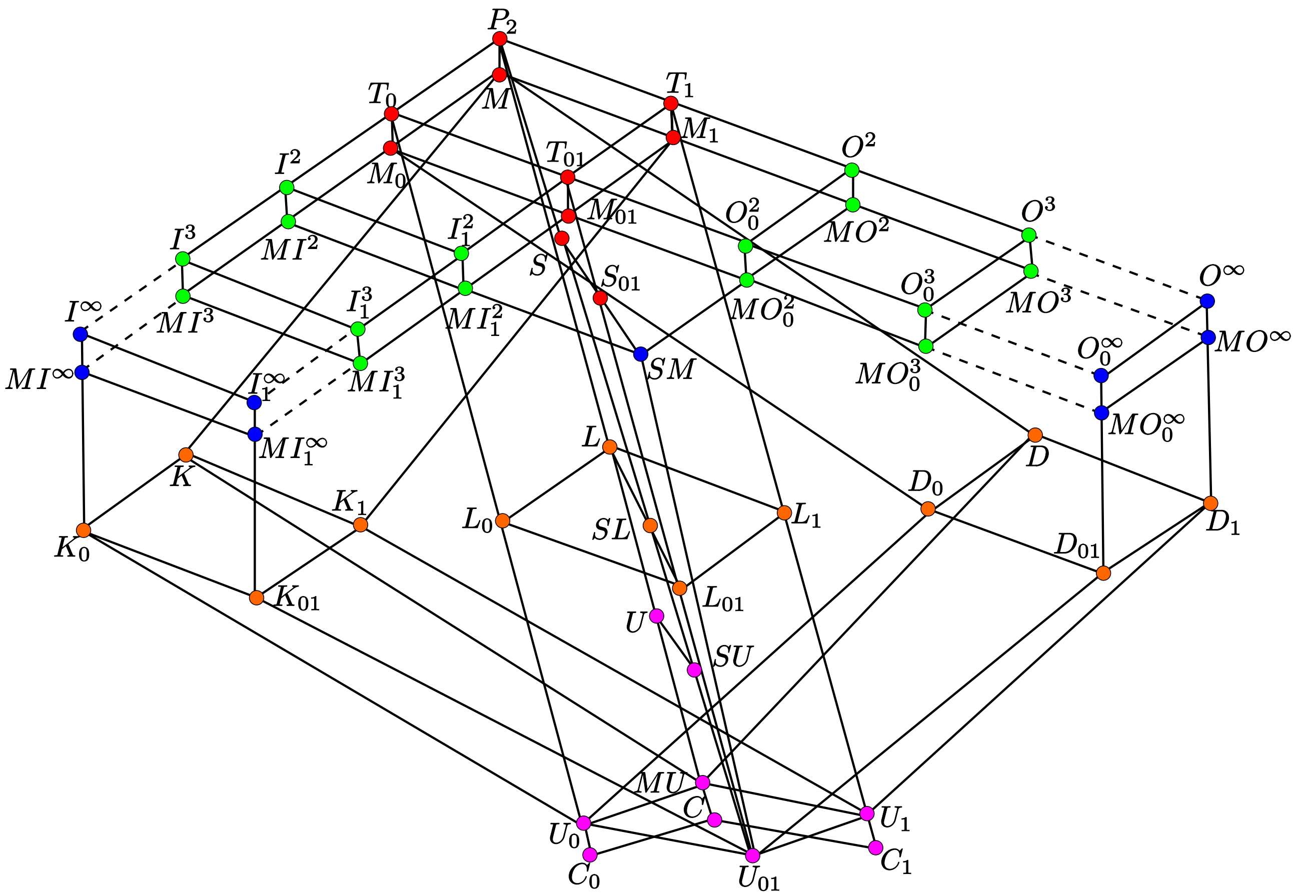


Figure 2

The described classification remains essentially unchanged when passing to the case of functions from closed classes applied to independent random variables whose distributions belong to some finite set of initial distributions (i. e. need not be identical).

One can easily check that for all classes containing MI_1^∞ or MO_0^∞ the set of approximable distributions for a given finite set of initial distributions is defined by just one element among those distributions — the one producing the largest set of approximable distributions.

For the class SM the presence of two distributions, having probability of 1 greater and less then $1/2$ respectively in the set of initial distributions, allows the approximation of an arbitrary distribution. Should this not be the case, just as described above the approximable distributions are then defined by just one element from the set of initial distributions.

The classes with an approximable set having no more than one limit point, considering a finite set of initial distributions does not increase the number of limit points.

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