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Y.V. Mashtakov, S.S. Tkachev

Reference angular motion synthesis for reaction wheels desaturation

Moscow — 2017

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Построение опорного углового движения для обеспечения разгрузки маховиков

В работе рассматривается задача бестопливной разгрузки маховичной системы управления в режиме солнечной ориентации. Показано, что всегда можно подобрать такую ориентацию спутника, чтобы происходила разгрузка маховиков как при использовании момента сил давления солнечного излучения, так и при использовании гравитационного момента. Получены точные выражения для нахождения ориентации КА, обеспечивающей оптимальную разгрузку маховиков, а также приведены более простые формулы для построения близкой к оптимальной ориентации

Ключевые слова: маховик, разгрузка, момент сил солнечного давления, гравитационный момент, ляпуновское управление

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Reference angular motion synthesis for reaction wheels desaturation

The paper considers the problem of fuelless reaction wheels desaturation for a satellite in solar acquisition mode. It is shown that there is always angular motion in the vicinity of required one when reaction wheels can be desaturated either with the aid of solar pressure torque or gravitational one. Optimal desaturation attitude is found in close form. Additionally, near optimal simplified expressions for desaturation attitude are also presented.

Key words: reaction wheel, desaturation, solar radiation pressure, gravitational torque, Lyapunov based attitude control

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Introduction

Most of all space missions involve special angular motion. It could be solar pointing for spacecraft (SC) battery recharging, remote sensing missions or stabilization for data transmitting. Usually gyroscopic actuators such as Reaction Wheels (RW) and Control Moment Gyroscopes (CMG) are used to control SC attitude. They can offer good accuracy and decent control torque, which allow us to perform fast and precise maneuvers. Unfortunately, these systems have substantial drawback: under certain conditions they are unable to produce the necessary control torque along the specific direction, therefore SC lost controllability. For RW this condition is named saturation, and for CMG it is the singularity problem. This paper considers the RW saturation problem only.

The saturation of RW occurs when its angular rate reaches the limit, , thus RW will not be able to produce the necessary control torque. In order to desaturate RW one can use additional attitude control systems, such as magnetorquers or thrusters. Magnetic attitude control systems can be used only in the presence of external magnetic field, i.e. only at sufficiently low Earth orbits. Thrusters require propellant, so their utilization might greatly affect the SC lifetime and/or maximum payload mass.

Main reason of RW saturation is the external torques influence, e.g. gravitational and solar radiation pressure (SRP) torque. These torques are usually accounted only in control algorithm, so RW must compensate them, but during required attitude motion construction they are usually omitted. In this paper we suggest an algorithm of angular motion synthesis that will allow us to simultaneously provide SC solar pointing and RW desaturation using SRP and gravitational torques.

Notice, that the similar problem has been already investigated in some papers. For example, in [1] the algorithm of angular motion synthesis for RW desaturation using gravitational torque is investigated, but the suggested technique is essentially differs from the one we describe here. In papers [2,3] another algorithm of RW desaturation using gravitational torque was suggested, but it requires unconstrained SC angular motion, while we consider solar pointed SC. There are also papers that suggest to use magnetic [4,5] and reactive [6] attitude control systems for RW desaturation.

The paper is organized as follows. In the first section, we present detailed problem statement. In the second section we obtain simplified expressions for SRP torque. The third section is dedicated to reference angular motion construction. In the fourth section we describe well-known Lyapunov-based attitude control algorithm. Finally, in the fifth section we provide some numerical simulations.

1. Problem Statement

Solar stabilization is one of the most common angular motion modes. It is used to recharge batteries installed on the spacecraft. The only constraint to be satisfied during this motion is that normal to SC solar panels must be aligned with vector from the satellite to the Sun (Sun direction).

As it was mention earlier, at Low Earth Orbits it is reasonable to use magnetic attitude control systems, such as magnetics coils, for desaturation. On the other hand, at high altitudes Earth magnetic field is too weak to use it. As an example of such missions, we can mention geostationary satellites and the ones that move along highly elliptical orbits, which are used for study the interactions between Earth magnetic field and solar radiation, e.g. missions Cluster II and Magnetospheric MultiScale [7,8]. At such orbits, we either have to use thrusters for desaturation or develop new methods of angular motion construction. We will investigate the last approach.

Consider the following mission:

- SC moves along highly elliptical keplerian orbit, where we can distinguish two different modes of angular motion: near the pericenter, where the gravitational torque prevails, and far from the pericenter, where gravitational torque is negligible and only SRP torque will affect the SC angular motion.
- There are two identical solar panels installed on the SC. Their parameters, such as coefficients of specularity and reflectivity, area and normals are supposed to be known. Solar panels are rigidly fixed.
- We know SC tensor of inertia and center of mass location.
- It is necessary to provide small angle between normal to solar panel and Sun direction for battery recharging.
- Attitude control system consists of three noncomplanar reaction wheels.

For this mission we have to obtain the algorithm of angular motion synthesis that provide RW desaturation.

2. Solar Radiation Pressure Torque

Consider the following model of SRP [9]:

$$d\mathbf{F} = -\frac{\Phi_0}{c}(\mathbf{r}_s,\tilde{\mathbf{n}})\left((1-\alpha)\mathbf{r}_s + 2\alpha\mu(\mathbf{r}_s,\tilde{\mathbf{n}})\tilde{\mathbf{n}} + \alpha(1-\mu)\left(\mathbf{r}_s + \frac{2}{3}\tilde{\mathbf{n}}\right)\right)dS,$$

where $\Phi_0 = 1367\ \mathrm{W/m^2}$ is solar constant, c is speed of light, $\mathbf{r}_s$ is Sun direction, i.e. unit vector from SC center of mass to the Sun, $\tilde{\mathbf{n}}$ is normal to the surface element dS, α, μ are reflectivity and specularity respectively. In accordance with this model, we can obtain the following expression for the solar radiation pressure torque:

$$\mathbf{M}_i = -S\frac{\Phi_0}{c}(\mathbf{r}_s,\mathbf{n}_i)\mathbf{R}_i \times \times\left((1-\alpha)\mathbf{r}_s + 2\alpha\mu(\mathbf{r}_s,\mathbf{n}_i)\mathbf{n}_i + \alpha(1-\mu)\left[\mathbf{r}_s + \frac{2}{3}\mathbf{n}_i\right]\right),\quad i = 1,2, \tag{1}$$

where $\mathbf{R}_i$ – is the radius-vector from SC center of mass to the center of i-th solar panel, S is the solar panel area (we consider identical solar panels, so their area, reflectivity and specularity are the same), $\mathbf{n}_i$ is the normal to the i-th solar panel surface. Normals to the solar panels are not necessarily coincide (Fig. 1), so angle between them can be arbitrary, but we will consider the case when this angle is rather small.

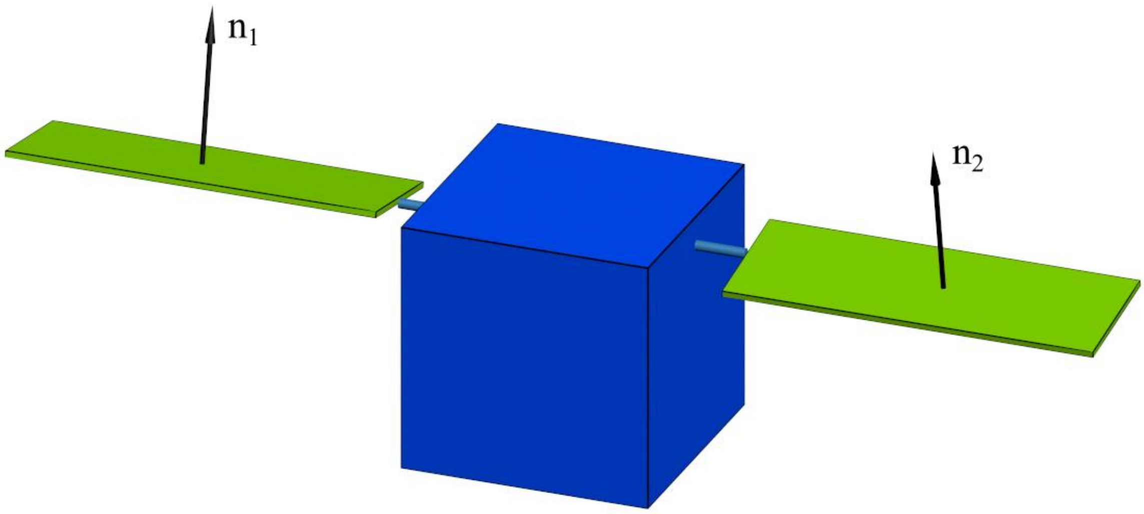

Fig. 1. Satellite Schematics

Total SRP torque is

$$\mathbf{M}_s = \mathbf{M}_1 + \mathbf{M}_2.$$

Introduce the following notation:

$$\mathbf{R} = \frac{1}{2}(\mathbf{R}_1 + \mathbf{R}_2), \quad \boldsymbol{\rho} = \frac{1}{2}(\mathbf{R}_1 - \mathbf{R}_2),$$

$$\mathbf{n} = \frac{1}{2}(\mathbf{n}_1 + \mathbf{n}_2), \quad \boldsymbol{\nu} = \frac{1}{2}(\mathbf{n}_1 - \mathbf{n}_2),$$

$$a = -S\frac{\Phi_0}{c}(1-\alpha\mu), \quad b = -S\frac{2\Phi_0}{3c}\alpha(1-\mu), \quad d = -2S\frac{\Phi_0}{c}\alpha\mu.$$

Take into account that angle between $\mathbf{n}_1, \mathbf{n}_2$ is small:

$$|\mathbf{n}| = \cos\delta \approx 1 - \delta^2 \approx 1, \quad |\boldsymbol{\nu}| = \sin\delta \approx \delta, \tag{2}$$

where $0 \le \delta \le \pi/2$ is the angle between $\mathbf{n}$ and $\mathbf{n}_1$ (the same as the angle between $\mathbf{n}$ and $\mathbf{n}_2$) (see Fig. 2). Further, we will suppose that the constraint implied on the SC attitude is

$$(\mathbf{r}_s, \mathbf{n}) \ge \cos\theta_{max} \tag{3}$$

where $\theta_{max} << 1$ is the maximum acceptable angle between the mean normal $\mathbf{n}$ and Sun direction.

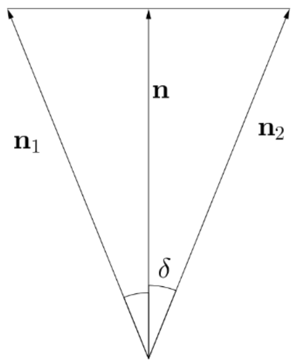

Fig. 2. Normals to solar panels surfaces

Let us rewrite expressions for SRP torque using the introduced notation:

$$\mathbf{M}_i = (\mathbf{R} \pm \boldsymbol{\rho}) \times \left[a\mathbf{r}_s (\mathbf{r}_s, \mathbf{n} \pm \mathbf{v}) + (\mathbf{n} \pm \mathbf{v})(\mathbf{r}_s, \mathbf{n} \pm \mathbf{v})(b + d(\mathbf{r}_s, \mathbf{n} \pm \mathbf{v})) \right], \; i = 1, 2,$$

where "+" is used for the first solar panel ($i = 1$), and "–" for the second one ($i = 2$). Notice that

$$(\mathbf{n}, \mathbf{v}) = \frac{1}{4}(\mathbf{n}_1 + \mathbf{n}_2, \mathbf{n}_1 - \mathbf{n}_2) = \frac{1}{4}\left(|\mathbf{n}_1|^2 - (\mathbf{n}_1, \mathbf{n}_2) + (\mathbf{n}_2, \mathbf{n}_1) - |\mathbf{n}_2|^2\right) = 0.$$

Since $(\mathbf{r}_s, \mathbf{n}) \approx 1$, the angle between $\mathbf{r}_s$ and $\mathbf{v}$ will be small. Hence, $(\mathbf{r}_s, \mathbf{v})$ is second order infinitesimal. Omitting all second order infinitesimals, we obtain the following expression for the SRP torque:

$$\mathbf{M}_s \approx \mathbf{R} \times 2a\mathbf{r}_s + [\mathbf{R} \times 2\mathbf{n} + \boldsymbol{\rho} \times 2\mathbf{v}](b + d). \tag{4}$$

Notice that SRP torque consists of two terms. The first one corresponds to the absorbed solar radiation and depends on the SC attitude. The second one will appear only when the reflectivity is not equal to zero. Moreover, its direction does not depend on attitude and always will be the same in the frame bounded to the SC. We have to mention that this result was obtained using tentative assumption that angle between mean normal and Sun direction, as well as angle between normal to solar panels, is rather small.

3. Angular Motion Synthesis

Before we start developing angular motion construction algorithm, consider the following assumptions. Firstly, reaction wheels principle of operation is based on angular momentum conservation law. Hence, when there is no external torquesor when fast maneuvers are perfomed the total angular momentum of the satellite and reaction wheels remains unchanged, even if SC changes its attitude. Secondly, when SC is Sun-pointed and its angular velocity along Sun vector equals to zero, SC angular momentum (without angular momentum stored in RW) is almost equal to zero. Thus, if we ensure that total angular momentum of the satellite (including RW) equals to zero, reaction wheels will be desaturated during Sun-stabilized motion.

Consider the equations that describe the total angular momentum evolution:

$$\frac{d\mathbf{K}}{dt}=\mathbf{M}, \quad (5)$$

where $\mathbf{K}$ is the total angular momentum of SC and RW, $\mathbf{M}$ is sum of all external torques, which corresponds to the SRP torque while SC moves far from pericenter and to the gravitational torque near the pericenter. Notice that

$$\frac{d}{dt}\left(K^2\right)=\frac{d}{dt}(\mathbf{K},\mathbf{K})=2(\mathbf{K},\mathbf{M}).$$

Hence, in order to decrease the total angular momentum we have to ensure $(\mathbf{M},\mathbf{K})<0$.

As it was mentioned above, SRP torque, as well as gravitational torque, depends on SC attitude. We can expect that we will not be always able to ensure $(\mathbf{M},\mathbf{K})<0$, but if for every moment

$$(\mathbf{M},\mathbf{K})\to\min, \quad (6)$$

the total angular momentum at least will increase slower. Problem (6) and restriction (3) together allow us to find the necessary attitude.

Obtained result can be generalized. For example, if solar stabilization is just temporary attitude motion mode and SC have some nominal one, where SC angular momentum without RW is $\mathbf{K}_0=\text{const}$ (e.g. inertial stabilization or spin stabilization), we can ensure that $\mathbf{K}=\mathbf{K}_0$, hence at the start of nominal motion RW angular momentum will be equal to zero, i.e. reaction wheels will be desaturated.

Let us prove that if $(\mathbf{M},\mathbf{K}-\mathbf{K}_0)<0$ we will achieve the nominal angular momentum:

$$\frac{d}{dt}\left|\mathbf{K}-\mathbf{K}_0\right|^2=\frac{d}{dt}\left(K^2+K_0^2-2(\mathbf{K}_0,\mathbf{K})\right)=\frac{d}{dt}(\mathbf{K},\mathbf{K})-2\left(\mathbf{K}_0,\frac{d}{dt}\mathbf{K}\right)=$$
$$=2(\mathbf{K},\mathbf{M})-2(\mathbf{K}_0,\mathbf{M})=2(\mathbf{M},\mathbf{K}-\mathbf{K}_0).$$

Hence, if RW can store enough angular momentum, we will be able to achieve any nominal angular momentum, so at the start of nominal angular motion RW will be completely desaturated.

3.1 Coordinate Systems

Let us introduce the following right-handed Cartesian coordinate systems:

$O_aY_1Y_2Y_3$ – Inertial Frame (IF): its origin O_a is located in the Earth center of mass, O_aY_1 directed to the Vernal equinox of the J2000 epoch, O_aY_3 is orthogonal to the ecliptic;

$Ox_1x_2x_3$ – Body Frame (BF): its origin O is located in the satellite center of mass, axes are its principal axes of inertia. We also suppose that mean normal $\mathbf{n}$ is aligned with Ox_3;

$Oz_1z_2z_3$ – Solar Frame (SF): Oz_3 is aligned with Sun direction, Oz_1 is aligned with $\mathbf{r}_s \times \mathbf{K}$ (if $\mathbf{r}_s \times \mathbf{K} = \mathbf{0}$ then Oz_1 is aligned with normal to ecliptic plane), thus $\mathbf{K} = \begin{pmatrix} 0 & K_2 & K_3 \end{pmatrix}^T$ in this coordinate system.

$Oy_1y_2y_3$ – Solar Panel Frame (SPF): Oy_3 is aligned with $\mathbf{n}$, Oy_1 is aligned with $\mathbf{n}\times\mathbf{p}$ where $\mathbf{p} = 2\mathbf{R}a$ (if $\mathbf{n}\times\mathbf{p} = \mathbf{0}$ then SPF coincide with BF), thus $\mathbf{p} = \begin{pmatrix} 0 & p_2 & p_3 \end{pmatrix}^T$ in this coordinate system.

$OZ_1Z_2Z_3$ – Orbital-Solar Frame: OZ_3 is aligned with the Sun direction, OZ_2 is aligned with $\mathbf{r}_s \times \mathbf{r}_{sat}$, where $\mathbf{r}_{sat}$ is the radius-vector from the Earth center to the SC center of mass (if $\mathbf{r}_s \times \mathbf{r}_{sat} = \mathbf{0}$ then OZ_2 is aligned with normal to the ecliptic plane), thus $\mathbf{r}_{sat} = \begin{pmatrix} r_1 & 0 & r_3 \end{pmatrix}^T, r_1 \geq 0$ in this coordinate system.

If we suppose that a vector is written in some specific frame, we will use superscript: $\mathbf{a}^Y, \mathbf{a}^x, \mathbf{a}^z, \mathbf{a}^y, \mathbf{a}^Z$ for IF, BF, SF, SPF and OSF respectively. Translations between these frames are described using rotation matrices $\mathbf{D}_{ij}$ so

$$\mathbf{a}^i = \mathbf{D}_{ij}\mathbf{a}^j,$$

where i, j can be Y, x, z, y, Z.

3.2 Desaturation Using Solar Radiation Pressure Torque

Let us consider the problem of desaturation at high altitudes, when SC moves far from the pericenter and SRP torque prevails. Rotation matrix from SF to SPF $\mathbf{D}_{zy}$ can be described by Euler angles (sequence 3-1-3 with angles ψ,θ,φ respectively):

$$D_{11} = \cos(\varphi)\cos(\psi) - \cos(\theta)\sin(\varphi)\sin(\psi),$$
$$D_{21} = -\cos(\psi)\sin(\varphi) - \cos(\varphi)\cos(\theta)\sin(\psi),$$
$$D_{31} = \sin(\psi)\sin(\theta),$$

$$D_{12} = \cos(\varphi)\sin(\psi) + \cos(\psi)\cos(\theta)\sin(\varphi), \quad D_{13} = \sin(\varphi)\sin(\theta),$$
$$D_{22} = \cos(\varphi)\cos(\psi)\cos(\theta) - \sin(\varphi)\sin(\psi), \quad D_{23} = \cos(\varphi)\sin(\theta),$$
$$D_{32} = -\cos(\psi)\sin(\theta), \qquad D_{33} = \cos(\theta).$$

as we can see, angle between $\mathbf{r}_s$ and $\mathbf{n}$ equals to θ, hence constraint (3) can be rewritten:

$$-\theta_{max} \leq \theta \leq \theta_{max}. \tag{7}$$

Further we will use only constraint in the form (7).

Additionally introduce notation

$$\mathbf{q} = \left[\mathbf{R} \times 2\mathbf{n} + \boldsymbol{\rho} \times 2\mathbf{v}\right](b + d).$$

Then expression for SRP torque is

$$\mathbf{M}_s = \mathbf{p} \times \mathbf{r}_s + \mathbf{q}.$$

Taking into account that $\theta_{max} << 1$, we can linearize w.r.t. θ expressions for $\mathbf{D}_{zy}$:

$$\mathbf{D}_{zy} = \begin{pmatrix} \cos(\varphi+\psi) & \sin(\varphi+\psi) & \theta\sin(\varphi) \\ -\sin(\varphi+\psi) & \cos(\varphi+\psi) & \theta\cos(\varphi) \\ \theta\cos(\psi) & -\theta\cos(\psi) & 1 \end{pmatrix}.$$

Now let us return to the $(\mathbf{M}_s, \mathbf{K})$. The simplest form vectors $\mathbf{K}$ and $\mathbf{r}_s$ have in SF, and vectors $\mathbf{p}, \mathbf{q}$ – in SPF. Hence,

$$(\mathbf{M}_s, \mathbf{K}) = (\mathbf{K}, \mathbf{p} \times \mathbf{r}_s + \mathbf{q}) = \left(\mathbf{D}_{zy}\left[\mathbf{r}_s^z \times \mathbf{K}^z\right], \mathbf{p}^y\right) + \left(\mathbf{D}_{zy}\mathbf{K}^z, \mathbf{q}^y\right).$$

After simplification

$$\begin{aligned}(\mathbf{M}_s, \mathbf{K}) = q_3K_3 + K_2\left(q_2\cos(\varphi+\psi) + \sin(\varphi+\psi)(q_1 + p_2)\right) + \\ + \theta\Big(K_3\left(q_1\sin(\varphi) + q_2\cos(\varphi)\right) - K_2\left(p_3\sin(\psi) + q_3\cos(\psi)\right)\Big).\end{aligned}$$

Thus, we obtain the following minimization problem:

$$\begin{gathered} q_3K_3 + K_2\left(q_2\cos(\varphi+\psi) + \sin(\varphi+\psi)(q_1 + p_2)\right) + \\ +\theta\Big(K_3\left(q_1\sin(\varphi) + q_2\cos(\varphi)\right) - K_2\left(p_3\sin(\psi) + q_3\cos(\psi)\right)\Big) \to \min_{\psi,\theta,\varphi}, \\ -\theta_{max} \le \theta \le \theta_{max}. \end{gathered} \tag{8}$$

In order to solve it we will use Lagrange multipliers method and Karush–Kuhn–Tucker conditions [10]. Lagrange function of this system is

$$\begin{aligned} L = q_3K_3 + K_2\left(q_2\cos(\varphi+\psi) + \sin(\varphi+\psi)(q_1 + p_2)\right) + \\ + \theta\Big(K_3\left(q_1\sin(\varphi) + q_2\cos(\varphi)\right) - K_2\left(p_3\sin(\psi) + q_3\cos(\psi)\right)\Big) \\ + \lambda_1(\theta_{max} - \theta) + \lambda_2(\theta_{max} + \theta) + q_3K_3. \end{aligned}$$

Where λ_i are Lagrange multipliers. Thus, the necessary conditions for the minimum are

$$\frac{\partial L}{\partial \theta} = K_3\left(q_1 \sin(\varphi) + q_2 \cos(\varphi)\right) - K_2\left(p_3 \sin(\psi) + q_3 \cos(\psi)\right) + \lambda_2 - \lambda_1 = 0,$$
$$\frac{\partial L}{\partial \varphi} = K_2\left((q_1 + p_2)\cos(\varphi+\psi) - \sin(\varphi+\psi) q_2\right) + \theta K_3\left(q_1 \cos(\varphi) - q_2 \sin(\varphi)\right) = 0,$$
$$\frac{\partial L}{\partial \psi} = K_2\left((q_1 + p_2)\cos(\varphi+\psi) - \sin(\varphi+\psi) q_2\right) - \theta K_2\left(p_3 \cos(\psi) - q_3 \sin(\psi)\right) = 0,$$
$$\lambda_1\left(\theta_{max} - \theta\right) = 0,$$
$$\lambda_2\left(\theta_{max} + \theta\right) = 0,$$
$$\lambda_i \geq 0.$$

Hence, we obtain the system of equations that allow us to find all extremum of the function (8). In general case, it is quite complex problem to solve. Notice, that vector $\mathbf{q}$ consists of two terms and one of them ($\boldsymbol{\rho} \times 2\mathbf{v}(b+d)$ to be exact) is the first order infinitesimal because $|\mathbf{v}| << 1$. In order to provide more energy, reflectivity of solar panels have to be small. Hence, since b, d are proportional to the reflectivity, $\boldsymbol{\rho} \times 2\mathbf{v}(b+d)$ is the second order infinitesimal and can be omitted. Of course, the solution that we obtain using this assumption will not be the exact solution of the problem (8), but it will be close to the one.

Consider vector $\mathbf{q}$ in SPF (without omitted term). In this frame

$$\mathbf{R}^y = \begin{pmatrix} 0 & R_2 & R_3 \end{pmatrix}^T, \quad \mathbf{n}^y = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T.$$

Hence

$$\mathbf{q}^y = 2\mathbf{R}^y \times \mathbf{n}^y \left(b+d\right) = \begin{pmatrix} 2R_2(b+d) & 0 & 0 \end{pmatrix}^T.$$

Thus, $q_3 = q_2 = 0$. Let

$$f = K_2(q_1 + p_2), \quad g = K_3 q_1, \quad h = K_2 p_3.$$

Then minimization problem (8) can be rewritten as

$$f \sin(\varphi+\psi) + \theta\left(g \sin(\varphi) - h \sin(\psi)\right) \to \min_{\psi,\theta,\varphi}, \qquad (9)$$
$$-\theta_{max} \leq \theta \leq \theta_{max}.$$

and the system of equations for the extremum is

$$\begin{aligned}
&\frac{\partial L}{\partial \theta} = g\sin(\varphi) - h\sin(\psi) + \lambda_2 - \lambda_1 = 0,\\
&\frac{\partial L}{\partial \varphi} = f\cos(\varphi+\psi) + \theta g\cos(\varphi) = 0,\\
&\frac{\partial L}{\partial \psi} = f\cos(\varphi+\psi) - \theta h\cos(\psi) = 0,\\
&\lambda_1(\theta_{max} - \theta) = 0,\\
&\lambda_2(\theta_{max} + \theta) = 0.
\end{aligned} \tag{10}$$

The exact solution of this system of algebraic equation presented in the appendix. Here we only notice that minima of problem (9) will be achieved at the border of permissible region, i.e. when $|\theta| = \theta_{max}$. When none of the f, g, h is equal to zero the total amount of extremum is not greater than twenty-four. Moreover, this function has a symmetry: if we replace θ by $-\theta$ and φ,ψ by $\varphi+\pi,\psi+\pi$ value of the function will not change. It is due to the fact that linearized matrix $\mathbf{D}_{zy}$ with the accuracy up to the second order infinitesimal does not change after the replacement. Therefore, we can consider only the case $\theta = \theta_{max}$, so total amount of extremum is not greater than twelve.

Notice that if

$$\psi = -\frac{\pi}{2} - \gamma_0, \quad \theta = \theta_{max}, \quad \varphi = \left(1 - \operatorname{sign}(f)\right)\frac{\pi}{2} + \gamma_0, \tag{11}$$

where

$$\sin(\gamma_0) = \frac{-g\operatorname{sign}(f)}{\sqrt{g^2+h^2}}, \quad \cos(\gamma_0) = \frac{-h}{\sqrt{g^2+h^2}}, \tag{12}$$

function (9) equals to

$$-|f| - \theta_{max}\sqrt{g^2+h^2}\,.$$

This value is not optimal, that can be easily verified by substitution (11) and (12) in (10), but it shows that (with accuracy up to second order infinitesimal) we can always construct the SC attitude that simultaneously provide solar pointing and RW desaturation using only SRP torque.

Let us prove that obtained simple approximate solution is close to the optimal one.

$$0 \le \frac{\min\left(f\sin(\varphi+\psi) + \theta\left(g\sin\varphi - h\sin\psi\right)\right)}{-|f| - \theta_{max}\sqrt{g^2+h^2}} \le \frac{|f| + \theta_{max}\left(|g|+|h|\right)}{|f| + \theta_{max}\sqrt{g^2+h^2}}.$$

If $f = 0, g = h$ then last expression is equal to $\sqrt{2}$. Let us prove that

$$\frac{|f|+\theta_{max}\left(|g|+|h|\right)}{|f|+\theta_{max}\sqrt{g^2+h^2}} \le \sqrt{2}\,.$$

We can rewrite it:

$$\left(1-\sqrt{2}\right)\frac{|f|}{\theta_{\max}} \le 2\left(\sqrt{\frac{g^2+h^2}{2}}-\frac{|g|+|h|}{2}\right).$$

Left part of this inequality is obviously nonpositive. Since

$$\sqrt{\frac{g^2+h^2}{2}} \ge \frac{|g|+|h|}{2},$$

the right part is nonnegative. Hence, we have proved that the approximate solution is worse than the optimal one by no more than $\sqrt{2}$ times.

3.3 Desaturation Using Gravitational Torque

Let us consider projection of SRP torque to the Sun direction $\mathbf{r}_s$:

$$\left(\mathbf{M}_s,\mathbf{r}_s\right)=\left(\mathbf{R}\times 2a\mathbf{r}_s+\left[\mathbf{R}\times 2\mathbf{n}+\boldsymbol{\rho}\times 2\mathbf{v}\right]\left(b+d\right),\mathbf{r}_s\right)=\left(\left[\mathbf{R}\times 2\mathbf{n}+\boldsymbol{\rho}\times 2\mathbf{v}\right]\left(b+d\right),\mathbf{r}_s\right).$$

As it was mentioned above, $\boldsymbol{\rho}\times 2\mathbf{v}(b+d)$ is second order infinitesimal. Since the angle between mean normal $\mathbf{n}$ and Sun direction $\mathbf{r}_s$ is small, $\left(\mathbf{R}\times 2\mathbf{n}(b+d),\mathbf{r}_s\right)$ is also second order infinitesimal. Hence, angular momentum along the Sun direction does not change when we use SRP torque. Considering this, we will use gravitational torque only to desaturate angular momentum along Sun direction. Thus, we obtain the following minimization problem:

$$\left(\mathbf{r}_s\left(\mathbf{K},\mathbf{r}_s\right),3\frac{\mu}{r_{sat}^5}\mathbf{r}_{sat}\times\mathbf{J}\mathbf{r}_{sat}\right)\to\min, \qquad (13)$$
$$(\mathbf{r}_s,\mathbf{n})\ge\cos\theta_{max},$$

where μ is the Earth gravitational parameter, $\mathbf{r}_{sat}$ is the SC radius-vector, $\mathbf{J}$ is its tensor of inertia. The only parameter we can vary is SC attitude, i.e. rotation matrix $\mathbf{D}_{Zx}$ from OSF to BF. As before, we will describe it using Euler angles (sequence 3-1-3 with angles Ψ,Θ,Φ accordingly). Constraint of the minimization problem then can be simplified:

$$-\theta_{max}\le\Theta\le\theta_{max}\,.$$

After linearization

$$\mathbf{D}_{Zx} = \begin{pmatrix} \cos(\Phi+\Psi) & \sin(\Phi+\Psi) & \Theta\sin(\Phi) \\ -\sin(\Phi+\Psi) & \cos(\Phi+\Psi) & \Theta\cos(\Phi) \\ \Theta\cos(\Psi) & -\Theta\cos(\Psi) & 1 \end{pmatrix}.$$

In OSF Sun direction and SC radius-vector are

$$\mathbf{r}_s^Z = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T, \quad \mathbf{r}_{sat}^Z = \begin{pmatrix} r_1 & 0 & r_3 \end{pmatrix}^T.$$

Function to be minimized is

$$\left(\mathbf{r}_s(\mathbf{K},\mathbf{r}_s), 3\frac{\mu}{r_{sat}^5}\mathbf{r}_{sat} \times \mathbf{J}\mathbf{r}_{sat}\right) = 3\frac{\mu}{r_{sat}^5}(\mathbf{K},\mathbf{r}_s)\left(\mathbf{r}_s^Z \times \mathbf{r}_{sat}^Z, \mathbf{D}_{Zx}^T \mathbf{J}^x \mathbf{D}_{Zx} \mathbf{r}_{sat}^Z\right), \tag{14}$$

where $\mathbf{J}^x$ is SC tensor of inertia in BF and

$$\mathbf{J}^x = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}.$$

After simplification, (14) became

$$3\frac{\mu}{r_{sat}^5}(\mathbf{K},\mathbf{r}_s)\left[r_1^2 \frac{A-B}{2}\sin(2\Psi+2\Phi) - \right.$$
$$\left. -\frac{1}{2} r_1 r_3 \Theta\left((A-B)\cos(2\Phi+\Psi) - \cos(\Psi)(A+B-2C)\right)\right].$$

Introduce notation

$$F = \frac{3\mu}{2r_{sat}^5}(\mathbf{K},\mathbf{r}_s) r_1^2 (B-A), \quad G = -\frac{3\mu}{2r_{sat}^5}(\mathbf{K},\mathbf{r}_s) r_1 r_3 (A-B),$$

$$H = -\frac{3\mu}{2r_{sat}^5}(\mathbf{K},\mathbf{r}_s) r_1 r_3 (A+B-2C),$$

$$\alpha = 2\Phi + \Psi + \frac{\pi}{2}, \quad \beta = \Psi + \frac{\pi}{2}.$$

Then problem (13) can be rewritten as follows

$$F\sin(\alpha+\beta) + \Theta\left(G\sin(\alpha) - H\sin(\beta)\right) \to \min_{\alpha,\beta,\theta},$$

$$-\theta_{max} \le \Theta \le \theta_{max}.$$

As we can see, the obtained problem is the same as (9), thus we can find solution in the same way.

Let us consider approximate solution (11) in more details. Considering notations, we can write exact expressions for γ_0 from the previous section:

$$\sin\gamma_0 = \frac{-G\,\mathrm{sign}(F)}{\sqrt{G^2+H^2}} = \frac{\frac{3\mu}{2r_{sat}^5}(\mathbf{K},\mathbf{r}_s)r_1r_3(A-B)\mathrm{sign}\left((\mathbf{K},\mathbf{r}_s)(B-A)\right)}{\left|\frac{3\mu}{2r_{sat}^5}(\mathbf{K},\mathbf{r}_s)r_1r_3\right|\sqrt{(A-B)^2+(A+B-2C)^2}} =$$

$$= \frac{\text{-sign}(r_3)|B-A|}{\sqrt{(A-B)^2+(A+B-2C)^2}},$$

$$\cos\gamma_0 = \frac{-H}{\sqrt{G^2+H^2}} = \frac{-\frac{3\mu}{2r_{sat}^5}(\mathbf{K},\mathbf{r}_s)r_1r_3(A+B-2C)}{\left|\frac{3\mu}{2r_{sat}^5}(\mathbf{K},\mathbf{r}_s)r_1r_3\right|\sqrt{(A-B)^2+(A+B-2C)^2}} =$$

$$= \frac{\text{-sign}\left((\mathbf{K},\mathbf{r}_s)r_3\right)(A+B-2C)}{\sqrt{(A-B)^2+(A+B-2C)^2}}.$$

Euler angles Φ,Ψ are defined by γ_0:

$$\Phi = \gamma_0 + \frac{2-\mathrm{sign}\left((\mathbf{K},\mathbf{r}_s)(B-A)\right)}{2},\ \ \Psi = -\gamma_0 - \pi\,.$$

Hence, SC attitude relative to OSF is defined by signs of $r_3 = (\mathbf{r}_{sat},\mathbf{r}_s),(\mathbf{K},\mathbf{r}_s)$ and its tensor of inertia. Therefore, if these signs do not change during the motion, the SC attitude will be fixed in OSF.

Let us obtain closed expressions for OSF angular velocity. Its axes are

$$\mathbf{e}_1 = \frac{\mathbf{r}_{sat} - \mathbf{r}_s(\mathbf{r}_{sat},\mathbf{r}_s)}{\left|\mathbf{r}_{sat} - \mathbf{r}_s(\mathbf{r}_{sat},\mathbf{r}_s)\right|},\quad \mathbf{e}_3 = \mathbf{r}_s,\quad \mathbf{e}_2 = \mathbf{e}_3\times\mathbf{e}_1.$$

Angular velocity can be found using Poisson equations for rotation matrices

$$\dot{\mathbf{D}}_{YZ} = -[\boldsymbol{\omega}_0^Z]_\times \mathbf{D}_{YZ},$$

where $\mathbf{D}_{YZ}$ is rotation matrix from IF to OSF, $\boldsymbol{\omega}_0^Z$ OSF angular velocity, and introduced notation for skew-symmetric matrix of cross product

$$[\mathbf{a}]_\times := \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}.$$

Hence, the angular velocity is

$$\boldsymbol{\omega}_0^Z = \left((\dot{\mathbf{e}}_1,\mathbf{e}_2)\quad -(\dot{\mathbf{e}}_3,\mathbf{e}_2)\quad (\dot{\mathbf{e}}_3,\mathbf{e}_1)\right)^T.$$

Since Sun direction changes slowly, we will neglect it, i.e. $\dot{\mathbf{e}}_3 = \dot{\mathbf{r}}_s = \mathbf{0}$. Time derivative of the first axis then

$$\dot{\mathbf{e}}_1 = \frac{\mathbf{v}_{sat} - \mathbf{r}_s(\mathbf{v}_{sat}, \mathbf{r}_s) - \mathbf{e}_1(\mathbf{v}_{sat} - \mathbf{r}_s(\mathbf{v}_{sat}, \mathbf{r}_s), \mathbf{e}_1)}{|\mathbf{r}_{sat} - \mathbf{r}_s(\mathbf{r}_{sat}, \mathbf{r}_s)|} = \frac{\mathbf{e}_2(\mathbf{e}_2, \mathbf{v}_{sat})}{|\mathbf{r}_{sat} \times \mathbf{r}_s|}.$$

Here we have taken into account that

$$\mathbf{v}_{sat} = \mathbf{e}_1(\mathbf{e}_1, \mathbf{v}_{sat}) + \mathbf{e}_2(\mathbf{e}_2, \mathbf{v}_{sat}) + \mathbf{e}_3(\mathbf{e}_3, \mathbf{v}_{sat}), \quad (\mathbf{r}_s, \mathbf{e}_1) = (\mathbf{e}_3, \mathbf{e}_1) = 0.$$

Hence, OSF angular velocity is

$$\boldsymbol{\omega}_0^Z = \begin{pmatrix} \dfrac{\mathbf{e}_2(\mathbf{e}_2, \mathbf{v}_{sat})}{|\mathbf{r}_{sat} \times \mathbf{r}_s|} & 0 & 0 \end{pmatrix}^T.$$

As we can see, if $\mathbf{r}_{sat}$ and $\mathbf{r}_s$ are collinear, OSF angular velocity became infinite. This fact have to be taken into account while we construct angular motion near the pericenter, i.e. if $|\mathbf{r}_{sat} \times \mathbf{r}_s|$ is small, desaturation using gravity torque will not work because RW will not be able to provide required angular velocity. Since Sun direction rotates in IF, time period when we can not use gravitational torque will be rather small, about several revolutions. Moreover, if pericenter located sufficiently far from the ecliptic we will always be able to desaturate RW using gravitational torque since $|\mathbf{r}_{sat} \times \mathbf{r}_s|$ will be large enough near the pericenter.

4. Attitude Control

In order to control SC attitude we will use well-known Lyapunov-based control algorithm [11–13] that provides asymptotic stability of the required (further we will call it reference) motion.

SC equations of motion are:

$$\begin{aligned} \mathbf{J}^x \dot{\boldsymbol{\omega}}^x + \boldsymbol{\omega}^x \times \mathbf{J}^x \boldsymbol{\omega}^x &= \mathbf{M}_{ctrl}^x + \mathbf{M}_{ext}^x, \\ \dot{\mathbf{D}}_{Yx} &= -[\boldsymbol{\omega}^x]_\times \mathbf{D}_{Yx}, \end{aligned} \qquad (15)$$

where $\mathbf{J}^x$ is SC tensor of inertia, $\boldsymbol{\omega}^x$ is its angular velocity, $\mathbf{M}_{ext}^x, \mathbf{M}_{ctrl}^x$ are external and control torques respectively, $\mathbf{D}_{Yx}$ is rotation matrix from IF to BF. Let reference motion be defined by rotation matrix $\mathbf{D}_{YR}$ from IF to Reference Frame (RF, further all vectors in this coordinate system will be noted with superscript R) and by reference angular velocity $\boldsymbol{\omega}_{ref}^R$, so

$$\dot{\mathbf{D}}_{YR} = -[\boldsymbol{\omega}_{ref}^R]_\times \mathbf{D}_{YR}. \qquad (16)$$

Attitude control have to ensure that RF and BF are coincide.

Consider Lyapunov-function candidate

$$V=\frac{1}{2}\left(\boldsymbol{\omega}_{rel}^{x},\mathbf{J}^{x}\boldsymbol{\omega}_{rel}^{x}\right)+k_{a}\left(3-\mathrm{tr}\mathbf{D}_{Rx}\right),\ k_{a}=\mathrm{const}>0, \qquad (17)$$

where $\mathbf{D}_{Rx}=\mathbf{D}_{Yx}\mathbf{D}_{YR}^{T}$ is rotation matrix from RF to BF, $\boldsymbol{\omega}_{rel}^{x}=\boldsymbol{\omega}^{x}-\mathbf{D}_{Rx}\boldsymbol{\omega}_{ref}^{R}$ is relative angular velocity, tr is trace of the matrix. Time derivative of V, using (15) and (16), is

$$\dot{V}=\frac{1}{2}\left(\boldsymbol{\omega}_{rel}^{x},\mathbf{J}^{x}\dot{\boldsymbol{\omega}}_{rel}^{x}\right)-k_{a}\,\mathrm{tr}\,\dot{\mathbf{D}}_{Rx}=\left(\boldsymbol{\omega}_{rel}^{x},\mathbf{J}^{x}\dot{\boldsymbol{\omega}}^{x}+\mathbf{J}\left[\boldsymbol{\omega}_{rel}^{x}\right]_{\times}\mathbf{D}_{Rx}\boldsymbol{\omega}_{ref}^{R}-\mathbf{J}^{x}\mathbf{D}_{Rx}\dot{\boldsymbol{\omega}}_{ref}^{R}+k_{a}\mathbf{S}\right),$$

where $\mathbf{S}=\left(d_{23}-d_{32}\quad d_{31}-d_{13}\quad d_{12}-d_{21}\right)$, d_{ij} are components of $\mathbf{D}_{Rx}$. If

$$\mathbf{J}^{x}\dot{\boldsymbol{\omega}}^{x}+\mathbf{J}\left[\boldsymbol{\omega}_{rel}^{x}\right]_{\times}\mathbf{D}_{Rx}\boldsymbol{\omega}_{ref}^{R}-\mathbf{J}\mathbf{D}_{Rx}\dot{\boldsymbol{\omega}}_{ref}^{R}+k_{a}\mathbf{S}=-k_{\omega}\boldsymbol{\omega}_{rel}^{x},\ k_{\omega}=\mathrm{const}>0,$$

then $\dot{V}\le 0$. Control torque in this case is

$$\mathbf{M}_{ctrl}^{x}=-\mathbf{M}_{ext}^{x}+\boldsymbol{\omega}^{x}\times\mathbf{J}^{x}\boldsymbol{\omega}^{x}-\mathbf{J}^{x}\left[\boldsymbol{\omega}_{rel}^{x}\right]_{\times}\mathbf{D}_{Rx}\boldsymbol{\omega}_{ref}^{R}+\mathbf{J}\mathbf{D}_{Rx}\dot{\boldsymbol{\omega}}_{ref}^{R}-k_{a}\mathbf{S}-k_{\omega}\boldsymbol{\omega}_{rel}^{x}. \qquad (18)$$

Let us show that this control torque ensures asymptotic stability of the reference motion. We will use Barbashin-Krasovsky theorem [14], therefore we have to show that there exist positive-definite Lyapunov-function and its time derivative is nonpositive and equals to zero only at the set that does not contain any trajectories of the system except the reference motion.

First part of this theorem obviously satisfied because (17) is positive definite. Its time derivative is nonpositive, and the set $M=\left\{\left(\mathbf{D}_{Rx},\boldsymbol{\omega}_{rel}\right):\dot{V}=0\right\}$ consist of all the points $\left(\mathbf{D}_{Rx},\boldsymbol{\omega}_{rel}\right)$ such as $\boldsymbol{\omega}_{rel}^{x}\equiv\mathbf{0}$. After substitution (18) in equations of motion, we obtain

$$\mathbf{J}^{x}\dot{\boldsymbol{\omega}}_{rel}^{x}+k_{\omega}\boldsymbol{\omega}_{rel}^{x}+k_{a}\mathbf{S}=\mathbf{0},$$
$$\dot{\mathbf{D}}_{Rx}=-[\boldsymbol{\omega}_{rel}^{x}]_{\times}\mathbf{D}_{Rx}.$$

We look for the trajectories only at the set M. As we can see, such trajectories are

$$\mathbf{S}\equiv\mathbf{0},\quad \boldsymbol{\omega}_{rel}^{x}\equiv\mathbf{0}.$$

$\mathbf{S}\equiv\mathbf{0}$ corresponds either to reference attitude or the one that differs from the reference to the rotation along any axis at the angle π. Hence, there are several equilibrium, and $\mathbf{D}_{Rx}\equiv\mathbf{E}_{3},\boldsymbol{\omega}_{rel}^{x}\equiv\mathbf{0}$ (reference motion) is globally (except other equilibrium) asymptotically stable, while other equilibrium are unstable. Detailed analysis of attitude accuracy that provides this algorithm presented in [15].

As it was mentioned above, we divide SC motion into two modes: motion near the pericenter and far from it. Let us take a closer look at expressions for $\mathbf{D}_{YR}$ and $\boldsymbol{\omega}_{ref}^{R}$ that correspond to each of these modes.

When SC moves far from pericenter its angular momentum changes very slow because SRP torque is rather small. Hence, attitude that ensures RW desaturation will also change slowly. Therefore, when SC moves far from pericenter its angular velocity will be almost the same as the inertial stabilization, i.e. its reference attitude will be constant and angular velocity will be equal to zero:

$$\mathbf{D}_{YR} = \mathbf{D}_{yx}\mathbf{D}_{zy}\mathbf{D}_{Yz}, \quad \boldsymbol{\omega}_{ref}^{R} \equiv \mathbf{0},$$

where $\mathbf{D}_{Yz}$ is rotation matrix from IF to SF, $\mathbf{D}_{zy}$ corresponds to rotation matrix determined by Euler angles that are the solution of the minimization problem (9), $\mathbf{D}_{yx}$ is constant rotation matrix from SPF to BF. $\mathbf{D}_{YR}$ changes slowly, so it will be necessary to redetermine it only once in several hours.

During motion near the pericenter assumption of small angular velocity does not work since rotation matrix from IF to OSF changes very fast. Since we determine the necessary attitude by solving cubic equations, the problem of reference angular velocity and acceleration determination is rather complex to be solved analytically, so we will determine them numerically using (16). Rotation matrix from IF to RF in this case is

$$\mathbf{D}_{YR} = \mathbf{D}_{Zx}\mathbf{D}_{YZ},$$

where $\mathbf{D}_{YZ}$ is rotation matrix from IF to OSF, $\mathbf{D}_{Zx}$ is rotation matrix determined by Euler angles that were found from minimization problem (13).

5. Numerical Simulation

In order to verify obtained results the following mission was simulated:

- SC moves along highly elliptical orbit with inclination equals to $60°$, its pericenter is 9 000 km and apocenter is 150 000 km, one revolution is about 62.7 h.
- Tensor of inertia is $\mathbf{J}^x = \text{diag}(150, 120, 200)\ \text{kg}\cdot\text{m}^2$.
- Solar panels area is $3\ \text{м}^2$.
- $\mathbf{R}_1^x = (0 \quad 0.75 \quad 0.15)^T$ m, $\mathbf{R}_2^x = (0 \quad -0.85 \quad 0.15)^T$ m.
- $\mathbf{n}_1^x = (0.075 \quad 0.075 \quad 0.996)^T$, $\mathbf{n}_2^x = (-0.075 \quad -0.075 \quad 0.996)^T$.
- Reflectivity $\alpha = 0.1$, specularity $\mu = 0.5$.
- Earth rotates along Sun with constant angular velocity, changes of Sun direction due to SC motion along the orbit is neglected, i.e. Sun direction rotates with the constant angular velocity in IF.
- $\theta_{max} = 10°$.
- Initial attitude $\mathbf{D}_{Yx} = \mathbf{E}_3$ and angular velocity $\boldsymbol{\omega}^x = (0 \quad 0 \quad 0)^T$ rad/s.
- Maximum control torque is $0.01\ \text{N}\cdot\text{m}$.
- Initial angular momentum stored in RW is $\mathbf{H} = (1 \quad 1 \quad 1)^T\ \text{N}\cdot\text{m}\cdot\text{s}$.
- Far from pericenter reference attitude redefined every 40 000 s.

For the reference motion construction we used equations (4) for SRP torque. Reference motion mode changes when SC radius-vector equals to 15 000 km. For the numerical simulation more precise expression (1) was used for SRP torque and gravitational torque considered to be always affecting SC attitude motion. In Fig. 3-8 results of the reference motion construction algorithm work are presented. Here we used exact solution of minimization problems (9) and (13).

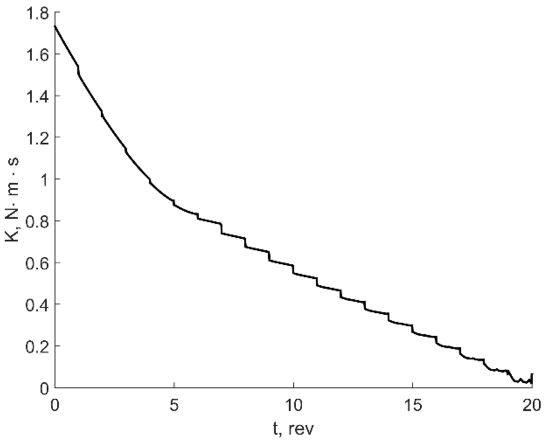

Fig. 3. Total angular momentum

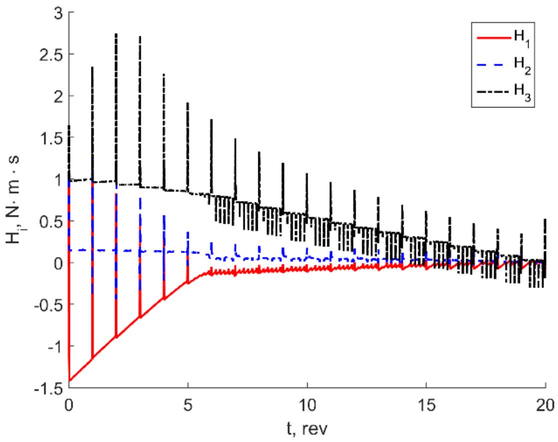

Fig. 4. RW angular momentum

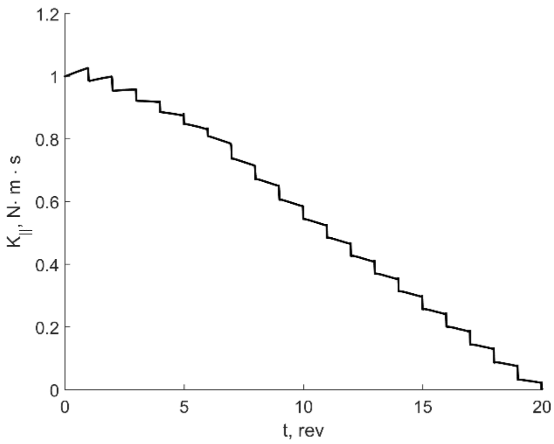

Fig. 5.Total angular momentum along Sun direction

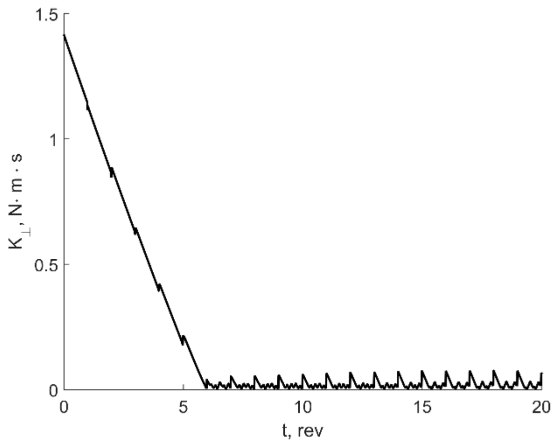

Fig. 6. Total angular momentum orthogonal to Sun direction

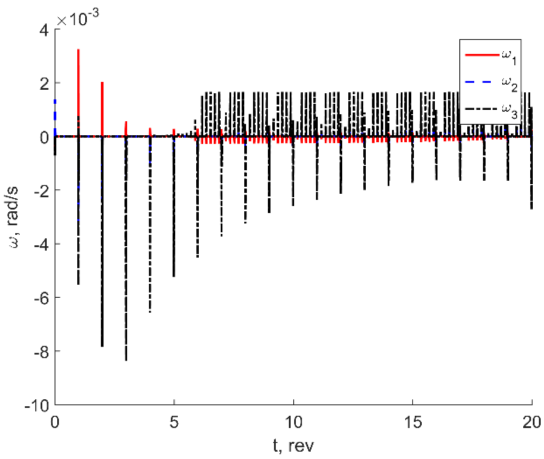

Fig. 7. SC angular velocity

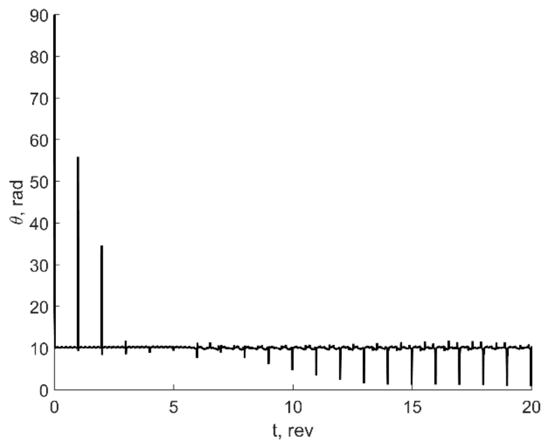

Fig. 8. Angle between mean normal and Sun direction

Peaks in Fig. 4 appear due to noticeable variation of the attitude in different modes of reference motion. In addition, near the pericenter SC angular velocity is not equal to zero, so RW have to store more angular momentum in order to spin the satellite. After the sixth revolution there are peaks during the motion far from pericenter. They appear as well as peaks in angular velocity because angular momentum orthogonal to the Sun direction is almost desaturated and optimal attitude changes greatly. Large angle between mean normal and Sun direction appears at small periods of time because SC have to be reorientated after switching reference motion modes and does not affect SC energy balance.

In Fig. 9-14 results of another simulation are presented, where we used not the exact solutions of minimization problems but their approximate ones (11).

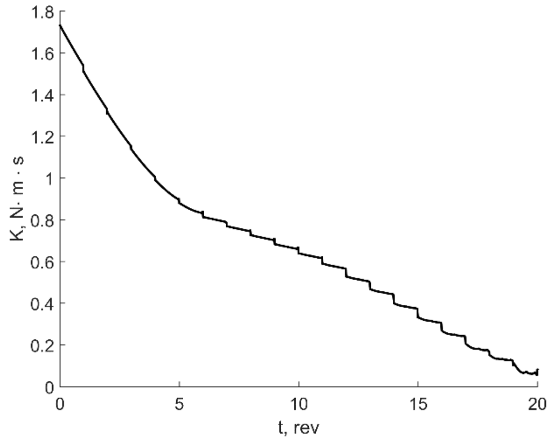

Fig. 9. Total angular momentum

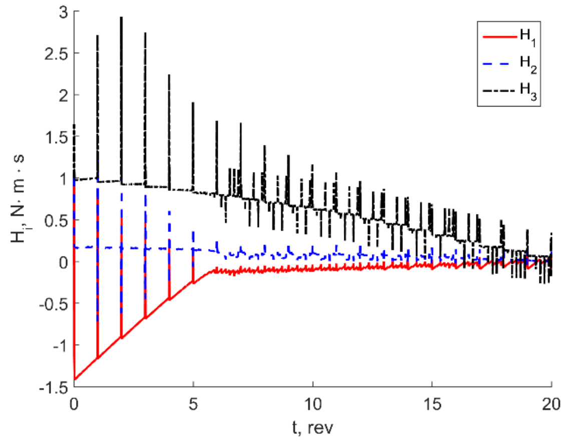

Fig. 10. RW angular momentum

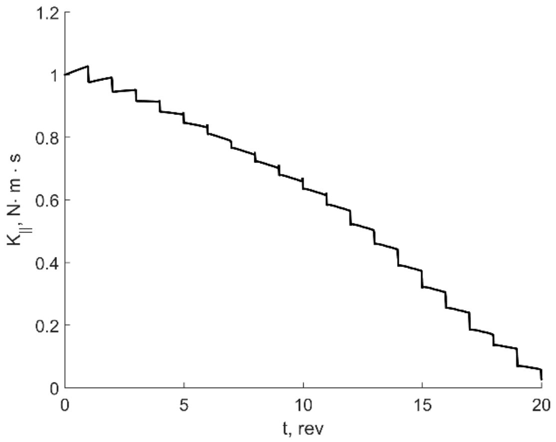

Fig. 11. Total angular momentum along Sun direction

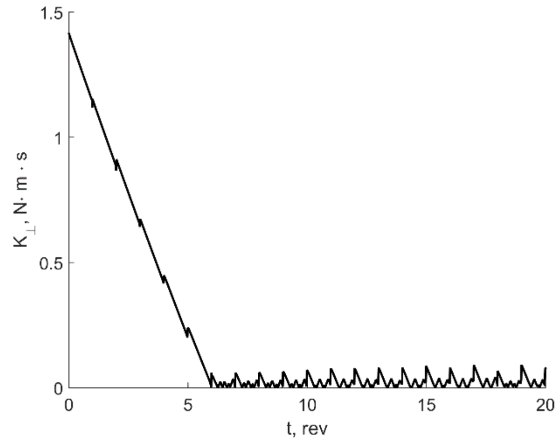

Fig. 12. Total angular momentum orthogonal to Sun direction

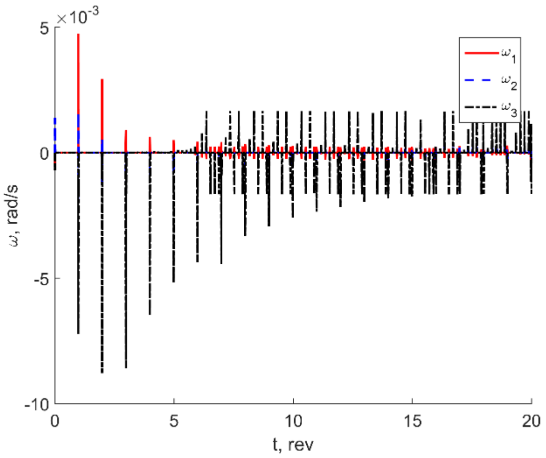

Fig. 13. SC angular velocity

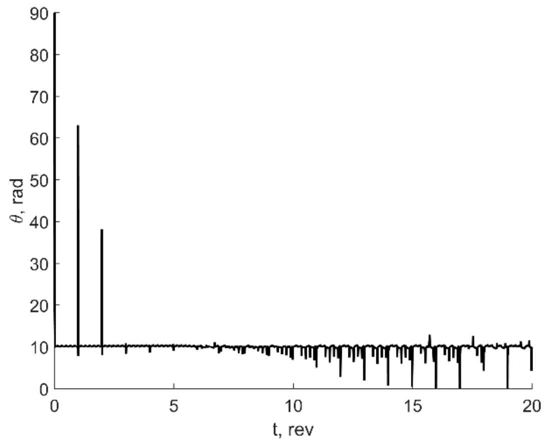

Fig. 14. Angle between mean normal and Sun direction

As we can see, angular momentum evolution, as well as other system parameters, are almost the same as the ones obtained using the exact solutions of minimization problems.

Conclusion

In the paper we have investigated the problem of fuelless reaction wheels desaturation. We have considered the satellite that moves along highly elliptical keplerian orbit with additional constraint: normal to solar panels, which are rigidly fixed to the satellite, have to be directed near to the Sun direction. In order to solve this problem we have suggested an algorithm of reference angular motion synthesis that allow us to use solar radiation pressure and gravitational torques fore desaturation.

We have obtained simplified model of solar radiation torque for the satellite with two identical solar panels in the case when normals to these solar panels are almost coincide and satellite is Sun-stabilized. Using this model, we have obtained expressions that allow us to find the satellite attitude that ensures reaction wheels desaturation.

We have shown that for both simplified model of solar radiation pressure torque and gravitational torque there is always an attitude that provide simultaneous solar pointing and reaction wheels desaturation. The problem of this attitude finding can be reduced to the solution of the system of trigonometric equation, which, in turn, can be reduced to the solution of one cubic equation. In addition, we have shown that this problem have simple approximate solution, which is worse than the exact one no more than by $\sqrt{2}$ times.

Suggested algorithms have been tested on the model problem. Results of the simulation have shown that there is no noticeable difference in reaction wheels angular momentum evolution using both the exact solution and approximate one. Hence, considering much simpler expressions for approximate solution, it is reasonable to use one.

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Appendix

Consider the following minimization problem

$$f\sin(\varphi+\psi)+\theta\big(g\sin(\varphi)-h\sin(\psi)\big)\to\min_{\psi,\theta,\varphi}, \\ -\theta_{max}\le\theta\le\theta_{max}. \quad (19)$$

Its solution depends on the values of f,g,h, thus it is reasonable to start looking for the solution from some simple special cases. They all are presented in the Table 1.

Table 1. Solutions for some special cases

Parameters value	φ_{opt}	θ_{opt}	ψ_{opt}	Function minimum value
$f=0,g=0,h\ne0$	any	θ_{max}	$\mathrm{sign}(h)\frac{\pi}{2}$	$-\theta_{max}\lvert h\rvert$
	any	$-\theta_{max}$	$-\mathrm{sign}(h)\frac{\pi}{2}$	$-\theta_{max}\lvert h\rvert$
$f=0,g\ne0,h=0$	$-\mathrm{sign}(g)\frac{\pi}{2}$	θ_{max}	any	$-\theta_{max}\lvert g\rvert$
	$\mathrm{sign}(g)\frac{\pi}{2}$	$-\theta_{max}$	any	$-\theta_{max}\lvert g\rvert$
$f=0,g\ne0,h\ne0$	$-\mathrm{sign}(g)\frac{\pi}{2}$	θ_{max}	$\mathrm{sign}(h)\frac{\pi}{2}$	$-\theta_{max}(\lvert g\rvert+\lvert h\rvert)$
	$\mathrm{sign}(g)\frac{\pi}{2}$	$-\theta_{max}$	$-\mathrm{sign}(h)\frac{\pi}{2}$	$-\theta_{max}(\lvert g\rvert+\lvert h\rvert)$
$f\ne0,g=0,h=0$	any	any	$-\mathrm{sign}(f)\frac{\pi}{2}-\varphi_{opt}$	$-\lvert f\rvert$
$f\ne0,g=0,h\ne0$	$-\mathrm{sign}(f)\frac{\pi}{2}-\psi_{opt}$	θ_{max}	$\mathrm{sign}(h)\frac{\pi}{2}$	$-\lvert f\rvert-\theta_{max}\lvert h\rvert$
	$-\mathrm{sign}(f)\frac{\pi}{2}-\psi_{opt}$	$-\theta_{max}$	$-\mathrm{sign}(h)\frac{\pi}{2}$	$-\lvert f\rvert-\theta_{max}\lvert h\rvert$
$f\ne0,g\ne0,h=0$	$-\mathrm{sign}(g)\frac{\pi}{2}$	θ_{max}	$-\mathrm{sign}(f)\frac{\pi}{2}-\varphi_{opt}$	$-\lvert f\rvert-\theta_{max}\lvert g\rvert$
	$\mathrm{sign}(g)\frac{\pi}{2}$	$-\theta_{max}$	$-\mathrm{sign}(f)\frac{\pi}{2}-\varphi_{opt}$	$-\lvert f\rvert-\theta_{max}\lvert g\rvert$

There are two more cases to be considered. The first one is trivial and correspond to the case when all parameters are equal to zero. In the second case all parameters are nonzero.

Before we start looking for the exact solution, notice that if

$$\varphi+\psi=-\mathrm{sign}(f)\frac{\pi}{2}.$$

then

$$-|f|+\theta\left(-g\,\mathrm{sign}(f)\cos(\psi)-h\sin(\psi)\right)=-|f|+\theta\sqrt{g^2+h^2}\sin(\psi+\gamma_0),$$

$$\sin(\gamma_0)=\frac{-g\,\mathrm{sign}(f)}{\sqrt{g^2+h^2}},\quad \cos(\gamma_0)=\frac{-h}{\sqrt{g^2+h^2}}.$$

Hence, if

$$\psi=-\frac{\pi}{2}-\gamma_0,\quad \theta=\theta_{max},\quad \varphi=\left(1-\mathrm{sign}(f)\right)\frac{\pi}{2}+\gamma_0 \qquad (20)$$

function is equal to

$$-|f|-\theta_{max}\sqrt{g^2+h^2}.$$

This value is not minimal, but shows that we always find φ,ψ,θ to ensure negative value of the function.

Solutions in permissible region

Firstly, we will look for the solution in permissible region, i.e. $|\theta|<\theta_{max}$. Then

$$\begin{aligned}&\lambda_1=\lambda_2=0,\\&g\sin(\varphi)-h\sin(\psi)=0,\\&f\cos(\varphi+\psi)+\theta g\cos(\varphi)=0,\\&f\cos(\varphi+\psi)-\theta h\cos(\psi)=0.\end{aligned} \qquad (21)$$

Let $\theta=0$. Thus,

$$\begin{aligned}&\cos(\varphi+\psi)=0,\\&g\sin(\varphi)-h\sin(\psi)=0.\end{aligned}$$

Solution of this system of equations is

$$\psi=\pi n-\gamma_0,\quad \varphi=\frac{\pi}{2}+\pi(\mathrm{k-n})+\gamma_0,\quad \sin(\gamma_0)=\frac{g(-1)^k}{\sqrt{g^2+h^2}},\quad \cos(\gamma_0)=\frac{-h}{\sqrt{g^2+h^2}}.$$

where $k,n\in\mathbb{Z}$. Notice that in this case function (19) only depends on sum of ψ and φ, and its minimum achieved if $\psi+\varphi=-\mathrm{sign}(f)\frac{\pi}{2}$ and equals to $-|f|$, which is more than the value obtained using (20). Hence, this solution is not the optimal one.

Let $\theta\neq 0$. From (21) we obtain

$$\begin{aligned}&g\sin(\varphi)-h\sin(\psi)=0,\\&g\cos(\varphi)+h\cos(\psi)=0.\end{aligned}$$

Square each of these equations and combine:

$$g^2 + h^2 + 2gh\cos(\varphi+\psi) = 0.$$

On the other hand, multiply the second equation by $\sin(\varphi)$, the first one by $-\cos(\varphi)$ and combine:

$$\sin(\varphi+\psi) = 0.$$

Hence, there will be a solution only if $|g| = |h|$. In this case we can obtain that

$$\varphi+\psi = \begin{cases} \pi + 2\pi k, & \text{if } g = h \\ 2\pi k, & \text{if } g = -h \end{cases}$$

This is sufficient to show that investigated function equals to zero, hence this solution is also not optimal. Therefore, there is no optimal solution in permissible area.

Solutions at the border of permissible region

Let us look for the solution at the border of permissible are, i.e. $|\theta| = \theta_{max}$. System of equation for extremum then is

$$\begin{aligned} &\lambda_2 = 0, \\ &g\sin(\varphi) - h\sin(\psi) - \lambda_1 = 0, \\ &f\cos(\varphi+\psi) + \theta^* g\cos(\varphi) = 0, \\ &f\cos(\varphi+\psi) - \theta^* h\cos(\psi) = 0, \end{aligned} \tag{22}$$

where $\theta^* = \theta_{max}$ or $\theta^* = -\theta_{max}$. Thus,

$$\cos(\varphi) = -\frac{h}{g}\cos(\psi), \quad \sin(\varphi) = \pm\sqrt{1 - \frac{h^2}{g^2}\cos^2(\psi)}.$$

Using this expression, rewrite the fourth equation from (22):

$$f\left(-\frac{h}{g}\cos^2(\psi) \mp \sin(\psi)\sqrt{1 - \frac{h^2}{g^2}\cos^2(\psi)}\right) = \theta^* h\cos(\psi).$$

After the mathematics we obtain

$$\cos^3(\psi) + \frac{1}{2}\cos^2(\psi)\left(\frac{g\theta^*}{f} + \frac{f}{g\theta^*} + \frac{fg}{h^2\theta^*}\right) - \frac{fg}{2h^2\theta^*} = 0. \tag{23}$$

We can solve this equation directly using Cardano method.

Let us prove that there will always be the real root of this equation which absolute value is less than one. Introduce notation

$$\beta = \frac{fg}{2h^2\theta^*}, \quad \alpha = \frac{fg}{h^2\theta^*}\left(\left(\frac{h\theta^*}{f}\right)^2 + \frac{h^2}{g^2} + 1\right), \quad x = \cos(\psi).$$

Thus, we can rewrite this equation:

$$q(x) = \beta, \quad q(x) = x^3 + \frac{1}{2}\alpha x^2.$$

For clarification purposes, graph of $q(x)$ for some α is presented in Fig. 15.

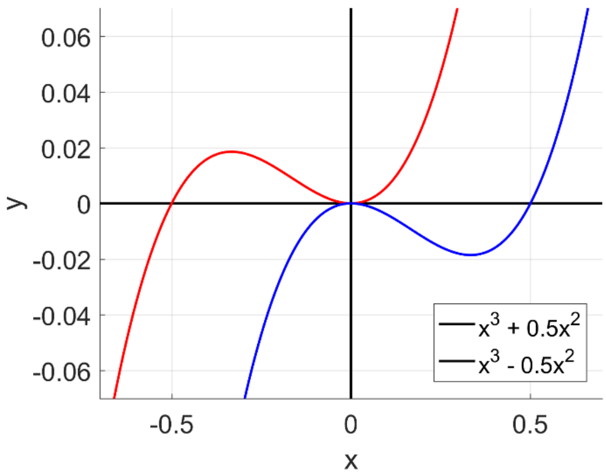

Fig. 15. $q(x)$ for $\alpha = \pm 1$

Now, the only thing to be proved is that β contained in the range of $q(x)$ at the segment $[-1,1]$. Consider $\alpha > 0$. Then $q(x)$ have two extremum

$$q(0) = 0, \quad q\left(-\frac{\alpha}{3}\right) = \frac{1}{54}\alpha^3.$$

$x = 0$ is local minima, and $x = -3^{-1}\alpha$ is local maxima. It is obvious that range of $q(x)$ is

$$\operatorname*{Im}_{-1\le x\le 1} q(x) = \begin{cases} \left[\min\left(0, -1+\frac{1}{2}\alpha, 1+\frac{1}{2}\alpha\right), \max\left(\frac{1}{54}\alpha^3, -1+\frac{1}{2}\alpha, 1+\frac{1}{2}\alpha\right)\right], & 0 < \alpha \le 3 \\ \left[\min\left(0, -1+\frac{1}{2}\alpha, 1+\frac{1}{2}\alpha\right), \max\left(-1+\frac{1}{2}\alpha, 1+\frac{1}{2}\alpha\right)\right], & \alpha \ge 3 \end{cases}$$

As we can see, this expression can be simplified

$$\operatorname*{Im}_{-1\le x\le 1} q(x) = \left[\min\left(-1+\frac{1}{2}\alpha, 0\right), 1+\frac{1}{2}\alpha\right], \quad \alpha > 0.$$

Using similar technique for the case $\alpha < 0$ we obtain

$$\operatorname*{Im}_{-1\le x\le 1} q(x) = \left[-1+\frac{1}{2}\alpha, \max\left(1+\frac{1}{2}\alpha, 0\right)\right], \quad \alpha < 0 .$$

Take into account that

$$\alpha = 2\beta\left(1+\left(\frac{h\theta^*}{f}\right)^2+\frac{h^2}{g^2}\right).$$

Thus, we just have to verify inequations

$$\beta \le \beta\left(1+\left(\frac{h\theta^*}{f}\right)^2+\frac{h^2}{g^2}\right)+1, \ \beta \ge 0,$$

$$\beta \ge \beta\left(1+\left(\frac{h\theta^*}{f}\right)^2+\frac{h^2}{g^2}\right)-1, \ \beta < 0,$$

that are obviously satisfied. Thus, we have shown that cubic equation always have at least one real root that belongs to $[-1,1]$. We have to notice that for every real root from this segment there are four solutions (at the interval $[0,2\pi)$) of the system (22).