ISSN 2071-2898 (Print)
ISSN 2071-2901 (Online)

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Recommended form of bibliographic references: Bruno A.D. Complicated and exotic expansions of solutions to the Painlevé equations // Keldysh Institute Preprints. 2018. No. 118. 44 p. doi:10.20948/prepr-2018-118-e
URL: http://library.keldysh.ru/preprint.asp?id=2018-118&lg=e

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ИНСТИТУТ ПРИКЛАДНОЙ МАТЕМАТИКИ
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Complicated and exotic expansions of solutions to the Painlevé equations

Moscow — 2018

УДК 517.925
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Complicated and exotic expansions of solutions to the Painlevé equations.

We consider the complicated and exotic asymptotic expansions of solutions to a polynomial ordinary differential equation (ODE). They are such series on integral powers of the independent variable, which coefficients are the Laurent series on decreasing powers either of the logarithm of the independent variable or on its pure imaginary power correspondingly. We propose an algorithm for writing ODEs for these coefficients. The first coefficient is a solution of a truncated equation. For some initial equations, it is a polynomial. Question: will the following coefficients be polynomials? Here the question is considered for the third (P_3), fifth (P_5) and sixth (P_6) Painlevé equations. We have found that second coefficients in seven of eight families of complicated expansions are polynomials, as well in two of four families of exotic expansions, but in other three families, polynomiality of the second coefficient demands some conditions. We give detailed proofs and calculations of these results.

Key words: expansions of solutions to ODE, complicated expansions, exotic expansions, polynomiality of coefficients, Painlevé equations.

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Сложные и экзотические разложения решений уравнений Пенлеве. Препринт Института прикладной математики им. М.В. Келдыша РАН, Москва, 2018.

Рассматриваются сложные и экзотические асимптотические разложения решений полиномиального обыкновенного дифференциального уравнения (ОДУ). Это такие ряды по целым степеням независимой переменной, коэффициенты которых суть ряды Лорана либо от логарифма этой переменной или от мнимой степени соответственно. Предлагается алгоритм составления ОДУ для этих коэффициентов. Первый коэффициент является решением укороченного уравнения. Для некоторых исходных уравнений он является многочленом. Спрашивается: будут ли многочленами следующие коэффициенты? Здесь этот вопрос изучается для третьего, пятого и шестого уравнений Пенлеве. Оказалось, что в семи из восьми семейств сложных разложений и в двух из четырёх семейств экзотических разложений вторые коэффициенты — многочлены. Но в трёх оставшихся семействах вторые коэффициенты являются многочленами только при определённых условиях. Здесь подробно изложены доказательства и вычисления этих результатов.

Ключевые слова: разложения решений ОДУ, сложные разложения, экзотические разложения, полиномиальность коэффициентов, уравнения Пенлеве.

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1. Introduction

In 2004 I proposed a method for calculation of asymptotic expansions of solutions to a polynomial ordinary differential equation (ODE) [1]. It allowed to compute power expansions and power-logarithmic expansions (or Dulac series) of solutions, where coefficients of powers of the independent variable x are either constants or polynomials of logarithm of x. I will remind the method lately. Later it is appeared that such equations have solutions with other expansions: they can have coefficients of powers of x as Laurent series either in increasing powers of $\log x$ or in increasing and decreasing imaginary powers of x. They are correspondingly complicated (psi-series) [2] or exotic [3] expansions. Methods from [1] are not suitable for their calculation. Now I have found a method to writing down ODE for each coefficient of such series (Section 2). The equations are linear and contain high and low variations from some parts of the initial equation. The first coefficient is a solution of the truncated equation, and usually it is a Laurent series in $\log x$ or in $x^{i\gamma}$. But it is a polynomial or a Laurent polynomial for some equations.

Question: *Will be the following coefficients of the same structure?*

I consider this question for three Painlevé equations P_3, P_5 and P_6, because among 6 Painlevé equations P_1–P_6 there are 3 equations P_3, P_5, P_6 having complicated and exotic expansions of solutions ([4–6]). First coefficients for equations P_3, P_5 and P_6 are polynomials in $\log x$ in complicated expansions and Laurent polynomials in $x^{i\gamma}$ in exotic expansions [4,6]. Each of the Painlevé equations P_3, P_5 and P_6 has 4 complex parameters a,b,c,d. Two of them are included into the truncated equation. These three Painlevé equations have 8 families of complicated expansions and 4 families of exotic expansions. I have calculated several first polynomial coefficients for all these 12 families, sometimes under some simplifications. Second coefficients in 7 of 8 families of complicated expansions are polynomials, as well in 2 families of exotic expansions, but one family of complicated and two families of exotic expansions demand some conditions for polynomiality of the second coefficient. The third coefficient is a polynomial ether always, either under some additional restrictions on parameters, or never. Results for equation P_3, P_5, P_6 are given in Section 3, 4 and 5, 6 correspondingly.

2. Writing ODEs for coefficients

2.1. Algebraic case. Let we have the polynomial

$$f(x,y) \tag{1}$$

and the series

$$y = \sum_{k=0}^{\infty} \varphi_k \, x^k \,, \tag{2}$$

where coefficients φ_k are functions of some quantities. Let we put the series (2) into the polynomial (1) and will select all addends with fixed power exponent of x. For that, we break up the polynomial (1) into the sum $f(x,y) = \sum_{i=0}^{m} f_i(y)\, x^i$, and we write the series (2) in the form $y = \varphi_0 + \sum_{k=1}^{\infty} \varphi_k\, x^k \stackrel{def}{=} \varphi_0 + \Delta$. Then $\Delta^j = \sum_{k=j}^{\infty} c_{jk}\, x^k$, where coefficients c_{jk} are definite sums of products of j coefficients φ_l and corresponding multinomial coefficients [7]. At last, each item $f_i(\varphi_0 + \Delta)$ can be expanded into the Taylor series

$$f_i = \sum_{j=0}^{\infty} \frac{1}{j!} \frac{d^j f_i}{dy^j}\Big|_{y=\varphi_0} \Delta^j \,.$$

So the result of the substitution of series (2) into the polynomial (1) can be written as the sum

$$\sum_{i=0}^{m} x^i \left[f_i(\varphi_0) + \sum_{j=1}^{\infty} \frac{1}{j!} \frac{d^j f_i(\varphi_0)}{dy^j} \sum_{k=j}^{\infty} c_{jk}\, x^k \right]$$

of items of the form

$$x^i \,\frac{1}{j!} \frac{d^j f_i(\varphi_0)}{dy^j}\, c_{jk}\, x^k \,. \tag{3}$$

Here integral indexes $i,j,k \geq 0$ are such

$$k \geq j; \;\; \text{if} \;\; j = 0, \;\; \text{then} \;\; k = 0 \,. \tag{4}$$

Set of such points $(i,j,k) \in \mathbb{Z}^3$ will be denoted as $\mathbf{M}$. At last, all items (3) with fixed power exponent x^n are selected by the equation $i + k = n$. The set $\mathbf{M}$ can be considered as a part of the integer lattice $\mathbb{Z}^3$ in $\mathbb{R}^3$ with points (i,j,k), which satisfy (4).

If we look for expansion (2) as a solution of the equation $f(x,y) = 0$ and want to use the method of indeterminate coefficients, then we obtain the equation $f_0(\varphi_0) = 0$ for the coefficient φ_0, and equation

$$\frac{df_0(\varphi_0)}{dy} \varphi_n x^n + \sum_{(i,j,k)\in\mathbf{N}(n)} x^i \frac{1}{j!} \frac{d^j f_i(\varphi_0)}{dy^j}\, c_{jk}\, x^k + x^n f_n(\varphi_0) = 0 \,, \tag{5}$$

for the coefficient φ_n with $n > 0$, where $\mathbf{N}(n) = \mathbf{M} \cap \{j > 0,\, i + k = n$ and $j > 1$, if $i = 0\}$. That equation can be canceled by x^n and be written in the form

$$\frac{df_0(\varphi_0)}{dy} \varphi_n + \sum_{(i,j,k)\in\mathbf{N}(n)} \frac{1}{j!} \frac{d^j f_i(\varphi_0)}{dy^j}\, c_{jk} + f_n(\varphi_0) = 0 \,. \tag{6}$$

Theorem 1 ([8]). *If $df_0(\varphi_0)/dy \neq 0$, then coefficients φ_n can be found from equations* (6) *successfully with increasing n.*

2.2. Case of ODE. If $f(x,y)$ is a differential polynomial, i.e. it contains derivatives $d^l y/dx^l$, then the job of derivatives $\dfrac{d^j f_i}{dy^j}$ play variations $\dfrac{\delta^j f_i}{\delta y^j}$, which are derivatives of Frechet or Gateaux. Here the j-variation $\dfrac{\delta^j f}{\delta y^j} = \dfrac{d^j f}{dy^j}$, if the polynomial does not contain derivatives, and variation of a derivation is $\dfrac{\delta}{\delta y}\left(\dfrac{d^k y}{dx^k}\right) = \dfrac{d^k}{dx^k}$, and for products

$$\frac{\delta(f \cdot g)}{\delta y} = f\,\frac{\delta g}{\delta y} + \frac{\delta f}{\delta y} \cdot g\ , \quad \frac{\delta}{\delta y}\left(\frac{d^k y}{dx^k} \cdot \frac{d^l}{dx^l}\right) = \frac{d^{k+l}}{dx^{k+l}}\ .$$

Analogue of the Taylor formula is correct for variations

$$f(y+\Delta) = \sum_{j=0}^{\infty} \frac{1}{j!}\,\frac{\delta^j f(y)}{\delta y^j}\,\Delta^j\ .$$

Let now we have the differential polynomial $f(x,y)$ and we look for solution of the equation $f(x,y) = 0$ in the form of expansion (2). Here the technique, described above for algebraic equation, can be used, but with the following refinements.

1) According to [1], differential polynomial $f(x,y)$ is a sum of differential monomials $a(x,y)$, which are products of a usual monomial const $\cdot\, x^r y^s$ and several derivatives $d^l y/dx^l$. Each monomial $a(x,y)$ corresponds to its vectorial power exponent $Q(a) = (q_1,q_2)$ under the following rules: $Q(\text{const}) = 0$, $Q(x^r y^s) = (r,s)$, $Q(d^l y/dx^l) = (-l,1)$, vectorial power exponent of a product of differential monomials is a vectorial sum of their vectorial power exponents $Q(ab) = Q(a)+Q(b)$. Set $\mathbf{S}(f)$ of all vectorial power exponents $Q(a)$ of all differential monomials $a(x,y)$ containing in $f(x,y)$ is called as *support* of f. Its convex hull $\Gamma(f)$ is a *Newton polygon* of f. Its boundary $\partial\Gamma$ consists of vertices $\Gamma_j^{(0)}$ and edges $\Gamma_j^{(1)}$. To each boundary element $\Gamma_j^{(d)}$ corresponds the *truncated equation* $\hat{f}_j^{(d)} = 0$, where $\hat{f}_j^{(d)}$ is a sum of all monomials with power exponents $Q \in \Gamma_j^{(d)}$. The first term of solution's expansion to the full equation is a solution to the corresponding truncated equation. Now the part $f_i(x,y)$ contains all such differential monomials $a(x,y)$, for which in $Q(a)$ the first coordinate $q_1 = i$. Besides, we assume that $f(x,y)$ has no monomials with $q_1 < 0$, and $f_0(y) \not\equiv 0$. Then all formula of the algebraic case with variations instead of derivations are correct.

2) Variations are operators, which are not commute with differential polynomials. So the formulae (5) takes the form

$$\frac{\delta f_0}{\delta y}\, x^n \varphi_n + \sum_{(i,j,k)\in \mathbf{N}(n)} x^i \frac{1}{j!}\, \frac{\delta^j f_i}{\delta y^j}\, x^k c_{jk} + x^n f_n = 0\ , \tag{7}$$

but in it we cannot cancel by x^n and obtain an analog of formulae (6). In (7) all $\delta^j f_i/\delta y^j$ are taken for $y = \varphi_0$.

Theorem 2 ([8]). *In the expansion* (2) *coefficient* φ_n *satisfies equation* (7).

3) Rules of commutation of variations with functions of different classes exist. If φ_k is a series in $\log x$, then $\xi = \log x$ and $x^s = e^{s\xi}$.

Lemma 1 ([4]).

$$\frac{d^n}{d\xi^n}\left[e^{s\xi}\varphi(\xi)\right] = e^{s\xi}\sum_{k=0}^{n}\binom{n}{k} s^{n-k}\varphi^{(k)}(\xi)\,,$$

where $\binom{n}{k}$ *are binomial coefficients and* $\varphi^{(k)}$ *is the* k*-th derivation of* $\varphi(\xi)$ *along* ξ.

Proof follows from the Leibniz's formula for derivation of a product.

Corollary 1.

$$\frac{d}{d\xi}\left[x^s\varphi(\xi)\right] = x^s[s\varphi(\xi) + \dot\varphi(\xi)]\,,$$

$$\frac{d^2}{d\xi^2}\left[x^s\varphi(\xi)\right] = x^s[s^2\varphi(\xi) + 2s\dot\varphi(\xi) + \ddot\varphi(\xi)]\,.$$

If φ_k is a series in $x^{i\gamma}$, then $\xi = x^{i\gamma}$ and $x^s = \xi^{s/(i\gamma)}$.

Lemma 2 ([9]).

$$\frac{d^n}{d\xi^n}\left[\xi^{s/(i\gamma)}\varphi(\xi)\right] = \\ = \xi^{s/(i\gamma)}\left[\sum_{k=0}^{n-1}\binom{n}{k}\frac{s}{i\gamma}\left(\frac{s}{i\gamma}-1\right)\dots\left(\frac{s}{i\gamma}-n+k+1\right)\varphi^{(k)}(\xi)\frac{1}{\xi^{n-k}} + \varphi^{(n)}\right].$$

Corollary 2.

$$\xi\frac{d}{d\xi}\left[x^n\varphi(\xi)\right] = x^n\left[\frac{n}{i\gamma}\varphi + \xi\dot\varphi\right],$$

$$\xi^2 \frac{d^2}{d\xi^2}\left[x^n \varphi(\xi)\right] = x^n \left[\frac{n}{i\gamma}\left(\frac{n}{i\gamma} - 1\right)\varphi + \frac{2n}{i\gamma}\xi\dot{\varphi} + \xi^2\ddot{\varphi}\right].$$

These Lemmas give rules of commutation of an operator with x^s. Applying them in equation (7), we can cancel the equation by x^n and obtain an equation without x, only with ξ. So the algorithm consists of the following steps.

Step 0 From the initial equation $f(x,y) = 0$, we select such truncated equation $\hat{f}_1^{(1)}(x,y) = 0$, which corresponds to edge $\Gamma_1^{(1)}$ of the polygon Γ of the differential sum $f(x,y)$ and has a complicated or exotic solution depending from $\log x$ or $x^{i\gamma}$, $\gamma \in \mathbb{R}$ correspondingly.

Step 1 We make a power transformation of the variables $y = x^l z$ to make the truncated equation correspond to the vertical edge.

Step 2 We divide the transformed equation $g(x,z) = 0$ into parts $g_i(x,y)x^i$, corresponding to different verticals of its support.

Step 3 In these parts $g_i(x,y)x^i$ we change the independent variable x by $\log x$ or by $x^{i\gamma}$.

Step 4 We write down equations for several first coefficients φ_k.

Step 5 Using the rules of commutation, we exclude powers of x from these equations and we obtain linear ODEs for coefficients with independent variable $\log x$ or $x^{i\gamma}$. Their solutions are power expansions and can be computed by known methods from [1].

3. The third Painlevé equation P_3

3.1. Truncated equation and its logarithmic solutions. The third Painlevé equation P_3 is

$$y'' = \frac{y'^2}{y} - \frac{y'}{x} + \frac{ay^2 + b}{x} + cy^3 + \frac{d}{y}.$$

Let multiply it by its denominator xy and translate the left hand side into right side. Then we obtain the equation P_3, written as a differential polynomial

$$f(x,y) \stackrel{def}{=} -xyy'' + xy'^2 - yy' + ay^3 + by + cxy^4 + dx = 0, \tag{8}$$

where a, b, c, d are complex parameters. Its support and polygon for $a,b,c,d \neq 0$ are shown in Fig. 1. The edge $\Gamma_1^{(1)}$ corresponds to the truncated equation

$$\hat{f}_1^{(1)} \stackrel{def}{=} -xyy'' + xy'^2 - yy' + by + dx = 0. \tag{9}$$

After the power transformation $y = xz$ and canceling by x, the full equation (8) became

$$g \stackrel{def}{=} -x^2zz'' + x^2z'^2 - xzz' + bz + d + ax^2z^3 + cx^4z^4 = 0. \tag{10}$$

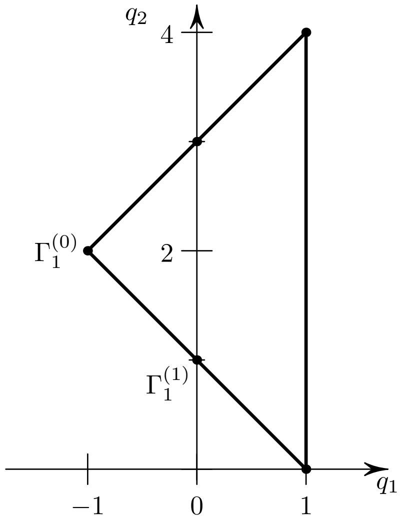

Figure 1. Support and polygon of the equation (8) for $a,b,c,d \neq 0$.

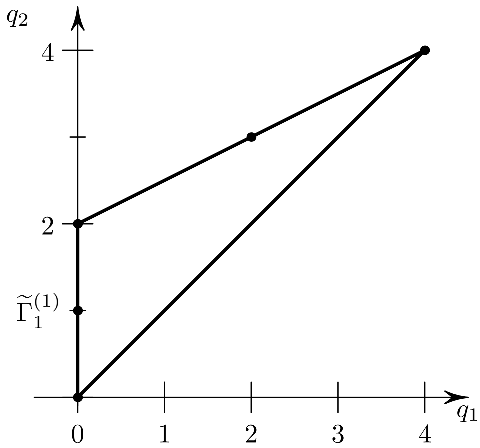

Figure 2. Support and polygon of the equation (10) for $a,b,c,d \neq 0$.

Here the truncated equation (9) takes the form

$$g_0 \stackrel{def}{=} -x^2zz'' + x^2z'^2 - xzz' + bz + d = 0\,. \tag{11}$$

Support and polygon of equation (10) are shown in Fig. 2. Here the truncated equation (11) corresponds to the vertical edge $\widetilde{\Gamma}_1^{(1)}$ at the axis $q_1 = 0$. Here $g_2 = az^3$, $g_4 = cz^4$.

After the logarithmic transformation $\xi = \log x$, equation (11) takes the form

$$h_0 \stackrel{def}{=} -z\ddot{z} + \dot{z}^2 + bz + d = 0, \tag{12}$$

where $\dot{z} = dz/d\xi$. Support and polygon of equation (12) are shown in Fig. 3 in the case $bd \neq 0$. Here $h_2 = az^3$, $h_4 = cz^4$.

Let $b \neq 0$. The edge $\widetilde{\Gamma}_1^{(1)}$ of Fig. 3 corresponds the truncated equation $\hat{h}_1^{(1)} \stackrel{def}{=} -z\ddot{z}+\dot{z}^2+bz = 0$. It has the power solution $z = -b\xi^2/2$. According to [1], extending

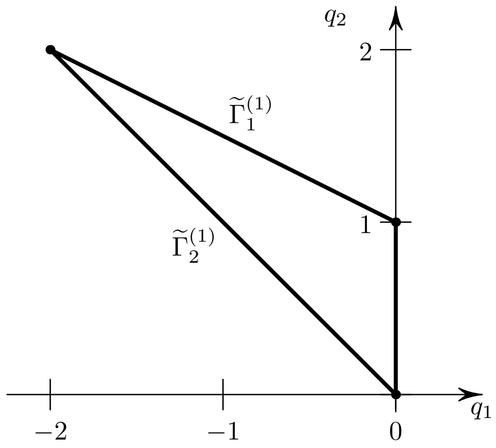

Figure 3. Support and polygon of the equation (12) with $bd \neq 0$.

it as expansion in decreasing powers of ξ, we obtain the solutions of equation (11)

$$z = -\frac{b}{2}(\log x + \tilde{c})^2 - \frac{d}{2b} = \varphi_0, \tag{13}$$

where $\tilde{c}$ is arbitrary constant.

Let us consider equation (11) in the case $b = 0$, $d \neq 0$. Then equation (12) has the form

$$h_0 \stackrel{def}{=} -z\ddot{z} + \dot{z}^2 + d = 0.$$

Its polygon coincides with the edge $\widetilde{\Gamma}_2^{(1)}$ in Fig. 3. The equation has solutions

$$z = \pm\sqrt{-d}\,(\log x + \tilde{c}) = \varphi_0. \tag{14}$$

Thus, we have proved

Theorem 3. *All nonconstant solutions to equation* (12)*, expanded into power series in decreasing powers of* ξ*, form two families:*
the main family (13) *for* $b \neq 0$*; and*
the additional family (14) *for* $b = 0, d \neq 0$.

Solutions to equation (10) have the form of expansion

$$z = \varphi_0(\xi) + \sum_{k=1}^{\infty} \varphi_{2k}(\xi) x^{2k}, \tag{15}$$

where φ_0 is given by (13) or (14).

In the first case $b \neq 0$, we call family of solutions (15) as **main**, and in the second case $b = 0, d \neq 0$, we call the family of solutions (15) as **additional**.

According to Theorem 2, equation for φ_2 is

$$\frac{\delta h_0}{\delta z}(x^2\varphi_2) + x^2 h_2(\varphi_0) = 0. \tag{16}$$

According to (12)

$$\frac{\delta h_0}{\delta z} = -\ddot{z} - z\frac{d^2}{d\xi^2} + 2\dot{z}\frac{d}{d\xi} + b, \quad \frac{\delta^2 h_0}{\delta z^2} = 0. \tag{17}$$

According to (10) $h_2 = az^3$ and according to Corollary 1

$$\frac{d}{d\xi}x^2\varphi_2 = x^2\left[2\varphi_2 + \dot{\varphi}_2\right], \quad \frac{d^2}{d\xi^2}x^2\varphi_2 = x^2\left[4\varphi_2 + 4\dot{\varphi}_2 + \ddot{\varphi}_2\right].$$

So, equation (16), after canceling x^2, takes the form

$$-z\left[4\varphi_2 + 4\dot{\varphi}_2 + \ddot{\varphi}_2\right] + 2\dot{z}\left[2\varphi_2 + \dot{\varphi}_2\right] + (b - \ddot{z})\varphi_2 + az^3 = 0, \tag{18}$$

where $z = \varphi_0$ from (13) or (14).

3.2. The additional complicated family. Let $\xi = \log x + \tilde{c}$, then, according to (14), $z = \varphi_0 = \beta\xi, \beta^2 = -d, \dot{z} = \beta, \ddot{z} = 0$, and equation (18) is

$$-\beta\,\xi\left[4\varphi_2 + 4\dot{\varphi}_2 + \ddot{\varphi}_2\right] + 2\,\beta\left[2\varphi_2 + \dot{\varphi}_2\right] + a(\beta\,\xi)^3 = 0\,.$$

Its support and polygon see in Fig. 4.

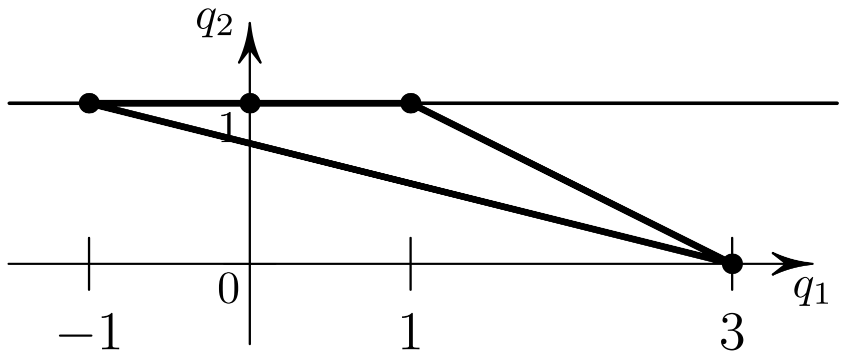

Figure 4. Support and polygon of equation for φ_2 in additional complicated expansion.

Cotangent of the angle of inclination of its right edge equals to -2. So we look for polynomial solution of degree 2. Indeed that equation has a polynomial solution:

$$\varphi_2 = -\frac{ad}{4}\left(\xi^2 - \xi + \frac{1}{2}\right).$$

Here a linear system of 4 algebraic equation is satisfied for 3 constant coefficients. According to Theorem 2, equation for φ_4 is

$$\frac{\delta h_0}{\delta z}\,x^4\varphi_4 + x^2\frac{\delta h_2}{\delta z}\,x^2\varphi_2 + x^4 h_4(\varphi_0) = 0\,. \tag{19}$$

According to Corollary 1

$$\frac{d^2}{d\xi^2}x^4\varphi_4 = x^4[16\varphi_4 + 8\dot{\varphi}_4 + \ddot{\varphi}_4]\,, \quad \frac{d}{d\xi}x^4\varphi_4 = x^4[4\varphi_4 + \dot{\varphi}_4]\,.$$

Here $\dfrac{\delta h_2}{\delta z} = \dfrac{dh_2}{dz} = 3az^2$, $h_4 = cz^4$.

So after canceling by x^4, equation (19) takes the form

$$-\beta\xi[16\varphi_4 + 8\dot{\varphi}_4 + \ddot{\varphi}_4] + 2\beta[4\varphi_4 + \dot{\varphi}_4] + 3a\beta^2\xi^2\varphi_2 + c(\beta\xi)^4 = 0\,. \tag{20}$$

Its support and polygon are shown in Fig. 5.

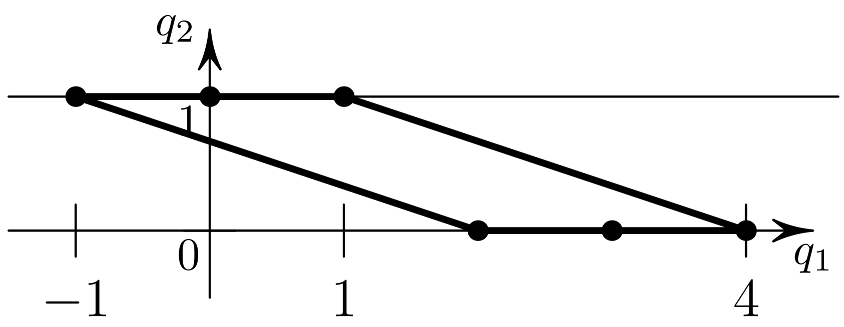

Figure 5. Support and polygon of the equation (20).

Cotangent of the angle of inclination of its right edge equals to -3. So the solution to equation (14) may be polynomial of order 3

$$\varphi_4 = A\xi^3 + B\xi^2 + C\xi + D\,.$$

Then the sum of two first addends in (19) is

$$-16\beta A\xi^4 + (-16B - 16A)\beta\xi^3 + (-16C - 8B)\beta\xi^2 + (-16D + 2B)\beta\xi + 2(4D + C)\beta\,.$$

Here coefficients near ξ^2, ξ^1 and $\xi^0 = 1$ for $\beta B, \beta C, \beta D$ form the matrix

$$\begin{pmatrix} -8 & -16 & 0 \\ 2 & 0 & -16 \\ 0 & 2 & 8 \end{pmatrix}$$

with zero determinant. From the other side, the sum of two last addends in (20) is

$$3a\beta^2\xi^2\varphi_2 + c(\beta\xi)^4 = \frac{3a^2d^2}{4}\left(\xi^4 - \xi^3 + \frac{1}{2}\xi^2\right) + c\beta^4\xi^4\,.$$

Coefficients of that sum near ξ^2, ξ^1 and 1 are $\frac{3}{8}a^2 d^2, 0$ and 0 correspondingly. Hence, the linear system of equations for A, B, C, D has a solution only if $\frac{3}{8}a^2 d^2 = 0$. As $d \neq 0$, then we obtain the condition $a = 0$ for existence A, B, C, D. Under the condition

$$\varphi_4 = \frac{c\beta^3}{16}\left(\xi^3 - \xi^2 + \frac{1}{2}\xi - \frac{1}{8}\right). \tag{21}$$

As $a = 0$, then $g = g_0 + x^4 g_4$, $\varphi_2 = 0$, and the expansion of solution contains powers of x, which are multiple to 4.

Theorem 2 gives for φ_8 the equation

$$\frac{\delta h_0}{\delta z}\, x^8 \varphi_8 + x^4 \frac{\delta h_4}{\delta z}\, x^4 \varphi_4 = 0\,. \tag{22}$$

According to (17), here $\dfrac{\delta h_0}{\delta z} = -z\dfrac{d^2}{d\xi^2} + \dot{z}\,\dfrac{d}{d\xi}$, $\dfrac{\delta h_4}{\delta z} = \dfrac{dg_4}{dz} = 4cz^3$. According to Corollary 1

$$\frac{d^2}{d\xi^2} x^8 \varphi_8 = x^8[64\varphi_8 + 16\dot{\varphi}_8 + \ddot{\varphi}_8]\,, \quad \frac{d}{d\xi} x^8 \varphi_8 = x^8[8\varphi_8 + \dot{\varphi}_8]\,.$$

As h_4 does not contain derivatives, then variation

$$\frac{\delta h_4}{\delta z} = \frac{dh_4}{dz} = 4c\,(\beta\xi)^3$$

and it commutes with $x^4\varphi_4$. Canceling equation (22) by x^4, we obtain equation

$$-\beta\xi\,[64\varphi_8 + 16\dot{\varphi}_8 + \ddot{\varphi}_8] + 2\beta\,[8\varphi_8 + \dot{\varphi}_8] + 4c\beta^3\xi^3\varphi_4 = 0\,.$$

It has the polynomial solution

$$\varphi_8 = \frac{c^2\beta^5}{16^2}\left(\xi^5 - 2\xi^4 + \frac{59}{32}\xi^3 - \frac{59}{64}\xi^2 + \frac{59}{4\cdot 64}\xi - \frac{59}{32\cdot 64}\right).$$

According to Theorem 2, we obtain the equation for φ_{12}

$$\frac{\delta h_0}{\delta z}\, x^{12}\varphi_{12} + x^4 \frac{\delta h_4}{\delta z}\, x^8 \varphi_8 + x^4\, \frac{1}{2}\,\frac{\delta^2 h_4}{\delta z^2}\, \left(x^4\varphi_4\right)^2 = 0\,.$$

According to Corollary 1, it has the form

$$-\beta\xi[144\varphi_{12} + 24\dot{\varphi}_{12} + \ddot{\varphi}_{12}] + 2\beta[12\varphi_{12} + \dot{\varphi}_{12}] + 4c(\beta\xi)^3\varphi_8 + \frac{1}{2}\cdot 6c(\beta\xi)^2\varphi_4^2 = 0\,. \tag{23}$$

If to look for solution of the equation as the polynomial of order 7

$$\varphi_{12} = E\xi^7 + F\xi^6 + G\xi^5 + H\xi^4 + I\xi^3 + J\xi^2 + K\xi + L\,,$$

then the sum of terms of small powers of ξ in the first two addends in (23) is

$$\beta(-144K - 24J)\xi^2 + \beta(-144L + 2J)\xi + \beta(24L + 2K)\,.$$

Matrix of coefficient near $\beta J, \beta K$ and βL is

$$\begin{pmatrix} -24 & -144 & 0 \\ 2 & 0 & -144 \\ 0 & 2 & 24 \end{pmatrix}.$$

It has zero determinant. From other side, terms of smallest power of ξ in the remaining part of equation (23) are

$$3c\beta^2 \left(\frac{c\beta^3}{16}\right)^2 \left(-\frac{1}{8}\right)^2 \xi^2 \qquad (24)$$

according to (21). The linear algebraic system of equations for $E, \ldots, L$ has a solution, if the coefficient in (24) equals to zero. As $\beta \neq 0$, then $c = 0$. In that case the full equation is degenerated into truncated one $g_0 = 0$, and in expansion $z = \sum_{k=0}^{\infty} \varphi_{4k}(\xi)\, x^{4k}$ all $\varphi_{4k} = 0$ for $k > 0$. That is the trivially degenerated integrable case with $a = c = 0$. So we have proved

Theorem 4. *In expansion* (15) *of the additional complicated family of solutions to the equation P_3, polynomial coefficients are φ_2 for any values of parameters a and c; also $\varphi_4, \varphi_6 = 0, \varphi_8$ are polynomials for $a = 0$. The fifth coefficient φ_8 never is a polynomial, if $|a| + |c| \neq 0$.*

3.3. The main complicated family. Let put $\xi = \log x + \tilde{c}$, then solution (13) is:

$$z = -\frac{b}{2}\xi^2 - \frac{d}{2b} = \varphi_0(\xi)\,.$$

Here $\dot{z} = -b\xi, \ddot{z} = -b$ and the equation (18) has the polynomial solution

$$\varphi_2 = \frac{ab^2}{16}\left[\xi^4 - 2\xi^3 + (2 + 2\lambda)\xi^2 - (1 + 2\lambda)\xi + \lambda^2\right],$$

where $\lambda = d/b^2$.

Theorem 5. *In expansion* (15) *of the main complicated family of solutions to the equation P_3, the second coefficient φ_2 is always a polynomial.*

Farther we consider the main family under the restriction $d = 0$. Then $\lambda = 0, z = -\frac{b}{2}\xi^2, \dot{z} = -b\xi, \ddot{z} = -b$ and

$$\varphi_2 = \frac{ab^2}{16}(\xi^4 - 2\xi^3 + 2\xi^2 - \xi)\,.$$

According to Theorem 2, equation for φ_4 is

$$\frac{\delta h_0}{\delta z}\,x^4\varphi_4 + x^2\frac{\delta h_2}{\delta z}\,x^2\varphi_2 + x^4 h_4 = 0\,.$$

According to (17) and Corollary 1,

$$\frac{\delta h_0}{\delta z}\,x^4\varphi_4 = x^4\frac{b}{2}\xi^2\,[16\varphi_4 + 8\dot{\varphi}_4 + \ddot{\varphi}_4] - x^4 2b\xi\,[4\varphi_4 + \dot{\varphi}_4] + x^4\cdot 2b\varphi_4\,,$$

$$\frac{\delta h_2}{\delta z}\,x^2\varphi_2 = x^2\frac{3ab^2}{4}\xi^4\varphi_2, \quad h_4 = \frac{c}{16}(b\xi^2)^4\,.$$

After canceling by x^4, we obtain the equation

$$\frac{b}{2}\,\xi^2\,[16\varphi_4 + 8\dot{\varphi}_4 + \ddot{\varphi}_4] - 2b\xi\,[4\varphi_4 + \dot{\varphi}_4] + 2\,b\varphi_4 + \frac{3}{4}ab^2\xi^4\varphi_2 + \frac{cb^4}{16}\xi^8 = 0\,.$$

It has the polynomial solution

$$\varphi_4 = a^2b^3\psi_1 + c\,b^3\psi_2\,,$$

where

$$\psi_1 = \frac{1}{2^9}\left(-3\xi^6 + \frac{15}{2}\,\xi^5 - \frac{91}{8}\,\xi^4 + \frac{115}{2}\,\xi^3 - \frac{115}{4}\,\xi^2 + \frac{115}{16}\,\xi\right)\,,$$

$$\psi_2 = \frac{1}{2^7}\left(-\xi^6 + 2\,\xi^5 - \frac{19}{2^3}\,\xi^4 + \frac{15}{2^3}\,\xi^3 - \frac{15}{2^4}\,\xi^2 + \frac{15}{2^6}\,\xi\right)\,.$$

According to Theorem 2, we have following equations for φ_6 and φ_8

$$\frac{\delta h_0}{\delta z}\,x^6\varphi_6 + x^2\frac{\delta h_2}{\delta z}\,x^4\varphi_4 + x^2\frac{1}{2}\,\frac{\delta^2 h_2}{\delta z^2}\,(x^2\varphi_2)^2 + x^4\frac{\delta h_4}{\delta z}\,x^2\varphi_2 = 0\,,$$

$$\frac{\delta h_0}{\delta z}\,x^8\varphi_8 + x^2\frac{\delta h_2}{\delta z}\,x^6\varphi_6 + x^2\frac{1}{2}\,\frac{\delta^2 h_2}{\delta z^2}\,2(x^2\varphi_2)\,(x^4\varphi_4) + x^4\frac{\delta h_4}{\delta z}\,x^4\varphi_4 + \\ + \frac{1}{2}\,\frac{\delta^2 h_4}{\delta z^2}\,(x^2\varphi_2)^2 = 0\,.$$

The equations have polynomial solutions for any parameters $b \neq 0, a, c$, because their parts, containing variations from h_2 and h_4, do not contain ξ^2,ξ and $\xi^0 = 1$.

Hypothesis 1 ([8]). *Coefficients $\varphi_{2k}(\xi)$ in expansion (15) of the main complicated family of solutions to the equation P_3 are polynomials in $\log x$, if the parameter of the equation $d = 0$.*

3.4. Exotic expansions for equation P_3. Now and to the end of the Section, we put $\xi = x^{i\gamma}, \gamma \in \mathbb{R}, \gamma \neq 0$. Then

$$x = \xi^{1/(i\gamma)}, \quad z' = \frac{i\gamma \dot{z}\xi}{x}, \quad z'' = -\frac{\gamma^2 \ddot{z}\,\xi^2 + i\gamma\dot{z}\xi + \gamma^2\dot{z}\xi}{x^2}\,.$$

So the truncated equation (11) takes the form

$$\gamma^2 z(\xi^2\ddot{z} + \xi\dot{z}) - \gamma^2\xi^2\dot{z}^2 + bz + d = 0\,.$$

Dividing it by γ^2, we obtain equation

$$h_0 \stackrel{def}{=} z(\xi^2\ddot{z} + \xi\dot{z}) - \xi^2\dot{z}^2 + \tilde{b}z + \tilde{d} = 0\,, \tag{25}$$

where $\tilde{b} = b/\gamma^2, \tilde{d} = d/\gamma^2$. In the full (nontruncated) equation $h_2 = \tilde{a}z^3, h_4 = \tilde{c}z^4$, where $\tilde{a} = a/\gamma^2, \tilde{c} = c/\gamma^2$.

Theorem 6. *All exotic solutions to equation (25) in the form of Laurent series*

$$z = A\xi + B + C\xi^{-1} + \cdots\,,$$

where $A, B, C = const \in \mathbb{C}$ are the Laurent polynomials

$$z = A\xi^{-1} + B + C\xi^{-1} = \varphi_0\,, \tag{26}$$

and form one family, where

$$B + \tilde{b} = 0, \quad 4AC - \tilde{b}^2 + \tilde{d} = 0\,. \tag{27}$$

Proof is based on [1]. Polygon Γ of the truncated equation (25) is the edge $q_1 = 0, 0 \leq q_2 \leq 2$. Its upper vertex $q_1 = 0, q_2 = 2$ corresponds to the truncated equation

$$\hat{h}_0 \stackrel{def}{=} z(\xi^2\ddot{z} + \xi\dot{z}) - \xi^2\dot{z}^2 = 0\,. \tag{28}$$

Its characteristic equation is

$$k(k-1) + k - k^2 \equiv 0\,.$$

So equation (28) has power solutions $z = A\xi^{\lambda}$ with any constants A and λ. In particular, $z = A\xi$ is its solution. We make substitution $z = A\xi + u$ into equation (25). Then it takes the form

$$A\xi\left(\xi^2\ddot{u} + u - \xi\dot{u} + \tilde{b}\right) + u\left(\xi^2\ddot{u} + \xi\dot{u}\right) - \xi^2\dot{u}^2 + \tilde{b}u + \tilde{d} = 0\,. \qquad (29)$$

Support and the polygon of equation (29) are shown in Fig. 6.

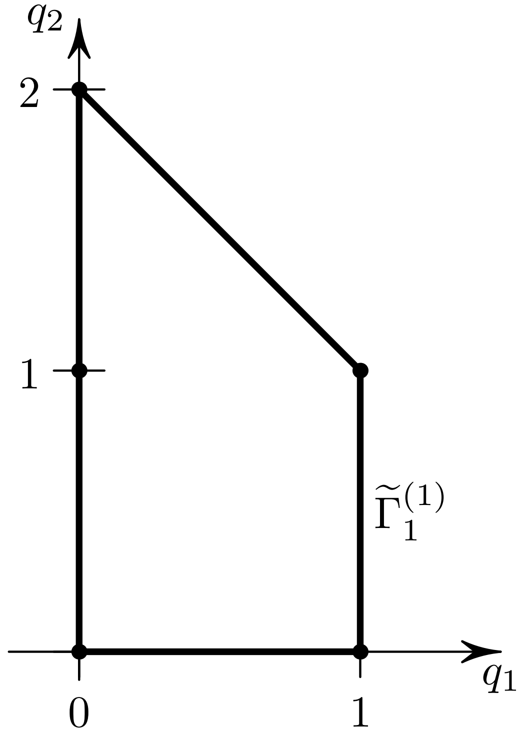

Figure 6. Support and polygon of equation (29).

It is a quadrangle with the edge $\widetilde{\Gamma}_1^{(1)}$ with normal $P = (1{,}0)$, corresponding to the truncated equation

$$A\xi(\xi^2\ddot{u} + u - \xi\dot{u} + \tilde{b}) = 0\,.$$

Its power solution $u = c\xi^2$ with $r = 0$ is $u = -\tilde{b}$. After substitution $u = -\tilde{b} + w$, the equation (29) takes the form

$$A\xi(\xi^2\ddot{w} - \xi\dot{w} + w) + (w - \tilde{b})(\xi^2\ddot{w} + \xi\dot{w}) - \xi^2\dot{w}^2 - \tilde{b}^2 + \tilde{b}w + \tilde{d} = 0\,. \qquad (30)$$

Its support and polygon Γ are shown in Fig. 7. Polygon Γ has the edge $\widetilde{\Gamma}_2^{(1)}$ with the normal $P = (1, -1)$, corresponding to the truncated equation

$$A\xi(\xi^2\ddot{w} - \xi\dot{w} + w) - \tilde{b}^2 + \tilde{d} = 0\,.$$

Constant C of its solution $w = C\xi^{(-1)}$ satisfies equation $4AC - \tilde{b}^2 + \tilde{d} = 0$. It is also a solution of equations $w(\xi^2\ddot{w} + \xi\dot{w}) - \xi^2\dot{w}^2 = 0$ and $\tilde{b}(w - \xi^2\ddot{w} - \xi\dot{w}) = 0$. So that

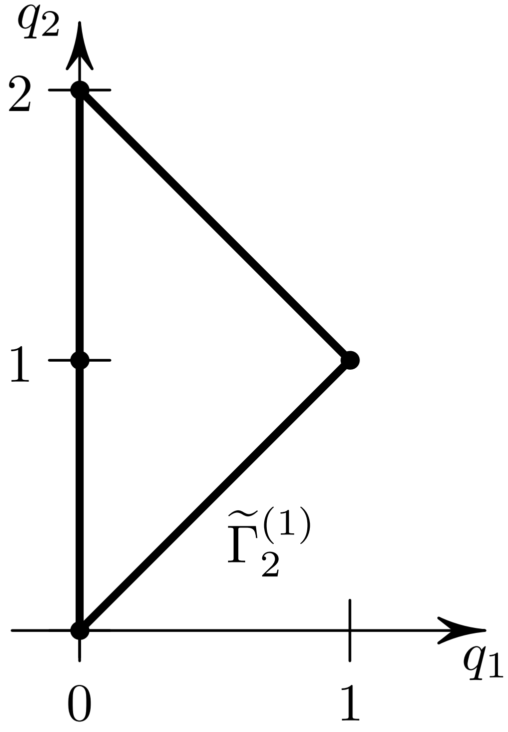

Figure 7. Support and polygon of equation (30).

solution $w = C\xi^{(-1)}$ is a solution of the equation (30). Hence, (26), (27) are solutions to equation (25).

Remark 1. Equation (25) is integrable and Theorem 6 follows from Theorem 1 [9], which describes all solutions of equation (25).

Exotic expansion of solutions to the full equation (10) again have the form (15). Let us find $\varphi_2(\xi)$. It is a solution to equation (16). But now according to (25),

$$\frac{\delta h_0}{\delta z} = z\xi^2\frac{d^2}{d\xi^2} + z\xi\frac{d}{d\xi} - 2\dot{z}\xi^2\frac{d}{d\xi} + \xi\dot{z} + \tilde{b} = z\xi^2\frac{d^2}{d\xi^2} + (z - 2\dot{z}\xi)\xi\frac{d}{d\xi} + \xi\dot{z} + \tilde{b}\,, \quad (31)$$

$$\frac{\delta^2 h_0}{\delta z^2} = -\xi^2\frac{d^2}{d\xi^2} + 2\xi\frac{d}{d\xi}, \quad \frac{\delta^3 h_0}{\delta z^3} = 0, \quad h_2 = \tilde{a}z^3.$$

According to (26) $\xi\dot{z} = A\xi - C\xi^{-1}$, $\xi^2\ddot{z} = 2C\xi^{-1}$. So, applying Corollary 2 to equation (16) and dividing it by x^2, we obtain equation

$$(A\xi + B + C\xi^{-1}) \left[\frac{2}{i\gamma}\left(\frac{2}{i\gamma} - 1\right)\varphi_2 + \frac{4}{i\gamma}\xi\dot{\varphi}_2 + \xi^2\ddot{\varphi}_2\right] +$$
$$+ (-A\xi + B + 3C\xi^{-1}) \left[\frac{2}{i\gamma}\varphi_2 + \xi\dot{\varphi}_2\right] +$$
$$+ (A\xi - B + C\xi^{-1})\varphi_2 + \tilde{a}(A\xi + B + C\xi^{-1})^3 = 0\,. \quad (32)$$

Its support and the polygon are shown in Fig. 8.

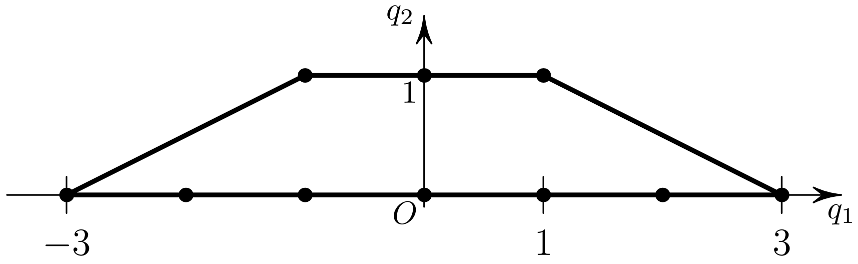

Figure 8. Support and polygon of equation (32).

Cotangents of angles of inclination of left and right edges are equal to ± 2. Hence, solution to equation (32) in form of a Laurent polynomial must have powers from -2 to $+2$, i.e.

$$\varphi_2 = D\xi^2 + E\xi + F + G\xi^{-1} + H\xi^{-2}\,, \tag{33}$$

where D,E,F,G,H — are constants. Then

$$\xi\dot{\varphi}_2 = 2D\xi^2 + E\xi - G\xi^{-1} - 2H\xi^{-2}\,,$$

$$\xi^2\ddot{\varphi}_2 = 2D\xi^2 + 2G\xi^{-1} + 6H\xi^{-2}\,.$$

Note that

$$\varphi_0^3 = (A\xi + B + C\xi^{-1})^3 = A^3\xi^3 + 3A^2B\xi^2 + 3(AB^2 + A^2C)\xi + B^3 + 6ABC +$$

$$+3(AC^2 + B^2C)\xi^{-1} + 3BC^2\xi^{-2} + C^3\xi^{-3}\,.$$

We substitute these expressions into equation (32) and nullity coefficients near ξ^3,ξ^2, ξ,ξ^0,ξ^{-1},ξ^{-2},ξ^{-3}. Then we obtain a system of 7 linear algebraic equations for 5 coefficients D,E, F,G,H. It has the unique solution

$$D = \frac{\tilde{a}A^2\gamma^2}{(2+i\gamma)^2},\ E = \frac{\tilde{a}AB\gamma^2}{2+i\gamma},\ F = \frac{\tilde{a}B^2\gamma^2}{4+\gamma^2} + \tilde{a}AC\frac{(8+6\gamma^2)\gamma^2}{(4+\gamma^2)^2}, \tag{34}$$
$$G = \frac{\tilde{a}BC\gamma^2}{2-i\gamma},\ H = \frac{\tilde{a}C^2\gamma^2}{(2-i\gamma)^2}\,.$$

According to Theorem 2, we have for φ_4 the equation

$$\frac{\delta h_0}{\delta z}x^4\varphi_4 + \frac{1}{2}\frac{\delta^2 h_0}{\delta z^2}(x^2\varphi_2)^2 + x^2\frac{\delta h_2}{\delta z}x^2\varphi_2 + x^4 h_4(\varphi_0) = 0\,, \tag{35}$$

Let us consider it in the case $a = 0$. Then according to (33), (34) $\varphi_2 = 0$ and equation (35) is

$$\frac{\delta h_0}{\delta z}x^4\varphi_4 + x^4 h_4 = 0\,, \tag{36}$$

where $\dfrac{\delta h_0}{\delta z}$ is in (31), $h_4 = \tilde{c} z^4$, $z = \varphi_0 = A\xi + B + C\xi^{-1}$. Using in equation (36) Corollary 2 and dividing it by x^4, we obtain equation

$$(A\xi + B + C\xi^{-1}) \left[\frac{4}{i\gamma} \left(\frac{4}{i\gamma} - 1 \right) \varphi_4 + \frac{8}{i\gamma} \xi \dot{\varphi}_4 + \xi^2 \ddot{\varphi}_4 \right] + \\ + (-A\xi + B + 3C\xi^{-1}) \left[\frac{4}{i\gamma} \varphi_4 + \xi \dot{\varphi}_4 \right] + \\ + (A\xi - B + C\xi^{-1}) \varphi_4 + \tilde{c} (A\xi + B + C\xi^{-1})^4 = 0 \,. \quad (37)$$

Its support and the Newton polygon are shown in Fig. 9.

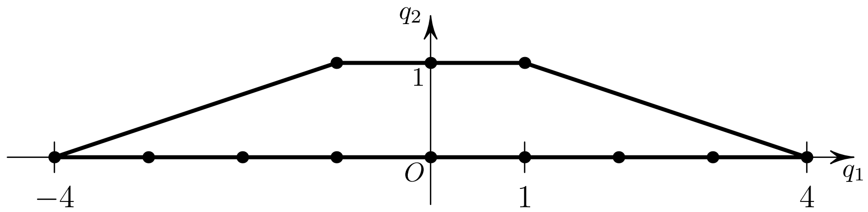

Figure 9. Support and polygon of equation (37).

Inclinations of its side edges are ± 3. Hence, solution to equation (37) in the form of Laurent polynomial must have powers of ξ from -3 up to $+3$.

$$\varphi_4 = I\xi^3 + J\xi^2 + K\xi + L + M\xi^{-1} + N\xi^{-2} + O\xi^{-3} \,. \quad (38)$$

Then

$$\xi \dot{\varphi}_4 = 3I\xi^3 + 2J\xi^2 + K\xi - M\xi^{-1} - 2N\xi^{-2} - 3O\xi^{-3} \,,$$
$$\xi^2 \ddot{\varphi}_4 = 6I\xi^3 + 2J\xi^2 + 2M\xi^{-1} + 6N\xi^{-2} + 12O\xi^{-3} \,.$$

Besides,

$$(A\xi + B + C\xi^{-1})^4 = A^4\xi^4 + 4A^3B\xi^3 + (6A^2B^2 + 4A^3C)\xi^2 + \\ + (4AB^3 + 12A^2BC)\xi + B^4 + 6A^2C^2 + 12AB^2C + (4B^3C + 12ABC^2)\xi^{-1} + \\ + (6B^2C^2 + 4AC^3)\xi^{-2} + 4BC^3\xi^{-3} + C^4\xi^{-4} \,.$$

Substituting these expressions in equation (37) and nullifying coefficients near ξ^4,ξ^3,ξ^2,ξ,ξ^0,ξ^{-1},ξ we obtain a system of 9 algebraic equations for 7 coefficients I,J,K,L,M,N,O. The system has solution

$$
\begin{gathered}
I = \frac{\tilde{c}A^3\gamma^2}{4(2+i\gamma)^2}, \quad J = \frac{2\tilde{c}A^2B\gamma^2(3+i\gamma)}{(2+i\gamma)(4+i\gamma)^2}, \\
K = \frac{\tilde{c}AB^2\gamma^2(12+5i\gamma)}{8(2+i\gamma)(4+i\gamma)} + \frac{\tilde{c}A^2C\gamma^2(3+2i\gamma)}{4(2+i\gamma)^2}, \\
L = \frac{\tilde{c}B^3\gamma^2}{16+\gamma^2} + \frac{2\tilde{c}ABC(48+5\gamma^2)}{(16+\gamma^2)^2}, \\
M = \frac{\tilde{c}B^2C\gamma^2(12-5i\gamma)}{8(2-i\gamma)(4-i\gamma)} + \frac{\tilde{c}AC^2\gamma^2(3-2i\gamma)}{4(2-i\gamma)^2}, \\
N = \frac{2\tilde{c}BC^2\gamma^2(3-i\gamma)}{(2-i\gamma)(4-i\gamma)^2}, \quad O = \frac{\tilde{c}C^3\gamma^2}{4(2-i\gamma)^2}\,.
\end{gathered}
\tag{39}
$$

Thus, we have proven

Theorem 7. *In the exotic expansions* (15) *of solutions to equation P_3, the second coefficient $\varphi_2(\xi)$ is always the Laurent polynomial* (33), (34)*, but the third coefficient φ_4 is a Laurent polynomial* (38), (39)*, if the parameter $a = 0$.*

The case $a \neq 0, c = 0$ should be studied separately, using equation (35).

4. The fifth Painlevé equation P_5 in Case I

4.1. Two cases for equation P_5. The fifth Painlevé equation P_5 is

$$
y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right) y'^2 - \frac{y'}{x} + \frac{(y-1)^2}{x^2}\left(ay + \frac{b}{y}\right) + \frac{cy}{x} + \frac{dy(y+1)}{y-1}\,, \tag{40}
$$

where a,b,c,d are complex parameters, x and y are independent and dependent variables, $y' = dy/dx$ [5]. To write equation (40) as a differential sum, multiply it by $x^2y(y-1)$ and carry all terms into right side. We obtain the equation

$$
\begin{aligned}
- x^2y(y-1)y'' + x^2(3y-1)y'^2/2 - xy(y-1)x' + (y-1)^3(ay^2+b)- \\
- cxy^2(y-1) + dx^2y^2(y+1)^2 = 0\,.
\end{aligned}
\tag{41}
$$

Its support and polygon are shown in Fig. 10.

After substitution $y = 1 + z$ into equation (41), we obtain equation

$$
\begin{aligned}
- x^2zz''(z+1) + x^2z'^2\left(\frac{3}{2}z+1\right) - xzz'(z+1) + az^3(z+1)^2 + bz^3+ \\
+ cxz(z+1)^2 + dx^2(z+1)^2(2+z) = 0\,.
\end{aligned}
\tag{42}
$$

Its support and polygon are shown in Fig. 11.

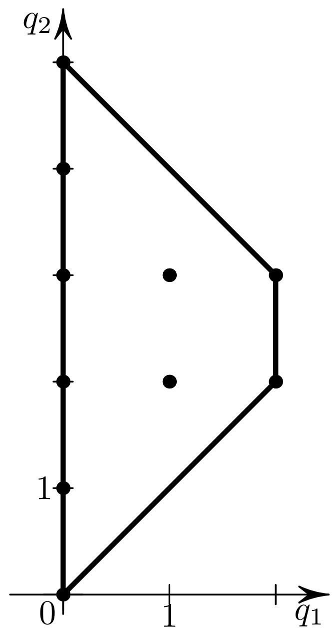

Figure 10. Support and polygon of equation (41).

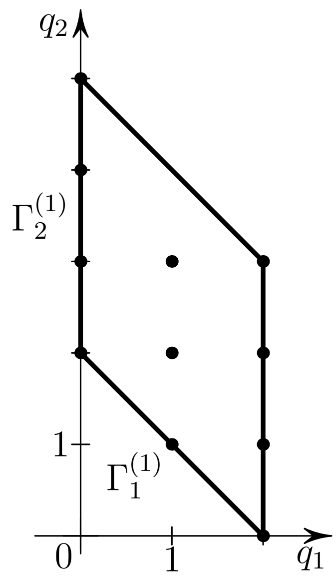

Figure 11. Support and polygon of equation (42).

We will differ two cases with different truncated equations:

Case I. Truncated equation corresponds to the low inclined edge $\Gamma_1^{(1)}$ in Fig. 11. It is

$$-z(z''x^2 + z'x) + x^2z/^2 + cxz + 2d = 0$$

and is similar to the truncated equation (9) of equation P_3.

Case II. Truncated equation corresponds to the left vertical edge $\Gamma_2^{(1)}$ in Fig. 11.

4.2. Preliminary transformations in Case I. To transform the edge $\Gamma_1^{(1)}$ in vertical one, we make the power transformation $z = xv$. Then $z' = v + xv'$, $z'' = 2v' + xv''$ and equation (42) divided by x^2 takes the form

$$\begin{aligned} g(x,v) \overset{def}{=} & - x^2vv''(1 + xv) + x^2v'^2\left(1 + \frac{3}{2}xv\right) - xv'v + \frac{1}{2}xv^3 + \\ & + a(xv^3 + 2x^2v^4 + x^3v^5) + bxv^3 + c(v + 2xv^2 + x^2v^3) + \\ & + d(2 + 5xv + 4x^2v^2 + x^3v^3) = 0\,. \end{aligned} \tag{43}$$

Its support and polygon are shown in Fig. 12.

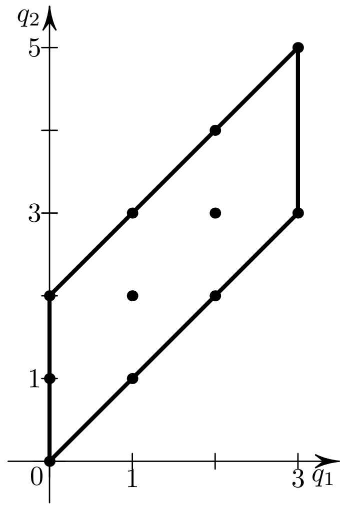

Figure 12. Support and polygon of equation (43).

If according to Section 2 to write

$$g(x,v) = g_0(x,v) + xg_1(x,v) + x^2g_2(x,v) + x^3g_3(x,v)\,,$$

then

$$g_0(x,v) = -x^2vv'' + x^2v'^2 - xv'v + cv + 2d\,,$$

$$g_1(x,v) = -x^2v^2v'' + \frac{3}{2}x^2vv'^2 + \left(\tfrac{1}{2} + a + b\right)v^3 + 2cv^2 + 5dv\,, \tag{44}$$

$$g_2(x,v) = 2av^4 + cv^3 + 4dv^2\,, \qquad g_3(x,v) = av^5 + dv^3\,.$$

Complicated and exotic expansions of solutions to equation (43) have the form

$$v = \varphi_0(\xi) + x\varphi_1(\xi) + x^2\varphi_2(\xi) + \cdots\,, \tag{45}$$

where $\xi = \log x + c_0$, c_0 is arbitrary constant. According to Theorem 2, equation for the second coefficient φ_1 is

$$\frac{\delta g_0}{\delta v}\,(x\varphi_1) + x g_1(\varphi_0) = 0\,. \tag{46}$$

4.3. Complicated expansions. In $g_j(x,v)$ from (44), we change the independent variable x by $\xi = \log x + c_0$, where c_0 is arbitrary constant. We obtain

$$\begin{aligned} g_0^*(\xi,v) =& g_0(x,v) = -v\ddot{v} + \dot{v}^2 + cv + 2d\,,\\ g_1^*(\xi,v) =& g_1(x,v) = -v^2(\ddot{v} - \dot{v}) + \frac{3}{2}v\dot{v}^2 + \omega v^3 + 2cv^2 + 5dv\,,\\ g_2^*(\xi,v) =& g_2(x,v) = 2av^4 + cv^3 + 4dv^2\,, \quad g_3^*(\xi,v) = g_3(x,v) = av^5 + dv^3\,, \end{aligned} \tag{47}$$

where $\omega = \dfrac{1}{2} + a + b$.

According to Theorem 3, solutions $v = \varphi_0(\xi)$ to equation $g_0(\xi,v) = 0$, which are the Laurent series in decreasing powers of ξ, form two families:
additional: $\varphi_0 = v = \beta\xi$ for $c = 0$, $\beta^2 = -2d$, $d \neq 0$, and
main: $\varphi_0 = v = -\dfrac{c}{2}\xi^2 - \dfrac{d}{c}$ for $c \neq 0$.

According to (47)

$$\frac{\delta g_0^*}{\delta v} = -v\frac{d^2}{d\xi^2} + 2\dot{v}\frac{d}{d\xi} + c - \ddot{v}\,.$$

According to Corollary 1

$$\frac{d}{d\xi}(x\varphi_1) = x[\varphi_1 + \dot{\varphi}_1], \quad \frac{d^2}{d\xi^2}(x\varphi_1) = x[\varphi_1 + 2\dot{\varphi}_1 + \ddot{\varphi}_1]\,.$$

First we consider the additional family. Then

$$\frac{\delta g_0^*}{\delta v} = -\beta\xi\frac{d^2}{d\xi^2} + 2\beta\frac{d}{d\xi}\,, \quad g_1 = \omega\beta^3\xi^3 + \beta^3\xi^2 + \left(\frac{3}{2}\beta^3 + 5d\beta\right)\xi$$

and equation (46) after dividing by x and using $2d = -\beta^2$ takes the form

$$-\beta\xi[\varphi_1 + 2\dot{\varphi}_1 + \ddot{\varphi}_1] + 2\beta[\varphi_1 + \dot{\varphi}_1] + \omega\beta^3\xi^3 + \beta^3\xi^2 - \beta^3\xi = 0\,.$$

It has the polynomial solution

$$\varphi_1 = -2\omega d(\xi^2 - 2\xi + 2) - 2d\xi + 2d\,. \tag{48}$$

Now we consider the main family. Then

$$\frac{\delta g_0^*}{\delta v} = \left(\frac{c}{2}\xi^2 + \frac{d}{c}\right)\frac{d^2}{d\xi^2} - 2c\xi\frac{d}{d\xi} + 2c\,,$$

$$g_1^* = \omega v^3 - c\xi v^2 + cv^2 + \frac{3}{2}c^2\xi^2 v + 2cv^2 + 5dv = \omega v^3 - c\xi v^2 + 2dv\,.$$

Equation (46) after division by x is

$$\left(\frac{c}{2}\xi^2 + \frac{d}{c}\right)[\varphi_1 + 2\dot\varphi_1 + \ddot\varphi_1] - 2c\xi[\varphi_1 + \dot\varphi_1] + 2c\varphi_1 + \omega v^3 - c\xi v^2 + 2dv = 0\,. \quad (49)$$

At first we consider auxiliary equation

$$\left(\frac{c}{2}\xi^2 + \frac{d}{c}\right)[\varphi_1 + 2\dot\varphi_1 + \ddot\varphi_1] - 2c\xi[\varphi_1 + \dot\varphi_1] + 2c\varphi_1 + \omega v^3 = 0\,.$$

It has the polynomial solution

$$\varphi_1 = -\omega\frac{c^2}{4}[\xi^4 - 4\xi^3 + (8 + 2\lambda)\xi^2 - (8 + 4\lambda)\xi + \lambda^2]\,,$$

where $\lambda = \dfrac{2d}{c^2}$.

Now we consider equation (49) with $\omega = 0$. We divide the equation by $c/2$ and put $\varphi_1 = c^2\psi_1/2$. Then the equation (49) takes the form

$$(\xi^2 + \lambda)[\psi_1 + 2\dot\psi_1 + \ddot\psi_1] - 4\xi[\psi_1 + \dot\psi_1] + 4\psi_1 - \xi(\xi^2 + \lambda)^2 + 2\lambda(\xi^2 + \lambda) = 0\,. \quad (50)$$

Its support and polygon are shown in Fig. 13.

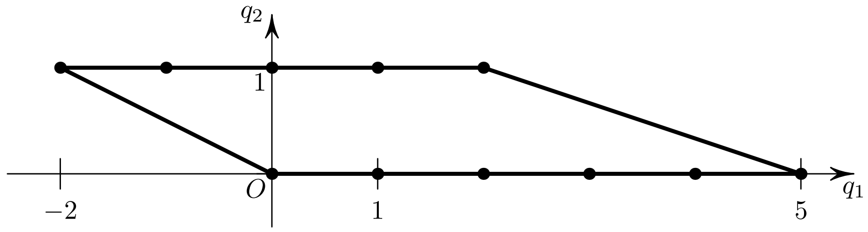

Figure 13. Support and polygon of equation (50).

As the inclination of the right edge is equal -3, then its solution in decreasing powers of ξ begins from ξ^3. So we look for its polynomial solution

$$\psi = \xi^3 + B\xi^2 + C\xi + D\,.$$

We substitute that expression in equation (50) and nullify coefficients for ξ^5, ξ^4, ξ^3, ξ^2, ξ^1, ξ^0. We obtain six linear algebraic equations for three coefficients B,C,D. Subsystem of first 4 equations for ξ^5,ξ^4,ξ^3,ξ^2 is triangle and has solution

$$B = -2, \quad C = 2 + \lambda, \quad D = -4\lambda.$$

Substitute there values in equation for ξ and ξ^0, we obtain equations $16\lambda = 0$ and $-16\lambda = 0$. Hence, $\lambda = 0$, i.e. $d = 0$. Thus, equation (50) has a polynomial solution only for $d = \lambda = 0$, and the solution is

$$\psi_1 = \xi^3 - 2\xi^2 + 2\xi\,.$$

Hence, the equation (50) has a polynomial solution only if $d = 0$, and the solution is

$$\varphi_1 = -\omega\frac{c^2}{4}[\xi^4 - 4\xi^3 + 8\xi^2 - 8\xi] + \frac{c^2}{2}[\xi^3 - 2\xi^2 + 2\xi]\,. \tag{51}$$

Theorem 8. *For the equation P_5 in Case I, the second coefficient $\varphi_1(\xi)$ in complicated expansions* (45) *of its solutions is polynomial* (48) *for the additional family always and* (51) *for the main family iff $d = 0$.*

4.4. Exotic expansions. We introduce new independent variable

$$\xi = x^{i\gamma},\ \gamma \in \mathbb{R},\ \gamma \neq 0\,. \tag{52}$$

Then

$$v' = i\gamma\dot{v}\frac{\xi}{x},\ v'' = \ddot{v}\left(i\gamma\frac{\xi}{x}\right)^2 + \dot{v}(i\gamma)^2\frac{\xi}{x^2} - \dot{v}i\gamma\frac{\xi}{x^2}\,, \tag{53}$$

where $\dot{v} = dv/d\xi$. Then

$$\begin{aligned} xv' &= i\gamma\xi\dot{v}\,, \\ x^2v'' &= -\gamma^2\xi^2\ddot{v} - \gamma^2\xi\dot{v} - i\gamma\xi\dot{v}\,. \end{aligned} \tag{54}$$

Hence, formulas (44) give

$$g_0 = \gamma^2 v(\xi^2\ddot{v} + \xi\dot{v}) - \gamma^2\xi^2\dot{v}^2 + cv + 2d\,,$$

$$g_1 = v^2(\gamma^2\xi^2\ddot{v} + \gamma^2\xi\dot{v} + i\gamma\dot{v}) - \frac{3}{2}\gamma^2\xi^2\dot{v}^2 + \omega v^3 + 2cv^2 + 5dv\,.$$

We put

$$\tilde{g}_0 = g_0/\gamma^2,\ \tilde{g}_1 = g_1/\gamma^2,\ \tilde{\omega} = \omega/\gamma^2,\ \tilde{c} = c/\gamma^2,\ \tilde{d} = d/\gamma^2\,.$$

Then these formulas give

$$\tilde{g}_0 = v(\xi^2\ddot{v} + \xi\dot{v}) - \xi^2\dot{v}^2 + \tilde{c}v + 2\tilde{d}\,,$$
$$\tilde{g}_1 = v^2\left[\xi^2\ddot{v} + \xi\dot{v}\left(1 - \frac{1}{i\gamma}\right)\right] - \frac{3}{2}v\xi^2\dot{v}^2 + \tilde{\omega}v^3 + 2\tilde{c}v^2 + 5\tilde{d}v\,. \tag{55}$$

From the first formulae (55) we have

$$\frac{\delta\tilde{g}_0}{\delta v} = v\xi^2\frac{d^2}{d\xi^2} + (v - 2\dot{v}\xi)\xi\frac{d}{d\xi} + \tilde{c} + \xi^2\ddot{v} + \xi\dot{v}\,.$$

According to Theorem 6, all solutions to equation $\tilde{g}_0 = 0$ in the form of Laurent series form one family of solutions

$$\varphi_0 = v = A\xi + B + C\xi^{-1}\,,$$

with following connections

$$B = -\tilde{c}, \qquad 4AC = \tilde{c}^2 - 2\tilde{d}\,.$$

As

$$v - 2\dot{v}\xi = -A\xi + B + 3C\xi^{-1}, \quad \tilde{c} + \xi^2\ddot{v} + \xi\dot{v} = A\xi - B + C\xi^{-1}\,,$$

$$\begin{aligned}
\tilde{g}_1 &= \tilde{\omega}v^3 + v^2\left[\ddot{v}\xi^2 + \dot{v}\xi\left(1 - \frac{1}{i\gamma}\right)\right] - \frac{3}{2}v\xi^2\dot{v}^2 + 2\tilde{c}v^2 + 5\tilde{d}v = \\
&= \tilde{\omega}\left[A^3\xi^3 + 3A^2B\xi^2 + 3(AB^2 + A^2C)\xi + B^3 + 6ABC + 3(AC^2 + B^2C)\xi^{-1} + \right. \\
&\quad \left. +3BC^2\xi^{-2} + C^3\xi^{-3}\right] - A^3\frac{2 + i\gamma}{2i\gamma}\xi^3 - A^2B\frac{4 + 3i\gamma}{2i\gamma}\xi^2 + \\
&\quad + \left(-A^2C\frac{2 + 11i\gamma}{2i\gamma} - AB^2\frac{2 + i\gamma}{2i\gamma}\right)\xi + \frac{1}{2}B^3 - 7ABC + \\
&\quad + \left(AC^2\frac{2 - 11i\gamma}{2i\gamma} + B^2C\frac{2 - i\gamma}{2i\gamma}\right)\xi^{-1} + BC^2\frac{4 - 3i\gamma}{2i\gamma}\xi^{-2} + C^3\frac{2 - i\gamma}{2i\gamma}\xi^{-3}\,,
\end{aligned}$$

then equation for $\varphi_1(\xi)$ is

$$(A\xi + B + C\xi^{-1})\left[\frac{1}{i\gamma}\left(\frac{1}{i\gamma} - 1\right)\varphi_1 + \frac{2}{i\gamma}\xi\dot{\varphi}_1 + \xi^2\ddot{\varphi}_1\right] +$$

$$+(-A\xi + B + 3C\xi^{-1})\left[\frac{1}{i\gamma}\varphi_1 + \xi\dot{\varphi}_1\right] + (A\xi - B + C\xi^{-1})\varphi_1 + \tilde{g}_1 = 0\,.$$

Its solution is the Laurent polynomial

$$\begin{aligned}\varphi_1(\xi) =&\tilde{\omega}\gamma^2\left[\frac{A^2}{(1+i\gamma)^2}\xi^2+\frac{2AB}{1+i\gamma}\xi+\frac{B^2}{1+\gamma^2}+\frac{AC(2+6\gamma^2)}{(1+\gamma^2)^2}+\right.\\ &\left.\frac{2BC}{1-i\gamma}\xi^{-1}+\frac{C^2}{(1-i\gamma)^2}\xi^{-2}\right]+\\ &+\gamma^2\left[-\frac{A^2(2+i\gamma)}{2i\gamma(1+i\gamma)^2}\xi^2-\frac{AB}{i\gamma(1+i\gamma)}\xi+\frac{B^2}{2(1+\gamma^2)}-\right.\\ &\left.\frac{AC(1-\gamma^2)}{(1+\gamma^2)^2}+\frac{BC}{i\gamma(1-i\gamma)}\xi^{-1}+\frac{C^2(2-i\gamma)}{2i\gamma(1-i\gamma)^2}\xi^{-2}\right].\end{aligned} \tag{56}$$

So, we have proved

Theorem 9. *In exotic expansion* (45) *solutions to equation P_5 in Case I, coefficient $\varphi_1(\xi)$ is the Laurent polynomial* (56).

5. The fifth Painlevé equation P_5 in Case II

5.1. Preliminary transformations. To obtain polynomial φ_0, we make in equation (42) the power transformation $z=\dfrac{1}{w}$. Then

$$z'=-\frac{w'}{w^2},\quad z''=\frac{2w'^2-ww''}{w^3},$$

and equation (42), multiplied by x^5, takes the form

$$\begin{aligned}h(x,w)\overset{def}{=}x^2ww''(1+w)-x^2w'^2\left(\frac{1}{2}+w\right)+xww'(1+w)+a(1+w)^2+\\ +bw^2+cxw^2(w+1)^2+dx^2w^2(w+1)^2(1+2w)=0\,.\end{aligned} \tag{57}$$

Its support and polygon are shown in Fig. 14. If write

$$h(x,w)=h_0(x,w)+xh_1(x,w)+x^2h_2(x,w)\,,$$

then

$$\begin{aligned}h_0(x,w) =&x^2ww''(w+1)-x^2w'^2\left(w+\frac{1}{2}\right)+xww'(w+1)+\\ &+a(w+1)^2+bw^2\,,\\ h_1(x,w) =&cw^2(1+w)^2\,,\\ h_2(x,w) =&dw^2(w+1)^2(2w+1)\,.\end{aligned} \tag{58}$$

Now formulas (45) and (46) are again correct if we put w instead of v.

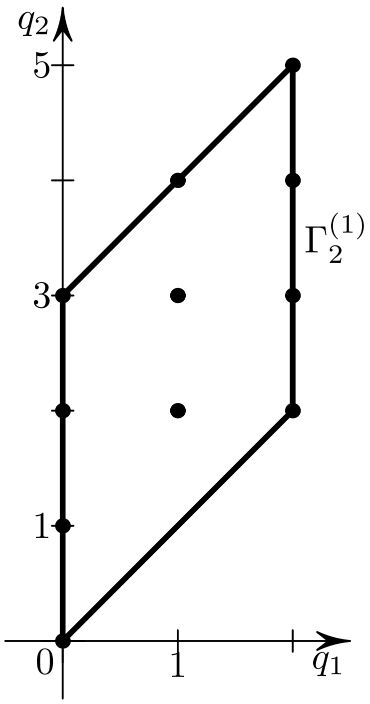

Figure 14. Support and polygon of equation (57).

5.2. Complicated expansions. In $h_j(x,w)$ from (58), we change the independent variable x by $\xi = \log x + c_0$. We obtain

$$\begin{aligned} h_0^*(\xi,w) &= h_0(x,w) = \ddot{w}w(w+1) - \dot{w}^2\left(w+\frac{1}{2}\right) + a(w+1)^2 + bw^2\,, \\ h_1^*(\xi,w) &= h_1(x,w) = cw^2(w+1)^2\,, \\ h_2^*(\xi,w) &= h_2(x,w) = dw^2(w+1)(2w+1)\,. \end{aligned} \tag{59}$$

Let us find all solutions of equation $h_0^*(\xi,w) = 0$ in the form of Laurent series.

Theorem 10. *All solutions $w = \varphi_0(\xi)$ of equation $h_0^*(\xi,w) = 0$ from* (59) *in the form of Laurent series and different from constant form two families:*
main (if $a + b \stackrel{def}{=} \alpha \neq 0$)

$$w = \varphi_0 = \frac{a+b}{2}\,(\xi + c_0)^2 - \frac{a}{a+b} = \frac{\alpha}{2}\,(\xi + c_0)^2 - \frac{a}{\alpha}\,, \tag{60}$$

and additional (if $\alpha = 0, a \neq 0$)

$$w = \varphi_0 = \beta\,(\xi + c_0)\,, \quad \beta^2 = 2a\,. \tag{61}$$

Here c_0 is arbitrary constant.

Proof. We will consider 3 cases: $1)\alpha \neq 0;\quad 2)\alpha = 0, a \neq 0;\quad 3)\alpha = a = 0.$

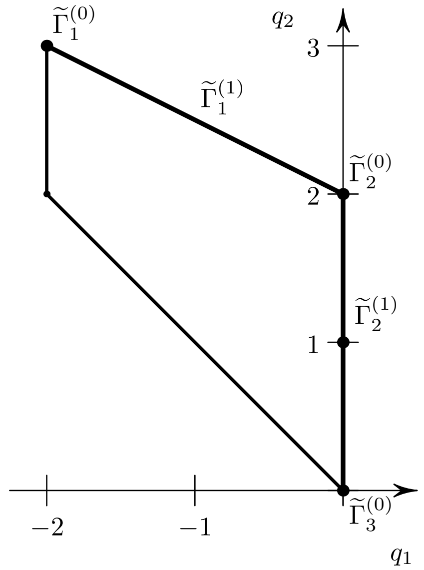

Figure 15. Support and polygon of equation $h_0^*(\xi,w) = 0$.

Case 1) $\alpha \neq 0$. Support and polygon Γ of equation $h_0^*(\xi,w) = 0$ are shown in Fig. 15.

Right side of the boundary $\partial\widetilde{\Gamma}$ of the polygon $\widetilde{\Gamma}$ consists of three vertices $\widetilde{\Gamma}_1^{(0)} = (-2,3)$, $\widetilde{\Gamma}_2^{(0)} = (0,2)$, $\widetilde{\Gamma}_3^{(0)} = 0$ and two edges $\widetilde{\Gamma}_1^{(1)}$ and $\widetilde{\Gamma}_2^{(1)}$. Corresponding truncations are

$$\hat{h}_1^{*(0)} = \ddot{w}w^2 - \dot{w}^2 w, \quad \hat{h}_2^{*(0)} = \alpha w^2, \quad \hat{h}_3^{*(0)} = a,$$
$$\hat{h}_1^{*(1)} = \ddot{w}w^2 - \dot{w}^2 w + \alpha w^2, \quad \hat{h}_2^{*(2)} = a(w+1)^2 + bw^2 .$$

Characteristic equation for truncation $\hat{h}_1^{*(0)}$ is $-r = 0$. It has unique solution $r = 0$. But vector $(1,0)$ does not belong to the normal cone

$$\mathbf{U}_1^{(0)} = \{P = \lambda_1(0,1) + \lambda_2(1,2), \quad \lambda_1,\lambda_2 \geq 0, \quad \lambda_1 + \lambda_2 > 0\}$$

Truncations $\hat{h}_2^{*(0)}$ and $\hat{h}_3^{*(0)}$ have trivial characteristic equations $\alpha = 0$ and $a = 0$, which have no solutions. Truncated equation $\hat{h}_1^{*(1)} = 0$ has the power solution $w = \alpha\xi^2/2$. According to Subsection 3.3 of [11], we will find critical numbers of that solution. We have

$$\frac{\delta \hat{h}_1^{*(1)}}{\delta w} = w^2 \frac{d^2}{d\xi^2} - 2\dot{w}w\frac{d}{d\xi} + 2\ddot{w}w - \dot{w}^2 + 2\alpha w .$$

On the curve $w = \alpha\xi^2/2$, that variation gives operator

$$\mathcal{L}(\xi) = \frac{\alpha^2\xi^4}{4}\frac{d^2}{d\xi^2} - \alpha^2\frac{d}{d\xi} + \alpha^2\xi^2 - \alpha^2\xi^2 + \alpha^2\xi^2 .$$

Characteristic polynomial of sum $\mathcal{L}(\xi)\xi^k$ is

$$\nu(k) = \frac{\alpha^2}{4}\left[k(k-1) - 4k + 4\right].$$

It has two roots $k_1 = 1$ and $k_2 = 4$. But $k_1 < 2$, and $k_2 > 2$ and it is not a critical number. So we have only one critical number $k_1 = 1$.

According to Subsection 3.4 of [11], the set

$$\mathbf{K}_1^{(1)} = \{2 - 2l,\, l \in \mathbb{N}\} = \{0, -2, -4, \cdots\}.$$

Now the critical number $k_1 = 1$ does not belong to the set $\mathbf{K}_1^{(1)}$. Thus, according to Theorem 3 [11], equation $h_0^*(\xi,w) = 0$ has a solution in the form of Laurent series

$$w = \alpha\xi^2/2 + \gamma_0 + \sum_{k=1}^{\infty} \gamma_2\xi^{-2k}, \qquad (62)$$

where $\gamma_i = const$. To find γ_0, we put $w = \alpha\xi^2/2 + \gamma_0$ into $h_0^*(\xi,w)$. We have $\dot{w} = \alpha\xi, \ddot{w} = \alpha$, hence,

$$h_0^*(\xi,\alpha\xi^2/2 + \gamma_0) = \alpha(\alpha\gamma_0 + a)\xi^2 + (2\gamma_0 + 1)(\alpha\gamma_0 + a).$$

Both coefficients near ξ^2 and ξ^0 are zero, iff $\gamma_0 = -a/\alpha$. So, solution (62) is indeed the polynomial

$$w = \alpha\xi^2/2 - a/\alpha.$$

Equation $h_0^*(\xi,w) = 0$ does not contain explicitly the independent variable ξ, so to its solution $w(\xi)$ there correspond solutions $w(\xi + c_0)$, where c_0 is arbitrary constant. Hence, we obtain family (60).

To finish that case, we must consider the last truncation $\hat{h}_2^{*(1)}$. It is the square polynomial $(a + b)\xi^2 + 2a\xi + a^2$. Its discriminant

$$\Delta = -4ab.$$

If $\Delta \neq 0$, then the polynomial has two roots. Each of them is the constant solution of the equation $h_0^*(\xi,w) = 0$ and cannot be continued into power expansion.

If $\Delta = 0$, i.e. $a = 0$ or $b = 0$, then the polynomial has one double solution $w = 0$ or $w = -1$. They are constant double solutions of the full equation $h_0^*(\xi,w) = 0$, and does not give nonconstant solutions to equation $h_0^*(\xi,w) = 0$. But we are looking for nonconstant solutions.

Case 2) $\alpha = 0, a \neq 0$. Support and polygon $\widetilde{\Gamma}$ of $h_0^*(\xi,w)$ are shown in Fig.16. Right side of the boundary $\partial\widetilde{\Gamma}$ of the polygon $\widetilde{\Gamma}$ consists of three vertices $\widetilde{\Gamma}_1^{(0)} = (-2,3)$,

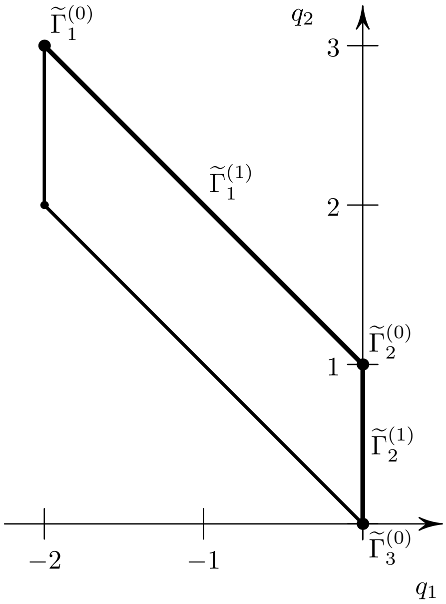

Figure 16. Support and polygon of equation $h_0^*(\xi,w) = 0$ in case 2).

$\widetilde{\Gamma}_2^{(0)} = (0,1)$, $\widetilde{\Gamma}_3^{(0)} = 0$ and two edges $\widetilde{\Gamma}_1^{(1)}$ and $\widetilde{\Gamma}_2^{(1)}$. As in case 1), truncated equations, corresponding to all vertices and edge $\widetilde{\Gamma}_2^{(1)}$ do not give us power expansions of solutions to equation $h_0^*(\xi,w) = 0$. So we consider the truncated equation

$$\hat{h}_1^{*(1)} \stackrel{def}{=} \ddot{w}w^2 - \dot{w}^2 w + 2aw = 0\,.$$

It has power solutions $w = \beta\xi$ with $\beta^2 = 2a$. The solution satisfies the equation

$$h_0^*(\xi,w) - \hat{h}_1^{*(1)}(\xi,w) \stackrel{def}{=} \ddot{w}w - \dot{w}^2/2 + a = 0\,.$$

Hence, the equation has family of solutions (61).

Case 3) $a = b = 0$. Here all solutions of equation $h_0^*(\xi,w) = 0$ belong to two-parameter family $w = \dfrac{[c_2 \exp(c_1\xi) - 1]^2}{4c_2 \exp(c_1\xi)}$, where c_1 and c_2 are arbitrary constants. No one of these solutions has a power expansion. □

Here

$$\frac{\delta h_0^*}{\partial w} = w(w+1)\frac{d^2}{d\xi^2} - 2\left(w + \frac{1}{2}\right)\dot{w}\frac{d}{d\xi} + 2a(w+1) + 2bw + \ddot{w}(2w+1) - \dot{w}^2\,.$$

Let us compute solution φ_1 to equation (46) for additional family (61). Here $\dot{w} = \beta, \ddot{w} = 0, \dot{w}^2 = 2a$ and equation (46) divided by x is

$$\beta\xi(\beta\xi+1)\left[\varphi_1 + 2\dot{\varphi}_1 + \ddot{\varphi}_1\right] - \beta(2\beta\xi+1)\left[\varphi_1 + \dot{\varphi}_1\right] + c\beta^2\xi^2(\beta\xi+1)^2 = 0\,.$$

It has polynomial solution

$$\varphi_1 = c\beta[-\beta\xi^2 + (2\beta-1)\xi + 1]\,. \qquad (63)$$

For the main family (60), equation (46) divided by x is

$$w(w{+}1)\left[\varphi_1+2\dot\varphi_1+\ddot\varphi_1\right]-\alpha\xi(2w{+}1)\left[\varphi_1+\dot\varphi_1\right]+\alpha(2w{+}1)\varphi_1+cw^2(w{+}1)^2=0\,,$$

where $w=\dfrac{\alpha}{2}\xi^2-\dfrac{a}{\alpha}$. It has the polynomial solution

$$\varphi_1(\xi)=-c\left[\frac{\alpha^2}{4}\xi^4-\alpha^2\xi^3+\left(2\alpha^2+\frac{\alpha}{2}-a\right)\xi^2-\right.\\ \left.-\left(2\alpha^2+\alpha-2a\right)\xi+\frac{a(a-\alpha)}{\alpha^2}\right]. \quad (64)$$

Thus, we have proven

Theorem 11. *In Case II of equation P_5 the second coefficient φ_1 of complicated expansions* (45) *is a polynomial* (63) *for the additional family and polynomial* (64) *for the main family.*

5.3. Exotic expansions. Let us introduce new independent variable $\xi=x^{i\gamma}$ according to (52). Then, according to (53), formulas (58), divided by γ^2, take the forms

$$\begin{aligned}\tilde h_0(\xi,w)=&\gamma^{-2}h_0(x,w)=-w(w+1)(\xi^2\ddot w+\xi\dot w)+\left(w+\frac{1}{2}\right)\xi^2\dot w^2+\\ &+\tilde a(w+1)^2+\tilde b w^2\,,\\ \tilde h_1(\xi,w)=&\gamma^{-2}h_1(x,w)=\tilde c w^2(w+1)^2\,,\end{aligned} \quad (65)$$

where $\tilde a=a/\gamma^2$, $\tilde b=b/\gamma^2$, $\tilde c=c/\gamma^2$.

Theorem 12. *All solutions $w=\varphi_0(\xi)$ to equation $\tilde h_0(\xi,w)=0$ from* (65) *in the form of Laurent series form one family*

$$w=\varphi_0=A\xi+B+C\xi^{-1}\,, \quad (66)$$

where parameters are connected by equalities

$$B=\tilde a+\tilde b-\frac{1}{2}\,,\qquad 4AC=(\tilde a+\tilde b)^2+\tilde a-\tilde b+\frac{1}{4}\,. \quad (67)$$

Proof. First we will show that parameters satisfy to (67) for solution (66) to equation $\tilde h_0(\xi,w)=0$. Let us denote

$$\alpha=A\xi+C\xi^{-1}\quad\text{and}\quad\beta=A\xi-C\xi^{-1}\,.$$

Then $\xi\dot{w} = A\xi - C\xi^{-1}, \xi^2\ddot{w} = 2C\xi^{-1}$ and $\xi\dot{w} + \xi^2\ddot{w} = \alpha$. So

$$\tilde{h}_0(\xi,w) = -(\alpha+B)(\alpha+B+1)\alpha+(\alpha+B+\frac{1}{2})\beta^2+\tilde{a}(\alpha+B+1)^2+\tilde{b}(\alpha+B)^2 =$$

$$= -\alpha^3+\alpha\beta^2-\alpha^2(2B+1)+(B+\frac{1}{2})\beta^2+\tilde{a}\alpha^2+\tilde{b}\alpha^2-B(B+1)\alpha+2\tilde{a}(B+1)\alpha+$$

$$+2\tilde{b}B\alpha+\tilde{a}(B+1)^2+\tilde{b}B^2 = \alpha[\beta^2-\alpha^2]+\alpha^2[\tilde{a}+\tilde{b}-(2B+1)]+(B+\frac{1}{2})\beta^2+$$

$$+\alpha[2\tilde{a}(B+1)+2\tilde{b}B-B(B+1)]+\tilde{a}(B+1)^2+\tilde{b}B^2\,.$$

We have

$$\beta^2-\alpha^2 = (\beta-\alpha)(\beta+\alpha) = 2A\xi(-2C\xi^{-1}) = -4AC\,.$$

Hence, $\beta^2 = \alpha^2 - 4AC$ and

$$\tilde{h}_0(\xi,w) = -4AC\alpha+\alpha^2[\tilde{a}+\tilde{b}-(2B+1)+B+\frac{1}{2}]-4AC(B+\frac{1}{2})+$$

$$+\alpha[2\tilde{a}(B+1)+2\tilde{b}B-B(B+1)]+\tilde{a}(B+1)^2+\tilde{b}B^2 = \alpha^2[\tilde{a}+\tilde{b}-B-\frac{1}{2}]+$$

$$+\alpha[2\tilde{a}(B+1)+2\tilde{b}B-B(B+1)-4AC]+\tilde{a}(B+1)^2+\tilde{b}B^2-4AC(B+\frac{1}{2})\,.$$

But $\alpha^2 = A^2\xi^2 + 2AC + C^2\xi^{-2}$, hence,

$$\tilde{h}_0(\xi,w) = (A^2\xi^2+C^2\xi^{-2})(\tilde{a}+\tilde{b}-B-\frac{1}{2})+\alpha[2\tilde{a}(B+1)+2\tilde{b}B-B(B+1)-4AC]+$$

$$+\tilde{a}(B+1)^2+\tilde{b}B^2-4AC(B+\frac{1}{2})+2AC(\tilde{a}+\tilde{b}-B-\frac{1}{2}) \equiv 0\,.$$

It means that coefficients for $\xi^{\pm 2}$, α and ξ^0 are zero. Exactly $\tilde{a}+\tilde{b}-B-\frac{1}{2}=0$, i.e.

$$\tilde{a}+\tilde{b} = B+\frac{1}{2}\,; \tag{68}$$

$$0 = 2\tilde{a}(B+1)+2\tilde{b}B-B^2-B-4AC = 2(B+\frac{1}{2})B+2\tilde{a}-B^2-B-4AC =$$
$$= B^2+2\tilde{a}-4AC$$

according to (68), i.e.

$$4AC = B^2+2\tilde{a}\,. \tag{69}$$

Finally,

$$\tilde{a}(B+1)^2+\tilde{b}B^2-4AC(B+\frac{1}{2}) = (\tilde{a}+\tilde{b})B^2+2\tilde{a}B+\tilde{a}-4AC(B+\frac{1}{2}) =$$

$$= (\tilde{a} + \tilde{b})[B^2 + 2\tilde{a} - 4AC] = 0$$

according to (68) and (69).

Now we will show that, for any solution

$$w = A\xi^1 + B + C\xi^{-1} + D\xi^{-l} + \ldots, \quad l \geq 2 \tag{70}$$

to equation $\tilde{h}_0(\xi,w) = 0$, coefficient $D = 0$. We insert (70) in $\tilde{h}_0(\xi,w)$ and find in it a term with maximal power ξ, containing D. Terms of the third order in $\tilde{h}_0(\xi,w)$ are

$$-w^2(\xi^2\ddot{w} + \xi\dot{w}) + w\xi^2\dot{w}^2 \stackrel{def}{=} \Omega_3 .$$

We assume that $w = A\xi + D\xi^{-l}$, then $\xi^2\ddot{w} + \xi\dot{w} = A\xi + l^2D\xi^{-l}$ and

$$\Omega_3 = -(A + D\xi^{-l})^2(A\xi + l^2D\xi^{-l}) + (A\xi + D\xi^{-l})(A\xi - lD\xi^{-l})^2 =$$
$$= -(l+1)^2A^2D\xi^{2-l} + \ldots$$

Coefficient before the ξ^{2-l} must be zero. But $(l+1)^2 \neq 0$, $A^2 \neq 0$, hence $D = 0$. □

According to (65)

$$\frac{\delta\tilde{h}_0}{\delta w} = -w(w+1)\xi^2\frac{d^2}{d\xi^2} - w(w+1)\xi\frac{d}{d\xi} + 2\left(w + \frac{1}{2}\right)\dot{w}\xi^2\frac{d}{d\xi} +$$

$$+2\tilde{a}(w+1) + 2\tilde{b}w - (2w+1)\left(\xi^2\ddot{w} + \xi\dot{w}\right) + \xi^2\dot{w}^2 .$$

Equation (46) for $\varphi_1(\xi)$ is

$$a_1\left[\frac{1}{i\gamma}\left(\frac{1}{i\gamma} - 1\right)\varphi_1 + \frac{2}{i\gamma}\xi\dot{\varphi}_1 + \xi^2\ddot{\varphi}_1\right] + a_2\left[\frac{1}{i\gamma}\varphi_1 + \xi\dot{\varphi}_1\right] + a_3\varphi_1 + \tilde{h}_1 = 0\,, \tag{71}$$

where

$$\begin{aligned}
a_1 =& - w(w+1) = -A^2\xi^2 - A(2B+1)\xi - 2AC - B(B+1) - \\
&- (2B+1)C\xi^{-1} - C^2\xi^{-2}\,, \\
a_2 =&(2w+1)\dot{w}\xi - w(w+1) = A^2\xi^2 - [2AC + B(B+1)] - 2(2B+1)C\xi^{-1} - \\
&- 3C^2\xi^{-2}\,, \\
a_3 =&2\tilde{a}(w+1) + 2\tilde{b}w - (2w+1)(\xi^2\ddot{w} + \xi\dot{w}) + \xi^2\dot{w}^2 = -A^2\xi^2 - 2AC + \\
&+ B(B+1) - C^2\xi^{-2}\,, \\
\frac{\tilde{h}_1}{\tilde{c}} =&w^2(w+1)^2 = A^4\xi^4 + 2A^3(2B+1)\xi^3 + [4A^3C + A^2\beta]\xi^2 + \\
&+ [6A^2(2B+1)C + 2AB(B+1)(2B+1)]\xi + 6A^2C^2 + 2A\beta C + \\
&+ B^2(B+1)^2 + [6A(2B+1)C^2 + 2B(B+1)(2B+1)C]\xi^{-1} + \\
&+ [4AC^3 + \beta C^2]\xi^{-2} + 2(2B+1)C^3\xi^{-3} + C^4\xi^{-4} \stackrel{def}{=} \tilde{h}_{15}\,, \\
\beta =&6B(B+1) + 1.
\end{aligned}$$

Support and the Newton polygon Γ for equation (69) are shown in Fig. 17. As

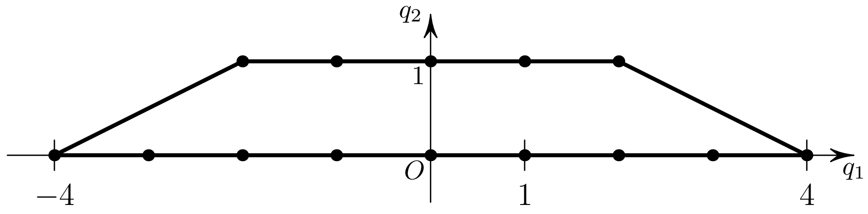

Figure 17. Support and polygon of equation (69).

inclinations of side edges of the polygon Γ are ±2, then polynomial solutions to equation (71) should be as

$$\varphi_1 = D\xi^2 + E\xi + F + G\xi^{-1} + H\xi^{-2}\,. \tag{72}$$

Inserting that φ_1 into equation (71), we obtain a linear system of 9 algebraic equations for 5 coefficients D,E,F,G,H. Equations correspond to vanish of coefficients near ξ^4, ξ^3, ξ^2, ξ, ξ^0, ξ^{-1}, ξ^{-2}, ξ^{-3}, ξ^{-4}. From coefficients near ξ^4, ξ^3, ξ^2, we find

$$\begin{gathered} D \stackrel{def}{=} D_1 = -c\frac{A^2}{(1+i\gamma)^2}\,, \quad E \stackrel{def}{=} E_1 = -c\frac{A(2B+1)}{1+i\gamma}\,, \\ F \stackrel{def}{=} F_1 = -c\frac{2AC(1-\gamma^2)}{(1+\gamma^2)^2} - c\frac{B(B+1)(1-3\gamma^2)}{(1+\gamma^2)^2}\,. \end{gathered} \tag{73}$$

From coefficients near $\xi^{-2}, \xi^{-3}, \xi^{-4}$, we find

$$\begin{gathered} H = H_2 = -\,c\frac{C^2}{(1-i\gamma)^2}\,, \quad G = G_2 = -c\frac{(2B+1)C}{1-i\gamma}\,, \\ F = F_2 = -\,c\frac{2AC(1+7\gamma^2)}{(1+\gamma^2)^2} - c\frac{B(B+1)(1+5\gamma^2)}{(1+\gamma^2)^2}\,. \end{gathered} \tag{74}$$

According to (73) and (74), equality $F_1 = F_2$ is possible, iff

$$2AC + B(B+1) = 0\,. \tag{75}$$

Then

$$F = -c\frac{4AC\gamma^2}{(1+\gamma^2)^2} = c\frac{2B(B+1)\gamma^2}{(1+\gamma^2)^2}\,. \tag{76}$$

Inserting found values (73), (74), (76) of coefficients D,E,F,G,H into equations near ξ and ξ^{-1}, we obtain, that for $A(2B+1)C \neq 0$ they are fulfilled, if $\gamma^4 = 1$, i.e. $\gamma^2 = \pm 1$. As $\gamma^2 > 0$, it means that $\gamma^2 = 1$. We have obtain the second condition

$$A(2B+1)C(\gamma^2-1) = 0\,. \tag{77}$$

Equation near ξ^0 is satisfied under substitution of find coefficients and condition (77). Thus, we have proven

Theorem 13. *In the exotic expansion* (45) *of solutions to equation P_5 in Case II, the second coefficient $\varphi_1(\xi)$ is a Laurent polynomial* (72)*,* (73)*,* (74)*,* (76)*, iff 2 conditions* (75) *and* (77) *are fulfilled.*

6. The sixth Painlevé equation P_6

6.1. Preliminary transformations. Usually the sixth Painlevé equation [6] is

$$y'' = \frac{y'^2}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right) - y'\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right) +$$

$$+a\frac{y(y-1)(y-x)}{x^2(x-1)^2} + b\frac{(y-1)(y-x)}{x(x-1)^2 y} + c\frac{y(y-x)}{x^2(x-1)(y-1)} + d\frac{y(y-1)}{x^2(x-1)^2(y-x)}\,.$$

We put $z = -y$, multiply the equation by its common denominator $x^2(x-1)^2 y(y-1)(y-x)$ and translate all terms into the right side of equation. So we obtain the equation

$$\begin{aligned}
g(x,z) &\stackrel{def}{=} -z''x^2(x-1)^2 z(z+1)(z+x)+\\
&+ \frac{1}{2}z'^2x^2(x-1)^2[(z+1)(z+x) + z(z+x) + z(z+1)]-\\
&- z'z(z+1)[x(x-1)^2(z+x) + x^2(x-1)(z+x) + x^2(x-1)(z+x) - x^2(x-1)^2]+\\
&+ az^2(z+1)^2(z+x)^2 + bx(z+1)^2(z+x)^2 + c(x-1)z^2(z-x)^2+\\
&+ dx(x-1)^2 z^2(z+1)^2 = 0\,.
\end{aligned} \tag{78}$$

Support and polygon of the equation are shown in Fig. 18.

If we write

$$g(x,z) = g_0(x,z) + xg_1(x,z) + x^2 g_2(x,z) + x^3 g_3(x,z)$$

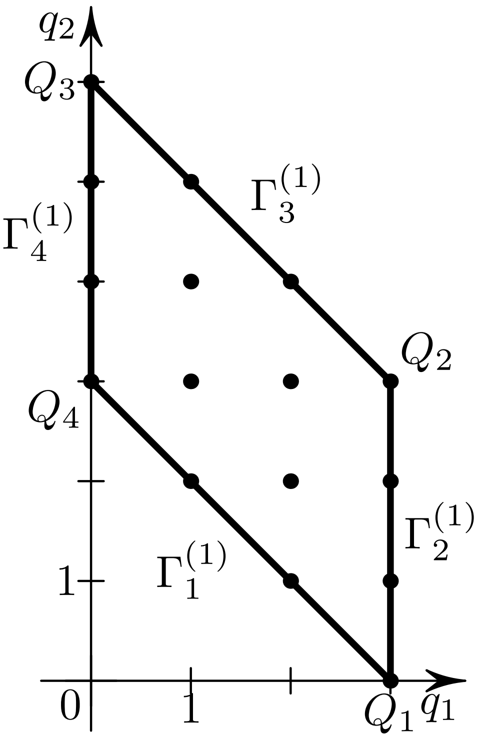

Figure 18. Support and polygon of equation (78).

according to Subsection 2.2, then

$$g_0(x,z) = -\, z''x^2z^2(z+1) + z'^2x^2z\left(\frac{3}{2}z+1\right) - z'xz^2(z+1) + az^4(z+1)^2 - cz^4\,,$$
$$g_1(x,z) = z''x^2(z+1)(2z-1) - z'^2x^2z\left(3z^2+z-\frac{1}{2}\right) + 3z'xz^2(z+1) +$$
$$+\, 2az^3(z+1)^2 + bz^2(z+1)^2 + cz^3(z-2) - dz^2(z+1)^2\,,$$
$$g_2(x,z) = -\, z''x^2z(z+1)(z-2) + z'^2x^2(3z^2-2z-2) - z'xz(z+1)(2z-1) -$$
$$-\, az^2(z+1)^2 - bz(z-1)^2 - cz^2(2z-1) - dz^2(z+1)^2\,,$$
$$g_3(x,z) = -\, 2z''x^2z(z+1) + z'^2x^2(2z+1) - z'xz(z+1) + b(z+1)^2 - cz^2\,.$$

Note, that $g_0(x,z)$ coincides with the upper line of formula (42), multiplied by z, if $-c$ change by b. Now in equation (78) we make the power transformation $z = \dfrac{1}{w}$. Then

$$z' = -\frac{w'}{w^2}, \quad z'' = \frac{2w'^2 - ww''}{w^3}\,,$$

Denote $h_i(x,w) = g_i\left(x,\dfrac{1}{w}\right)\cdot w^6$, $i = 0,1,2,3$. Then

$$\begin{aligned} h_0(x,w) =& ww''x^2(1+w) - w'^2x^2\left(w+\frac{1}{2}\right) + w'xw(w+1) + a(w+1)^2 - cw^2\,,\\ h_1(x,w) =& ww''x^2(w+1)(w-2) + w'^2x^2\left(-\frac{3}{2}w^2+w+1\right) - 3w'xw(w+1)+\\ &+2aw(w+1)^2 + cw^2(1-2w) + (b-d)w^2(w+1)^2\,. \end{aligned} \tag{79}$$

After change $-c$ by b, $h_0(x,w)$ coincides with $h_0(x,w)$ from (58), but in $h_1(x,w)$ here only one term $(b-d)w^2(w+1)^2$ coincides with $h_1(x,w)$ in (58), but now h_1 has several other terms.

6.2. Complicated expansions. In $h_i(x,w)$ from (79), we change independent variable $\xi = \log x + c_0$ and obtain

$$\begin{aligned} h_0^*(\xi,w) =& h_0(x,w) = \ddot{w}w(w+1) - \dot{w}^2\left(w+\frac{1}{2}\right) + a(w+1)^2 - cw^2\,,\\ h_1^*(\xi,w) =& h_1(x,w) = \ddot{w}w(w+1)(w-2) - \dot{w}^2\left(\frac{3}{2}w^2 - w - 1\right) - \dot{w}w(w+1)^2+\\ &+2aw(w+1)^2 - cw^2(2w-1) + \omega w^2(w+1)^2\,, \end{aligned}$$

where $\omega = b - d$.

According to Theorem 10 all nonconstant power series solutions to equation $h_0^*(\xi,w) = 0$ form two families:
main (if $\alpha \overset{def}{=} a - c \neq 0$)

$$w = \varphi_0 = \frac{\alpha}{2}(\xi + c_0)^2 - \frac{a}{\alpha}\,, \tag{80}$$

and additional (if $\alpha = 0$, $a \neq 0$)

$$w = \varphi_0 = \beta(\xi + c_0), \quad \beta^2 = 2a\,, \tag{81}$$

where c_0 is arbitrary constant. Let us compute the second coefficient $\varphi_1(\xi)$ of expansion (45), using equation (46). Here

$$\begin{aligned} \frac{\delta h_0^*}{\delta z} =& w(w+1)\frac{d^2}{d\xi^2} - (2w+1)\dot{w}\frac{d}{d\xi} + 2a(w+1) - 2cw + \ddot{w}(2w+1)-\\ &-\dot{w}^2 \overset{def}{=} a_1\frac{d^2}{d\xi^2} + a_2\frac{d}{d\xi} + a_3\,. \end{aligned}$$

According to Corollary 1 equation (46) for φ_1 is equivalent to equation

$$a_1[\varphi_1 + 2\dot{\varphi}_1 + \ddot{\varphi}_1] + a_2[\varphi_1 + \dot{\varphi}_1] + a_3\varphi_1 + h_1^* = 0\,. \qquad (82)$$

Denote $\xi = \log x + c_0$. For the additional family (81)

$$a_1 = \beta\xi(\beta\xi + 1), \quad a_2 = -\beta(2\beta\xi + 1), \quad a_3 = 0\,,$$
$$h_1 = 2a(w + 1)^2 - \beta w(w + 1)^2 + \omega w^2(w + 1)^2\,,$$

because here $a = c$. Equation (82) has polynomial solution

$$\varphi_1 = 2\omega a\xi^2 + [\omega(4a - \beta) + 2a]\xi + \omega(\beta - 4a) + \beta - 2a\,. \qquad (83)$$

Calculation of φ_2 see in [9].

For the main family (80)

$$a_1 = w(w + 1), \quad a_2 = -\alpha\xi(2w + 1), \quad a_3 = \alpha(2w + 1)\,,$$
$$h_1^* = \omega w^2(w + 1)^2 - \alpha\xi w(w + 1)^2 + 2a(w + 1)^2\,.$$

If in equation (82) $h_1^* = \omega w^2(w+1)^2$, then according to Theorem 11, it has polynomial solution (64) with ω instead of c. Now we consider equation (82) for $\omega = 0$. We look for its polynomial solution in the form

$$\varphi_1 = A\xi^4 + B\xi^3 + C\xi^2 + D\xi + E\,. \qquad (84)$$

For 5 coefficients A,B,C,D,E we obtain a system of 9 linear algebraic equations. They correspond to vanishing coefficients near ξ^8, ξ^7, $\cdots$, ξ^0, which arrive after substitution of expression (84) into equation (82). From coefficients near ξ^8, ξ^7, $\cdots$, ξ^4, we obtain

$$A = 0, \quad B = \alpha^2/2, \quad C = -\alpha^2, \quad D = \alpha^2 + \alpha - a, \quad E = 0.$$

Inserting these values into coefficient near ξ^3, ξ^2, ξ^1, ξ^0, we obtain the zeroes. And polynomial solution (84) of the full equation (82) has

$$A = -\omega\frac{\alpha^2}{4},\, B = \omega\alpha^2 + \frac{\alpha^2}{2},\, C = -\omega\left(2\alpha^2 + \frac{\alpha}{2} - a\right) - \alpha^2, \qquad (85)$$
$$D = \omega\left(2\alpha^2 + \alpha - 2a\right) + \alpha^2 + \alpha - a,\, E = -\omega\frac{a(a - \alpha)}{\alpha^2}\,.$$

Thus, we have proven

Theorem 14. *The second coefficient φ_1 of the complicated expansion* (45) *of solution to equation P_6 is a polynomial* (84)*,* (85) *for the main family and is a polynomial* (83) *for the additional family.*

Calculation of φ_2 see in [9].

6.3. Exotic expansions. Let us introduce new independent variable $\xi = x^{i\gamma}$ according to (52), (53), (54). Then expressions (79) after division by γ^2 take forms

$$\begin{aligned}\tilde{h}_0(\xi,w) &= \gamma^{-2}h_0(x,w) = -(\dot{w}\xi + \ddot{w}\xi^2)w(w+1) + \dot{w}^2\xi^2(w+\tfrac{1}{2}) + \tilde{a}(w+1)^2 - \tilde{c}w^2,\\ \tilde{h}_1(\xi,w) &= \gamma^{-2}h_1(x,w) = -(\dot{w}\xi + \ddot{w}\xi^2)w(w+1)(w-2) + \dot{w}^2\xi^2(\tfrac{3}{2}w^2 - w - 1) +\\ &+\frac{1}{i\gamma}\xi\dot{w}w(w+1)^2 + 2\tilde{a}w(w+1)^2 - \tilde{c}w^2(2w-1) + \tilde{\omega}w^2(w+1)^2\,,\end{aligned} \tag{86}$$

where

$$\tilde{a} = a/\gamma^2,\ \tilde{b} = b/\gamma^2,\ \tilde{c} = c/\gamma^2,\ \tilde{\omega} = \omega/\gamma^2,.$$

In (86) $\tilde{h}_0(\xi,w)$ coincides with $\tilde{h}_0(\xi,w)$ from (65), if $-c$ change by b. So according to Theorem 12, all power series solutions to equation $\tilde{h}_0(\xi,w) = 0$ from (86) are

$$w = \varphi_0 = A\xi + B + C\xi^{-1}\,,$$

where

$$B = \tilde{a} - \tilde{c} - \frac{1}{2},\quad 4AC = (\tilde{a} - \tilde{c})^2 + \tilde{a} + \tilde{c} + \frac{1}{4}\,.$$

According to (86),

$$\frac{\delta\tilde{h}_0}{\delta w} = -w(w+1)\xi^2\frac{d^2}{d\xi^2} + [(2w+1)\dot{w}\xi - w(w+1)]\frac{d}{d\xi} + 2\tilde{a}(w+1) -$$

$$-2\tilde{c}w - (2w+1)(\xi\dot{w} + \xi^2\ddot{w}) + \xi^2\dot{w}^2\,.$$

According to Corollary 2, equation (46) for $\varphi_1(\xi)$ is (71) with following changes: a_1,a_2 and a_3 are the same as in Subsection 5.3, with $-2\tilde{c}$ instead of $2\tilde{b}$, $\tilde{h}_1 = \tilde{h}_{16} + \tilde{\omega}\tilde{h}_{15}$, where $\tilde{h}_{15}$ is from Subsection 5.3 and

$$\tilde{h}_{16} = \frac{2+i\gamma}{2i\gamma}A^4\xi^4 + \left(\frac{3+2i\gamma}{i\gamma}B + \frac{2+i\gamma}{i\gamma}\right)A^3\xi^3 +$$

$$+\left(10AC + \frac{2+4i\gamma}{i\gamma}B^2 + \frac{3+5i\gamma}{i\gamma}B + \frac{2+i\gamma}{2i\gamma}\right)A^2\xi^2 +$$

$$+\left(20ABC + 4AC - \frac{6i\gamma - 1}{2i\gamma}B^3 + \frac{i\gamma - 1}{2i\gamma}B^2 - \frac{5}{2}B\right)A\xi +$$

$$+20AB^2C + 14ABC + \frac{1}{4}B(B+1)(9B^2 + 13B + 2) +$$

$$+\left(20ABC + 4AC - \frac{1+6i\gamma}{2i\gamma}B^3 + \frac{1+i\gamma}{2i\gamma}B^2 - \frac{5}{2}B\right)C\xi^{-1} +$$

$$+\left(10AC-\frac{2-4i\gamma}{i\gamma}B^2-\frac{3-5i\gamma}{i\gamma}B-\frac{2-i\gamma}{2i\gamma}\right)C^2\xi^{-2}+$$
$$+\left(-\frac{3-2i\gamma}{i\gamma}B-\frac{2-i\gamma}{i\gamma}\right)C^3\xi^{-3}-\frac{2-i\gamma}{2i\gamma}C^4\xi^{-4}.$$

Polynomial solution φ_1 to new equation (71) we look for in the form (72). Again we obtain a system of 9 linear algebraic equations for 5 coefficients. Let us consider case $\tilde{\omega}=0$. From vanishing coefficients near ξ^4,ξ^3,ξ^2, we find

$$D=-\frac{(2+i\gamma)\gamma^2}{2i\gamma(1+i\gamma)^2}A^2,$$
$$E=-\left[\frac{B}{i\gamma(1+i\gamma)}+\frac{2+i\gamma}{2i\gamma(1+i\gamma)}\right]\gamma^2A=-\left[\frac{\Omega}{2i\gamma(1+i\gamma)}+\frac{1}{2i\gamma}\right]\gamma^2A,$$
$$F_1=2AC\gamma^2\left[\frac{(2+i\gamma)(1+4i\gamma-\gamma^2)}{2i\gamma(1+i\gamma^2)^2}-\frac{5}{(1-i\gamma)^2}\right]+$$
$$+B(B+1)\gamma^2\left[\frac{(2+i\gamma)(1+4i\gamma+\gamma^2)}{2i\gamma(1+i\gamma^2)^2}-\frac{4}{(1-i\gamma)^2}\right], \tag{87}$$

where $\Omega=2B+1$.

From vanishing coefficients near ξ^{-4},ξ^{-3},ξ^{-2}, we obtain

$$H=-\frac{(2-i\gamma)\gamma^2}{2i\gamma(1-i\gamma)^2}C^2,$$
$$G=\left[\frac{B}{i\gamma(1-i\gamma)}+\frac{2-i\gamma}{2i\gamma(1-i\gamma)}\right]\gamma^2C=\left[\frac{\Omega}{2i\gamma(1-i\gamma)}+\frac{1}{2i\gamma}\right]\gamma^2C,$$
$$F_2=-2AC\gamma^2\left[\frac{(2-i\gamma)(1-4i\gamma-9\gamma^2)}{2i\gamma(1+i\gamma^2)^2}+\frac{5}{(1+i\gamma)^2}\right]-$$
$$-B(B+1)\gamma^2\left[\frac{(2-i\gamma)(1-4i\gamma-7\gamma^2)}{2i\gamma(1+i\gamma^2)^2}+\frac{4}{(1+i\gamma)^2}\right]. \tag{88}$$

Equality $F_1=F_2$ is possible, iff $2AC+B(B+1)=0$, see (75). Then

$$F=-2AC\frac{\gamma^2}{(1+\gamma^2)^2}=B(B+1)\frac{\gamma^2}{(1+\gamma^2)^2}. \tag{89}$$

Coefficients near ξ^1 and ξ^{-1} vanish for values (87), (88), (89). Coefficient near ξ^0 vanishes if

$$AC(6B^2 - B - 3) = 0\,. \tag{90}$$

If $\tilde{\omega} \neq 0$, we have additional condition (77) for polynomiality of $\varphi_1(\xi)$, i.e.

$$\omega A(2B + 1)C(\gamma^2 - 1) = 0\,. \tag{91}$$

Thus, we have proven

Theorem 15. *In the exotic expansion* (45) *of solutions to equation P_6, the second coefficient $\varphi_1(\xi)$ never is a Laurent polynomial* (72), (73)+ (87), (74)+ (88), (76)+ (89) *with $\omega = b - d$ instead of c, if 3 conditions* (75), (90) *and* (91) *are fulfilled.*

Usually the equation for $\varphi_k(\xi)$ has two solutions: with increasing and with decreasing powers of ξ. But they coincide if the solution is an usual or Laurent polynomial. If all coefficients $\varphi_k(\xi)$ are polynomials then there is one family of exotic expansions. In another case there are two different families. Details see in [10].

7. Conclusion

In both cases: complicated and exotic expansions we have its own alternative. In complicated expansion the coefficient $\varphi_k(\xi)$ is either a polynomial or a divergent Laurent series. In exotic expansion the coefficient $\varphi_k(\xi)$ is either a Laurent polynomial, in that case it is unique, or a Laurent series, then there are two different coefficients in form of convergent series. The convergence follows from [12].

In all considered cases, when coefficient $\varphi_k(\xi) = D\xi^m + E\xi^{m-1} + F\xi^{m-2} + \ldots$ of the complicated or exotic expansion is an usual or Laurent polynomial, its coefficients D,E,$F, \ldots$, satisfy to a system of linear algebraic equations. And number of equations is more then number of these coefficients. Such linear systems have solutions only in degenerated cases when rank of the extended matrix of the system is less then the maximal possible. Existence of such situations in the Painlevé equations shows their degeneracy or their inner symmetries.

We have considered 4 cases: equations P_3, Case I of P_5, Case II of P_5, P_6. In each of them there are 3 families: additional complicated, main complicated and exotic. Among these 12 families, 9 have polynomial second coefficient, but 3 families demand for that some conditions on parameters. Namely, main complicated family for Case I of P_5 demands one condition; exotic families for Case II of P_5 and for P_6 demand 2 conditions and 3 conditions correspondingly. In all cases number of conditions is less than difference between number of equations and number of unknowns.

All these calculations were made by hands. Further computations should be made using Computer Algebra.

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