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Dudnikova T.V.

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ИНСТИТУТ ПРИКЛАДНОЙ МАТЕМАТИКИ
имени М. В. КЕЛДЫША
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T. V. Dudnikova

Virial identities and energy–momentum relation for solitary waves of nonlinear Dirac equations

Москва — 2018

Дудникова Т.В.

Вириальные равенства и соотношение энергии–импульса для солитонных волн нелинейных уравнений Дирака

Рассматриваются солитонные волны нелинейных уравнений Дирака, Максвелла–Дирака и Клейна–Гордона–Дирака. Мы выводим некоторые вириальные равенства и проверяем, что соотношение энергии-импульса для солитонных волн совпадает с соотношением Эйнштейна энергии-импульса для точечных частиц.

Ключевые слова: Нелинейное уравнение Дирака, уравнения Максвелла–Дирака и Клейна–Гордона–Дирака, солитонные волны, соотношение Эйнштейна энергии-импульса

Tatiana Vladimirovna Dudnikova

Virial identities and energy–momentum relation for solitary waves of nonlinear Dirac equations

Solitary waves of nonlinear Dirac, Maxwell–Dirac and Klein–Gordon–Dirac equations are considered. We deduce some virial identities and check that the energy-momentum relation for solitary waves coincides with the Einstein energy-momentum relation for point particles.

Key words: Nonlinear Dirac equation, Maxwell–Dirac and Klein–Gordon–Dirac equations, solitary waves, Einstein energy-momentum relation

Contents

1. Introduction

The paper concerns the old problem of mathematically describing elementary particles in field theory. Einstein and Grommer [13] suggested that particles could be described as singularities of solutions to the field equations. The generalization of this result to interacting systems of particles was given by Einstein, Infeld and Hoffmann [14]. Rosen [27] was the first who proposed a description of particles for the coupled Klein–Gordon–Maxwell equations, which are invariant with respect to the Lorentz group. Namely, the particle at rest is described by a finite energy solution that has "Schrödinger's" form $\varphi(x)e^{-i\omega t}$ ("nonlinear eigenfunctions" or "solitary waves"). The particle with the nonzero velocity v, $|v| < 1$, is obtained by the corresponding Lorentz (or Poincaré) transformation. The existence of solitary waves has been analyzed by many authors for diverse Lagrangian field theories [15, 22, 26, 27, 31, 34], such that nonlinear Dirac fields, the Maxwell–Dirac (MD) and Klein–Gordon–Dirac (KGD) equations. We describe briefly some results.

Nonlinear Dirac equations occur in the attempt to construct relativistic models of extended particles by means of nonlinear Dirac fields. The review of such models can be found in [25]. The stationary solutions of nonlinear Dirac equation were extensively studied in the literature used variational methods [17] and a dynamical systems approach [6, 23, 2]. For details, see the survey papers [18, 16, 25] and the references therein.

The (MD) equations (see, e.g., [4, 30]) describing the interaction of an electron with its own electromagnetic field have been widely studied by many authors. The first results on the local existence and uniqueness of solutions was obtained by Gross [19], Chadam [7], Chadam and Glassey [9]. The stationary (localized) solutions of the classical (MD) system were studied numerically by Wakano [34] and Lisi [22]. Using variational methods, Esteban, Georgiev and Séré [15] have proved the existence of stationary solutions with $\omega \in (-m,0)$. These results were extended by Abenda [1] for $\omega \in (-m, m)$.

For the (KGD) equations, the local existence and uniqueness of solutions were proved by Chadam and Glassey [8]. Numerical results on the stationary states were obtained by Ranada and Vazquez in [26]. The rigorous proof of the existence for the stationary solutions was given by Esteban *et al.* [15]. For some Lorentz invariant complex scalar fields theories, the particle-like solutions was studied by Rosen [28, 29].

Note that it would be of importance to develop a particle-like dynamics for moving solitons. We make a step in this direction for relativistic-invariant nonlinear Dirac, (MD) and (KGD) equations. Namely, we prove that the energy-momentum relation coincides with that of a relativistic particle.

Now we outline the main result in the case of nonlinear Dirac equations. We consider the Dirac equations of the form

$$i\dot{\psi} = -i\alpha \cdot \nabla\psi + m\beta\psi - g(\bar{\psi}\psi)\beta\psi. \tag{1.1}$$

We use natural units, in which we have rescaled length and time so that $\hbar = c = 1$. Here unknown function $\psi \equiv \psi(t,x) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^4$ is four-component Dirac spinor field, $m > 0$, $\dot{\psi} = \partial_t\psi$, $x = (x_1, x_2, x_3)$, $\nabla = (\partial_1, \partial_2, \partial_3)$, $\partial_k = \partial/(\partial x_k)$, $k = 1,2,3$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. α_k, β are the 4×4 complex Pauli-Dirac matrices (in the standard 2×2 blocks representation)

$$\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \quad (k = 1,2,3), \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

where I denotes the 2×2 unit matrix, and σ_k are Pauli matrices defined as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

One verifies that $\sigma_k\sigma_l + \sigma_l\sigma_k = 2\delta_{kl}I$, $\sigma_k^* = \sigma_k$, $k = 1,2,3$. Then

$$\beta^* = \beta, \;\; \alpha_k^* = \alpha_k, , \;\; \alpha_k^2 = \beta^2 = I, \;\; \alpha_k\alpha_j + \alpha_j\alpha_k = 0 \;\text{ for }\; j \neq k, \;\; \alpha_k\beta + \beta\alpha_k = 0. \tag{1.2}$$

Let us fix the following notations. Given two vectors of $\mathbb{C}^4$, $\psi\phi := \psi \cdot \phi$ is the inner product in $\mathbb{C}^4$, $*$ denotes the complex conjugate. By definition, the "adjoint spinor" is $\bar{\psi} = \psi^*\beta$.

The particular nonlinearity $g(s) = \lambda s$ corresponds to the so-called *Soler model of extended fermions [31, 2]*. In the general case of $g(s)$, Eqn (1.1) is often called the *generalized Soler model* (see [4, 17]). The review of models of extended particles by means of nonlinear Dirac fields can be found in [25].

The stationary solutions of nonlinear Dirac equation are considered as particle-like solutions. They are the solutions of a form $\psi_0(t,x) = e^{-i\omega t}\varphi(x)$, where φ is non-zero localized solution of the stationary nonlinear Dirac equation (2.1), see Definition 2.1 below.

Denote by $\psi_{\mathbf{v}}(t,x)$ *the moving solitary waves* with velocity $\mathbf{v} \in \mathbb{R}^3$, $|\mathbf{v}| < 1$,

$$\psi_{\mathbf{v}}(t,x) = S(\Lambda_{\mathbf{v}})\psi_0\left(\Lambda_{\mathbf{v}}^{-1}(t,x)\right), \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R},$$

where $\Lambda_{\mathbf{v}}$ is a Lorentz transformation (see formula (3.5) below), $S(\Lambda_{\mathbf{v}})$ is a matrix defined in (3.6). Put $G(s) = \int_0^s g(p)\,dp$. The energy functional is given by

$$\mathcal{E}(\psi) = \int_{\mathbb{R}^3} \Big(-i\psi^*\alpha \cdot \nabla\psi + m\bar{\psi}\psi - G(\bar{\psi}\psi) \Big)\,dx. \tag{1.3}$$

Using equalities (1.2), it is easy to check that $\mathcal{E}(\psi(t,\cdot)) = \text{const}$.

Our main objective is to check that the energy-momentum relation coincides with one of relativistic point particle, namely,

$$\mathcal{E}(\psi_{\mathbf{v}}) = \gamma\mathcal{E}(\psi_0), \quad \gamma = (1 - |\mathbf{v}|^2)^{-1/2}. \tag{1.4}$$

The paper is organized as follows. In Sections 2 and 3, we check (1.4) for nonlinear Dirac equations (1.1). Section 4 concerns the Dirac equations in $\mathbb{R}^1$. For (MD) and (KGD) equations, the result is obtained in Sections 5 and 6, respectively.

2. Standing solitary waves for Dirac equations

Denote by $H^s(\mathbb{R}^3)$, $s \in \mathbb{R}$, the Sobolev space, i.e., the Hilbert space of distributions $\varphi \in S'(\mathbb{R}^3)$ endowed with the norm $\|\varphi\|_{H^s} = \|\Lambda^s\varphi\|_{L^2}$, where $\Lambda^s\varphi := F^{-1}_{\xi\to x}((1+|\xi|^2)^{s/2}\hat{\varphi}(\xi))$, and $\hat{\varphi} := F\varphi$ denotes Fourier transform.

Let $W^{1,q}(\mathbb{R}^3)$, $q \geq 2$, denote the space of distributions $\varphi \in S'(\mathbb{R}^3)$ endowed with the norm $\|\varphi\|_{W^{1,q}} = \|\nabla\varphi\|_{L^q} + \|\varphi\|_{L^q}$. In particular, $W^{1,2}(\mathbb{R}^3) = H^1(\mathbb{R}^3)$.

Definition 2.1. *The stationary states or localized solutions of Eqn (1.1) are the solutions of the form $\psi_0(t,x) = e^{-i\omega t}\varphi_\omega(x)$, $\omega \in \mathbb{R}$, such that $\varphi_\omega \in H^1(\mathbb{R}^3;\mathbb{C}^4)$, and $\varphi \equiv \varphi_\omega$ is a nonzero localized solution of the following stationary nonlinear Dirac equation*

$$i\alpha\cdot\nabla\varphi + \omega\varphi - m\beta\varphi + g(\bar{\varphi}\varphi)\beta\varphi = 0, \quad x \in \mathbb{R}^3. \tag{2.1}$$

The existence of solutions of Eqn (2.1) has been proved in [2, 3, 6, 17, 23] under some restrictions on G for $\omega \in (0,m)$. In particular, in [17] the following conditions were imposed.

G1. $G \in C^2(\mathbb{R};\mathbb{R})$

G2. For any $s \in \mathbb{R}$, $g(s)s \geq \theta G(s)$ with some $\theta > 1$, $(g(s) = G'(s))$

G3. $G(0) = G'(0) = 0$

G4. $G(s) \geq 0$ for any $s \in \mathbb{R}$, and $G(A_0) > 0$ for some $A_0 > 0$.

Theorem 2.2. *(see [17, Theorem 1]) Let conditions* **G1**–**G4** *hold and $\omega \in (0,m)$. Then there is an infinity of solutions of Eqn (2.1) in $\bigcap\limits_{2\leq q<\infty} W^{1,q}(\mathbb{R}^3;\mathbb{C}^4)$. Each of them are critical points of the functional I^ω_D,*

$$I^\omega_D(\varphi) = -\frac{1}{2}\int_{\mathbb{R}^3}\Big(i\varphi^*\alpha\cdot\nabla\varphi - m\bar{\varphi}\varphi + \omega|\varphi|^2 + G(\bar{\varphi}\varphi)\Big)\,dx.$$

These solutions $\varphi \equiv \varphi_\omega$ are of the form (in the spherical coordinates (r,ϕ,θ) of $x \in \mathbb{R}^3$)

$$\varphi_\omega(x) = \begin{pmatrix} v(r)\begin{pmatrix}1\\0\end{pmatrix} \\ iu(r)\begin{pmatrix}\cos\theta\\ \sin\theta e^{i\phi}\end{pmatrix}\end{pmatrix}, \qquad \begin{array}{rcl} x_1 &=& r\cos\phi\sin\theta, \\ x_2 &=& r\sin\phi\sin\theta, \\ x_3 &=& r\cos\theta,\ r = |x|. \end{array} \tag{2.2}$$

Thus they correspond to classical solutions of the O.D.E. system

$$\begin{cases} u' + \frac{2u}{r} = v[g(v^2 - u^2) - (m - \omega)], \\ v' = u[g(v^2 - u^2) - (m + \omega)]. \end{cases}$$

Finally, the solutions decrease exponentially at infinity, together with their first derivatives.

Remarks 2.3. (i) Denote by $\mathcal{L}_D$ the Lagrangian density for considered Dirac fields,

$$\mathcal{L}_D(\psi) = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + G(\bar{\psi}\psi), \tag{2.3}$$

where $\gamma^\mu \partial_\mu = \gamma^0 \partial_t + \gamma \cdot \nabla$ with Dirac matrices γ^μ ($\gamma^0 = \beta$, $\gamma^k = \beta \alpha_k$, $k = 1,2,3$). It is easy to check that the Euler–Lagrange equations applied to (2.3) give Eqn (1.1). In particular, for stationary solutions $\psi_0(t, x)$ we have

$$\mathcal{L}_D(\psi_0) = \varphi^*(\omega + i\alpha \cdot \nabla - m\beta)\varphi + G(\bar{\varphi}\varphi).$$

Note that $I_D^\omega(\varphi) = -(1/2) \int \mathcal{L}_D(\psi_0)\, dx$. Here and below, for simplicity, we omit the symbol $\mathbb{R}^3$ in the notation of the integral $\int_{\mathbb{R}^3} \ldots dx$.

(ii) In [3], the existence of solutions of the form (2.2) have been proved for singular self-interactions $g(s) \sim s^{-\alpha}$ with some $\alpha \in (0,1)$.

(iii) The stationary nonlinear Dirac equations of the form

$$i\alpha \cdot \nabla\varphi + \omega\varphi - m\beta\varphi + \nabla F(\varphi) = 0, \quad x \in \mathbb{R}^3, \tag{2.4}$$

has been studied by Esteban and Séré in [17]. If $F(\varphi) = G(\bar{\varphi}\varphi)$, then Eqn (2.4) coincides with (2.1). For a more general class of nonlinearities F, which do not satisfy condition $F(\varphi) = G(\bar{\varphi}\varphi)$, the ansatz (2.2) is no more valid. In this case, the existence of solutions of (2.4) has been proved in [17, Theorems 2,3] with nonlinearities as (1) $F(\varphi) = \lambda(|\bar{\varphi}\varphi|^{\kappa_1} + b|\bar{\varphi}\gamma^5\varphi|^{\kappa_2})$ with $1 < \kappa_1, \kappa_2 < 3/2$, $\gamma^5 = -i\alpha_1\alpha_2\alpha_3$, $\lambda, b > 0$; and
(2) $F'(0) = F''(0) = 0$, $0 \le F(\varphi) \le a(|\bar{\varphi}|^{\kappa_3} + |\varphi|^{\kappa_4})$ with $a > 0$, $2 < \kappa_3 \le \kappa_4 < 3$.

The following virial identity (or so-called Pokhozhaev identity [24]) was proved in [17, Proposition 3.1].

Lemma 2.4. *Let $\varphi \in H^1(\mathbb{R}^3; \mathbb{C}^4)$ be a solution to Eqn (2.1). Then $\varphi(x)$ satisfies*

$$i \int \varphi^* \alpha \cdot \nabla\varphi\, dx = \frac{3}{2} \int \Big(m\bar{\varphi}\varphi - \omega\varphi^*\varphi - G(\bar{\varphi}\varphi)\Big)\, dx. \tag{2.5}$$

Introduce the following notations

$$
\begin{aligned}
I_k &\equiv I_k(\varphi) = -i \int \varphi^* \alpha_k \partial_k \varphi \, dy, \quad k = 1,2,3, \\
Q &\equiv Q(\varphi) = \int \varphi^* \varphi \, dx, \quad V \equiv V(\varphi) = \int \Big(m\bar{\varphi}\varphi - G(\bar{\varphi}\varphi) \Big) \, dx.
\end{aligned} \tag{2.6}
$$

Then the equality (2.5) is rewritten as

$$
\omega Q = V + \frac{2}{3}(I_1 + I_2 + I_3). \tag{2.7}
$$

Remark 2.5. Formally, the identity (2.5) can be proved used Derrick's technique [10, p.1253]. Indeed, introducing $\varphi_\lambda(x) = \varphi(x/\lambda)$ gives

$$
\begin{aligned}
0 &= \frac{d}{d\lambda}\Big|_{\lambda=1} I_D^\omega(\varphi_\lambda) = \frac{1}{2}\frac{d}{d\lambda}\Big|_{\lambda=1} \Big[I_1(\varphi_\lambda) + I_2(\varphi_\lambda) + I_3(\varphi_\lambda) + V(\varphi_\lambda) - \omega Q(\varphi_\lambda) \Big] \\
&= \frac{1}{2}\frac{d}{d\lambda}\Big|_{\lambda=1} \Big[\lambda^2 I_1(\varphi) + \lambda^2 I_2(\varphi) + \lambda^2 I_3(\varphi) + \lambda^3 V(\varphi) - \lambda^3 \omega Q(\varphi) \Big] \\
&= I_1(\varphi) + I_2(\varphi) + I_3(\varphi) + \frac{3}{2}(V - \omega Q).
\end{aligned}
$$

This gives the identity (2.5). The similar Derrick's technique has been used in [11] for relativistic-invariant nonlinear wave equations. Using the similar reasonings with $\varphi_\lambda(x) = \varphi(x_1/\lambda, x_2, x_3)$, $\varphi_\lambda(x) = \varphi(x_1, x_2/\lambda, x_3)$, $\varphi_\lambda(x) = \varphi(x_1, x_2, x_3/\lambda)$, it is easy to check that

$$
I_1 = I_2 = I_3 = \frac{1}{3}(I_1 + I_3 + I_3) = \frac{1}{2}(\omega Q - V). \tag{2.8}
$$

Corollary 2.6. *Let φ be a solution of (2.1). Then the following relations hold.*

$$
I_1 + I_2 + I_3 = \omega Q + \int \Big(g(\bar{\varphi}\varphi) - m \Big) \bar{\varphi}\varphi \, dx. \tag{2.9}
$$

$$
I_1 + I_2 + I_3 = 3 \int \Big(g(s)s - G(s) \Big)\Big|_{s=\bar{\varphi}\varphi} dx > 0. \tag{2.10}
$$

$$
\mathcal{E}_0 \equiv I_1 + I_2 + I_3 + V > 0. \tag{2.11}
$$

Proof By (2.1), we have

$$
\int \varphi^* i\alpha \cdot \nabla \varphi \, dx = \int \varphi^* \Big(-\omega\varphi + m\beta\varphi - g(\bar{\varphi}\varphi)\beta\varphi \Big) \, dx.
$$

This implies the identity (2.9). Then, by (2.7) and (2.9), we obtain

$$\omega Q = V + \frac{2}{3}(I_1 + I_2 + I_3) = \int \Big(m - g(\bar\varphi\varphi)\Big)\bar\varphi\varphi\, dx + I_1 + I_2 + I_3.$$

Hence,

$$\frac{1}{3}(I_1 + I_2 + I_3) = \int \Big(g(s)s - G(s)\Big)\Big|_{s=\bar\varphi\varphi} dx. \tag{2.12}$$

Therefore, (2.10) follows from (2.12) and condition **G2**, since

$$g(s)s - G(s) > g(s)s - \theta G(s) \geq 0 \quad \text{for all } \ s \in \mathbb{R}.$$

By (1.3) and (2.9), the energy $\mathcal{E}_0 := \mathcal{E}(\psi_0(t,\cdot))$ associated with particle-like solutions ψ_0 is expressed by

$$\mathcal{E}_0 \equiv \mathcal{E}_0(\varphi) = I_1 + I_2 + I_3 + V = \omega \int |\varphi(x)|^2\, dx + \int \Big(g(s)s - G(s)\Big)\Big|_{s=\bar\varphi\varphi} dx > 0,$$

by condition **G2**. ■

Denote by $(\cdot,\cdot)$ the inner scalar product in L^2.

Lemma 2.7. *Let φ be a solution of Eqn (2.1), $\varphi \in H^1(\mathbb{R}^3;\mathbb{C}^4)$. Then*

$$\omega(\varphi^*, \alpha_k\varphi) = -i(\varphi^*, \partial_k\varphi), \quad k = 1{,}2{,}3. \tag{2.13}$$

Proof Multiply (2.1) on the left by α_1 and obtain

$$i\partial_1\varphi + i\alpha_1\alpha_2\partial_2\varphi + i\alpha_1\alpha_3\partial_3\varphi + \omega\alpha_1\varphi - m\alpha_1\beta\varphi + g(\bar\varphi\varphi)\alpha_1\beta\varphi = 0.$$

Hence

$$\begin{aligned} i(\varphi^*, \partial_1\varphi) + i(\varphi^*, \alpha_1\alpha_2\partial_2\varphi) + i(\varphi^*, \alpha_1\alpha_3\partial_3\varphi) + \omega(\varphi^*, \alpha_1\varphi) \\ - m(\varphi^*, \alpha_1\beta\varphi) + (\varphi^*, g(\bar\varphi\varphi)\alpha_1\beta\varphi) = 0. \end{aligned} \tag{2.14}$$

On the other hand, taking the adjoint of Eqn (2.1) and multiplying on the right by α_1, one obtains

$$-i\partial_1\varphi^* - i\partial_2\varphi^*\alpha_2\alpha_1 - i\partial_3\varphi^*\alpha_3\alpha_1 + \omega\varphi^*\alpha_1 - m\varphi^*\beta\alpha_1 + g(\bar\varphi\varphi)\varphi^*\beta\alpha_1 = 0.$$

Hence,

$$\begin{aligned} -i(\partial_1\varphi^*, \varphi) - i(\partial_2\varphi^*\alpha_2\alpha_1, \varphi) - i(\partial_3\varphi^*\alpha_3\alpha_1, \varphi) + \omega(\varphi^*\alpha_1, \varphi) \\ - m(\varphi^*\beta\alpha_1, \varphi) + (\varphi^*\beta\alpha_1, g(\bar\varphi\varphi)\varphi) = 0. \end{aligned} \tag{2.15}$$

By (1.2), summing Eqns (2.14) and (2.15) gives (2.13) for $k = 1$. For $k \neq 1$ the proof is similar. ■

2.1. A particular ansatz for the solutions of Dirac equations. As in [22], we choose to orient the angular momentum along the x_3-axis and consider four families of solutions of Eqn (2.1) which in spherical coordinates (r, ϕ, θ) (i.e. $x_1 = r\cos\phi\sin\theta$, $x_2 = r\sin\phi\sin\theta$, $x_3 = r\cos\theta$) are of a form

$$\varphi^1(x) = \begin{pmatrix} v_+(r)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ iu_+(r)\begin{pmatrix} \cos\theta \\ \sin\theta e^{i\phi} \end{pmatrix} \end{pmatrix}, \quad \varphi^2(x) = \begin{pmatrix} v_-(r)\begin{pmatrix} \cos\theta \\ \sin\theta e^{i\phi} \end{pmatrix} \\ iu_-(r)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix},$$

$$\varphi^3(x) = \begin{pmatrix} v_+(r)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ iu_+(r)\begin{pmatrix} \sin\theta e^{-i\phi} \\ -\cos\theta \end{pmatrix} \end{pmatrix}, \quad \varphi^4(x) = \begin{pmatrix} v_-(r)\begin{pmatrix} -\sin\theta e^{-i\phi} \\ \cos\theta \end{pmatrix} \\ iu_-(r)\begin{pmatrix} 0 \\ -1 \end{pmatrix} \end{pmatrix}.$$

If $\varphi^1, ..., \varphi^4$ are substituted into Eqn (2.1), then this equation reduces to the following O.D.E. system for radial functions $u_\pm$ and $v_\pm$:

$$\begin{cases} u'_\pm + \dfrac{2u_\pm}{r} = v[g(v_\pm^2 - u_\pm^2) - (m \mp \omega)], \\ v'_\pm = u_\pm[g(v_\pm^2 - u_\pm^2) - (m \pm \omega)]. \end{cases}$$

The existence of the solutions $u_\pm$ and $v_\pm$ follows from results [6, 23, 2, 17].

The total angular momentum operator is $\mathbf{M} = \mathbf{L} + \mathbf{S}$, where $\mathbf{L} = x \times (-i\nabla)$ is the orbital angular momentum, $\mathbf{S} = \Sigma/2$ is the spin angular momentum, $\Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$. Here $\mathbf{M} = (M_1, M_2, M_3)$, $\mathbf{L} = (L_1, L_2, L_3)$, $x = (x_1, x_2, x_3)$. In particular, in the spherical coordinates, the third component of $\mathbf{L}$ is $L_3 = -i\partial_\phi$. It is easy to check the following properties of $\varphi^a(x)$, $a = 1,2,3,4$.

(i) φ^a are eigenfunctions of the third component of $\mathbf{M}$ with eigenvalue $m_3 = \pm 1/2$. More exactly, $M_3\varphi^a = 1/2\varphi^a$ for $a = 1,2$, $M_3\varphi^a = -1/2\varphi^a$ for $a = 3,4$. Since $\mathbf{M}\varphi^1 = (1/2)(\varphi^3, i\varphi^3, \varphi^1)$, $\mathbf{M}\varphi^2 = (1/2)(-\varphi^4, -i\varphi^4, \varphi^2)$, $\mathbf{M}\varphi^3 = (1/2)(\varphi^1, -i\varphi^1, -\varphi^3)$, $\mathbf{M}\varphi^4 = (1/2)(-\varphi^2, i\varphi^2, -\varphi^4)$, then $M_k^2\varphi^a = 1/4\varphi^a$ for all $k = 1,2,3$, and $\mathbf{M}^2\varphi^a = 3/4\varphi^a = j(j+1/2)\varphi^a$ for all a. Hence, the quantum number $j = 1/2$ for all φ^a.

(ii) For the "spin–orbit" operator $\mathbf{K} = \beta\Sigma\cdot\mathbf{M} - 1/2\beta = \beta(\Sigma\cdot\mathbf{L} + 1)$ (see [12, p.19]), we have $\mathbf{K}^2 = \mathbf{M}^2 + 1/4$. Then the eigenvalues of $\mathbf{K}$ are $\kappa = \pm(j + 1/2)$. Hence, for all a, φ^a are eigenfunctions of $\mathbf{K}$ with eigenvalues $\kappa = \pm 1$, where the quantum number $\kappa = 1$ for $a = 1,3$ and $\kappa = -1$ for $a = 2,4$.

(iii) For any solution φ from the four families $\{\varphi^1, \ldots, \varphi^4\}$, the following equalities hold. At first, $\varphi^*(x)\alpha_3\varphi(x) \equiv 0$. Secondly,

$$\int \varphi^*(x)\nabla\varphi(x)\,dx = 0, \tag{2.16}$$

$$\int \varphi^*(x)\alpha_k\,\partial_l\varphi(x)\,dx = 0 \quad \text{for any} \quad k \neq l. \tag{2.17}$$

(iv) For stationary states $\psi_0(t,x) = e^{-i\omega t}\varphi(x)$ with φ from these particular families of solutions we have $Q(\psi_0) = 4\pi \int_0^{+\infty} \left(v_\pm^2 + u_\pm^2\right) r^2\,dr$,

$$\mathcal{E}_0 \equiv \mathcal{E}(\psi_0) = 4\pi\omega \int_0^{+\infty} \left(v_\pm^2 + u_\pm^2\right) r^2 dr + 4\pi \int_0^{+\infty} \Big(g(s)s - G(s)\Big)\Big|_{s=v_\pm^2-u_\pm^2} r^2 dr,$$

where $v_\pm = v_\pm(r)$, $u_\pm = u_\pm(r)$. Moreover, the current $\mathbf{J}(x) := \psi_0^*(t,x)\alpha\psi_0(t,x)$ equals

$$\mathbf{J}(x) = 4\kappa_\pm m_3 u_\pm v_\pm(-\sin\phi, \cos\phi, 0),$$

where the quantum numbers $m_3 = \pm 1/2$, $\kappa_\pm = \pm 1$ are introduced above.

3. Moving solitary waves for nonlinear Dirac equations

As shown, e.g., in [4, 12, 32], the Dirac equation (1.1) with $g \equiv 0$ is Lorentz invariant. Namely, let $\Lambda = (\Lambda_{\mu\nu})_{\mu,\nu=0}^4$ be a Lorentz transformation and $\psi(t,x)$ be a solution of (1.1) with $g \equiv 0$. Then there exists a matrix $S(\Lambda)$ such that $\psi'(t',x') = S(\Lambda)\psi(t,x)$ satisfies the same equation in the terms of the new variables $(t',x') = \Lambda(t,x)$. It requires the following conditions on $S \equiv S(\Lambda)$:

$$\alpha_\mu = \sum_{\nu=0}^{4} \beta S \beta \Lambda_{\mu\nu}\alpha_\nu S^{-1} \quad \text{with} \quad \alpha_0 \equiv I, \tag{3.1}$$

or $S^{-1}\gamma^\nu S = \sum_{\mu=0}^{3} \Lambda_{\nu\mu}\gamma^\mu$, $\nu = 0,1,2,3$, where $\gamma^0 := \beta$, $\gamma^k := \beta\alpha_k$, $k = 1,2,3$ (see, e.g., [4]). Here and below by I we denote the unit 4×4 (or 2×2) matrix. The nonlinear equation (1.1) is Lorentz invariant, if condition (3.1) holds and

$$S^*\beta S = \beta. \tag{3.2}$$

The conditions (3.1) and (3.2) can be rewritten in the form (cf formulas (23) and (27) from [12])

$$S^*\beta S = \beta, \quad S^*\alpha_\mu S = \sum_{\nu=0}^{4} \Lambda_{\mu\nu}\alpha_\nu \quad \text{with} \quad \alpha_0 \equiv I. \tag{3.3}$$

The existence of the matrix S satisfying conditions (3.3) follows from Pauli's Fundamental Theorem.

Let $\Lambda_{\mathbf{v}} : \mathbb{R}^4 \to \mathbb{R}^4$ be a Lorentz transformation (boost) with velocity $\mathbf{v} \in \mathbb{R}^3$, $|\mathbf{v}| < 1$: $\Lambda_{\mathbf{v}}(t,x) = \Big(\gamma(t + \mathbf{v}\cdot x), \gamma(x^{\|} + \mathbf{v}t) + x^{\perp}\Big)$, where $x^{\|} + x^{\perp} = x$, $x^{\|} \| \mathbf{v}$, $x^{\perp} \perp \mathbf{v}$, $\gamma = (1 - \mathbf{v}^2)^{-1/2}$. Hence (see, e.g., [32, formula (2.14)]),

$$\Lambda_{\mathbf{v}} = \begin{pmatrix} \gamma & \gamma \mathbf{v}^T \\ \gamma \mathbf{v} & I + \frac{\gamma-1}{|\mathbf{v}|^2}\mathbf{v}\mathbf{v}^T \end{pmatrix}, \quad \text{where } \mathbf{v}\mathbf{v}^T = (v_i v_j)_{i,j=1}^3, \tag{3.4}$$

i.e.,

$$\Lambda_{\mathbf{v}}(t,x) = \Big(\gamma(t + \mathbf{v}\cdot x), x + (\gamma - 1)\mathbf{v}\frac{x\cdot\mathbf{v}}{|\mathbf{v}|^2} + \gamma \mathbf{v} t\Big), \quad (t,x) \in \mathbb{R}^4. \tag{3.5}$$

Note that $\det \Lambda_v = 1$ and $\Lambda_{\mathbf{v}}^{-1} = \Lambda_{-\mathbf{v}}$. The matrix $S_{\mathbf{v}} \equiv S(\Lambda_{\mathbf{v}})$ can be chosen as

$$S_{\mathbf{v}} = \sqrt{\frac{\gamma+1}{2}}\left(I + \alpha\cdot\mathbf{v}\frac{\gamma}{\gamma+1}\right) = \exp\left(\frac{\xi}{2}\frac{\alpha\cdot\mathbf{v}}{|\mathbf{v}|}\right), \tag{3.6}$$

where $\mathrm{ch}(\xi/2) = \sqrt{(\gamma+1)/2}$ or $\mathrm{th}(\xi) = |\mathbf{v}|$. It is easy to verify that

$$\begin{aligned} &S_0 = I, \quad S_{\mathbf{v}}^* = S_{\mathbf{v}}, \quad S_{-\mathbf{v}} = S_{\mathbf{v}}^{-1}, \quad S_{\mathbf{v}}^2 = \gamma(\alpha\cdot\mathbf{v} + I), \\ &S_{\mathbf{v}}^*\beta S_{\mathbf{v}} = \beta, \quad S_{\mathbf{v}}^*\alpha_j S_{\mathbf{v}} = \alpha_j + \gamma v_j I + v_j\frac{\gamma-1}{|\mathbf{v}|^2}\,\alpha\cdot\mathbf{v}, \quad j = 1,2,3, \end{aligned} \tag{3.7}$$

and conditions (3.3) hold. In particular,

$$\begin{aligned} &\gamma(I - \alpha\cdot\mathbf{v})S_{\mathbf{v}} = S_{\mathbf{v}}^{-1}, \\ &\alpha\cdot S_{\mathbf{v}}\left(\nabla\varphi + \mathbf{v}\frac{\gamma-1}{|\mathbf{v}|^2}\nabla\varphi\cdot\mathbf{v}\right) - \gamma S_{\mathbf{v}}\nabla\varphi\cdot\mathbf{v} = S_{\mathbf{v}}^{-1}\alpha\cdot\nabla\varphi. \end{aligned} \tag{3.8}$$

Let $\omega \in (0,m)$, and $\psi_0(t,x) = e^{-i\omega t}\varphi(x)$, be a standing solitary wave. By $\psi_{\mathbf{v}}(t,x)$ we denote a *(moving) solitary wave* with velocity $\mathbf{v} \in \mathbb{R}^3$, $|\mathbf{v}| < 1$:

$$\psi_{\mathbf{v}}(t,x) = S_{\mathbf{v}}\psi_0(\Lambda_{\mathbf{v}}^{-1}(t,x)).$$

In other words,

$$\psi_{\mathbf{v}}(t,x) = e^{-i\omega\gamma(t - \mathbf{v}\cdot x)}S_{\mathbf{v}}\varphi\Big(x + (\gamma-1)\mathbf{v}\frac{x\cdot\mathbf{v}}{|\mathbf{v}|^2} - \gamma\mathbf{v}t\Big). \tag{3.9}$$

Remark 3.1. *(i) Let $\varphi(x)$ be a non-zero solution of Eqn (2.1). Then solitary waves $\psi_{\mathbf{v}}(t,x)$ satisfies Eqn (1.1). This follows from (3.8) and (3.9). Indeed, substituting $\psi_{\mathbf{v}}(t,x)$ in Eqn (1.1) and using (3.7) and (3.8) we obtain*

$$\Big(i\partial_t + i\alpha\cdot\nabla - m\beta + g(\bar{\psi}_{\mathbf{v}}\psi_{\mathbf{v}})\beta\Big)\psi_{\mathbf{v}}(t,x) = e^{-i\omega\gamma(t-\mathbf{v}\cdot\mathbf{x})}\Big[\omega\gamma(I-\alpha\cdot\mathbf{v})S_{\mathbf{v}}\varphi(y)$$
$$+ i\Big(\alpha\cdot S_{\mathbf{v}}\Big(\nabla\varphi + \mathbf{v}\tfrac{\gamma-1}{|\mathbf{v}|^2}\nabla\varphi(y)\cdot\mathbf{v}\Big) - \gamma S_{\mathbf{v}}\nabla\varphi(y)\cdot\mathbf{v}\Big) - m\beta S_{\mathbf{v}}\varphi(y)$$
$$+ g(\bar{\varphi}\varphi)\beta S_{\mathbf{v}}\varphi\Big] = e^{-i\omega\gamma(t-\mathbf{v}\cdot\mathbf{x})}S_{\mathbf{v}}^{-1}\Big[\omega + i\alpha\cdot\nabla - m\beta + g(\bar{\varphi}\varphi)\beta\Big]\varphi(y) = 0,$$

with $y = x + \mathbf{v}(\gamma-1)x\cdot\mathbf{v}/|\mathbf{v}|^2 - \gamma\mathbf{v}t$.
(ii) Let $\psi'(t',x') = S(\Lambda)\psi(t,x)$, where Λ is a Lorentz transformation. Denote by $J(t,x)$ the 4-current, $J^\mu(t,x) = \psi^*(t,x)\alpha_\mu\psi(t,x)$ (with $\alpha_0 \equiv I$) and let $J'^\mu(t,x) = \psi'(t,x)^*\alpha_\mu\psi'(t,x)$. Then $J'(t',x') = \Lambda J(t,x)$, where $(t',x') = \Lambda(t,x)$. In particular, if Λ is a boost, i.e., $\Lambda = \Lambda_{\mathbf{v}}$ with $\mathbf{v}\in\mathbb{R}^3$, and $\psi = \psi_{\mathbf{v}}$ with $\psi_{\mathbf{v}}$ from (3.9), then

$$J_{\mathbf{v}}(t,x) = \Lambda_{\mathbf{v}}J_0(y) = \Lambda_{\mathbf{v}}\begin{pmatrix}\varphi^*(y)\varphi(y)\\ \varphi^*(y)\alpha\varphi(y)\end{pmatrix}, \tag{3.10}$$

where $J_{\mathbf{v}} = (J^\mu_{\mathbf{v}})^4_{\mu=1}$, $J^\mu_{\mathbf{v}} = \psi^*_{\mathbf{v}}\alpha_\mu\psi_{\mathbf{v}}$, $y = x + \mathbf{v}(\gamma-1)x\cdot\mathbf{v}/|\mathbf{v}|^2 - \gamma\mathbf{v}t$.

For simplicity, put $\mathbf{v} = (0,0,v)\in\mathbb{R}^3$. In this case, we denote by Λ_v the Lorentz transformation (boost) $\Lambda_{\mathbf{v}}$:

$$\Lambda_v : (t,x) \to (\gamma(t+vx_3), x_1, x_2, \gamma(x_3+vt)), \quad |v|<1; \tag{3.11}$$

the solitary waves $\psi_v(t,x) := \psi_{\mathbf{v}}(t,x)|_{\mathbf{v}=(0,0,v)}$ are

$$\psi_v(t,x) = S_v\psi_0(\Lambda_v^{-1}(t,x)) = e^{-i\omega(t-vx_3)\gamma}S_v\varphi(x_1,x_2,\gamma(x_3-vt)); \tag{3.12}$$

the matrix S_v ($S_v := S_{\mathbf{v}}$ if $\mathbf{v} = (0,0,v)$) is defined as

$$S_v = \sqrt{\frac{\gamma+1}{2}}\left(I + \alpha_3\frac{v\gamma}{\gamma+1}\right) = \sqrt{\frac{\gamma+1}{2}}\begin{pmatrix} I & \frac{v\gamma}{\gamma+1}\sigma_3\\ \frac{v\gamma}{\gamma+1}\sigma_3 & I\end{pmatrix}, \quad v\in\mathbb{R}^1. \tag{3.13}$$

Using the explicit formulas (3.13), we obtain the following properties of S_v (cf (3.7), (3.8)).

$$\begin{gathered} S_0 = I, \quad S_v^* = S_v, \quad S_{-v} = S_v^{-1}, \quad S_v^*\beta S_v = \beta,\\ S_v^*\alpha_3 S_v = \gamma(vI+\alpha_3), \ \ S_v^*S_v = \gamma(v\alpha_3 + I), \ \ S_v^*\alpha_k S_v = \alpha_k, \ \ k = 1,2. \end{gathered} \tag{3.14}$$

In particular, $\gamma S_v^*(\alpha_3 - vI)S_v = \alpha_3$, $\gamma S_v^*(I-\alpha_3 v)S_v = I$.

Given $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$, we impose the following conditions on $\varphi(x)$.

C1 $\displaystyle\int \varphi^* \nabla\varphi \, dx \cdot \mathbf{v} = 0.$

C2 $\displaystyle\sum_{k,j\colon k\neq j} v_k v_j \int \varphi^* \alpha_k \partial_j \varphi \, dx = 0.$

Theorem 3.2. *Let $\mathbf{v} \in \mathbb{R}^1$ with $|\mathbf{v}| < 1$, $\psi_{\mathbf{v}}(t,x)$ be a solitary wave of the form (3.9), and φ satisfy conditions* **C1** *and* **C2**. *Then*

$$\mathcal{E}_{\mathbf{v}} := \mathcal{E}(\psi_{\mathbf{v}}) = \gamma \mathcal{E}_0. \tag{3.15}$$

Proof We first consider the particular case $\mathbf{v} = (0,0,v) \in \mathbb{R}^3$ when $\psi_v(t,x)$ is defined in (3.12). Substitute the function ψ_v into (1.3) and apply equalities (3.14):

$$\begin{aligned}
\mathcal{E}_{\mathbf{v}} :=& \int \Big(-i \sum_{k=1}^{2} \varphi^* S_v^* \alpha_k S_v \partial_k \varphi - i\varphi^* S_v^* \alpha_3 S_v \gamma (i\omega v \varphi + \partial_3 \varphi) + m\varphi^* S_v^* \beta S_v \varphi \\
&- G(\bar\varphi\varphi) \Big) \, dx \\
=& \int \Big(\gamma^2 \varphi^* (v + \alpha_3)(\omega v \varphi - i\partial_3 \varphi) - i \sum_{k=1}^{2} \varphi^* \alpha_k \partial_k \varphi + m\bar\varphi\varphi - G(\bar\varphi\varphi) \Big) \, dx,
\end{aligned}$$

where $\varphi \equiv \varphi(x_1, x_2, \gamma(x_3 - vt))$. Changing variables $x = (x_1, x_2, x_3) \to y := (x_1, x_2, \gamma(x_3 - vt))$, we obtain

$$\mathcal{E}_{\mathbf{v}} = \omega\gamma v (\varphi^*, (v + \alpha_3)\varphi) - i\gamma v (\varphi^*, \partial_3 \varphi) + \gamma I_3 + \frac{1}{\gamma}(I_1 + I_2) + \frac{1}{\gamma} V.$$

In particular,

$$\mathcal{E}_0 \equiv \mathcal{E}(\psi_0) = I_1 + I_2 + I_3 + V = 3I_3 + V, \tag{3.16}$$

since $I_1 = I_2 = I_3$. Applying equalities (2.7) and (2.13), one obtains

$$\omega(\varphi^*, (v + \alpha_3)\varphi) = v\,\omega Q + \omega(\varphi^*, \alpha_3 \varphi) = v(V + 2I_3) - i(\varphi^*, \partial_3 \varphi).$$

Therefore,

$$\begin{aligned}
\mathcal{E}_{\mathbf{v}} &= v\gamma \Big(vV + 2vI_3 - i(\varphi^*, \partial_3 \varphi) \Big) - i\gamma v (\varphi^*, \partial_3 \varphi) + \gamma I_3 + \tfrac{2}{\gamma} I_3 + \tfrac{1}{\gamma} V \\
&= \gamma \mathcal{E}_0 - 2\gamma v i (\varphi^*, \partial_3 \varphi).
\end{aligned}$$

Hence identity (3.15) holds iff $(\varphi^*, \partial_3 \varphi) = 0$ what follows from condition **C1**.

In the general case of $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$, we substitute $\psi_{\mathbf{v}}$ from (3.9) in (1.3), apply equalities (3.7), change variables $x \to y := x + \mathbf{v}(\gamma - 1)x \cdot \mathbf{v}/|\mathbf{v}|^2 - \gamma \mathbf{v}t$, use formulas (2.7) and (2.13) and obtain $\mathcal{E}_{\mathbf{v}} = \gamma \mathcal{E}_0 + \eta_{\mathbf{v}}$, where, by definition,

$$\eta_{\mathbf{v}} := -2i\gamma(\varphi^*, \nabla\varphi) \cdot \mathbf{v} - i\gamma \sum_{j,k:j\neq k} (\varphi^*, \alpha_k \partial_j \varphi) v_k v_j. \tag{3.17}$$

Since, by conditions **C1** and **C2**, $\eta_{\mathbf{v}} = 0$, then identity (3.15) follows. ■

Let $\mathbf{v} \in \mathbb{R}^3$ and $\psi_{\mathbf{v}}$ be of the form (3.9). Write $P_{\mathbf{v}} := P(\psi_{\mathbf{v}})$, where $P(\psi)$ stands for the momentum operator,

$$P(\psi) := -i \int \psi^*(t, x) \nabla \psi(t, x)\, dx.$$

To prove the next result for $P_{\mathbf{v}}$, we impose conditions **C1'** and **C2'** which are stronger than conditions **C1** and **C2**.

C1' $\displaystyle\int \varphi^* \nabla \varphi \, dx = 0.$

C2' For $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ and any $j = 1,2,3$, $\displaystyle\sum_{k:k\neq j} v_k \int \varphi^* \alpha_k \partial_j \varphi \, dx = 0.$

Lemma 3.3. *Let φ be a solution to Eqn (2.1) and conditions* **C1'** *and* **C2'** *hold. Then*

$$P_{\mathbf{v}} = \gamma \mathbf{v} \mathcal{E}_0, \tag{3.18}$$

where $\mathcal{E}_0$ is defined in (3.16).

Proof By (3.7) and (3.9), we have

$$P_{\mathbf{v}} = -i \int \varphi^* \gamma(\alpha \cdot \mathbf{v} + I)\Big(i\omega\gamma\mathbf{v}\varphi + \nabla\varphi + \mathbf{v}\kappa\nabla\varphi \cdot \mathbf{v}\Big)\, dx, \quad \text{with } \kappa := \frac{\gamma - 1}{|\mathbf{v}|^2},$$

where $\varphi \equiv \varphi(y)$ with $y := x + \mathbf{v}\kappa(x \cdot \mathbf{v}) - \gamma \mathbf{v} t$. Since $dy = \gamma dx$, changing variables $x \to y$ gives

$$P_{\mathbf{v}} = -i \int \varphi^* (\alpha \cdot \mathbf{v} + I)\Big(i\omega\gamma\mathbf{v}\varphi + \nabla\varphi + \mathbf{v}\kappa\nabla\varphi \cdot \mathbf{v}\Big)\, dy.$$

Using (2.5) and (2.13), we obtain $P_{\mathbf{v}} = \gamma \mathbf{v} \mathcal{E}_0 - i\xi_{\mathbf{v}}$, where, by definition,

$$\xi_{\mathbf{v}} = \mathbf{v}\Big(\gamma + \kappa\Big) \int \varphi^* \nabla\varphi \cdot \mathbf{v}\, dy + \int \varphi^* \nabla \varphi\, dy + \mathbf{v}\kappa \sum_{k,j:\, k\neq j} \int \varphi^* \alpha_k \partial_j \varphi \, dy\, v_k v_j$$
$$+ \Big(\sum_{k\neq 1} v_k \int \varphi^* \alpha_k \partial_1 \varphi\, dy, \sum_{k\neq 2} v_k \int \varphi^* \alpha_k \partial_2 \varphi\, dy, \sum_{k\neq 3} v_k \int \varphi^* \alpha_k \partial_3 \varphi\, dy\Big). \tag{3.19}$$

In particular, if $\mathbf{v} = (0,0,v) \in \mathbb{R}^3$,

$$\xi_{\mathbf{v}} = \Big(\int \varphi^*(v\alpha_3 + 1)\partial_1\varphi \, dy, \int \varphi^*(v\alpha_3 + 1)\partial_2\varphi \, dy, \gamma(v^2+1) \int \varphi^*\partial_3\varphi \, dy \Big).$$

By conditions **C1'** and **C2'**, $\xi_{\mathbf{v}} = 0$. Hence, identity (3.18) holds. ■

Remark 3.4. (i) Conditions **C1'** and **C2'** (and also **C1** and **C2**) are fulfilled with any $\mathbf{v} \in \mathbb{R}^3$ for four families of solutions considered in Section 2.1, see formulas (2.16) and (2.17).
(ii) Let $\psi_{\mathbf{v}}(t,x)$ be of the form (3.9) and condition **C1'** hold. Then, by (3.7), the charge functional is

$$\begin{aligned} Q(\psi_{\mathbf{v}}) &= \int \psi^*_{\mathbf{v}}(t,x)\psi_{\mathbf{v}}(t,x) \, dx = \int \varphi^*(y)(\alpha \cdot \mathbf{v} + I)\varphi(y) \, dy \\ &= (\varphi^*, \varphi) + (\varphi^*, \alpha\varphi) \cdot \mathbf{v} = \int |\varphi(y)|^2 \, dy. \end{aligned}$$

The last equality follows from (2.13) and condition **C1'**. Hence, $Q(\psi_{\mathbf{v}}) = Q(\psi_0)$ for any $\mathbf{v} \in \mathbb{R}^3$.
(iii) Let $\varphi = \varphi_\omega$ be a solution of Eqn (2.1) from one of four families of solutions considered in Remark 2.1. Then applying the total angular momentum operator M_3 to ψ_v, we have

$$M_3\psi_v = e^{-i\omega\gamma(t - vx_3)} S_v M_3 \varphi(x_1, x_2, \gamma(x_3 - vt)).$$

Hence, if $\varphi \in \{\varphi^1, \varphi^2\}$ (see Remark 2.1), then $M_3\psi_v = 1/2\psi_v$. For $\varphi \in \{\varphi^3, \varphi^4\}$, $M_3\psi_v = -1/2\psi_v$.

4. Solitary waves in $1+1$ dimensions

We consider the nonlinear Dirac equation in $\mathbb{R}^1$,

$$i\dot{\psi} = -i\alpha\psi' + m\beta\psi - \beta g(\bar{\psi}\psi)\psi, \quad x \in \mathbb{R}^1, \quad t \in \mathbb{R}. \tag{4.1}$$

Here $\psi' := \partial_x\psi$, $\psi(t,x) \in \mathbb{C}^2$, $\alpha = -\sigma_2$, $\beta = \sigma_3$. In the case when $g(s) = s$, Eqn (4.1) is called *the massive Gross–Neveu model* (or the 1D Soler model). The stationary states or localized solutions of (4.1) are the solutions of the form $\psi(t,x) = e^{-i\omega t}\varphi_\omega$, $\omega \in (0,m)$, such that $\varphi_\omega \in H^1(\mathbb{R}^1)$, and $\varphi \equiv \varphi_\omega$ is a nonzero localized solution of the following stationary nonlinear Dirac equation

$$i\alpha\varphi' + \omega\varphi - m\beta\varphi + g(\bar{\varphi}\varphi)\beta\varphi = 0, \quad x \in \mathbb{R}^1. \tag{4.2}$$

The solitary wave solutions have been studied, e.g., in [20, 21]. Write

$$I = -i\int_{\mathbb{R}^1} \varphi^*\alpha\varphi'\,dy, \quad Q = \int_{\mathbb{R}^1} \varphi^*\varphi\,dx, \quad V = \int_{\mathbb{R}^1} (m\bar\varphi\varphi - G(\bar\varphi\varphi))\,dy.$$

Note that

$$\omega Q = V. \tag{4.3}$$

This equality can be proved similarly to (2.7).

For $v \in \mathbb{R}^1$, $|v| < 1$, introduce the "moving solitary waves"

$$\psi_v(t,x) = e^{-i\omega\gamma(t-vx)} S_v \varphi(\gamma(x-vt)), \quad S_v = \sqrt{\frac{\gamma+1}{2}}\left(I + \alpha\frac{v\gamma}{\gamma+1}\right), \ \ x \in \mathbb{R}^1.$$

Note that $\alpha^* = \alpha$, $\beta^* = \beta$, $\alpha^2 = \beta^2 = I$, $\alpha\beta + \beta\alpha = 0$. Hence $S_v^*\beta S_v = \beta$, $S_v^* S_v = \gamma(v\alpha + I)$, $S_v^*\alpha S_v = \gamma(vI + \alpha)$. Consider

$$\mathcal{E}_v := \mathcal{E}(\psi_v) = \int_{\mathbb{R}^1} \left(-i\psi_v^*\alpha\psi_v' + m\bar\psi_v\psi_v - G(\bar\psi_v\psi_v)\right)\,dx.$$

Using the properties S_v, we obtain

$$\begin{aligned}
\mathcal{E}_v &= \int_{\mathbb{R}^1} \Big(-i\varphi^* S_v^*\alpha S_v(i\omega\gamma v\varphi + \gamma\varphi') + m\bar\varphi\varphi - G(\bar\varphi\varphi)\Big)\Big|_{\varphi=\varphi(\gamma(x-vt))} dx \\
&= -i\gamma\int_{\mathbb{R}^1} \varphi^*(y)(vI+\alpha)(i\omega v\varphi(y) + \varphi'(y))dy + \frac{1}{\gamma}V.
\end{aligned}$$

In the last integral we changed variable $x \to y = \gamma(x - vt)$. Hence,

$$\mathcal{E}_v = \gamma v^2\omega Q + \gamma v\omega(\varphi^*, \alpha\varphi) - i\gamma v(\varphi^*, v\varphi') + \gamma I + \frac{1}{\gamma}V.$$

In particular,

$$\mathcal{E}_0 = \int_{\mathbb{R}^1} (-i\varphi^*\alpha\varphi + m\bar\varphi\varphi - G(\bar\varphi\varphi))\,dx = I + V.$$

We apply equalities (4.3) and $\omega(\varphi^*, \alpha\varphi) = -i(\varphi^*, \varphi')$ (cf (2.13)) and obtain

$$\mathcal{E}_v = \gamma\mathcal{E}_0 - 2i\gamma v(\varphi^*, \varphi').$$

Assuming that φ satisfies the property $(\varphi^*, \varphi') = 0$ (cf condition **C1** or **C1'**), we have $\mathcal{E}_v = \gamma\mathcal{E}_0$. Moreover, under the same condition on φ, one obtains

$$P_v = -i\int_{\mathbb{R}^1} \psi_v^*(t,x)\psi_v'(t,x)dx = \gamma v\mathcal{E}_0 - \gamma i(v^2+1)\int_{\mathbb{R}^1} \varphi^*(x)\varphi'(x)\,dx = \gamma v\mathcal{E}_0.$$

5. Maxwell–Dirac equations

We use natural units, in which we have rescaled length and time so that $\hbar = c = e = 1$. Then, in the Lorentz gauge, the (MD) system reads

$$i\dot{\psi} = \Phi\psi - i\alpha\cdot\nabla\psi - \alpha\cdot\mathbf{A}\psi + m\beta\psi, \quad x\in\mathbb{R}^3, \quad t\in\mathbb{R}, \tag{5.1}$$

$$\left.\begin{aligned}\ddot{\Phi} - \Delta\Phi &= 4\pi\rho \\ \ddot{\mathbf{A}} - \Delta\mathbf{A} &= 4\pi\mathbf{J}\end{aligned}\right| \tag{5.2}$$

$$\nabla\cdot\mathbf{A} + \dot{\Phi} = 0. \tag{5.3}$$

Here ψ describes the charged Dirac spinor, $\psi \equiv \psi(t,x) \in \mathbb{C}^4$ for $(t,x) \in \mathbb{R}^3\times\mathbb{R}$, $\mathbf{A} \equiv \mathbf{A}(t,x) = (A^1, A^2, A^3)$ and $\Phi \equiv \Phi(t,x)$ are the classical electromagnetic potentials, $m > 0$, $\rho \equiv \rho(t,x)$ is charge density, $\mathbf{J} \equiv \mathbf{J}(t,x)$ is electric current. By definition, $\rho = \psi^*\psi$, $\mathbf{J} = \psi^*\alpha\psi$, β, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ are Pauli–Dirac matrices. We also introduce notation $J = (\rho, \mathbf{J})$ for the 4-electromagnetic current ($J^\mu = \bar{\psi}\gamma^\mu\psi$), and $A = (A^\mu) = (\Phi, \mathbf{A})$ for the 4-potential of the electromagnetic field. The equation (5.3) is called the Lorentz gauge condition.

The magnetic and electric fields $\mathbf{H} \equiv \mathbf{H}(t,x)$ and $\mathbf{E} \equiv \mathbf{E}(t,x)$ are given by

$$\mathbf{H} = \mathrm{rot}\mathbf{A} \equiv \nabla\times\mathbf{A}, \quad \mathbf{E} = -\dot{\mathbf{A}} - \nabla\Phi. \tag{5.4}$$

Then, by condition (5.3), equations (5.2) become classical Maxwell's equations of electrodynamics

$$\dot{\mathbf{H}} = -\mathrm{rot}\mathbf{E}, \quad \dot{\mathbf{E}} = \mathrm{rot}\mathbf{H} - 4\pi\mathbf{J}, \quad \nabla\cdot\mathbf{E} = 4\pi\rho, \quad \nabla\cdot\mathbf{H} = 0.$$

As shown, e.g., in [4, 30], this model is based on the Lagrangian density $\mathcal{L}_Q = \mathcal{L}_D + \mathcal{L}_M + \mathcal{L}_I$. Here $\mathcal{L}_D$ and $\mathcal{L}_M$ are Lagrangian densities for the free Dirac and electromagnetic fields, resp., $\mathcal{L}_I$ is extra term describing the interaction between ψ and the electromagnetic field,

$$\begin{aligned}&\mathcal{L}_D = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi, \quad \mathcal{L}_M = -\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} = \frac{1}{8\pi}(\mathbf{E}^2 - \mathbf{H}^2),\\ &\mathcal{L}_I = -J_\mu A^\mu \equiv -\rho\Phi + \mathbf{J}\cdot\mathbf{A},\end{aligned}$$

where $F_{\mu\nu}$ stands for the electromagnetic field tensor, $F_{\mu\nu} := \partial_\nu A_\mu - \partial_\mu A_\nu$, $\partial_\mu = \partial/\partial x^\mu$, $\mu,\nu = 0,1,2,3$. Other words,

$$\begin{aligned}\mathcal{L}_Q \equiv \mathcal{L}_Q(\psi, A) \;&=\; \psi^*\Big(i\partial_t + i\alpha\cdot\nabla - \Phi + \alpha\cdot\mathbf{A} - m\beta\Big)\psi \\ &\quad + \frac{1}{8\pi}\left(|\dot{\mathbf{A}} + \nabla\Phi|^2 - |\mathrm{rot}\mathbf{A}|^2\right).\end{aligned} \tag{5.5}$$

It is easy to check that the Euler–Lagrange equations applied to (5.5) give Eqn (5.1) and $(\partial_t^2 - \Delta)A^\mu - \partial^\mu(\partial_\nu A^\nu) = J^\mu$. Due to the Lorentz gauge (5.3), we obtain Eqn (5.2).

Since $\partial\mathcal{L}_Q/(\partial\dot{\Phi}) = 0$, the Hamiltonian density equals

$$\begin{aligned}\mathcal{H}(\psi, A) &= \tfrac{\partial\mathcal{L}_Q}{\partial\dot{\psi}}\cdot\dot{\psi} + \tfrac{\partial\mathcal{L}_Q}{\partial\dot{\mathbf{A}}}\cdot\dot{\mathbf{A}} - \mathcal{L}_Q = i\psi^*\cdot\dot{\psi} + \tfrac{1}{4\pi}(\dot{\mathbf{A}} + \nabla\Phi)\cdot\dot{\mathbf{A}} - \mathcal{L}_Q \\ &= \psi^*\left[\alpha\cdot(-i\nabla - \mathbf{A}) + \Phi + m\beta\right]\psi + \tfrac{1}{4\pi}\mathbf{E}\cdot\nabla\boldsymbol{\Phi} + \tfrac{1}{8\pi}(\mathbf{E}^2 + \mathbf{H}^2).\end{aligned}$$

Hence the energy functional of the system (5.1)–(5.3) reads (cf [4])

$$\begin{aligned}\mathcal{E}(t) &\equiv \mathcal{E}(\psi(t,\cdot), A(t,\cdot)) = \int \mathcal{H}(\psi(t,\cdot), A(t,\cdot))\,dx \\ &= \int \psi^*\left[\alpha\cdot(-i\nabla - \mathbf{A})\psi + m\beta\psi\right]dx + \frac{1}{8\pi}\int(\mathbf{E}^2 + \mathbf{H}^2)\,dx, \quad (5.6)\end{aligned}$$

where $\mathbf{E}$ and $\mathbf{H}$ are defined in (5.4). Here we use the fact that $\int \mathbf{E}\cdot\nabla\Phi\,dx = -4\pi\int\rho\Phi\,dx$. Evidently, $\dot{\mathcal{E}}(t) = 0$.

5.1. Standing solitary waves. Let $\omega \in (-m, m)$. Consider a *stationary* solution (ψ, A) of system (5.1)–(5.3) such that $\psi(t,x) = e^{-i\omega t}\varphi(x)$ and $A = (\Phi, \mathbf{A})$ does not depend on t. Such stationary solutions we denote by (ψ_0, A_0). Substituting these solutions in system (5.1)–(5.2) we obtain

$$(-\omega + \Phi_0 - i\alpha\cdot\nabla - \alpha\cdot\mathbf{A}_0 + m\beta)\,\varphi = 0, \quad x \in \mathbb{R}^3, \quad (5.7)$$

$$\left.\begin{aligned}-\Delta\Phi_0 &= 4\pi\rho_0 = 4\pi\varphi^*\varphi, \\ -\Delta\mathbf{A}_0^k &= 4\pi J_0^k = 4\pi\varphi^*\alpha_k\varphi, \quad k = 1,2,3\end{aligned}\right| \quad (5.8)$$

By (5.8), $A_0^\mu = \varphi^*\alpha_\mu\varphi * (1/|x|)$ (with $\alpha_0 \equiv I$), $\mu = 0,1,2,3$, i.e.,

$$\Phi_0(x) = \int \frac{\rho_0(y)}{|x-y|}\,dy, \quad \mathbf{A}_0(x) = \int \frac{\mathbf{J}_0(y)}{|x-y|}\,dy, \quad \text{with } \rho_0 = |\varphi|^2, \quad \mathbf{J}_0 = \varphi^*\alpha\varphi. \quad (5.9)$$

Note that the Lorentz condition (5.3) becomes

$$\nabla\cdot\mathbf{A}_0 = 0, \quad (5.10)$$

what follows from (5.7) and (5.9). Using (5.8) and (5.10), we rewrite the energy associated with stationary states (ψ_0, A_0) as

$$\begin{aligned}\mathcal{E}_0 &:= \mathcal{E}(\psi_0, A_0) = \int \varphi^*\left[\alpha\cdot(-i\nabla - \mathbf{A}_0) + m\beta\right]\varphi\,dx \\ &\quad + \frac{1}{8\pi}\int\left(|\nabla\Phi_0|^2 + |\mathrm{rot}\mathbf{A}_0|^2\right)dx \\ &= \int \varphi^*\left[-i\alpha\cdot\nabla + m\beta\right]\varphi\,dx + \frac{1}{2}\int(\rho_0\Phi_0 - \mathbf{J}_0\cdot\mathbf{A}_0)\,dx. \quad (5.11)\end{aligned}$$

The last integral in (5.11) is $\frac{1}{2}\int J_\mu(x)A_0^\mu(x)\,dx = \frac{1}{2}\iint \frac{J_\mu(x)J^\mu(y)}{|x-y|}\,dxdy.$

Definition 5.1. *The stationary states or standing waves* $(\psi_0(t,x), A_0(x)) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^4 \times \mathbb{R}^4$ *are the solutions of the (MD) system of a form*

$$\begin{aligned} \psi_0(x,t) &= e^{-i\omega t}\varphi_\omega(x), \\ A_0^\mu(x) &= J^\mu * \frac{1}{|x|} = \int \frac{J^\mu(y)}{|x-y|}\,dy, \quad \mu = 0,1,2,3. \end{aligned} \tag{5.12}$$

Here $\omega \in (-m,m)$, $(J^\mu) = (\varphi_\omega^* \alpha_\mu \varphi_\omega) = (\rho_0, \mathbf{J}_0)$, *and* $\varphi \equiv \varphi_\omega$ *is a solution of (5.7).*

The stationary solutions of the (MD) system were studied numerically by Lisi [22]. Using variational methods, Esteban Georgiev and Séré [15] have proved the existence of stationary solutions with $\omega \in (-m,0)$. To state this result we introduce a functional

$$\begin{aligned} I_Q^\omega(\varphi) &:= \frac{1}{2}\int \mathcal{L}_Q(\psi_0, A_0)\,dx \\ &= \frac{1}{2}\int \varphi^*\left(i\alpha\cdot\nabla - m\beta + \omega\right)\varphi\,dx - \frac{1}{4}\iint \frac{J_\mu(x)J^\mu(y)}{|x-y|}\,dxdy. \end{aligned}$$

Note that if (ψ_0, A_0) is a solution of the (MD) system of the form (5.12), then (formally) φ_ω is a critical point of $I_Q^\omega(\varphi)$.

Theorem 5.2. *(see [15, Theorem 1]) For any* $\omega \in (-m,0)$, *there exists a non-zero critical point* $\varphi \equiv \varphi_\omega \in H^{1/2}(\mathbb{R}^3;\mathbb{C}^4)$ *of the functional* $I_Q^\omega(\varphi)$. *Moreover,* φ_ω *is a smooth function of* x *exponentially decreasing at infinity with all its derivatives, and* $\psi(x,t) = e^{-i\omega t}\varphi_\omega(x)$, $A^\mu(x,t) = J^\mu * (1/|x|)$ *are the solutions of the (MD) system.*

5.2. Virial identities.

The following virial identity was proved in [17, Proposition 3.1].

Lemma 5.3. *Let* $\varphi \in H^1(\mathbb{R}^3;\mathbb{C}^4)$ *be a solution to Eqn (5.7). Then* $\varphi(x)$ *satisfies*

$$i\int \varphi^*\alpha\cdot\nabla\varphi\,dx = \frac{3}{2}\int\left(m\bar\varphi\varphi - \omega\varphi^*\varphi + \frac{5}{6}J_\mu(x)A^\mu(x)\right)dx, \tag{5.13}$$

where $J_\mu A^\mu = \rho_0\Phi_0 - \mathbf{J}_0\cdot\mathbf{A}_0$, $\rho_0 = |\varphi|^2$, $\mathbf{J}_0 = \varphi^*\alpha\varphi$.

Let functionals $I_k(\varphi)$, $k = 1,2,3$, and $Q(\varphi)$ be as in (2.6). Also, we put

$$\begin{aligned} T &\equiv T(\varphi) = \int\left(\rho_0(x)\Phi_0(x) - \mathbf{J}_0(x)\cdot\mathbf{A}_0(x)\right)dx \\ &= (2\pi)^{-3}4\pi\int \frac{|\hat\rho_0(k)|^2 - |\mathbf{J}_0(k)|^2}{k^2}\,dk, \\ m_0 &\equiv m_0(\varphi) = m\int \bar\varphi\varphi\,dx. \end{aligned} \tag{5.14}$$

Remark 5.4. Formally, the identity (5.13) can be proved used Derrick's technique [10, p.1253]. Indeed, using notations (5.14), we rewrite $I_Q^\omega(\varphi)$ as

$$I_Q^\omega(\varphi) = \frac{1}{2}\Big(\omega Q(\varphi) - m_0(\varphi) - I_1(\varphi) - I_2(\varphi) - I_3(\varphi)\Big) - \frac{1}{4}T(\varphi), \qquad (5.15)$$

and introduce $\varphi_\lambda(x) = \varphi(x/\lambda)$. Then $T(\varphi_\lambda) = \lambda^5 T(\varphi)$, $I_k(\varphi_\lambda) = \lambda^2 I_k(\varphi)$, $Q(\varphi_\lambda) = \lambda^3 Q(\varphi)$, $m_0(\varphi_\lambda) = \lambda^3 m_0(\varphi)$. Hence,

$$\begin{aligned} 0 &= \frac{d}{d\lambda}\Big|_{\lambda=1} I_Q^\omega(\varphi_\lambda) = \frac{1}{2}\frac{d}{d\lambda}\Big|_{\lambda=1}\Big[\omega Q(\varphi_\lambda) - m_0(\varphi_\lambda) - I_1(\varphi_\lambda) - I_2(\varphi_\lambda) - I_3(\varphi_\lambda) \\ &\quad -\frac{1}{2}T(\varphi_\lambda)\Big] = \frac{3}{2}\Big(\omega Q(\varphi) - m_0(\varphi)\Big) - I_1(\varphi) - I_2(\varphi) - I_3(\varphi) - \frac{5}{4}T(\varphi). \end{aligned}$$

Hence,

$$I_1 + I_2 + I_3 = \frac{3}{2}(\omega Q - m_0) - \frac{5}{4}T, \qquad (5.16)$$

and the identity (5.13) holds.

Corollary 5.5. *(i) Eqn (5.7) implies the following equality,*

$$\omega Q - m_0 - (I_1 + I_2 + I_3) = T$$

(cf [1, p.238] or formula (3.10) in [15]). Hence, by (5.16),

$$\omega Q - m_0 = \frac{1}{2}T. \qquad (5.17)$$

Moreover,

$$I_1 + I_2 + I_3 = -\frac{1}{2}T. \qquad (5.18)$$

In particular, $I_Q^\omega(\varphi) = -\frac{1}{2}(I_1 + I_2 + I_3) = \frac{1}{4}T$, *by (5.15), (5.17) and (5.18).*
(ii) Using (5.11) and (5.14), we rewrite $\mathcal{E}_0$ *as*

$$\mathcal{E}_0 = I_1 + I_2 + I_3 + m_0 + \frac{1}{2}T. \qquad (5.19)$$

By (5.18) and (5.19), we obtain $\mathcal{E}_0 = m_0$.

Lemma 5.6. *The following identity holds,*

$$I_j = \frac{1}{2}(\omega Q - m_0) - \frac{3}{4}T + T_j, \quad j = 1,2,3, \qquad (5.20)$$

where

$$\begin{aligned} T_j &:= 4\pi(2\pi)^{-3}\int \frac{k_j^2(|\hat\rho_0(k)|^2 - |\hat{\mathbf{J}}_0(k)|^2)}{k^4}\,dk \\ &= \frac{1}{4\pi}\int \Big(|\partial_j\Phi_0(x)|^2 - |\partial_j\mathbf{A}_0(x)|^2\Big)\,dx. \end{aligned} \qquad (5.21)$$

Proof Introduce $\varphi_\lambda(x) = \varphi(x_1/\lambda, x_2, x_3)$. Then $I_1(\varphi_\lambda) = I_1(\varphi)$, $I_k(\varphi_\lambda) = \lambda I_k(\varphi)$, $k = 2,3$, $Q(\varphi_\lambda) = \lambda Q(\varphi)$, $m_0(\varphi_\lambda) = \lambda m_0(\varphi)$, and

$$T(\varphi_\lambda) = (2\pi)^{-3} 4\pi\lambda \int \frac{|\hat{\rho}_0(k)|^2 - |\hat{\mathbf{J}}_0(k)|^2}{k_1^2\lambda^{-2} + k_2^2 + k_3^2}\, dk.$$

Hence,

$$0 = \frac{d}{d\lambda}\Big|_{\lambda=1} I_Q^\omega(\varphi_\lambda) = \frac{1}{2}\Big[\omega Q - m_0 - (I_2 + I_3) - \frac{1}{2}(T + 2T_1)\Big].$$

Therefore, $I_2 + I_3 = \omega Q - m_0 - (T + 2T_1)/2$. Together with (5.16), the last identity implies (5.20) with $j = 1$. Similarly, introducing $\varphi_\lambda(x) = \varphi(x_1, x_2/\lambda, x_3)$ or $\varphi_\lambda(x) = \varphi(x_1, x_2, x_3/\lambda)$, we can verify (5.20) with $j = 2,3$. ■

Corollary 5.7. *Since $T_1 + T_2 + T_3 = T$, (5.20) implies identity (5.16). Moreover, by (5.17), we have $I_j = -T/2 + T_j$, $j = 1,2,3$.*

Remark 5.8. *(cf Lemma 2.4)* Let φ be a solution of Eqn (5.7). Then

$$i\int \varphi^*(x)\nabla\varphi(x)\,dx + \omega\int \varphi^*(x)\alpha\varphi(x)\,dx = \int \Big(\mathbf{J}_0(x)\Phi_0(x) - \rho_0(x)\mathbf{A}_0(x)\Big)\,dx.$$

Since $(\Phi_0, \mathbf{A}_0)$ is of the form (5.9), then

$$\int \mathbf{J}_0(x)\Phi_0(x)\,dx = \int \rho_0(x)\mathbf{A}_0(x)\,dx. \tag{5.22}$$

Hence, if $(\varphi, \Phi_0, \mathbf{A}_0)$ is a solution of the system (5.7)–(5.8), then (cf formula (2.5))

$$i\int \varphi^*(x)\nabla\varphi(x)\,dx = -\omega\int \varphi^*(x)\alpha\varphi(x)\,dx. \tag{5.23}$$

5.3. A particular ansatz of stationary solutions. Abenda [1, Theorem A] extended the results of Theorem 5.2 for $\omega \in (-m, m)$ and proved the existence of the particular ansatz of solutions to Eqn (5.7)–(5.8) in the form

$$\varphi_\omega(x) = \begin{pmatrix} u_1(r,z)e^{i(m_3-1/2)\phi} \\ u_2(r,z)e^{i(m_3+1/2)\phi} \\ -iu_3(r,z)e^{i(m_3-1/2)\phi} \\ -iu_4(r,z)e^{i(m_3+1/2)\phi} \end{pmatrix}, \quad \text{with } m_3 = \pm\frac{1}{2}, \tag{5.24}$$

$$\Phi_0(x) = \Phi_*(r,z), \quad \mathbf{A}_0(x) = A_*(r,z)(-\sin\phi, \cos\phi, 0), \tag{5.25}$$

where (r, z, ϕ) are the cylindric coordinates of $x \in \mathbb{R}^3$. Moreover, $u_1, u_2, u_3, u_4, \Phi_*$ and A_* are scalar real–valued smooth functions exponentially decreasing at infinity with all its derivatives. The system of equations for $u_1, u_2, u_3, u_4, \Phi_*, A_*$ was derived by Lisi [22].

Remark 5.9. *The solutions* $(\varphi_\omega, \Phi_0, \mathbf{A}_0)$ *of the form (5.24) and (5.25) have the following properties. (i)* φ_ω *are eigenfunctions of the third component of the total angular momentum* $\mathbf{M}$ *(see Section 2.1) with eigenvalues* $m_3 = \pm 1/2$.

(ii) $\displaystyle\int \varphi_\omega^*(x)\nabla\varphi_\omega(x)\,dx = 0.$

(iii) $\displaystyle\int \varphi_\omega^*(x)\alpha_k\partial_j\varphi_\omega(x)\,dx = 0$ *for* $k \neq j$*,* $k, j = 1,2,3.$

(iv) For $i \neq j$*,* $\displaystyle\int \partial_i\mathbf{\Phi}_0(x)\partial_j\mathbf{\Phi}_0(x)\,dx = 0$ *and* $\displaystyle\int \partial_i\mathbf{A}_0(x)\cdot\partial_j\mathbf{A}_0(x)\,dx = 0.$

(v) The charge density is $\rho_0 \equiv \rho_0(r,z) = u_1^2 + u_2^2 + u_3^2 + u_4^2$*, the current* $\mathbf{J}_0(x)$ *is*

$$\mathbf{J}_0(x) = \psi_0^*(t,x)\alpha\psi_0(t,x) = \varphi_\omega^*(x)\alpha\varphi_\omega(x) = 2(u_1u_4 - u_2u_3)(\sin\phi, -\cos\phi, 0).$$

Moreover, by (5.25),

$$\begin{aligned}\mathbf{E}_0(x) &= -(\cos\phi\,\partial_r\Phi_*, \sin\phi\,\partial_r\Phi_*, \partial_z\Phi_*),\\ \mathbf{H}_0(x) &= (-\cos\phi\,\partial_z A_*, -\sin\phi\,\partial_z A_*, \partial_r A_* + A_*/r).\end{aligned}$$

5.4. Moving solitary waves. Consider *travelling solutions* $(\psi_{\mathbf{v}}, A_{\mathbf{v}})$, where $A_{\mathbf{v}} = (\Phi_{\mathbf{v}}, \mathbf{A}_{\mathbf{v}})$, with velocity $\mathbf{v} \in \mathbb{R}^3$, $|\mathbf{v}| < 1$:

$$\begin{aligned}\psi_{\mathbf{v}}(t,x) &= S_{\mathbf{v}}\psi_0(\Lambda_{\mathbf{v}}^{-1}(t,x)),\\ A_{\mathbf{v}}(t,x) &= \Lambda_{\mathbf{v}}A_0(y) \quad \text{with } y = x + \mathbf{v}\frac{(\gamma-1)}{|\mathbf{v}|^2}x\cdot\mathbf{v} - \gamma\mathbf{v}t.\end{aligned} \tag{5.26}$$

Here the stationary solutions (ψ_0, A_0) are introduced in Definition 5.1, $S_{\mathbf{v}}$ is defined in (3.6), $\Lambda_{\mathbf{v}}$ is a Lorentz transformation defined in (3.4). It is easily to check that $(\psi_{\mathbf{v}}, A_{\mathbf{v}})$ is a solution of the (MD) system. Indeed, first, similarly to Remark 3.1 (i), we obtain

$$i\dot\psi_{\mathbf{v}} + i\alpha\cdot\nabla\psi_{\mathbf{v}} - m\beta\psi_{\mathbf{v}} = e^{-i\omega\gamma(t-\mathbf{v}\cdot x)}S_{\mathbf{v}}^{-1}\left(\omega + \alpha\cdot\nabla - m\beta\right)\varphi(y).$$

Here and below y stands for the expression $y = x + \mathbf{v}(\gamma-1)x\cdot\mathbf{v}/|\mathbf{v}|^2 - \gamma\mathbf{v}t$ (as in (5.26)). On the other hand, $S_{\mathbf{v}}(\Phi_{\mathbf{v}}(t,x) - \alpha\cdot\mathbf{A}_{\mathbf{v}}(t,x))S_{\mathbf{v}} = \Phi_0(y) - \alpha\cdot\mathbf{A}_0(y)$, hence

$$(\Phi_{\mathbf{v}}(t,x) - \alpha\cdot\mathbf{A}_{\mathbf{v}}(t,x))\psi_{\mathbf{v}}(t,x) = e^{-i\omega\gamma(t-\mathbf{v}\cdot x)}S_{\mathbf{v}}^{-1}(\Phi_0(y) - \alpha\cdot\mathbf{A}_0(y))\varphi(y),$$

and Eqn (5.1) follows. To verify Eqn (5.2), we put $J_{\mathbf{v}}^\mu = \psi_{\mathbf{v}}\alpha_\mu\psi_{\mathbf{v}}$. Then, by (5.26), (5.8), and Remark 3.1 (ii), one obtains

$$(\partial_t^2 - \Delta)A_{\mathbf{v}}(t,x) = \Lambda_{\mathbf{v}}(\partial_t^2 - \Delta_x)A_0(y) = \Lambda_{\mathbf{v}}(-\Delta_y A_0(y)) = 4\pi\Lambda_{\mathbf{v}}J_0(y) = 4\pi J_{\mathbf{v}}(t,x),$$

and Eqn (5.2) follows. Moreover, $\dot{\Phi}_{\mathbf{v}}(t,x) + \nabla_x\cdot\mathbf{A}_{\mathbf{v}}(t,x) = \nabla_y\cdot\mathbf{A}_0(y) = 0$, i.e., the Lorentz gauge condition (5.3) is fulfilled.

Remark 5.10. *Denote* $\mathbf{E}_0 = -\nabla\Phi_0$, $\mathbf{H}_0 = \nabla \times \mathbf{A}_0$, *and* $\mathbf{E}_\mathbf{v} = -\dot{\mathbf{A}}_\mathbf{v} - \nabla\Phi_0$, $\mathbf{H}_\mathbf{v} = \nabla \times \mathbf{A}_\mathbf{v}$, $\mathbf{v} \in \mathbb{R}^3$. *Then*

$$\left.\begin{array}{rcl} \mathbf{E}_\mathbf{v}(t,x) &=& \gamma\mathbf{E}_0(y) - \mathbf{v}\kappa\mathbf{v}\cdot\mathbf{E}_0(y) - \gamma\mathbf{v}\times\mathbf{H}_0(y) \\ \mathbf{H}_\mathbf{v}(t,x) &=& \gamma\mathbf{H}_0(y) - \mathbf{v}\kappa\mathbf{v}\cdot\mathbf{H}_0(y) + \gamma\mathbf{v}\times\mathbf{E}_0(y) \end{array}\right| \quad \kappa := \frac{\gamma-1}{|\mathbf{v}|^2}. \qquad (5.27)$$

We impose conditions **C1** and **C2** on φ_ω (see Section 3). Moreover, we assume the additional condition **C0** on $(\Phi_0, \mathbf{A}_0)$.

C0 For $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$, the following relation holds

$$\sum_{i,j:\, i\neq j} v_i v_j \int \Big(\partial_i \mathbf{\Phi}_0(x)\partial_j\mathbf{\Phi}_0(x) - \partial_i\mathbf{A}_0(x)\cdot\partial_j\mathbf{A}_0(x)\Big)\, dx = 0.$$

Remark 5.11. *(i) By Fourier transform and formulas (5.8), condition* **C0** *can be rewritten in the form*

$$\sum_{i,j:\, i\neq j} v_i v_j \int \frac{k_i k_j}{k^4}\Big(|\hat\rho_0(k)|^2 - |\hat{\mathbf{J}}_0(k)|^2\Big)\, dk = 0.$$

(ii) Conditions **C0**–**C2** *are fulfilled, for instance, for the particular family of solutions considered in Section 5.3 (see Remark 5.9 (ii)–(iv)). Obviously, conditions* **C0** *and* **C2** *are valid in the particular case when* $\mathbf{v} = (0,0,v)$.

Put $\mathcal{E}_\mathbf{v} = \mathcal{E}(\psi_\mathbf{v}, A_\mathbf{v})$, $\mathbf{v} \in \mathbb{R}^3$, where $\mathcal{E}$ is defined in (5.6), i.e.,

$$\mathcal{E}_\mathbf{v} = \int \Big(\psi^*_\mathbf{v}(-i\alpha\cdot\nabla + m\beta)\psi_\mathbf{v} - \mathbf{J}_\mathbf{v}\cdot\mathbf{A}_\mathbf{v}\Big)\, dx + \frac{1}{8\pi}\int(\mathbf{E}^2_\mathbf{v} + \mathbf{H}^2_\mathbf{v})\, dx.$$

Then the following result holds.

Theorem 5.12. *Let* $(\psi_\mathbf{v}, A_\mathbf{v})$ *be a solitary wave of the (MD) system and conditions* **C0**–**C2** *hold. Then the "particle-like" energy relation holds,* $\mathcal{E}_\mathbf{v} = \gamma\mathcal{E}_0$.

Proof First we rewrite the term in $\mathcal{E}_\mathbf{v}$ corresponding to the Dirac field,

$$\begin{aligned} e_D &:= \int \Big(\psi^*_\mathbf{v}(-i\alpha\cdot\nabla + m\beta)\psi_\mathbf{v}\, dx \\ &= \int \Big(-i\varphi^* S_\mathbf{v}\alpha\cdot S_\mathbf{v}\Big(i\gamma\omega\mathbf{v}\varphi + \nabla\varphi + \mathbf{v}\frac{\gamma-1}{|\mathbf{v}|^2}\nabla\varphi\cdot\mathbf{v}\Big) + m\varphi^* S_\mathbf{v}\beta S_\mathbf{v}\varphi\Big) dx \\ &= \int \Big(\gamma^2\omega\varphi^*[\alpha\cdot\mathbf{v}+\mathbf{v}^2]\varphi - i\varphi^*\left[\gamma^2(\alpha\cdot\mathbf{v}+1)\mathbf{v}\cdot\nabla\varphi + \alpha\cdot\nabla\varphi\right] + m\bar\varphi\varphi\Big) dx, \end{aligned}$$

where $\varphi \equiv \varphi(y)$ with y from (5.26). Here we apply formulas (3.7) and (3.8). We change variables $x \to y = x + \mathbf{v}(\gamma-1)x\cdot\mathbf{v}/|\mathbf{v}|^2 - \gamma\mathbf{v}t$, $dx = dy/\gamma$. Using (5.23), we obtain

$$\begin{aligned} e_D &= \omega\gamma(\varphi^*, [\alpha\cdot\mathbf{v}+\mathbf{v}^2]\varphi) - i\gamma(\varphi^*, (\alpha\cdot\mathbf{v}+1)\mathbf{v}\cdot\nabla\varphi) + \frac{1}{\gamma}(I_1+I_2+I_3) + \frac{1}{\gamma}m_0 \\ &= \omega Q\gamma\mathbf{v}^2 + \frac{1}{\gamma}(I_1+I_2+I_3) + \frac{1}{\gamma}m_0 + \gamma\sum_{j=1}^{3} v_j^2 I_j + \eta_{\mathbf{v}}, \end{aligned} \tag{5.28}$$

where $\eta_{\mathbf{v}}$ is defined in (3.17). Applying 'virial' identities (5.16) and (5.20), we obtain $e_D = \gamma(I_1+I_2+I_3+m_0) + \frac{1}{2}\gamma\mathbf{v}^2 T + \gamma\sum\limits_{j=1}^{3} v_j^2 T_j + \eta_{\mathbf{v}}$. Moreover, by (5.18),

$$e_D = \gamma m_0 - \frac{1}{2\gamma}T + \gamma\sum_{j=1}^{3} v_j^2 T_j + \eta_{\mathbf{v}}. \tag{5.29}$$

Second, we rewrite the "magnetic" term in $\mathcal{E}_{\mathbf{v}}$, i.e., the term corresponding to the interaction. Since

$$\left.\begin{array}{rcl} \mathbf{A}_{\mathbf{v}}(t,x) &=& \gamma\mathbf{v}\Phi_0(y) + \mathbf{A}_0(y) + \mathbf{v}\kappa\,\mathbf{A}_0(y)\cdot\mathbf{v} \\ \mathbf{J}_{\mathbf{v}}(t,x) &\equiv& \psi_{\mathbf{v}}\alpha\psi_{\mathbf{v}} = \gamma\mathbf{v}\rho_0(y) + \mathbf{J}_0(y) + \mathbf{v}\kappa\,\mathbf{J}_0(y)\cdot\mathbf{v} \end{array}\right| \quad \kappa = \frac{\gamma-1}{|\mathbf{v}|^2},$$

then, by (5.22), we have

$$\begin{aligned} e_I &= -\int \mathbf{A}_{\mathbf{v}}(t,x)\cdot\mathbf{J}_{\mathbf{v}}(t,x)dx \qquad (5.30) \\ &= -\int \Big(\gamma\mathbf{v}^2\Phi_0(y)\rho_0(y) + \frac{1}{\gamma}\mathbf{A}_0(y)\cdot\mathbf{J}_0(y) + \gamma(\mathbf{v}\cdot\mathbf{A}_0(y))(\mathbf{v}\cdot\mathbf{J}_0(y))\Big)\,dy - R_{\mathbf{v}}, \end{aligned}$$

where $R_{\mathbf{v}}$ stands for the integral $R_{\mathbf{v}} = 2\gamma\displaystyle\int \rho_0(y)\mathbf{v}\cdot\mathbf{A}_0(y)\,dy$.

Further, using (5.27), we rewrite the energy corresponding to the electromagnetic field,

$$\begin{aligned} e_M := \frac{1}{8\pi}\int(\mathbf{E}_{\mathbf{v}}^2 + \mathbf{H}_{\mathbf{v}}^2)\,dx = \frac{4\gamma}{8\pi}\int \mathbf{v}\cdot(\mathbf{H}_0\times\mathbf{E}_0)\,dy + \frac{\gamma}{8\pi}\int\Big(\mathbf{E}_0^2 + \mathbf{H}_0^2 + \\ +(\mathbf{v}\times\mathbf{E}_0)^2 + (\mathbf{v}\times\mathbf{H}_0)^2 - (\mathbf{v}\cdot\mathbf{E}_0)^2 - (\mathbf{v}\cdot\mathbf{H}_0)^2\Big)\,dy. \end{aligned} \tag{5.31}$$

Since $\displaystyle\int \mathbf{H}_0\times\mathbf{E}_0\,dy = 4\pi\int \rho_0\mathbf{A_0}\,dy$, the first term in the r.h.s. of (5.31) equals $R_{\mathbf{v}}$. Using the formula $|a|^2|b|^2 = (a\cdot b)^2 + (a\times b)^2$ for $a, b\in\mathbb{R}^3$, the second term in e_M can be rewritten as

$$\frac{\gamma(1+\mathbf{v}^2)}{8\pi}\int(\mathbf{E}_0^2+\mathbf{H}_0^2)\,dy-\frac{\gamma}{4\pi}\int\Big((\mathbf{v}\cdot\mathbf{E}_0)^2+(\mathbf{v}\cdot\mathbf{H}_0)^2\Big)\,dy.$$

Using formulas $\mathbf{E}_0=-\nabla\Phi_0$, $\mathbf{H}_0=\nabla\times\mathbf{A}_0$ and (5.8), we obtain

$$e_M=\frac{\gamma(1+\mathbf{v}^2)}{2}\int\Big(\rho_0\Phi_0+\mathbf{J}_0\cdot\mathbf{A}_0\Big)\,dy-\frac{\gamma}{4\pi}\int\Big((\mathbf{v}\cdot\nabla\Phi_0)^2+(\mathbf{v}\cdot(\nabla\times\mathbf{A}_0))^2\Big)dy+R_{\mathbf{v}}. \tag{5.32}$$

Finally, substituting (5.29), (5.30) and (5.32) in $\mathcal{E}_{\mathbf{v}}=e_D+e_I+e_M$ and using notations (5.14) and (5.21), we have

$$\begin{aligned}
\mathcal{E}_{\mathbf{v}}=&\,\gamma m_0-\frac{1}{2\gamma}\int\Big(\rho_0\Phi_0-\mathbf{J}_0\cdot\mathbf{A}_0\Big)dy+\frac{\gamma}{4\pi}\sum_{j=1}^{3}v_j^2\int\Big(|\partial_j\Phi_0|^2-|\partial_j\mathbf{A}_0|^2\Big)dy+\\
&+\eta_{\mathbf{v}}-\int\Big(\gamma\mathbf{v}^2\Phi_0\rho_0+\frac{1}{\gamma}\mathbf{A}_0\cdot\mathbf{J}_0+\gamma(\mathbf{v}\cdot\mathbf{A}_0)(\mathbf{v}\cdot\mathbf{J}_0)\Big)dy-R_{\mathbf{v}}+\frac{\gamma(1+\mathbf{v}^2)}{2}\times\\
&\times\int\Big(\rho_0\Phi_0+\mathbf{J}_0\cdot\mathbf{A}_0\Big)\,dy-\frac{\gamma}{4\pi}\int\Big((\mathbf{v}\cdot\nabla\Phi_0)^2+(\mathbf{v}\cdot(\nabla\times\mathbf{A}_0))^2\Big)dy+R_{\mathbf{v}} \quad (5.33)\\
=&\,\gamma m_0+\eta_{\mathbf{v}}+\frac{\gamma}{4\pi}\Big(\sum_{j=1}^{3}v_j^2\int|\partial_j\Phi_0|^2dy-\int(\mathbf{v}\cdot\nabla\Phi_0)^2\,dy\Big)+\\
&+\frac{\gamma}{4\pi}\int\Big(4\pi\mathbf{v}^2\mathbf{A}_0\cdot\mathbf{J}_0-4\pi(\mathbf{v}\cdot\mathbf{A}_0)(\mathbf{v}\cdot\mathbf{J}_0)-(\mathbf{v}\cdot(\nabla\times\mathbf{A}_0))^2-\sum_{j=1}^{3}v_j^2|\partial_j\mathbf{A}_0|^2\Big)dy
\end{aligned}$$

Using Fourier transform, relation $\hat{\mathbf{J}}_0(k)=k^2\hat{\mathbf{A}}_0(k)/(4\pi)$ and formula $|a|^2|b|^2=|a\cdot b|^2+|a\times b|^2$, we rewrite the last integral in (5.33) in the form

$$\frac{\gamma}{4\pi}(2\pi)^{-3}\int\Big(|k\times(\mathbf{v}\times\hat{\mathbf{A}}_0)|^2-\sum_{j=1}^{3}v_j^2k_j^2|\hat{\mathbf{A}}_0|^2\Big)\,dk.$$

By condition (5.10), the last expression is

$$\frac{\gamma}{4\pi}(2\pi)^{-3}\int\Big((k\cdot\mathbf{v})^2-\sum_{j=1}^{3}v_j^2k_j^2\Big)|\hat{\mathbf{A}}_0|^2dk=\frac{\gamma}{4\pi}\sum_{i,j:\,i\neq j}v_iv_j\int\partial_i\mathbf{A}_0(x)\cdot\partial_j\mathbf{A}_0(x)dx.$$

Hence, by (5.33), $\mathcal{E}_{\mathbf{v}}=\gamma m_0+\eta_{\mathbf{v}}+\tilde{\eta}_{\mathbf{v}}$, where, by definition,

$$\tilde{\eta}_{\mathbf{v}}:=-\frac{\gamma}{4\pi}\sum_{i,j;\,i\neq j}v_iv_j\int\Big(\partial_i\mathbf{\Phi}_0(x)\partial_j\mathbf{\Phi}_0(x)-\partial_i\mathbf{A}_0(x)\cdot\partial_j\mathbf{A}_0(x)\Big)\,dx.$$

Finally, conditions **C0**–**C2** yield $\eta_{\mathbf{v}} = \tilde{\eta}_{\mathbf{v}} = 0$. Therefore, $\mathcal{E}_{\mathbf{v}} = \gamma m_0 = \gamma \mathcal{E}_0$, by Corollary 5.5 (ii). ∎

Denote by $P = (P^1, P^2, P^3)$ the momentum operator for the (MD) system,

$$\begin{aligned} P(\psi, A) &= -i \int \psi^*(t,x) \nabla \psi(t,x)\, dx \\ &\quad + \frac{1}{4\pi} \int \Big(\dot{\Phi}(t,x) \nabla \Phi(t,x) - \sum_{k=1}^{3} \dot{A}^k(t,x) \nabla A^k(t,x) \Big)\, dx. \end{aligned}$$

Put $P_{\mathbf{v}} := P(\psi_{\mathbf{v}}, A_{\mathbf{v}})$, $\mathbf{v} \in \mathbb{R}^3$. We impose conditions **C1'** and **C2'** on φ_ω (see Section 3). Moreover, we impose a stronger condition **C0'** on A_0 than **C0**.
C0' Let $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$. For any $k = 1,2,3$, the following relation holds

$$\sum_{j:\, j \neq k} v_j \int \Big(\partial_j \mathbf{\Phi}_0(x) \partial_k \mathbf{\Phi}_0(x) - \partial_j \mathbf{A}_0(x) \cdot \partial_k \mathbf{A}_0(x) \Big)\, dx = 0.$$

Note that conditions **C0'**–**C2'** are fulfilled for the particular ansatz of solutions $\varphi \equiv \varphi_\omega$ considered in Section 5.3, see Remark 5.9 (iv).

Lemma 5.13. *Let conditions* **C0'**–**C2'** *hold. Then* $P_{\mathbf{v}} = \gamma \mathbf{v} \mathcal{E}_0$.

Proof Using (5.26) and (3.7), we rewrite the term in $P_{\mathbf{v}}$ corresp. to $\psi_{\mathbf{v}}$,

$$\begin{aligned} P(\psi_{\mathbf{v}}) &:= -i \int \psi^*_{\mathbf{v}}(t,x) \nabla \psi_{\mathbf{v}}(t,x)\, dx \\ &= -i \int \varphi^*(y)(\alpha \cdot \mathbf{v} + I)\Big(i\omega\gamma \mathbf{v} \varphi(y) + \nabla \varphi(y) + \mathbf{v}\kappa \nabla \varphi(y) \cdot \mathbf{v} \Big)\, dy, \end{aligned}$$

where $\kappa := (\gamma - 1)/(|\mathbf{v}|^2)$. Using notations (2.6) and formula (5.23), we have

$$P(\psi_{\mathbf{v}}) = \mathbf{v}\gamma\omega Q + (I_1 v_1, I_2 v_2, I_3 v_3) + \mathbf{v}\kappa \sum_{j=1}^{3} v_j^2 I_j - i\xi_{\mathbf{v}},$$

where $\xi_{\mathbf{v}}$ is defined in (3.19). Applying (5.20) and (5.16), we obtain

$$P(\psi_{\mathbf{v}}) = \mathbf{v}\gamma \mathcal{E}_0 + (T_1 v_1, T_2 v_2, T_3 v_3) + \mathbf{v}\kappa \sum_{j=1}^{3} v_j^2 T_j - i\xi_{\mathbf{v}}. \tag{5.34}$$

By conditions **C1'** and **C2'**, $\xi_{\mathbf{v}} = 0$.

Further, the second term in $P_{\mathbf{v}}$ corresponding to $A_{\mathbf{v}}$ is

$$\begin{aligned} P(A_{\mathbf{v}}) &:= \frac{1}{4\pi}\int \Big(\dot\Phi_{\mathbf{v}}(t,x)\nabla\Phi_{\mathbf{v}}(t,x) - \dot{\mathbf{A}}_{\mathbf{v}}(t,x)\cdot\nabla\mathbf{A}_{\mathbf{v}}(t,x)\Big)\,dx \\ &= -\frac{1}{4\pi}\int \Big((\mathbf{v}\cdot\nabla\Phi_0(y))\nabla\Phi_0(y) + \mathbf{v}\kappa(\mathbf{v}\cdot\nabla\Phi_0(y))^2 \\ &\quad -\sum_{n=1}^{3}\Big((\mathbf{v}\cdot\nabla A_0^n(y))\nabla A_0^n(y) + \mathbf{v}\kappa(\mathbf{v}\cdot\nabla A_0^n(y))^2\Big)\Big)\,dy \\ &= -(T_1v_1, T_2v_2, T_3v_3) - \mathbf{v}\kappa\sum_{j=1}^{3} v_j^2 T_j - \tilde\xi_{\mathbf{v}}, \end{aligned}$$

where $\tilde\xi_{\mathbf{v}}$ stands for the following vector

$$\tilde\xi_{\mathbf{v}} := \Big(\sum_{j\neq 1} v_jT_{1j}, \sum_{j\neq 2} v_jT_{2j}, \sum_{j\neq 3} v_jT_{3j}\Big) + \mathbf{v}\kappa\sum_{i,j:\ i\neq j} v_iv_jT_{ij}.$$

Here by T_{ij} we denote the integral

$$T_{ij} := \frac{1}{4\pi}\int \Big(\partial_i\mathbf{\Phi}_0(y)\partial_j\mathbf{\Phi}_0(y) - \partial_i\mathbf{A}_0(y)\cdot\partial_j\mathbf{A}_0(x)\Big)\,dy.$$

By condition **C0**', $\tilde\xi_{\mathbf{v}} = 0$. Hence, formulas (5.34) and (5.35) yield

$$P_{\mathbf{v}} = P(\psi_{\mathbf{v}}) + P(A_{\mathbf{v}}) = \gamma\mathbf{v}\mathcal{E}_0. \quad \blacksquare$$

6. Klein–Gordon–Dirac equations

We consider the Klein–Gordon–Dirac (KGD) system arising in the Yukawa model (see, for instance, [4, §49]) and describes the interaction between the Dirac and scalar (or pseudoscalar) fields. This system is based on the Lagrangian density

$$\mathcal{L}(\psi,\chi) = \mathcal{L}_D(\psi) + \mathcal{L}_{KG}(\chi) + \mathcal{L}_I(\psi,\chi). \tag{6.1}$$

Here $\mathcal{L}_D(\psi)$ and $\mathcal{L}_{KG}(\chi)$ are the Lagrangian densities for the nonlinear Dirac field ψ and for the free Klein–Gordon field χ, respectively, "extra" term $\mathcal{L}_I$ describes the Yukawa interaction between the fields. $\mathcal{L}_D(\psi)$ is defined in (2.3),

$$\mathcal{L}_{KG}(\chi) = \frac{1}{2}\Big(|\dot\chi|^2 - |\nabla\chi|^2 - M^2\chi^2\Big), \quad \mathcal{L}_I(\psi,\chi) = \eta\bar\psi\Gamma\psi\chi,$$

where χ is a (real) scalar field, $M > 0$, η is a constant, and Γ is some 4×4 matrix. This model with $G \equiv 0$ and $\Gamma = I$ has been studied by Chadam and Glassey [8] and Esteban *et al.* [15]. In another model presented by Ranada and Vazquez [26] the self–coupling is $G(\bar\psi\psi)=\lambda(\bar\psi\psi)^2$ (as in the Soler model) and $\Gamma=i\gamma^5$ with $\gamma^5=i\gamma^0\gamma^1\gamma^2\gamma^3=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$.

For simplicity, we consider the case $\Gamma=I$. Then applied the Lagrange–Euler equations to (6.1) we come to the following system

$$(-i\gamma^\mu\partial_\mu+m-g(\bar\psi\psi))\psi \;=\; \eta\chi\psi,\quad x\in\mathbb{R}^n,\quad t\in\mathbb{R}, \tag{6.2}$$

$$(\partial_t^2-\Delta+M^2)\chi \;=\; \eta\bar\psi\psi, \tag{6.3}$$

where $g(s)=G'(s)$, $n=1{,}3$ (cf [8, p.5]). If $n=1$, we put $\psi=(\psi_1,\psi_2)$, $\beta=\sigma_3$, $\alpha=-\sigma_2$. Below we consider the case $n=3$ only. The case $n=1$ can be studied by a similar way. By (6.1), the Hamiltonian density reads

$$\mathcal{E}(\psi,\chi)=\int\Big(\psi^*(-i\alpha\cdot\nabla+m\beta)\psi-G(\bar\psi\psi) + \tfrac12(|\dot\chi|^2+|\nabla\chi|^2+M^2|\chi|^2)-\eta\bar\psi\psi\chi\Big)\,dx$$

If $G\equiv 0$, the local existence and uniqueness of solutions to system (6.2)–(6.3) were obtained by Chadam and Glassey [8]. The existence for the stationary solutions was given by Esteban *et al.* [15, Th.2] also only in the case when $G\equiv 0$. In spite of this fact we verify below the identity (1.4) for (KGD) system assuming that either the self-coupling G vanishes or G satisfies the conditions **G1**–**G4** (see Section 2).

6.1. Standing waves for (KGD) equations.

Definition 6.1. *Let $\omega\in(-m,m)$. The standing waves of the (KGD) system are the stationary solutions (ψ_0,χ_0) of the form*

$$\psi_0(t,x)=e^{-i\omega t}\varphi(x),\quad \chi_0(x)=\frac{e^{-M|x|}}{4\pi|x|}*f_\varphi,\quad \textit{with}\;\; f_\varphi:=\eta\bar\varphi\varphi, \tag{6.4}$$

where $\varphi\in H^1(\mathbb{R}^3;\mathbb{C}^4)$ satisfies the following equation

$$\Big(\omega+i\alpha\cdot\nabla-m\beta+\eta\chi_0\beta+g(\bar\varphi\varphi)\beta\Big)\varphi=0. \tag{6.5}$$

Put $I^\omega(\varphi)=\dfrac12\displaystyle\int\mathcal{L}(\psi_0,\chi_0)\,dx$. Then, by (6.1) and (6.4),

$$I^\omega(\varphi) \;=\; \frac12\int\Big(\varphi^*\,(i\alpha\cdot\nabla-m\beta+\omega)\,\varphi+G(\bar\varphi\varphi)\Big)\,dx + \frac{1}{16\pi}\iint\frac{e^{-M|x-y|}}{|x-y|}f_\varphi(x)f_\varphi(y)\,dxdy.$$

Note that if (ψ_0, χ_0) is a stationary solution of the (KGD) equations, then (formally) $\varphi \equiv \varphi_\omega$ is a critical point of $I^\omega(\varphi)$.

A particular family of solutions. In the spherical coordinates (r, ϕ, θ) of $x \in \mathbb{R}^3$, the particular family of the stationary solutions (ψ_0, χ_0) is given by

$$\psi_0(t,x) = e^{-i\omega t}\varphi_\omega(x), \;\; \varphi_\omega(x) = \begin{pmatrix} v\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ iu\begin{pmatrix} \cos\theta \\ e^{i\phi}\sin\theta \end{pmatrix} \end{pmatrix},$$

$$\chi_0(x) = \begin{cases} \chi_* \cos\theta, & \text{if } \Gamma = i\gamma^5, \\ \chi_*, & \text{if } \Gamma = I, \end{cases}$$

where u, v, χ_* being radial functions. In the case $\Gamma = i\gamma^5$, this ansatz has been studied numerically in [26]. In the case when $\Gamma = I$, the functions u, v are classical solutions of the following system:

$$\begin{cases} u' + \dfrac{2u}{r} = v[g(v^2 - u^2) - (m - \omega) + \eta\chi_*], \\ v' = u[g(v^2 - u^2) - (m + \omega) + \eta\chi_*]. \end{cases}$$

The function χ_* is a solution of the equation $-\chi_*'' - \dfrac{2}{r}\chi_*' + M^2\chi_* = \eta(v^2 - u^2)$ or $\chi_*(|x|) = \eta \displaystyle\int \frac{e^{-M|x-y|}}{4\pi|x-y|}\Big(v^2(|y|) - u^2(|y|)\Big)\,dy.$

6.2. A virial identity. Let $I_k \equiv I_k(\varphi)$, $V \equiv V(\varphi)$, $Q \equiv Q(\varphi)$ be as in (2.6),

$$\begin{aligned} R \equiv R(\varphi) &:= \int \chi_0(x) f_\varphi(x)\,dx = \iint \frac{e^{-M|x-y|}}{4\pi|x-y|} f_\varphi(x) f_\varphi(y)\,dxdy, \\ R_1 \equiv R_1(\varphi) &:= \frac{1}{4\pi}\int e^{-M|x-y|} f_\varphi(x) f_\varphi(y)\,dxdy. \end{aligned} \tag{6.6}$$

Note that by Parseval identity and formulas (6.4),

$$\begin{aligned} R(\varphi) &= (2\pi)^{-3}\int \frac{|\hat{f}_\varphi(k)|^2}{k^2 + M^2}\,dk, \\ R_1(\varphi) &= 2M(2\pi)^{-3}\int \frac{|\hat{f}_\varphi(k)|^2}{(k^2 + M^2)^2}\,dk = 2M\int |\chi_0(x)|^2\,dx, \end{aligned} \tag{6.7}$$

where $\hat{f}_\varphi$ denotes the Fourier transform of f_φ. Using (6.7), we rewrite $I^\omega(\varphi)$ as

$$I^\omega(\varphi) = \frac{1}{2}\left(\omega Q(\varphi) - V(\varphi) - I_1(\varphi) - I_2(\varphi) - I_3(\varphi) + \frac{1}{2}R(\varphi)\right).$$

Lemma 6.2. *Let $\varphi \in H^1(\mathbb{R}^3;\mathbb{C}^4)$ be a solution to Eqn (6.5). Then*

$$\omega Q = \frac{2}{3}(I_1 + I_2 + I_3) + V - \frac{1}{6}(5R - M\,R_1). \tag{6.8}$$

Moreover,

$$I_j(\varphi) = \frac{1}{2}(\omega Q(\varphi) - V(\varphi)) + \frac{3}{4}R(\varphi) - \frac{M}{4}R_1(\varphi) - P_j(\varphi), \quad j = 1,2,3, \tag{6.9}$$

where $P_j(\varphi)$ stands for the following functional

$$P_j \equiv P_j(\varphi) = (2\pi)^{-3}\int \frac{k_j^2|\hat{f}_\varphi(k)|^2}{(k^2+M^2)^2}\,dk = \int |\partial_j\chi_0(x)|^2\,dx. \tag{6.10}$$

Remark 6.3. (i) The virial identity (6.8) was derived in [15] in the case when $G \equiv 0$. Formally, this identity can be proved used Derrick's technique. Indeed, introduce $\varphi_\lambda(x) = \varphi(x/\lambda)$. Then $R(\varphi_\lambda) = \lambda^5 \iint \frac{e^{-\lambda M|x-y|}}{4\pi|x-y|} f_\varphi(x) f_\varphi(y)\,dxdy$, $I_k(\varphi_\lambda) = \lambda^2 I_k(\varphi)$, $Q(\varphi_\lambda) = \lambda^3 Q(\varphi)$, $V(\varphi_\lambda) = \lambda^3 V(\varphi)$.

$$\begin{aligned} 0 &= \frac{d}{d\lambda} I^\omega(\varphi_\lambda)\Big|_{\lambda=1} \\ &= \frac{1}{2}\frac{d}{d\lambda}\Big[\omega Q(\varphi_\lambda) - V(\varphi_\lambda) - I_1(\varphi_\lambda) - I_2(\varphi_\lambda) - I_3(\varphi_\lambda) + \frac{1}{2}R(\varphi_\lambda)\Big]\Big|_{\lambda=1} \\ &= \frac{3}{2}\Big(\omega Q(\varphi) - V(\varphi)\Big) - I_1(\varphi) - I_2(\varphi) - I_3(\varphi) + \frac{1}{4}(5R(\varphi) - M\,R_1(\varphi)), \end{aligned}$$

and the identity (6.8) holds.

(ii) Introduce $\varphi_\lambda(x) = \varphi(x_1/\lambda, x_2, x_3)$. Then $I_1(\varphi_\lambda) = I_1(\varphi)$, $I_k(\varphi_\lambda) = \lambda I_k(\varphi)$ for $k = 2,3$, $R(\varphi_\lambda) = (2\pi)^{-3}\lambda \int \frac{|\hat{f}_\varphi(k)|^2\,dk}{k_1^2\lambda^{-2} + k_2^2 + k_3^2 + M^2}$, $Q(\varphi_\lambda) = \lambda Q(\varphi)$, $V(\varphi_\lambda) = \lambda V(\varphi)$. By (6.10), we have $\frac{d}{d\lambda}R(\varphi_\lambda)|_{\lambda=1} = R + 2P_1$. Hence

$$0 = \frac{d}{d\lambda}\Big|_{\lambda=1} I^\omega(\varphi_\lambda) = \frac{1}{2}\Big(\omega Q(\varphi) - V(\varphi) - I_2(\varphi) - I_3(\varphi) + \frac{1}{2}R(\varphi) + P_1(\varphi)\Big).$$

Therefore,

$$I_2(\varphi) + I_3(\varphi) = \omega Q(\varphi) - V(\varphi) + \frac{1}{2}R(\varphi) + P_1(\varphi). \tag{6.11}$$

Therefore, identities (6.8) and (6.11) imply (6.9) with $j = 1$. Similarly, introducing $\varphi_\lambda(x) = \varphi(x_1, x_2/\lambda, x_3)$ or $\varphi_\lambda(x) = \varphi(x_1, x_2, x_3/\lambda)$ gives (6.9) with $j = 2,3$. Note that (6.8) follows from (6.9), since $P_1 + P_2 + P_3 = R - MR_1/2$.

Corollary 6.4. *(cf Corollary 2.6, Corollary 5.5) Let φ be a solution of Eqn (6.5). Then the following relations hold. (i) By (6.5), we have*

$$I_1 + I_2 + I_3 = \omega Q + \int (g(\bar{\varphi}\varphi) - m)\bar{\varphi}\varphi \, dx + R. \tag{6.12}$$

(ii) Using identities (6.8) and (6.12), we obtain

$$\omega Q = \frac{2}{3}(I_1+I_2+I_3)+V-\frac{1}{6}(5R-MR_1) = I_1+I_2+I_3+\int (m-g(\bar{\varphi}\varphi))\bar{\varphi}\varphi \, dx - R.$$

Hence,

$$I_1 + I_2 + I_3 = 3\int (g(s)s - G(s))|_{s=\bar{\varphi}\varphi} \, dx + \frac{1}{2}(R + MR_1) > 0, \tag{6.13}$$

by (6.7) and condition **G2**.
(iii) The total energy associated to particle-like solutions (ψ_0, χ_0) is

$$\mathcal{E}_0 := \mathcal{E}(\psi_0, \chi_0) = I_1 + I_2 + I_3 + V - \frac{1}{2}R. \tag{6.14}$$

Using (6.12), we have

$$\mathcal{E}_0 = \omega \int |\varphi(x)|^2 \, dx + \int (g(s)s - G(s))|_{s=\bar{\varphi}\varphi} \, dx + \frac{1}{2}R > 0,$$

by condition **G2**.
(iv) Similarly to Lemma 2.7 it can be proved that identity (2.13) holds.

6.3. Moving waves for (KGD) equations. Consider *travelling solutions* $(\psi_{\mathbf{v}}, \chi_{\mathbf{v}})$ with velocity $\mathbf{v} \in \mathbb{R}^3$, $|\mathbf{v}| < 1$:

$$\begin{cases} \psi_{\mathbf{v}}(t,x) &= S_{\mathbf{v}}\psi_0(\Lambda_{\mathbf{v}}^{-1}(t,x)), \\ \chi_{\mathbf{v}}(t,x) &= \chi_0(y) \quad \text{with } y = x + \mathbf{v}\frac{(\gamma-1)}{|\mathbf{v}|^2}x\cdot\mathbf{v} - \gamma\mathbf{v}t. \end{cases} \tag{6.15}$$

It is easy to check that $(\psi_{\mathbf{v}}, \chi_{\mathbf{v}})$ is a solution of (6.2)–(6.3).

Denote by $\mathcal{E}_{\mathbf{v}} := \mathcal{E}(\psi_{\mathbf{v}}, \chi_{\mathbf{v}})$ the energy of the moving solitary waves $(\psi_{\mathbf{v}}, \chi_{\mathbf{v}})$,

$$\mathcal{E}_{\mathbf{v}} = \int \Big(\psi_{\mathbf{v}}^*(-i\alpha\cdot\nabla + m\beta)\psi_{\mathbf{v}} - G(\bar{\psi}_{\mathbf{v}}\psi_{\mathbf{v}}) + \frac{1}{2}\Big(|\dot{\chi}_{\mathbf{v}}|^2 + |\nabla\chi_{\mathbf{v}}|^2 + M^2|\chi_{\mathbf{v}}|^2\Big) - \\ - \eta\chi_{\mathbf{v}}\bar{\psi}_{\mathbf{v}}\psi_{\mathbf{v}}\Big) \, dx.$$

Assume that conditions **C1** and **C2** hold (see Section 3). Moreover, we impose the additional condition **C3**.
C3 For given $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$, $|\mathbf{v}| < 1$,

$$\sum_{i,j:\, i\neq j} v_i v_j \int \partial_i\chi_0(x)\partial_j\chi_0(x)\, dx \, = 0. \tag{6.16}$$

The integral in (6.16) equals $(2\pi)^{-3}\displaystyle\int \frac{k_i k_j |\hat{f}_\varphi(k)|^2}{(k^2+M^2)^2}\, dk$. Then condition **C3** holds, for instance, if $\bar\varphi\varphi(x)$ is an even function in $x \in \mathbb{R}^3$. In particular, conditions **C1**–**C3** are fulfilled for the particular family of solutions considered in Section 6.1 (see also Section 2.1 and formulas (2.16) and (2.17)).

Lemma 6.5. *Let conditions* **C1**–**C3** *hold,* $\mathbf{v} \in \mathbb{R}^3$ *with* $|\mathbf{v}| < \mathbf{1}$*. Then* $\mathcal{E}_\mathbf{v} = \gamma\mathcal{E}_0$*.*

Proof At first, consider the term in $\mathcal{E}_\mathbf{v}$ corresponding to the Dirac field (cf formula (5.28)),

$$\begin{aligned}\mathbf{e}_D \;&:=\; \int \Big(\psi^*_\mathbf{v}(-i\alpha\cdot\nabla + m\beta)\psi_\mathbf{v} - G(\bar\psi_\mathbf{v}\psi_\mathbf{v})\Big)\, dx \\ &=\; \omega Q\gamma\mathbf{v}^2 + \tfrac{1}{\gamma}(I_1+I_2+I_3) + \tfrac{1}{\gamma}V + \gamma\sum_{j=1}^{3} v_j^2 I_j + \eta_\mathbf{v},\end{aligned}$$

where $\eta_\mathbf{v}$ is defined in (3.17). Applying formula (6.9) and then the identity (6.8), we obtain

$$\mathbf{e}_D = \gamma(I_1+I_2+I_3+V) - \frac{1}{2}\gamma\mathbf{v}^2 R - \gamma\sum_{j=1}^{3} v_j^2 P_j + \eta_\mathbf{v}. \tag{6.17}$$

Second, we rewrite the term in $\mathcal{E}_\mathbf{v}$ corresponding to the Klein–Gordon field,

$$\begin{aligned}\mathbf{e}_{KG} \;&:=\; \frac{1}{2}\int (|\dot\chi_\mathbf{v}(t,x)|^2 + |\nabla\chi_\mathbf{v}(t,x)|^2 + M^2|\chi_\mathbf{v}(t,x)|^2)\, dx \\ &=\; \frac{1}{2\gamma}\int (|\nabla\chi_0(y)|^2 + 2\gamma^2|\mathbf{v}\cdot\nabla\chi_0(y)|^2 + M^2|\chi_0(y)|^2)\, dy \\ &=\; \frac{1}{2\gamma}\int \chi_0(y)(-\Delta+M^2)\chi_0(y)\, dy + \gamma\sum_{j=1}^{3} v_j^2 \int |\partial_j\chi_0(y)|^2\, dy + \eta'_\mathbf{v} \\ &=\; \frac{1}{2\gamma}R + \gamma\sum_{j=1}^{3} v_j^2 P_j + \eta'_\mathbf{v},\end{aligned} \tag{6.18}$$

where, by definition, $\eta'_{\mathbf{v}} := \gamma \sum_{i,j:\, i\neq j} v_i v_j \int \partial_i\chi_0(y)\partial_j\chi_0(y)\,dy$. Further, the term in $\mathcal{E}_{\mathbf{v}}$ corresponding to the interaction is

$$\mathbf{e}_I := -\eta \int \chi_{\mathbf{v}}(t,x)\bar{\psi}_{\mathbf{v}}(t,x)\psi_{\mathbf{v}}(t,x)\,dx = -\frac{\eta}{\gamma}\int \chi_0(y)\bar{\varphi}(y)\varphi(y)\,dy = -\frac{1}{\gamma}R(\varphi). \tag{6.19}$$

Applying (6.17)–(6.19), and (6.14), we obtain

$$\mathcal{E}_{\mathbf{v}} = \mathbf{e}_D + \mathbf{e}_{KG} + \mathbf{e}_I = \gamma(I_1 + I_2 + I_3 + V) - \frac{1}{2}\gamma R + \eta_{\mathbf{v}} + \eta'_{\mathbf{v}} = \gamma\mathcal{E}_0 + \eta_{\mathbf{v}} + \eta'_{\mathbf{v}}.$$

Finally, by conditions **C1**–**C3**, $\eta_{\mathbf{v}} = \eta'_{\mathbf{v}} = 0$. Therefore, $\mathcal{E}_{\mathbf{v}} = \gamma\mathcal{E}_0$. ■

Remark 6.6. *If* $\mathbf{v} = (0{,}0,v)$ *with* $|v| < 1$, *then* $\eta_{\mathbf{v}} = -2i\gamma v(\varphi^*, \partial_3\varphi)$ *and* $\eta'_{\mathbf{v}} = 0$. *In this case, conditions* **C2** *and* **C3** *are fulfilled, and condition* **C1** *is equivalent to the condition* $(\varphi^*, \partial_3\varphi) = 0$.

Denote by $P = (P^1, P^2, P^3)$ the momentum operator, where $P^\beta = \int T^{0\beta}\,dx$, and $T^{\alpha\beta}$ $(\alpha, \beta = 0{,}1{,}2{,}3{,}4)$ denotes the energy–momentum tensor for the (KGD) model. Using formula (13) from [26] for the tensor $T^{\alpha\beta}$, we have

$$P \equiv P(\psi,\chi) = -\int \Big(i\psi^*(t,x)\nabla\psi(t,x) + \dot{\chi}(t,x)\nabla\chi(t,x)\Big)\,dx.$$

Put $P_{\mathbf{v}} := P(\psi_{\mathbf{v}}, \chi_{\mathbf{v}})$, $\mathbf{v} \in \mathbb{R}^3$. We impose conditions **C1'** and **C2'** on φ_ω (see Section 3). Moreover, we assume the additional condition **C3'** on χ_0.

C3' For $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ and $k = 1{,}2{,}3$, $\sum_{j:\, j\neq k} v_j \int \partial_k\chi_0(y)\partial_j\chi_0(y)\,dy = 0$.

Note that conditions **C1'**–**C3'** are fulfilled for the particular ansatz of solutions $\varphi \equiv \varphi_\omega$ considered in Section 6.1.

Lemma 6.7. *Let conditions* **C1'**–**C3'** *hold. Then* $P_{\mathbf{v}} = \gamma\mathbf{v}\mathcal{E}_0$.

Proof Using (6.15) and (3.7), we rewrite the term in $P_{\mathbf{v}}$ corresp. to $\psi_{\mathbf{v}}$,

$$\begin{aligned} P(\psi_{\mathbf{v}}) &:= -i\int \psi^*_{\mathbf{v}}(t,x)\nabla\psi_{\mathbf{v}}(t,x)\,dx \\ &= -i\int \varphi^*(y)(\alpha\cdot\mathbf{v} + I)\Big(i\omega\gamma\mathbf{v}\varphi(y) + \nabla\varphi(y) + \mathbf{v}\kappa\nabla\varphi(y)\cdot\mathbf{v}\Big)\,dy, \end{aligned}$$

where $\kappa = (\gamma - 1)/(|\mathbf{v}|^2)$. Applying (6.8) and (6.9), we obtain (cf (5.34))

$$P(\psi_{\mathbf{v}}) = \mathbf{v}\gamma\mathcal{E}_0 - (P_1 v_1, P_2 v_2, P_3 v_3) - \mathbf{v}\kappa\sum_{j=1}^{3} v_j^2 P_j - i\xi_{\mathbf{v}}, \tag{6.20}$$

where $\xi_{\mathbf{v}}$ is defined in (3.19). By conditions **C1'** and **C2**', $\xi_{\mathbf{v}} = 0$.

Further, the second term in $P_{\mathbf{v}}$ corresponding to $\chi_{\mathbf{v}}$ is

$$P(\chi_{\mathbf{v}}) := -\int \dot{\chi}_{\mathbf{v}}(t,x)\nabla\chi_{\mathbf{v}}(t,x)\,dx = \int(\nabla\chi_0(y)\cdot\mathbf{v})\Big(\nabla\chi_0(y) + \mathbf{v}\kappa\nabla\chi_0(y)\cdot\mathbf{v}\Big)\,dy = (P_1v_1, P_2v_2, P_3v_3) + \mathbf{v}\kappa\sum_{j=1}^{3} v_j^2 P_j + \xi'_{\mathbf{v}}, \quad (6.21)$$

where $\xi'_{\mathbf{v}}$ stands for the following vector

$$\xi'_{\mathbf{v}} := \Big(\sum_{j\neq 1} v_j \int \partial_j\chi_0\partial_1\chi_0\,dy,\ \sum_{j\neq 2} v_j \int \partial_j\chi_0\partial_2\chi_0\,dy,\ \sum_{j\neq 3} v_j \int \partial_j\chi_0\partial_3\chi_0\,dy\Big) + \mathbf{v}\kappa \sum_{k,j:\,k\neq j} v_k v_j \int \partial_k\chi_0\partial_j\chi_0\,dy.$$

By condition **C3**', $\xi'_{\mathbf{v}} = 0$. Hence, formulas (6.20) and (6.21) yield

$$P_{\mathbf{v}} = -\int \Big(i\psi^*_{\mathbf{v}}(t,x)\nabla\psi_{\mathbf{v}}(t,x) + \dot{\chi}_{\mathbf{v}}(t,x)\nabla\chi_{\mathbf{v}}(t,x)\Big)dx = P(\psi_{\mathbf{v}}) + P(\chi_{\mathbf{v}}) = \gamma\mathbf{v}\mathcal{E}_0.$$

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