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The asymptotic behavior of solutions to
the Cauchy problem with periodic initial data
for the nonlinear Lamb system

Москва — 2018

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Асимптотическое поведение решений задачи Коши с периодическими начальными данными для нелинейной системы Лэмба

Рассматривается модель, состоящая из бесконечной струны, взаимодействующей с нелинейным осциллятором. Для нее изучается задача Коши с периодическими начальными данными. Основная цель – доказать сходимость решений при $t \rightarrow \infty$ к периодическим по времени решениям.

Ключевые слова: нелинейная система Лэмба, задача Коши, периодические начальные данные, принцип предельной амплитуды

Tatiana Vladimirovna Dudnikova

The asymptotic behavior of solutions to the Cauchy problem with periodic initial data for the nonlinear Lamb system

A model consisting of an infinite string coupled to a nonlinear oscillator is considered. The Cauchy problem for the system with the periodic initial data is studied. The main goal is to prove the convergence of the solutions as $t \rightarrow \infty$ to time periodic solutions.

Key words: the nonlinear Lamb system, the Cauchy problem, periodic initial data, the limit amplitude principle

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1. Introduction

Consider the following problem for a function $u(x, t) \in C(\mathbb{R}^2)$:

$$(\mu + m\delta(x))\ddot{u}(x, t) = \kappa u''(x, t) + \delta(x)F(u(x, t)), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}. \quad (1.1)$$

Here $m \geq 0$, $\mu, \kappa > 0$; $\dot{u} \equiv \partial u / \partial t$, $u' \equiv \partial u / \partial x$, $\delta(x)$ is the Dirac δ -function. The initial data (as $t = 0$) for Eq. (1.1) are assumed to be periodic, see Definition 1.5 below.

By definition, Eq. (1.1) is equivalent to the following system:

$$\mu\ddot{u}(x, t) = \kappa u''(x, t), \quad t \in \mathbb{R}, \quad x \in \mathbb{R} \setminus \{0\}, \quad (1.2)$$

$$m\ddot{y}(t) = F(y(t)) + \kappa [u'(0+, t) - u'(0-, t)], \quad t \in \mathbb{R}, \quad (1.3)$$

where

$$y(t) = u(0-, t) = u(0+, t), \quad t \in \mathbb{R}. \quad (1.4)$$

Physically, the system describes small crosswise oscillations of an infinite string stretched parallel to the Ox -axis. μ is the line density of the string, κ is its tension, $F(y)$ is an external (nonlinear, in general) force field perpendicular to Ox . In the case $m = 0$, the string is coupled to a spring of a rigidity $F(y)$. In the case $m > 0$, a ball of mass m is attached to the string at the point $x = 0$, and the field $F(y)$ subjects the ball.

The system (1.2)–(1.4) was introduced first by Lamb [4] for the linear case, i.e., when $F(y) = -ry$ with a positive constant r . This system can be considered as a simple model for the phenomenon of radiation damping experienced by a vibrating body in an energy conducting medium, for example, vibrations of an elastic sphere in a gaseous medium, relativistic radiation of energy from a concentrated mass by gravity waves and so on. For general nonlinear functions $F(y)$, this model was studied by Komech [10] for finite energy solutions. In the present paper, the solutions of infinite energy with space-periodic initial data are considered. Main goal is to prove that each solution $u(x, t)$ to the system for large times is close to a time-periodic solution (see Theorem 1.6 below).

Let us describe our assumptions on the external force $F(y)$.

Denote by $V(y) = -\int F(y) dy$ the potential energy of the external field, $F(y) = -V'(y)$, $y \in \mathbb{R}$. We assume that

$$F(y) \in C^1(\mathbb{R}), \quad F(y) \rightarrow \mp\infty \quad \text{as } y \rightarrow \pm\infty. \quad (1.5)$$

Obviously, condition (1.5) implies that

$$V \in C^2(\mathbb{R}), \quad V(y) \rightarrow \infty \quad \text{as } |y| \rightarrow \infty. \quad (1.6)$$

Let us introduce a class \mathcal{E} of solutions $u(x, t)$ to Eq. (1.1) with locally finite energy.

Definition 1.1. *A function $u(x, t)$ belongs to \mathcal{E} if $u \in C(\mathbb{R}^2)$ and $\dot{u}, u' \in L_{\text{loc}}^2(\mathbb{R}^2)$, where the derivatives are understood in the sense of distributions.*

For $u(x, t) \in \mathcal{E}$, the system (1.2)–(1.3) is understood as follows (see [10]).

For $u \in C(\mathbb{R}^2)$, Eq. (1.2) is understood in the sense of distributions in the region $(x, t) \in \mathbb{R}^2$, $x \neq 0$. Moreover, Eq. (1.2) is equivalent to the d’Alembert decomposition

$$u(x, t) = f_{\pm}(x - at) + g_{\pm}(x + at), \quad \pm x > 0, \quad t \in \mathbb{R}, \quad (1.7)$$

where $a = \sqrt{\kappa/\mu}$, $f_{\pm}, g_{\pm} \in C(\mathbb{R})$, since $u(x, t) \in C(\mathbb{R}^2)$.

We now explain Eq. (1.3). Equality (1.7) implies

$$u'(x, t) = f'_{\pm}(x - at) + g'_{\pm}(x + at) \quad \text{for } \pm x > 0, \quad t \in \mathbb{R},$$

where all derivatives are understood in the sense of distributions. For $u(x, t) \in C(\mathbb{R}^2)$ satisfying (1.2), write

$$u'(0\pm, t) := f'_{\pm}(-at) + g'_{\pm}(at). \quad (1.8)$$

Note that condition $u(x, t) \in \mathcal{E}$ implies that $f'_{\pm}, g'_{\pm} \in L_{\text{loc}}^2(\mathbb{R})$. The derivative $\ddot{y}(t)$ of $y(t) \in C(\mathbb{R})$ is understood in the sense of distributions. Moreover, for $m \neq 0$, Eq. (1.3) and condition (1.5) imply that $\ddot{y}(t) = \ddot{u}(0\pm, t) \in L_{\text{loc}}^2(\mathbb{R})$. Hence, if $m \neq 0$, then $y(t) \in C^1(\mathbb{R})$ for any solution $u \in \mathcal{E}$.

We study the Cauchy problem for the system (1.2)–(1.3) with the initial conditions

$$u|_{t=0} = u_0(x), \quad \dot{u}|_{t=0} = u_1(x), \quad x \in \mathbb{R}, \quad (1.9)$$

$$\dot{y}|_{t=0} = y_1 \quad (\text{if } m \neq 0). \quad (1.10)$$

We assume that $y_1 \in \mathbb{R}$ and the initial data $u_0(x), u_1(x)$ belong to the space \mathcal{H} .

Definition 1.2. *The pair of functions (u_0, u_1) belongs to the space \mathcal{H} if $u_0 \in C(\mathbb{R})$, $u'_0, u_1 \in L_{\text{loc}}^2(\mathbb{R})$.*

Proposition 1.3. *Let condition (1.6) hold and $(u_0, u_1) \in \mathcal{H}$, $y_1 \in \mathbb{R}$. Then the Cauchy problem (1.2)–(1.4), (1.9), (1.10) has a unique solution $u(x, t) \in \mathcal{E}$.*

This proposition is proved in Section 2.

To prove the main result we impose additional conditions on the initial data (u_0, u_1) . At first, for an $\omega > 0$, we introduce a class P^ω of the space periodic functions.

Definition 1.4. For $\omega > 0$, we say that $u \in P^\omega$ if $u(x \pm \omega) = u(x)$ for $\pm x > 0$.

Definition 1.5. For $\omega > 0$, $(u_0, u_1) \in \mathcal{H}^\omega$ if $u_0 \in C^1(\mathbb{R})$, $u_1 \in C(\mathbb{R})$ and $u_0, u_0', u_1 \in P^\omega$.

In the case $m = 0$, the following result holds.

Theorem 1.6. Let $m = 0$, condition (1.5) hold and $(u_0, u_1) \in \mathcal{H}^\omega$ for some $\omega > 0$. Then for every solution $u(x, t) \in \mathcal{E}$ of the Cauchy problem (1.2)–(1.4), (1.9) there exists a solution $u_p(x, t) \in \mathcal{E}$ to Eq. (1.1) such that

$$u_p(x, t + \omega/a) = u_p(x, t) \quad \text{for} \quad (x, t) \in \mathbb{R}^2 : |t| > |x|/a, \quad (1.11)$$

and for every $R > 0$,

$$\begin{aligned} & \int_{|x| < R} (|\dot{u}(x, t) - \dot{u}_p(x, t)|^2 + |u'(x, t) - u_p'(x, t)|^2) dx \\ & + \max_{|x| < R} |u(x, t) - u_p(x, t)| \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty. \end{aligned} \quad (1.12)$$

This theorem is proved in [3]. We prove the similar result in the case $m \neq 0$ under additional restrictions on the function $F(y)$ (see Section 3).

In Appendix, we consider Eq. (1.1) for $t > 0$ under the initial condition

$$u(x, t)|_{t \leq 0} = p(x + at), \quad x \in \mathbb{R}, \quad (1.13)$$

where the function $p(z) \in P^\omega$, $p \in C^1(\mathbb{R})$, $p(x) = p_0$ for $x \leq 0$, and $F(p_0) = 0$. In this case, the convergence (1.12) holds, i.e., the solution $u(x, t)$ of the problem (1.2)–(1.4), (1.13) either is a time-periodic for $|x| \leq at$ with period ω/a or converges to a function $u_p(x, t) \in \mathcal{E}$ satisfying (1.11). Moreover, the function $u_p(x, t)$ is a solution of Eq. (1.1) for $t > 0$ under the condition $u_p(x, t)|_{t \leq 0} = q(x + at)$. Here $q(x) = q_0$ for $x \leq 0$ and $q(x) = q_0 + p(x) - p_0$ for $x > 0$, with some point $q_0 \in \mathbb{R}$ depending on p_0 .

We outline the strategy of the proof of (1.12). At first, using the d'Alembert method, we reduce the problem (1.2)–(1.4), (1.9), (1.10) to the study of the following Cauchy problem for the function $y(t)$,

$$m\ddot{y} + (2\kappa/a)\dot{y} - F(y(t)) = 2\kappa p'(at), \quad t \in \mathbb{R}, \quad (1.14)$$

with some ω -periodic function p (see formula (2.4) below) and with the initial conditions

$$\begin{aligned} y|_{t=0} &= y_0 = u_0(0), \\ \dot{y}|_{t=0} &= y_1 \quad (\text{if } m \neq 0). \end{aligned} \quad (1.15)$$

Further, for $m = 0$, we show (see Theorem 3.1) that any solution of Eq. (1.14) either ω/a -periodic or tends to an ω/a -periodic solution $y_p(t)$, i.e., $|y(t) - y_p(t)| \rightarrow 0$ as $t \rightarrow \infty$. Finally, using the explicit formula (2.3) for $u(x, t)$ we derive the results of Theorem 1.6.

If $m \neq 0$, then the behavior of solutions to Eq. (1.14) is more complex. In the case when $F(y) = -ax - by^3$, the equation of a form (1.14) is called the *Duffing equation with damping*, see for example, [8, 17]. Eq. (1.14) is a particular case of the generalized Liénard equations with a forcing term $e(t) = 2\kappa p'(at)$,

$$\ddot{y} + f(y)\dot{y} + g(y) = e(t). \quad (1.16)$$

Eq. (1.16) with $g(y) = y$ and $e(t) \equiv 0$ was studied first by Liénard [7]. A class of equations of the form (1.16) has been widely investigated in the literature, see, for example, Cartwright [1], Littlewood [2], Levinson [6], Loud [8, 9], Reuter [15]. We refer the reader to the survey works [5, 12, 13, 14, 16] for a detailed discussion of the results and methods concerning these equations. Some results concerning Eq. (1.14) are given in Section 3. In particular, condition (1.6) implies that for large times the pairs $Y(t) = (y(t), \dot{y}(t))$ (where $y(t)$ is a solution of (1.14)) belong to a fixed bounded region of \mathbb{R}^2 . Denote by $U(t, 0)$ the solving operator to the Cauchy problem (1.14), (1.15). By the Pliss results [12, 13], there exists a set $I \subset \mathbb{R}^2$ which is invariant w.r.t. $U(\omega/a, 0)$. Moreover, the set I is not empty and has zero Lebesgue measure. Introduce an integral set $\mathcal{S} \subset \{(Y(t), t) \in \mathbb{R}^3\}$ consisting of the solutions of Eq. (1.14) with the initial values $(y_0, y_1) \in I$. Let \mathcal{S}_τ denote the intersection of \mathcal{S} and the hyperplane $t = \tau$, and $\rho(Y, \mathcal{S}_\tau)$ stand for the distance between a point $Y \in \mathbb{R}^2$ and the set \mathcal{S}_τ . In Section 3 we check that every solution of Eq. (1.14) tends to the set \mathcal{S} as $t \rightarrow \infty$, i.e., $\rho(Y(\tau), \mathcal{S}_\tau) \rightarrow 0$ as $\tau \rightarrow \infty$. Hence the explicit formula (2.3) for the solutions $u(x, t)$ implies that for any $R > 0$,

$$\inf \left\{ \int_{|x| < R} (|\dot{u}(x, t) - \dot{u}_p(x, t)|^2 + |u'(x, t) - u'_p(x, t)|^2) dx + \max_{|x| < R} |u(x, t) - u_p(x, t)| \right\} \rightarrow 0 \quad \text{as } t \rightarrow \infty, \quad (1.17)$$

where the infimum is taken over all solutions $u_p(x, t) \in \mathcal{E}$ of the problem (1.2)–(1.3) such that $u_p(0\pm, t) = y_p(t)$ and $(y_p(t), \dot{y}_p(t)) \in \mathcal{S}_t$.

We give additional restrictions on the function $F(y)$ (see Examples 3.5–3.7 below) under which the set I has a unique point and then Eq. (1.14) has a unique stable periodic solution. In this case, every solution of Eq. (1.14) tends to a ω/a -periodic solution $y_p(t)$ as $t \rightarrow \infty$, and convergence (1.12) holds.

2. Existence of solutions

In this section we prove Proposition 1.3. The method of construction of finite energy solutions to the Cauchy problem (1.2)–(1.4), (1.9), (1.10) was given by Komech [10]. We apply this method to the infinite energy solutions. For simplicity, we consider only the case $t > 0$. Substituting (1.7) into initial conditions (1.9), we have

$$\begin{aligned} f_{\pm}(z) &= \frac{u_0(z)}{2} - \frac{1}{2a} \int_0^z u_1(y) dy + C_{\pm} \quad \text{for } \pm z > 0, \\ g_{\pm}(z) &= \frac{u_0(z)}{2} + \frac{1}{2a} \int_0^z u_1(y) dy - C_{\pm} \quad \text{for } \pm z > 0, \end{aligned} \quad (2.1)$$

where we can put constants $C_{\pm} = 0$. On the other hand, substituting (1.7) into the condition (1.4), we have

$$y(t) = f_-(-at) + g_-(at) = f_+(-at) + g_+(at) \quad \text{for } t \in \mathbb{R}.$$

Hence, we can determinate $g_-(z)$ with $z > 0$ and $f_+(z)$ with $z < 0$ as follows:

$$g_-(z) = y(z/a) - f_-(-z), \quad f_+(-z) = y(z/a) - g_+(z) \quad \text{for } z > 0. \quad (2.2)$$

Therefore, for $t > 0$ we obtain

$$u(x, t) = \begin{cases} f_+(x - at) + g_+(x + at) & \text{for } x \geq at \\ y(t - x/a) + g_+(x + at) - g_+(at - x) & \text{for } 0 \leq x < at \\ y(t + x/a) + f_-(x - at) - f_-(-at - x) & \text{for } -at \leq x < 0 \\ f_-(x - at) + g_-(x + at) & \text{for } x < -at \end{cases} \quad (2.3)$$

where $f_{\pm} \in C(\mathbb{R}_{\pm})$, $f'_{\pm} \in L^2_{\text{loc}}(\mathbb{R}_{\pm})$ with $\mathbb{R}_{\pm} = \{x \in \mathbb{R} : \pm x > 0\}$. Moreover, by definition (1.8), we have

$$\begin{aligned} u'(0+, t) &:= f'_+(-at) + g'_+(at) = 2g'_+(at) - \dot{y}(t)/a, \\ u'(0-, t) &:= f'_-(-at) + g'_-(at) = 2f'_-(-at) + \dot{y}(t)/a. \end{aligned}$$

Hence, Eq. (1.3) writes

$$m\ddot{y}(t) = F(y(t)) + 2\kappa [g'_+(at) - f'_-(-at) - \dot{y}(t)/a], \quad t > 0.$$

Denote

$$p(z) := \frac{u_0(z) + u_0(-z)}{2} + \frac{1}{2a} \int_{-z}^z u_1(y) dy, \quad z \in \mathbb{R}. \quad (2.4)$$

Therefore,

$$p(0) = u_0(0), \quad p'(at) = g'_+(at) - f'_-(-at) \in L^2_{\text{loc}}(\mathbb{R}_+),$$

and we obtain the following evolution equation for $y(t)$, $t > 0$:

$$\dot{y}(t) = (a/2\kappa)F(y(t)) + ap'(at), \quad t > 0, \quad \text{if } m = 0, \quad (2.5)$$

$$m\ddot{y}(t) = F(y(t)) - (2\kappa/a)\dot{y}(t) + 2\kappa p'(at), \quad t > 0, \quad \text{if } m > 0. \quad (2.6)$$

Eq. (2.1) implies the following initial condition for the function $y(t)$:

$$y(0) = f_{\pm}(0) + g_{\pm}(0) = u_0(0). \quad (2.7)$$

Eqs (2.5) and (2.6) are rewritten in the equivalent integral form,

$$y(t) = \frac{a}{2\kappa} \int_0^t F(y(s)) ds + p(at) - p(0) + y(0), \quad t \geq 0, \quad \text{if } m = 0, \quad (2.8)$$

$$my(t) = \int_0^t ds \int_0^s F(y(\tau)) d\tau + \frac{2\kappa}{a} \int_0^t (p(as) - y(s)) ds \\ + my(0) + m\dot{y}(0)t + \frac{2\kappa}{a} (y(0) - p(0))t, \quad t \geq 0, \quad \text{if } m > 0. \quad (2.9)$$

Lemma 2.1 below implies Proposition 1.3 immediately.

Lemma 2.1. (i) Let $m = 0$ and all assumptions of Proposition 1.3 hold. Then for any $y_0 \in \mathbb{R}$, Eq. (2.5) has a unique solution $y(t) = U(t,0)y_0 \in C(\mathbb{R}_+)$.

(ii) Let $m > 0$. Then for any $(y_0, y_1) \in \mathbb{R}^2$, Eq. (2.6) has a unique solution $(y(t), \dot{y}(t)) = U(t,0)(y_0, y_1)$, and $y(t) \in C^1(\mathbb{R}_+)$.

(iii) For $m \geq 0$, the following bound holds,

$$\sup_{[0,\tau]} [m|\dot{y}(t)| + |y(t)|] \leq C_1\tau + C_2 \quad \text{for any } \tau > 0. \quad (2.10)$$

Proof. We prove Lemma 2.1 only in the case $m > 0$. For $m = 0$ the proof is similarly. It follows from (2.9), condition (1.5) and from the contraction mapping principle that for any fixed initial data $y(0+)$ and $\dot{y}(0+)$, the solution $y(t)$ to Eq. (2.9) has a unique solution on a certain interval $t \in [0, \varepsilon)$ with an $\varepsilon, \varepsilon > 0$. Let us derive an a priori estimate for $y(t)$. This estimate will imply the existence and uniqueness of the global solution of (2.6) for any $y(0+)$ and $\dot{y}(0+)$. We multiply Eq. (2.6) by $\dot{y}(t)$. Using $\frac{d}{dt}V(y(t)) = -F(y(t))\dot{y}(t)$, we obtain

$$\frac{d}{dt} \left(\frac{m\dot{y}^2(t)}{2} + V(y(t)) \right) = 2\kappa p'(at)\dot{y}(t) - \frac{2\kappa}{a}\dot{y}^2(t) \leq \frac{a\kappa}{2}(p'(at))^2.$$

Let us integrate this inequality and obtain

$$\frac{m\dot{y}^2(t)}{2} + V(y(t)) \leq \frac{m\dot{y}^2(0)}{2} + V(y(0)) + \frac{a\kappa}{2} \int_0^t |p'(as)|^2 ds, \quad t > 0.$$

Hence, for any $\tau > 0$, there exist constants $C_1, C_2 > 0$ such that

$$\sup_{t \in [0, \tau]} \left[\frac{m\dot{y}^2(t)}{2} + V(y(t)) \right] \leq C_1\tau + C_2. \quad (2.11)$$

Condition (1.6) implies the estimate (2.10). Lemma 2.1 is proved. \blacksquare

The following result follows from the Gronwall inequality and from a priori estimate (2.10) (see [10]).

Lemma 2.2. *Let $m = 0$ and $y_1(t)$ and $y_2(t)$ be two solutions of Eq. (2.5) with the initial values $y_1(0)$ and $y_2(0)$, respectively. Then for every $\tau > 0$,*

$$\|\dot{y}_1(t) - \dot{y}_2(t)\|_{L^2(0, \tau)} + \max_{[0, \tau]} |y_1(t) - y_2(t)| \leq C(\tau) |y_1(0) - y_2(0)|, \quad (2.12)$$

where a constant $C(\tau)$ is bounded for bounded $y_1(0), y_2(0)$. The similar result holds for Eq. (2.6) in the case $m \neq 0$.

3. The proof of the main result

Since $(u_0, u_1) \in \mathcal{H}^\omega$, the function p defined in (2.4) has the following properties:

$$p \in C^1(\mathbb{R}), \quad p(z \pm \omega) = p(z), \quad \pm z > 0.$$

Then the function $p'(at)$ in Eqs (2.5) and (2.6) is periodic with ω/a -period, and $p'(at) \in C(\mathbb{R}_+)$.

3.1. The string–spring system ($m = 0$). At first, we study the behavior of solutions to Eq. (2.5).

Theorem 3.1. *Let condition (1.5) hold. Then the following assertions are true.*

- (i) *All solutions of Eq. (2.5) are bounded.*
- (ii) *Eq. (2.5) has at least one ω/a -periodic solution.*
- (iii) *Any solution $y(t)$ of Eq. (2.5) either is ω/a -periodic or tends to an ω/a -periodic solution $y_p(t)$ as $t \rightarrow \infty$ such that for every $R > 0$,*

$$\int_t^{t+R} |\dot{y}(s) - \dot{y}_p(s)|^2 ds + \sup_{s \in [t, t+R]} |y(s) - y_p(s)| \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (3.1)$$

Assertions (i) and (ii) follows from the results of [12, §9]. These assertions imply item (iii) by Theorem 9.1 from [12].

Proof of Theorem 1.6. Let $u(x, t)$ be a solution of the problem (1.2)–(1.4), (1.9). Then, $u(0, t) = y(t)$ is the solution of Eq. (2.5) with the initial condition $y(0) = u_0(0)$. By [3, Lemma 3.2], for any $y_0 \in \mathbb{R}$ there exists the limit of $U(n\omega/a, 0)y_0$ as $n \rightarrow \infty$. Write $\bar{y}_0 := \lim_{n \rightarrow \infty} U(n\omega/a, 0)u_0(0)$. Then, $y_p(t) = U(t, 0)\bar{y}_0$ is the ω/a -periodic solution of Eq. (2.5) and convergence (3.1) holds.

Put $\bar{u}_0(x) = u_0(x) - u_0(0) + \bar{y}_0$ and define functions $\bar{f}_\pm(x)$ and $\bar{g}_\pm(x)$ so as $f_\pm(x)$ and $g_\pm(x)$ in (2.1) but with $\bar{u}_0(x)$ instead of $u_0(x)$. Introduce a function $u_p(x, t)$ as follows

$$u_p(x, t) = \begin{cases} \bar{f}_+(x - at) + \bar{g}_+(x + at) & \text{for } x \geq at \\ y_p(t - x/a) + \bar{g}_+(x + at) - \bar{g}_+(at - x) & \text{for } 0 \leq x < at \\ y_p(t + x/a) + \bar{f}_-(x - at) - \bar{f}_-(-at - x) & \text{for } -at \leq x < 0 \\ \bar{f}_-(x - at) + \bar{g}_-(x + at) & \text{for } x < -at \end{cases} \quad (3.2)$$

Then $u_p(x, t)$ is the solution of (1.2)–(1.4) with the initial data (\bar{u}_0, u_1) and $u_p(0, t) = y_p(t)$. Since $(\bar{u}_0, u_1) \in \mathcal{H}^\omega$, the functions $\bar{f}_\pm(\pm x - at)$ and $\bar{g}_\pm(\pm x + at)$ in (3.2) are ω/a -periodic in t . Then the equality (1.11) holds, and the convergence (1.12) follows from (2.3) and (3.1), since

$$\begin{aligned} \bar{g}_+(x + at) - \bar{g}_+(at - x) &= g_+(x + at) - g_+(at - x) \\ \bar{f}_-(x - at) - \bar{f}_-(-at - x) &= f_-(x - at) - f_-(-at - x). \quad \blacksquare \end{aligned}$$

Remark 3.2. Let us consider the problem (1.1) for $t > 0$ with initial data (1.9), satisfying the following conditions: $(u_0, u_1) \in \mathcal{H}$ and u_1 has a form

$$u_1(x) = \begin{cases} a(2p'_+(x) - u'_0(x)), & x \geq 0, \\ a(u'_0(x) - 2p'_-(x)), & x < 0, \end{cases} \quad (3.3)$$

where $p_\pm \in C^1(\mathbb{R}_\pm)$ and $p_\pm(x)$ is ω -periodic for $\pm x > 0$, but $u_0(x), u_1(x)$ are not ω -periodic, in general. Then $f_-(z) = p_-(z)$ for $z < 0$ and $g_+(z) = p_+(z)$ for $z > 0$. Hence, by formula (2.3), the solution $u(x, t)$ for $t > 0$ has the form

$$u(x, t) = \begin{cases} u_0(x - at) - p_+(x - at) + p_+(x + at) & \text{for } x > at \\ y(t - x/a) + p_+(x + at) - p_+(at - x) & \text{for } 0 < x < at \\ y(t + x/a) + p_-(x - at) - p_-(-at - x) & \text{for } -at < x < 0 \\ p_-(x - at) + u_0(x + at) - p_-(x + at) & \text{for } x < -at \end{cases}$$

where $y(t)$ is a solution of Eq. (2.5) with the ω -periodic function $p(x) := p_+(x) + p_-(-x)$, $x > 0$, and satisfies the initial condition (2.7). Then the results of Theorems 3.1 and 1.6 remain true as $t \rightarrow +\infty$.

3.2. The string–oscillator system ($m > 0$). Put

$$c = 1/m, \quad k = 2\kappa/(am) = 2\sqrt{\kappa\mu}/m.$$

Then Eq. (2.6) is equivalent to the following system

$$\begin{cases} \dot{y} = v, \\ \dot{v} = cF(y) - kv + kap'(at). \end{cases} \quad (3.4)$$

Denote by $Y(t, Y_0, t_0) = (y(t, Y_0, t_0), \dot{y}(t, Y_0, t_0)) = U(t, t_0)Y_0$ the solution of the Cauchy problem for the system (3.4) with the initial data

$$Y_0 = (y, \dot{y})|_{t=t_0} = (y_0, y_1). \quad (3.5)$$

Definition 3.3. *The system is called dissipative (or D-system) if for any $(Y_0, t_0) \in \mathbb{R}^3$ there exists a $R, R > 0$, such that $\lim_{t \rightarrow \infty} \|Y(t, Y_0, t_0)\| < R$.*

Lemma 3.4. *Let condition (1.5) hold. Then the following assertions hold.*

(i) *The system (3.4) is dissipative, and there exist constants $M, N > 0$ such that for large time the solutions of the system (3.4) belong to a bounded set*

$$\{(y_0, y_1) \in \mathbb{R}^2 : |y_0| \leq M, |y_1| \leq N\}, \quad (3.6)$$

and M and N are independent on the parameters k and c of the system (3.4).

(ii) *The system (3.4) has at least one ω/a -periodic solution.*

Assertion (i) of Lemma 3.4 follows from the results of Cartwright and Littlewood, Reuter and others (see [1, 2, 15] and the review works [16, Ch. VII], [5, Ch. XI, §4], and [14, Theorem 5.5.4]). According to the Opial theorem (see, e.g., [14, Theorem 5.3.6]) instead of condition (1.5) it suffices to assume that

$$\lim_{y \rightarrow +\infty} F(y) < -r, \quad \lim_{y \rightarrow -\infty} F(y) > r, \quad \text{where } r = \max_{t \in \mathbb{R}} |p'(at)|.$$

Assertion (i) implies assertion (ii) by the Brouwer Fixed Point Theorem (see [12, Ch. 1, §2]).

Introduce a mapping $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ as $T = U(\omega_0, 0)$, $\omega_0 := \omega/a$. The map T is called *the Poincaré transformation* associated with the periodic system (3.4). Lemma 3.4 and the Pliss results (see [13, Ch. 2, §2]) imply that there exists an invariant set I w.r.t. T , i.e., $TI = I$. This set is called *characteristical set* of the dissipative system (3.4) or a *global attractor* of the diffeomorphism T . The set I has the following properties (see [12]–[14]):

- I is closed and bounded.

- I is *stable w.r.t. T* , i.e., for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\rho(Y_0, I) < \delta$ then $\rho(T^m Y_0, I) < \varepsilon$ for every $m \in \mathbb{N}$.
- For all $Y_0 \in \mathbb{R}^2$, $\rho(T^n Y_0, I) \rightarrow 0$, $n \rightarrow \infty$.
- There exists a fixed point of the mapping T belonging to I , i.e., there exists an $\omega_0 = \omega/a$ -periodic solution (or *harmonics*) of the system (3.4).
- The set I has zero Lebesgue measure by Theorem 1.9 from [13].

Define a set \mathcal{S} as

$$\mathcal{S} := \{(Y, t) \in \mathbb{R}^3 : Y = Y(t, Y_0, t_0), Y_0 \in I, t \in \mathbb{R}\}.$$

The set \mathcal{S} has the following properties:

- \mathcal{S} is bounded and closed.
- \mathcal{S} is ω_0 -periodic, i.e., for $(Y, t) \in \mathcal{S}$, $(Y, t + n\omega_0) \in \mathcal{S}$, $\forall n \in \mathbb{N}$.
- \mathcal{S} is invariant, i.e., if $(Y_0, t_0) \in \mathcal{S}$, then $(Y(t, Y_0, t_0), t) \in \mathcal{S}$ for all $t \geq t_0$.
- \mathcal{S} is stable, i.e., $\forall \varepsilon > 0 \exists \delta > 0$ such that if $\rho(Y_0, \mathcal{S}_{t_0}) < \delta$, then

$$\rho(Y(t, Y_0, t_0), \mathcal{S}_t) < \varepsilon, \quad \forall t \geq t_0,$$

where $\mathcal{S}_\tau = \mathcal{S} \cap \{t = \tau\}$.

- \mathcal{S} is stable in whole, i.e., for all $Y(t, Y_0, t_0) \in \mathbb{R}^2$ we have

$$\lim_{t \rightarrow \infty} \rho(Y(t, Y_0, t_0), \mathcal{S}_t) = 0.$$

However, these properties of \mathcal{S} do not imply, in general, the convergence (3.1). Now we consider the particular case of the system (3.4) when I has a unique point. Then (3.4) is called *the system with convergence* (see [12, §7, Definition 7.1]). In this case, the system (3.4) has a unique stable ω_0 -periodic solution $Y_p(t)$, and any another solution $Y(t, Y_0, t_0)$ tends to this periodic solution, i.e.,

$$\lim_{t \rightarrow \infty} \|Y(t, Y_0, t_0) - Y_p(t)\| = 0,$$

and the result (3.1) follows.

Below we give examples of the restrictions on the function $F(y)$ when the system (3.4) has convergence property.

Example 3.5. Assume that

(F1) $F(y) = -ry$ with a constant $r > 0$.

Then by the Levinson theorem (see [6], [12, Theorem 8.1], [14, Theorem 5.2.1]), Eq. (2.6) has a unique ω_0 -periodic solution and all other solutions tend to this periodic solution as $t \rightarrow +\infty$.

Example 3.6. Assume that for $y_1 \neq y_2$, we have

(F2) $k^2/2 - 1 \leq -c \frac{F(y_2) - F(y_1)}{y_2 - y_1} \leq 1$, $1 < k^2/2 \leq 2$, where $k = \sqrt{\kappa\mu}/m$ is the constant in (3.4).

Then according to the Zlamál theorem (see, e.g., [14, Theorem 5.3.2], [18]) all solutions tend exponentially to a unique periodic solution as $t \rightarrow +\infty$.

Example 3.7. (see [12, Theorem 8.4], [5, Ch.XI, §5] or [17]) Assume that

(F3) $F \in C^2(\mathbb{R})$, $F'(y) < 0$ for $|y| \leq M$; $\exists \beta > 0$ such that $F(y) \operatorname{sgn} y \leq -\beta$ for $|y| \geq M$ with the constant M from the bound (3.6). Moreover, the constant k from (3.4) is enough large,

$$k > (1/2)N \max_{|y| \leq M} (|F''(y)|/|F'(y)|),$$

where the constant N is defined in the bound (3.6).

Then the system (3.4) has convergence property. For instance, the function $F(y) = -ay^3 - by$ with constants $a, b > 0$ satisfies these conditions.

Note that condition **(F1)** is a particular case of **(F3)**.

Corollary 3.8. *Let condition **(F2)** or **(F3)** be true. Then the following assertions hold.*

(i) *There exists a unique ω_0 -periodic solution $y_p(t)$ of Eq. (2.6), and for any another solution $y(t)$ the convergence (3.1) holds.*

(ii) *The convergence (1.12) holds with the function $u_p(x, t)$ satisfying (1.11).*

Proof. Assertion (i) follows from the results mentioned above. Now we check assertion (ii). Indeed, let $u(x, t)$ be a solution of the problem (1.2)–(1.4), (1.9), (1.10). Then there exists the limit

$$\lim_{n \rightarrow \infty} T^n(u_0(0), y_1) =: (\bar{y}_0, \bar{y}_1),$$

and (\bar{y}_0, \bar{y}_1) is a unique point of the set I . Hence $(y_p(t), \dot{y}_p(t)) = U(t, 0)(\bar{y}_0, \bar{y}_1)$ is the unique ω/a -periodic solution of the system (3.4) and the convergence (3.1) holds. Put $\bar{u}_0(x) = u_0(x) - u_0(0) + \bar{y}_0$ and define functions $\bar{f}_\pm(x)$ and $\bar{g}_\pm(x)$ by formulas (2.1) but with $\bar{u}_0(x)$ instead of $u_0(x)$. Define $u_p(x, t)$ by (3.2). Then $u_p(x, t)$ is the solution of the problem (1.2)–(1.4) with the initial data $(\bar{u}_0, u_1, \bar{y}_1)$. Since $(\bar{u}_0, u_1) \in \mathcal{H}^\omega$, the functions $\bar{f}_-(\pm x - at)$ and $\bar{g}_+(\pm x + at)$ in (3.2) are ω/a -periodic in t . Hence the equality (1.11) holds, and the convergence (1.12) follows from (2.3) and (3.1). \blacksquare

4. Appendix: Limit amplitude principle

Here we apply the results to the following problem for a function $u(x, t) \in C(\mathbb{R}^2)$:

$$(\mu + m\delta(x))\ddot{u}(x, t) = \kappa u''(x, t) + \delta(x)F(u(x, t)), \quad t > 0, \quad x \in \mathbb{R}, \quad (4.1)$$

$$u(x, t)|_{t \leq 0} = p(x + at), \quad x \in \mathbb{R}. \quad (4.2)$$

Here $m \geq 0$, $a = \sqrt{\kappa/\mu}$. In the case $m > 0$ we assume that either condition **(F2)** or **(F3)** holds. The function p from Eq. (4.2) satisfies the following conditions:

P1 $p \in C^1(\mathbb{R})$.

P2 There exist numbers $\omega > 0$ and $p_0 \in \mathbb{R}$ such that $F(p_0) = 0$ and

$$p(z + \omega) = p(z) \text{ for } z > 0, \quad p(z) = p_0 \text{ for } z \leq 0.$$

Note that the function $p(x + at)$ is a solution of Eq. (4.1) for $t < 0$. Therefore, we can consider Eq. (4.1) for $t \in \mathbb{R}$. In particular, we have

$$\begin{aligned} u_0(x) = u|_{t=0} &= p(x), & u_1(x) = \dot{u}|_{t=0} &= ap'(x), & x \in \mathbb{R}, \\ y(0) = u_0(0) &= p_0, & \dot{y}(0) &= 0. \end{aligned}$$

Then $f_{\pm}(z) = 0$ and $g_{\pm}(z) = p(z)$ for $\pm z > 0$. Therefore, by (2.3),

$$u(x, t) = \begin{cases} p(x + at) & \text{for } x > at, \\ y(t - x/a) - p(at - x) + p(x + at) & \text{for } 0 < x < at, \\ y(t + x/a) & \text{for } -at < x < 0, \\ p_0 & \text{for } x < -at. \end{cases} \quad (4.3)$$

where $y(t)$ is a solution to Eq. (2.6) (or Eq. (2.5)) for $t > 0$, and $y(t) = p_0$ for $t \leq 0$. By Proposition 1.3, the Cauchy problem (4.1)–(4.2) has a unique solution $u(x, t) \in \mathcal{E}$ for every function $p \in C^1(\mathbb{R})$.

Let $m \geq 0$ and $y_p(t)$ be the ω/a -periodic solution of Eq. (2.5) or (2.6) with the initial data $\bar{p}_0 = \lim_{n \rightarrow \infty} T^n p_0$ (if $m = 0$) or with the initial data $(\bar{p}_0, \bar{y}_1) = \lim_{n \rightarrow \infty} T^n(p_0, 0)$ (if $m > 0$). We extend $y_p(t) \equiv \bar{p}_0$ for $t < 0$ and define

$$u_p(x, t) = \begin{cases} y_p(t - x/a) - p(at - x) + p(x + at) & \text{for } x > 0, \quad t > 0, \\ y_p(t + x/a) & \text{for } x < 0, \quad t > 0. \end{cases} \quad (4.4)$$

Then $u_p(x, t) \in \mathcal{E}$, $u_p(x, t)$ is the solution of Eq. (4.1) under the condition

$$u_p(x, t)|_{t \leq 0} = \bar{p}(x + at),$$

where $\bar{p}(x) = \bar{p}_0 + p(x) - p_0$ for $x \in \mathbb{R}$. Moreover, the identity (1.11) holds. Then convergence (1.12) follows from equality (4.3) and bound (3.1).

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