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ИНСТИТУТ ПРИКЛАДНОЙ МАТЕМАТИКИ
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Differential equations for the radial limits in $\mathbb{Z}_+^2$ of the solutions of a discrete integrable system

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Аптекарев А. И., Кожан Р.

Дифференциальные уравнения для радиальных пределов в $\mathbb{Z}_+^2$ решений одной дискретной интегрируемой системы

Изучается предельное свойство коэффициентов рекуррентных соотношений до ближайших соседей. Конкретно, предполагая наличие пределов вдоль лучей решетки коэффициентов, мы описываем этот предел в теминах решений некоторой системы обыкновенных дифференциальных уравнений. Для систем Анжелеско этот результат иллюстрируется числено.

Ключевые слова: Спектральная теория разностных операторов, матрицы Якоби, совместно ортогональные многочлены, рекуррентные соотношения до ближайших соседей.

Aptekarev A. I., Kozhan R.

Differential equations for the radial limits in $\mathbb{Z}_+^2$ of the solutions of a discrete integrable system

A limiting property of the coefficients of the nearest-neighbor recurrence coefficients for the multiple orthogonal polynomials is studied. Namely, assuming the existence of the limits along rays of the lattice nearest-neighbor coefficients, we describe the limit in terms of the solution of a system of ordinary differential equations. For Angelesco systems, the result is illustrated numerically.

Key words: Spectral theory difference operators; Jacobi matrices, multiple orthogonal polynomials, nearest-neighbor recurrence relations.

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Contents

1. Introduction

1.1. Orthogonal polynomials on the real line and the Jacobi matrices. It is a well-known fact (see [12]) that given a probability measure μ on $\mathbb{R}$ with infinite support, the sequence of its orthonormal polynomials $\{p_j\}_{j=0}^{\infty}$ satisfies the three-term recurrence relation

$$xp_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + A_{n-1} p_{n-1}(x)\,, \tag{1.1}$$

where the recurrence coefficients $\{A_j, B_j\}_{j=0}^{\infty}$ satisfy $A_j > 0$, $B_j \in \mathbb{R}$, and $p_{-1} = 0$. The corresponding Jacobi matrix is

$$\mathcal{J}(\mu) = \begin{pmatrix} B_0 & A_0 & 0 & \\ A_0 & B_1 & A_1 & \ddots \\ 0 & A_1 & B_2 & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}. \tag{1.2}$$

1.2. Multiple orthogonal polynomials and the nearest neighbor recurrence relations. Let us now describe multiple orthogonality situation with respect to the vector-measure $\vec{\mu} := \{\mu_i\}_{i=1}^{d}$ on $\mathbb{R}$. For any $\vec{n} := (n_1, \ldots, n_d) \in \mathbb{Z}_+^d$, let $Q_{\vec{n}}$ be the monic polynomial of degree $\leq |\vec{n}| := n_1 + \ldots + n_d$ which satisfies

$$\int Q_{\vec{n}}(x) x^k d\mu_i = 0, \quad k \in \{0, \ldots, n_i - 1\}, \quad i \in \{1, \ldots, d\}. \tag{1.3}$$

The polynomial $Q_{\vec{n}}(x)$ is called the type II *multiple orthogonal polynomial* (MOP). We say that $\vec{n}$ is a *normal* multi-index if $Q_{\vec{n}}$ is unique (it is equivalent to $\deg Q_{\vec{n}} = |\vec{n}|$). If all multi-indices of the lattice $\mathbb{Z}_+^d$ are normal then the system of measures $\{\mu_i\}_{i=1}^{d}$ are called *perfect*. It is known [13, 14, 3], that (similarly to the case with one measure) MOPs for the perfect systems satisfy the following nearest neighbor recurrence relations (NNRR) system

$$zQ_{\vec{n}}(z) = Q_{\vec{n}+\vec{e}_j}(z) + \beta_{\vec{n},j} Q_{\vec{n}}(z) + \sum_{i=1}^{d} \alpha_{\vec{n},i} Q_{\vec{n}-\vec{e}_i}(z). \tag{1.4}$$

Here we have d recurrence relations for $j = 1, \ldots, d$. Thus for each $\vec{n} \in \mathbb{Z}_+^d$ we have two sets of the coefficients for NNRR, namely $\{\beta_{\vec{n},j}\}_{j=1}^{d}$ and $\{\alpha_{\vec{n},i}\}_{i=1}^{d}$. In order to define by means of (1.4) the polynomials $\{Q_{\vec{n}}(z\}$ in unique way the NNRR coefficients cannot be taken arbitrary. As it was shown in [14](see also [1]), the recurrence coefficients must satisfy the compatibility conditions (CC):

$$\begin{aligned} &\nabla_j \beta_{\vec{n},i} = \nabla_i \beta_{\vec{n},j} \\ &\beta_{\vec{n},i}\nabla_j\beta_{\vec{n},i} - \beta_{\vec{n},j}\nabla_i\beta_{\vec{n},j} = \Big\langle (\vec{\nabla}_j - \vec{\nabla}_i),\, \vec{\alpha}_{\vec{n}} \Big\rangle \\ &(\nabla_i \ln)\alpha_{\vec{n},j} = (\nabla_j \ln)\left(\beta_{\vec{n}-e_j,i} - \beta_{\vec{n}-e_j,j}\right) \end{aligned} \tag{1.5}$$

where we denote

$$\nabla_j \beta_{\vec{n},i} := \beta_{\vec{n}+e_j,i} - \beta_{\vec{n},i}, \quad \vec{\nabla}_i := (\nabla_i, \cdots, \nabla_i), \quad (\nabla_i \ln)\alpha_{\vec{n}} := \left(\frac{\alpha_{\vec{n}+e_j}}{\alpha_{\vec{n}}} - 1\right).$$

Here we present CC in a form taken from [2] which is equivalent to CC from [14]. The system of difference equations (1.5) is also called *Discrete Integrable System* (DIS) for details see [1]. The boundary problem for DIS (1.5) in $\mathbb{Z}_+^d$ means the following. Given the boundary data: coefficients of the d-collections of the three-terms recurrence relations, corresponding to usual orthogonal polynomials with respect to each $\{\mu_i\}_{i=1}^d$ measure. Then solving equations (1.5) we have to find all NNRR coefficients $\{\beta_{\vec{n},j}\}_{j=1}^d$ and $\{\alpha_{\vec{n},i}\}_{i=1}^d$.

1.3. Zero asymptotics and limits of the recurrence coefficients. Our goal is to investigate the asymptotic behavior of the recurrence coefficients $\{\alpha_{\vec{n},i}, \beta_{\vec{n},i}\}$ as $|\,\vec{n}\,|$ grows. This behavior is intimately connected to the *asymptotic zero distribution* of multiple orthogonal polynomials $Q_{\vec{n}}$. To state the problem, we need to place some restrictions on the way $|\,\vec{n}\,|$ approaches infinity as well as the measures μ_i. The same time we have to be in a class of the perfect systems to keep NNRR.

The important example of a perfect system of measures $\{\mu_i\}$ is the so-called *Angelesco system* defined by*

$$\operatorname{supp}(\mu_i) = [a_i, b_i], \quad \text{and} \quad [a_i, b_i] \cap [a_j, b_j] = \varnothing \quad \text{when} \quad i \neq j. \tag{1.6}$$

Multiple orthogonal polynomial with respect to Angelesco system has the form:

$$Q_{\vec{n}}(z) =: \prod_{i=1}^{d}\prod_{l=1}^{n_i}(z - x_{\vec{n},i,l}), \quad x_{\vec{n},i,l} \in [a_i, b_i].$$

Moreover, we restrict our attention to sequences of multi-indices such that

$$n_i = t_i|\,\vec{n}\,| + o\left(|\,\vec{n}\,|\right), \qquad \vec{t} = (t_1, \ldots, t_d) \in (0,1)^d, \quad |\,\vec{t}\,| = 1. \tag{1.7}$$

*If supports of measures are intervals with nonintersecting interiors then system $\{\mu_i\}$ is perfect as well.

Asymptotic zero distribution for $Q_{\vec{n}}(z)$ (or *limiting zero counting measure*):

$$\omega(x) := \lim_{|\vec{n}|\to\infty} \frac{1}{|\,\vec{n}\,|} \sum_{i=1}^{d} \sum_{l=1}^{n_i} \delta(x - x_{\vec{n},i,l}), \tag{1.8}$$

for Angelesco systems (1.6) with $\mu_i' > 0$ a.e. on $[a_i, b_i]$ in the regime (1.7) obtained by Gonchar and Rakhmanov [10]. To state their result we fix $\vec{t}$ as in (1.7), and denote

$$M_{\vec{t}}\big(\{a_i, b_i\}_1^d\big) := \big\{\vec{\nu} = (\nu_1, \ldots, \nu_d) :\ \nu_i \in M_{t_i}(a_i, b_i),\ i \in \{1, \ldots, d\}\big\},$$

where $M_t(a, b)$ is the set of positive Borel measures of mass t supported on $[a, b]$.

Theorem 1 ([10])**.** ***1)*** *There exists the unique vector of measures* $\vec{\omega} \in M_{\vec{t}}\big(\{a_i, b_i\}_1^d\big)$:

$$I[\,\vec{\omega}\,] = \min_{\nu \in M_{\vec{t}}(\{a_i,b_i\}_1^d)} I[\,\vec{\nu}\,], \qquad I[\,\vec{\nu}\,] := \sum_{i=1}^{d} \Big(2I[\nu_i] + \sum_{k\neq i} I[\nu_i, \nu_k]\Big), \tag{1.9}$$

where $I[\nu_i] := I[\nu_i, \nu_i]$ *and* $I[\nu_i, \nu_k] := -\int\int \log|z - x| d\nu_i(x) d\nu_k(z)$.
2) *Moreover, for the limiting counting measure* (1.8) *it holds:* $\omega = |\vec{\omega}|$.

An important feature of the case $d > 1$ (in comparison with classic $d = 1$) is the fact that measures ω_i might no longer be supported on the whole intervals $[a_i, b_i]$ (the so-called *pushing effect*), but in general it holds that

$$\operatorname{supp}(\omega_i) = [a_{\vec{t},i}, b_{\vec{t},i}] \subseteq [a_i, b_i], \qquad i \in \{1, \ldots, d\}. \tag{1.10}$$

Namely the supports of the extremal measures (not the supports of the multiple orthogonality measures†) define the recurrence coefficients limits.

To describe the asymptotics of the recurrence coefficients, we shall need a $(d+1)$-sheeted compact Riemann surface, say $\mathfrak{R}$, that we realize in the following way. Take $d + 1$ copies of $\overline{\mathbb{C}}$. Cut one of them along the union $\bigcup_{i=1}^d \big[a_{\vec{t},i}, b_{\vec{t},i}\big]$, which henceforth is denoted by $\mathfrak{R}^{(0)}$. Each of the remaining copies cut along only one interval $\big[a_{\vec{t},i}, b_{\vec{t},i}\big]$ so that no two copies have the same cut and denote them by $\mathfrak{R}^{(i)}$. To form $\mathfrak{R}$, take $\mathfrak{R}^{(i)}$ and glue the banks of the cut $\big[a_{\vec{t},i}, b_{\vec{t},i}\big]$ crosswise to the banks of the corresponding cut on $\mathfrak{R}^{(0)}$. It can be easily verified that thus constructed Riemann surface has genus 0. Denote by π the natural projection from $\mathfrak{R}$ to $\overline{\mathbb{C}}$. We also shall employ the notations $\mathbf{z}$ for a point on $\mathfrak{R}$ and $z^{(i)}$ for a point on $\mathfrak{R}^{(i)}$ with $\pi(\mathbf{z}) = \pi(z^{(i)}) = z$.

†For $d = 1$ both these notions coincides.

Since $\mathfrak{R}$ has genus zero, one can arbitrarily prescribe zero/pole multisets of rational functions on $\mathfrak{R}$ as long as the multisets have the same cardinality. Hence, we define Υ_i, $i \in \{1,\dots,d\}$, to be the rational function on $\mathfrak{R}$ with a simple zero at $\infty^{(0)}$, a simple pole at $\infty^{(i)}$, and otherwise non-vanishing and finite. We normalize it so that $\Upsilon_i(z^{(i)})/z \to 1$ as $z \to \infty$. Then the following theorem holds.

Theorem 2 ([2]). *Let* $\{\mu_i\}_{i=1}^d$ *be a system of measures satisfying* (1.6) *and such that*

$$d\mu_i(x) = \rho_i(x)dx, \tag{1.11}$$

where ρ_i *is holomorphic and non-vanishing in some neighborhood of* $[a_i, b_i]$. *Further, let* $\mathcal{N}_{\vec{t}} = \{\vec{n}\}$ *be a sequence of multi-indices as in* (1.7) *for some* $\vec{t} \in (0,1)^d$. *Then the recurrence coefficients* $\{\alpha_{\vec{n},j}, \beta_{\vec{n},j}\}$ *given by* (1.4) *and* (1.3) *satisfy*

$$\lim_{\mathcal{N}_{\vec{t}}} \alpha_{\vec{n},i} = \alpha_{\vec{t},i} \quad \text{and} \quad \lim_{\mathcal{N}_{\vec{t}}} \beta_{\vec{n},i} = \beta_{\vec{t},i}, \quad i \in \{1,\dots,d\}, \tag{1.12}$$

where $\alpha_{\vec{t},i}$ *and* $\beta_{\vec{t},i}$ *are constants:* $z^2\Upsilon_i(z^{(0)}) = \alpha_{\vec{t},i}(z + \beta_{\vec{c},i}) + \mathcal{O}(z^{-1})$ *as* $z \to \infty$.

Remarks. 1) We note that Theorem 2 is valid for $d = 1$ as well.
2) It is not too difficult to extend the proof (from [10]) of Theorem 1 to include the case of touching intervals.
3)We also can affirm (at least for $d = 2$) that Theorem 2 remains valid for the case of touching intervals (technicalities can be taken from [5]) and for weight functions (1.11) singularities of the types: Jacobi and Fisher-Hartwig weights [16]. ■

We say that a probability measure on $\mathbb{R}$ belongs to the Nevai class (see [12] and references therein for more details) $N(\alpha, \beta)$ if its Jacobi coefficients (in (1.1)) satisfy $B_n \to \beta$ and $A_n \to \alpha$ as $n \to \infty$.

Weyl's theorem on compact perturbations says that any measure in $N(\alpha, \beta)$ has $\sigma_{ess}(\mu) = [\beta - 2\alpha, \beta + 2\alpha]$. For the (partial) converse, we have the Denisov–Rakhmanov theorem stating that if $\sigma_{ess}(\mu) = [\beta - 2\alpha, \beta + 2\alpha]$ and $\frac{d\mu}{dx} > 0$ a.e. on $[\beta - 2\alpha, \beta + 2\alpha]$ then $\mu \in N(\alpha, \beta)$.

By the analogy, let us say that an perfect system of measures $\{\mu_i\}_{i=1}^d$ belongs to the *multiple Nevai class* if the nearest neighbor recursion coefficients have limits along each ray of $\mathbb{Z}_+^d$, starting at origin, that is: for every $\vec{t}$ in (1.7). Thus, Theorem 2 is a partial analogue of the Denisov–Rakhmanov and Andeleso system (from Theorem 2) belongs to the multiple Nevai class.

In the next Section 2 we state and proof our main result: a conditional theorem on ODE (with respect to variable t) for the limiting value (in the regime (1.7)) of the NNRR coefficients ($d = 2$). Then in Section 3 we present numeric illustrations. Finally in Section 4 we consider the case of touching intervals.

2. ODE for the limits of NNRR coefficients

2.1. Preliminaries for $(d = 2)$**.** Now let us restrict ourselves to the special case when two measures μ_1 and μ_2 form a perfect system.

We rewrite NNRR (1.4) for d=2 (changing the notations for the recurrence coefficients):

$$xQ_{n,m}(x) = Q_{n+1,m}(x) + c_{n,m}Q_{n,m}(x) + a_{n,m}Q_{n-1,m} + b_{n,m}Q_{n,m-1} \tag{2.1}$$
$$xQ_{n,m}(x) = Q_{n,m+1}(x) + d_{n,m}Q_{n,m}(x) + a_{n,m}Q_{n-1,m} + b_{n,m}Q_{n,m-1} \tag{2.2}$$

for some sequences of coefficients $a_{n,m}, b_{n,m}, c_{n,m}, d_{n,m}$ satisfying $a_{0,m} = b_{n,0} = 0$.

Note that $\{c_{n,0}\}_{n=0}^{\infty}$ and $\{\sqrt{a_{n,0}}\}_{n=0}^{\infty}$ are the diagonal and off-diagonal coefficients of the Jacobi matrix $\mathcal{J}(\mu_1)$, and $\{d_{0,m}\}_{m=0}^{\infty}$ and $\{\sqrt{b_{0,m}}\}_{m=0}^{\infty}$ are the diagonal and off-diagonal coefficients of the Jacobi matrix of $\mathcal{J}(\mu_2)$.

As for the first time was shown in [14] (see also [1]), the recurrence coefficients must satisfy the compatibility conditions (CC):

$$\frac{a_{n,m+1}}{a_{n,m}} = \frac{c_{n,m} - d_{n,m}}{c_{n-1,m} - d_{n-1,m}}, \tag{2.3}$$
$$\frac{b_{n+1,m}}{b_{n,m}} = \frac{c_{n,m} - d_{n,m}}{c_{n,m-1} - d_{n,m-1}}, \tag{2.4}$$
$$d_{n+1,m} - d_{n,m} = c_{n,m+1} - c_{n,m}, \tag{2.5}$$
$$c_{n,m+1} - c_{n,m} = \frac{a_{n+1,m} + b_{n+1,m} - a_{n,m+1} - b_{n,m+1}}{c_{n,m} - d_{n,m}}, \tag{2.6}$$

together with the boundary-type conditions $a_{0,m} = 0$ and $b_{n,0} = 0$ for all n, m. In other words the coefficients of NNRR (2.1)–(2.2) are solutions of the BVP for DIS (2.3)–(2.6).

Note that one can take $p_{2n}(x) = Q_{n,n}(x)$, $p_{2n+1}(x) = Q_{n+1,n}(x)$, which results in the "recurrence relation along the step-line":

$$xp_n(x) = p_{n+1}(x) + \kappa_n p_n(x) + \gamma_n p_{n-1}(x) + \delta_n p_{n-2}(x), \tag{2.7}$$

where

$$\kappa_{2n} = c_{n,n}, \qquad \kappa_{2n+1} = d_{n+1,n},$$
$$\gamma_{2n} = a_{n,n} + b_{n,n}, \qquad \gamma_{2n+1} = a_{n+1,n} + b_{n+1,n},$$
$$\delta_{2n} = a_{n,n}(c_{n-1,n-1} - d_{n-1,n-1}), \qquad \delta_{2n+1} = b_{n+1,n}(d_{n,n-1} - c_{n,n-1}),$$

see, e.g., [6, 9].

In this section we consider perfect systems (μ_1, μ_2) belonging to the multiple Nevai class $N(A, B, C, D)$: the nearest neighbour recursion coefficients have limits along each ray, that is: for every $t \in [0,1]$

$$\lim_{n+m\to\infty, \frac{n}{n+m}\to t} a_{n,m} = A(t) \tag{2.8}$$

$$\lim_{n+m\to\infty, \frac{n}{n+m}\to t} b_{n,m} = B(t) \tag{2.9}$$

$$\lim_{n+m\to\infty, \frac{n}{n+m}\to t} c_{n,m} = C(t) \tag{2.10}$$

$$\lim_{n+m\to\infty, \frac{n}{n+m}\to t} d_{n,m} = D(t) \tag{2.11}$$

for some real-valued functions $A(t), B(t), C(t), D(t) : [0,1] \to \mathbb{R}$.

By the discussion in the previous section, Angelesco systems, satisfying to the condition of Theorem2 belong to the multiple Nevai class $MN(A, B, C, D)$. Thus $MN(A, B, C, D)$ is non empty and the Angelesco systems (mentioned above) have μ_1 and μ_2 in the Nevai class $N(\sqrt{A(1)}, C(1))$ and $N(\sqrt{B(0)}, D(0))$, respectively. Therefore we get the application of Weyl's theorem: these Angelesco systems from $MN(A, B, C, D)$ have

$$\sigma_{ess}(\mu_1) = [\beta_1 - 2\alpha_1, \beta_1 + 2\alpha_1] \text{ and } \sigma_{ess}(\mu_2) = [\beta_2 - 2\alpha_2, \beta_2 + 2\alpha_2],$$

where

$$\alpha_1 = \sqrt{A(1)}, \qquad \beta_1 = C(1), \tag{2.12}$$

$$\alpha_2 = \sqrt{B(0)}, \qquad \beta_2 = D(0). \tag{2.13}$$

It is an interesting open problem to generalize the above analogue of Denisov–Rakhmanov result (i.e. Theorem2) to more general measures (i.e. to Angelesco systems with $\mu_i > 0$ a.e. on $\sigma_{ess}(\mu_i)$).

In this paper we investigate the possibility of describing functions A, B, C, D through differential equations. This is done in Theorem 3 below.

For the perfect systems from the multiple Nevai classessee also, [15]. Note that if an Angelesco system is in the Nevai class, then the coefficients $\{\kappa_j\}, \{\gamma_j\}, \{\delta_j\}$ of the step-line recurrence (2.7) are asymptotically two-periodic (see [11, 6, 8, 7], and references therein).

Regarding to asymptotic zero distribution of MOP from Angelesco class, for the case of $d = 2$ we can add to the statement of Theorem1 the fact, that support of the limiting zero density is $[\beta_1 - 2\alpha_1, e_1] \cup [e_2, \beta_2 + 2\alpha_2]$, where $e_1 \le \beta_1 + 2\alpha_1$ and $e_2 \ge \beta_2 - 2\alpha_2$. Moreover, $e_1 - e_2 \to 0$ (that is, $e_1 \to e$ and $e_2 \to e$ for some e) in the limiting case of touching supports $(\beta_2 - 2\alpha_2) - (\beta_1 + 2\alpha_1) \to 0$.

2.2. Statement. Before to state the main result we introduce approximations of the limiting functions of the Nevai class $N(A,B,C,D)$, see (2.8)–(2.11).

For each $k \in \mathbb{Z}_+$, let us "pack" the diagonal sequence $\{a_{n,k-n}\}_{n=0}^{k}$ into a piecewise linear function A_k as follows. For any k, we let $m = k-n$, $t_n^{(k)} = \frac{n}{n+m} = \frac{n}{k}$, $\varepsilon^{(k)} = \frac{1}{n+m} = \frac{1}{k}$. Define

$$A_k(t_n^{(k)}) = a_{n,k-n}$$

and connect these points to make $A_k(t)$ piecewise linear on [0,1].

Similarly for $B_k(t), C_k(t), D_k(t)$.

Then (2.8)–(2.11) is equivalent to A_k, B_k, C_k, D_k converging pointwise on [0,1] to $A(t), B(t), C(t), D(t)$ as $k \to \infty$.

Let us **assume**

(i) A, B, C, D are piecewise continuously differentiable on [0,1];

(ii) The convergence is uniform and fast enough:

$$|A_k(t) - A(t)| \leq o(\tfrac{1}{k}) = o(\varepsilon^{(k)}), \quad k \to \infty, \tag{2.14}$$

and similarly for B, C, D.

Theorem 3. *1) Given a perfect system $(\mu_1, \mu_2) \in N(A,B,C,D)$ satisfying the condition (i), (ii) above. Then the limiting functions $A(t), B(t), C(t), D(t)$ satify to the following system of differential equations:*

$$\begin{pmatrix} tE(t) & 0 & (1-t)A(t) \\ 0 & (1-t)E(t) & tB(t) \\ \frac{1}{t(1-t)} & \frac{1}{t(1-t)} & E(t) \end{pmatrix} \begin{pmatrix} A'(t) \\ B'(t) \\ E'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.15}$$

where $E(t) = C(t) - D(t)$.

2) Suppose a system of Angelesco satisfies conditions of Theorem 2. Then there exist $c_1, c_2 \in (0,1)$ such that the functions $A, B, C, D : [0,1] \to \mathbb{R}$ are smooth on $[0, c_1)$ and $(c_2, 1]$, and satisfy the system of differential equations

$$\begin{cases} (1+t)tK'(t) + 4tK(t) + (2-t)(1-t)L'(t) - 4(1-t)L(t) = 0 \\ \dfrac{t^2K'(t)}{K(t)} + 2t = \dfrac{(1-t)^2L'(t)}{L(t)} - 2(1-t) \end{cases} \tag{2.16}$$

with initial/boundary conditions

$$\begin{cases} K(0) = (\beta_1 - \beta_2 - 2\alpha_1)^2 - \alpha_2^2 \\ L(0) = \alpha_2^2, \end{cases} \qquad \begin{cases} K(1) = \alpha_1^2 \\ L(1) = (\beta_1 - \beta_2 - 2\alpha_2)^2 - \alpha_1^2. \end{cases} \tag{2.17}$$

where

$$A(t) = t^2 K(t), \quad B(t) = (1-t^2)L(t), \quad C(t) - D(t) = \sqrt{K(t)+L(t)}.$$

Moreover, A, B, C, D are constant on the interval $[c_1, c_2]$.

Remarks. 1) We note that general (and conditional) part 1) of Theorem 3 admits presence inside $[0,1]$ subdomain, where $A(t), B(t), C(t), D(t)$ are constant. For Angelesco systems it is a generic situation which happens when "pushing " is not active, see [10].
2) Conditions i), ii) are fulfilled for Angelesco systems from Theorem 2. Namely, condition i) follows directly from (1.12) and for ii) (from the proof of Theorem 2) we even have in RHS of (2.14) the bound $O(\frac{1}{k^2})$.
3) We note that known information about support of zero counting measure of MOP for Angelesco system (see [10]) allows us to identify the subdomain where $A(t)$, $B(t)$, $C(t)$, $D(t)$ are constant, i.e. interval $[c_1, c_2]$. Then it is possible, using BC (2.17) to solve ODE system (2.16) on $[0, c_1]$ and $[c_2, 1]$.
4) It would be valuable to to show inductively, using the compatibility conditions (2.3)–(2.6) and additional assumptions on the speed of convergence of marginal Jacobi coefficients, that $|A_k(t) - A_{k+1}(t)| = O(\frac{1}{k^2})$ ‡. Note that if successful this would generalize Denisov–Rakhmanov theorem from Theorem 2 to non-analytic weights.

2.3. Proof. We start with limiting relations (2.8)–(2.8). Then by linearity (or Taylor theorem/mean-value theorem also work),

$$\begin{aligned} a_{n,m+1} &= A_{k+1}(\tfrac{n}{k+1}) = A_{k+1}(\tfrac{n}{k}) + A'_{k+1}(\tfrac{n}{k})\left(\tfrac{n}{k+1} - \tfrac{n}{k}\right) = A_{k+1}(\tfrac{n}{k}) - A'_{k+1}(\tfrac{n}{k})\tfrac{n}{k(k+1)} \\ &= A_{k+1}(t_n^{(k)}) - t_n^{(k)}\varepsilon^{(k)} A'_{k+1}(t_n^{(k)}) + o(\varepsilon^{(k)}) \end{aligned}$$

Similarly,

$$\begin{aligned} a_{n+1,m} &= A_{k+1}(\tfrac{n+1}{k+1}) = A_{k+1}(\tfrac{n}{k}) + A'_{k+1}(\tfrac{n}{k})\left(\tfrac{n+1}{k+1} - \tfrac{n}{k}\right) = A_{k+1}(\tfrac{n}{k}) \\ &+ A'_{k+1}(\tfrac{n}{k})\left(\tfrac{1}{k+1} - \tfrac{n}{k(k+1)}\right) = A_{k+1}(t_n^{(k)}) + (1-t_n^{(k)})\varepsilon^{(k)} A'_{k+1}(t_n^{(k)}) + o(\varepsilon^{(k)}), \end{aligned}$$

as well as

$$\begin{aligned} a_{n-1,m} &= A_{k-1}(\tfrac{n-1}{k-1}) = A_{k-1}(\tfrac{n}{k}) + A'_{k-1}(\tfrac{n}{k})\left(\tfrac{n-1}{k-1} - \tfrac{n}{k}\right) = A_{k-1}(\tfrac{n}{k}) \\ &+ A'_{k-1}(\tfrac{n}{k})\left(-\tfrac{1}{k-1} + \tfrac{n}{k(k-1)}\right) = A_{k-1}(t_n^{(k)}) - (1-t_n^{(k)})\varepsilon^{(k)} A'_{k-1}(t_n^{(k)}) + o(\varepsilon^{(k)}), \end{aligned}$$

‡The same for B_k, C_k, D_k.

and

$$a_{n,m-1} = A_{k-1}(\tfrac{n}{k-1}) = A_{k-1}(\tfrac{n}{k}) + A'_{k-1}(\tfrac{n}{k})\left(\tfrac{n}{k-1} - \tfrac{n}{k}\right) = A_{k-1}(\tfrac{n}{k}) + A'_{k-1}(\tfrac{n}{k})\tfrac{n}{k(k-1)}$$
$$= A_{k-1}(t_n^{(k)}) + t_n^{(k)}\varepsilon^{(k)}A'_{k-1}(t_n^{(k)}) + o(\varepsilon^{(k)}),$$

Similar equalities hold for B, C, D.

Plugging these equalities into (2.3), and multiplying the terms out, we get

$$(A_{k+1}C_{k-1} - A_kC_k - A_{k+1}D_{k-1} + A_kD_k)+$$
$$\varepsilon^{(k)}\left(-A_{k+1}(C'_{k-1} - D'_{k-1})(1-t) - A'_{k+1}(C_{k-1} - D_{k-1})t\right) + o(\varepsilon^{(k)}) = 0,$$

(everything is evaluated at $t = t_n^{(k)}$). The first bracket is $o(\varepsilon^{(k)})$ by (2.14), so dividing by $\varepsilon^{(k)}$ and taking the limit $k \to \infty$, we get

$$A(t)(C'(t) - D'(t))(1-t) + A'(t)(C(t) - D(t))t = 0 \tag{2.18}$$

Similar arguments applied to (2.4)–(2.6) lead to three more ODE's

$$B(t)(C'(t) - D'(t))t + B'(t)(C(t) - D(t))(1-t) = 0 \tag{2.19}$$
$$C'(t)t + D'(t)(1-t) = 0 \tag{2.20}$$
$$-C'(t)t = \frac{A'(t) + B'(t)}{C(t) - D(t)}. \tag{2.21}$$

Let us simplify this system a bit. First of all, let

$$E(t) = C(t) - D(t).$$

Then from (2.21), $C' = -\frac{A'+B'}{tE}$, from (2.20), $D' = -\frac{t}{1-t}C' = \frac{A'+B'}{(1-t)E}$, so $E' = C' - D' = -\frac{A'+B'}{t(1-t)E}$. Thus we end up with the following ODE system:

$$\begin{pmatrix} tE(t) & 0 & (1-t)A(t) \\ 0 & (1-t)E(t) & tB(t) \\ \frac{1}{t(1-t)} & \frac{1}{t(1-t)} & E(t) \end{pmatrix} \begin{pmatrix} A'(t) \\ B'(t) \\ E'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.22}$$

Part 1) of the Theorem 3 is proved.

For Angelesco systems we have BC (2.12)– (2.13):

$$C(1) = \beta_1, \qquad A(1) = \alpha_1^2 \tag{2.23}$$
$$D(0) = \beta_2, \qquad B(0) = \alpha_2^2 \tag{2.24}$$

and the natural marginal BC

$$A(0) = 0, \qquad B(1) = 0.$$

Let us divide interval [0,1] into two disjoint sets:

$$I_1 = \{t \in [0,1] : A'(t) = B'(t) = E'(t) = 0\} \quad \text{and} \quad I_2 = [0,1] \setminus I_1.$$

From [10] we know that: I_1 consists of one point if Δ_1 and Δ_2 are touching, and otherwise I_1 is an interval $[c_1, c_2]$ inside (0,1).

For $t \in I_2$, we have that the determinant of the matrix in (2.22) must be zero, i.e.,

$$t(1-t)E(t)^3 - \tfrac{1-t}{t}A(t)E(t) - \tfrac{t}{1-t}B(t)E(t) = 0 \tag{2.25}$$

which implies

$$E(t)^2 = \tfrac{1}{t^2}A(t) + \tfrac{1}{(1-t)^2}B(t) \tag{2.26}$$

on the set where $E(t) \neq 0$. This means that

$$2E(t)E'(t) = \tfrac{1}{t^2}A'(t) + \tfrac{1}{(1-t)^2}B'(t) - \tfrac{2}{t^3}A(t) + \tfrac{2}{(1-t)^3}B(t)$$

Plugging this into the third equation of (2.22), we get

$$\tfrac{2}{t(1-t)}(A'(t) + B'(t)) + \tfrac{1}{t^2}A'(t) + \tfrac{1}{(1-t)^2}B'(t) - \tfrac{2}{t^3}A(t) + \tfrac{2}{(1-t)^3}B(t) = 0$$

which simplifies to

$$\tfrac{1+t}{t}A'(t) + \tfrac{2-t}{1-t}B'(t) - \tfrac{2(1-t)}{t^2}A(t) + \tfrac{2t}{(1-t)^2}B(t) = 0 \tag{2.27}$$

The first two equations can be solved for $\frac{E'(t)}{E(t)}$ giving us

$$\tfrac{t}{1-t}\tfrac{A'(t)}{A(t)} = \tfrac{1-t}{t}\tfrac{B'(t)}{B(t)}. \tag{2.28}$$

So our new system of two ODE's is

$$\tfrac{1+t}{t}A'(t) + \tfrac{2-t}{1-t}B'(t) - \tfrac{2(1-t)}{t^2}A(t) + \tfrac{2t}{(1-t)^2}B(t) = 0 \tag{2.29}$$

$$\tfrac{t}{1-t}\tfrac{A'(t)}{A(t)} = \tfrac{1-t}{t}\tfrac{B'(t)}{B(t)}. \tag{2.30}$$

for $t \in I_2$, and with four boundary conditions

$$A(1) = \alpha_1^2 \qquad B(1) = 0, \tag{2.31}$$

$$A(0) = 0 \qquad B(0) = \alpha_2^2. \tag{2.32}$$

Warning: it might seem that this system is independent on β_1, β_2, but it's not correct. β_1, β_2 will change the interval I_2!

To end the proof we involve an extra boundary conditions. As discussed in the introduction, the limiting zero distribution of $Q_{n,m}(z)$ is supported on

$[\beta_1 - 2\alpha_1, e_1]$, $[e_2, \beta_2 + 2\alpha_2]$ for some $e_1 \leq \beta_1 + 2\alpha_1$ and $e_2 \geq \beta_2 - 2\alpha_2$. In the limit this should give us the following two extra boundary conditions:

$$C(0) = \beta_1 - 2\alpha_1 \tag{2.33}$$
$$D(1) = \beta_2 + 2\alpha_2 \tag{2.34}$$

Thus we have the system

$$\begin{pmatrix} tE(t) & 0 & (1-t)A(t) \\ 0 & (1-t)E(t) & tB(t) \\ \frac{1}{t(1-t)} & \frac{1}{t(1-t)} & E(t) \end{pmatrix} \begin{pmatrix} A'(t) \\ B'(t) \\ E'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.35}$$

with the boundary/initial conditions

$$\begin{aligned} A(0) &= 0, & A(1) &= \alpha_1^2, \\ B(0) &= \alpha_2^2, & B(1) &= 0, \\ E(0) &= \beta_1 - \beta_2 - 2\alpha_1, & E(1) &= \beta_1 - \beta_2 - 2\alpha_2. \end{aligned}$$

Now recall that on the non-constant region I_2 where $E \neq 0$, we have

$$E(t)^2 = \tfrac{1}{t^2}A(t) + \tfrac{1}{(1-t)^2}B(t) \tag{2.36}$$

Putting $t = 0$, we therefore should expect $A(0) = A'(0) = 0$ and

$$E(0)^2 = \tfrac{1}{2}A''(0) + B(0),$$

which implies

$$A''(0) = 2E(0)^2 - 2B(0) = 2(\beta_1 - \beta_2 - 2\alpha_1)^2 - 2\alpha_2^2 \tag{2.37}$$

Similarly $t = 1$ gives us $B(1) = B'(1) = 0$ and

$$E(1)^2 = A(1) + \tfrac{1}{2}B''(1),$$

which implies

$$B''(1) = 2E(1)^2 - 2A(1) = 2(\beta_1 - \beta_2 - 2\alpha_2)^2 - 2\alpha_1^2 \tag{2.38}$$

We need to remove double zeros of A and B at 0 and 1, respectively, to reduce our system to a standard initial value problem. So let

$$K(t) = \frac{A(t)}{t^2}, \qquad L(t) = \frac{B(t)}{(1-t)^2}.$$

Then our system (2.35) becomes:

$$\begin{pmatrix} tE(t) & 0 & (1-t)K(t) \\ 0 & (1-t)E(t) & tL(t) \\ \frac{t}{(1-t)} & \frac{1-t}{t} & E(t) \end{pmatrix} \begin{pmatrix} K'(t) \\ L'(t) \\ E'(t) \end{pmatrix} = \begin{pmatrix} -2E(t)K(t) \\ 2E(t)L(t) \\ -\frac{2}{1-t}K(t) + \frac{2}{t}L(t) \end{pmatrix} \tag{2.39}$$

with the boundary/initial conditions

$$K(0) = \tfrac{1}{2}A''(0) = (\beta_1 - \beta_2 - 2\alpha_1)^2 - \alpha_2^2, \qquad K(1) = A(1) = \alpha_1^2,$$
$$L(0) = B(0) = \alpha_2^2, \qquad L(1) = \tfrac{1}{2}B''(1) = (\beta_1 - \beta_2 - 2\alpha_2)^2 - \alpha_1^2,$$
$$E(0) = \beta_1 - \beta_2 - 2\alpha_1, \qquad E(1) = \beta_1 - \beta_2 - 2\alpha_2.$$

The determinant of the matrix in (2.39) is equal to $t(1-t)E(t)(E(t)^2 - K(t) - L(t))$ which should be 0 on I_2 due to (2.36), so numerical simulation of this system is not likely to work well. So we need to eliminate $E(t)$ using $E(t)^2 = K(t) + L(t)$. This leads to:

$$(1+t)tK'(t) + 4tK(t) + (2-t)(1-t)L'(t) - 4(1-t)L(t) = 0 \tag{2.40}$$
$$\frac{t^2K'(t)}{K(t)} + 2t = \frac{(1-t)^2L'(t)}{L(t)} - 2(1-t) \tag{2.41}$$

with initial/boundary conditions (2.17)

$$\begin{cases} K(0) = (\beta_1 - \beta_2 - 2\alpha_1)^2 - \alpha_2^2 \\ L(0) = \alpha_2^2, \end{cases} \qquad \begin{cases} K(1) = \alpha_1^2 \\ L(1) = (\beta_1 - \beta_2 - 2\alpha_2)^2 - \alpha_1^2. \end{cases}$$

Theorem is proved.

This system of ODE's can be simulated. See the pictures below. Note that essentially it's two initial value problems – one at $t = 0$ and one at $t = 1$.

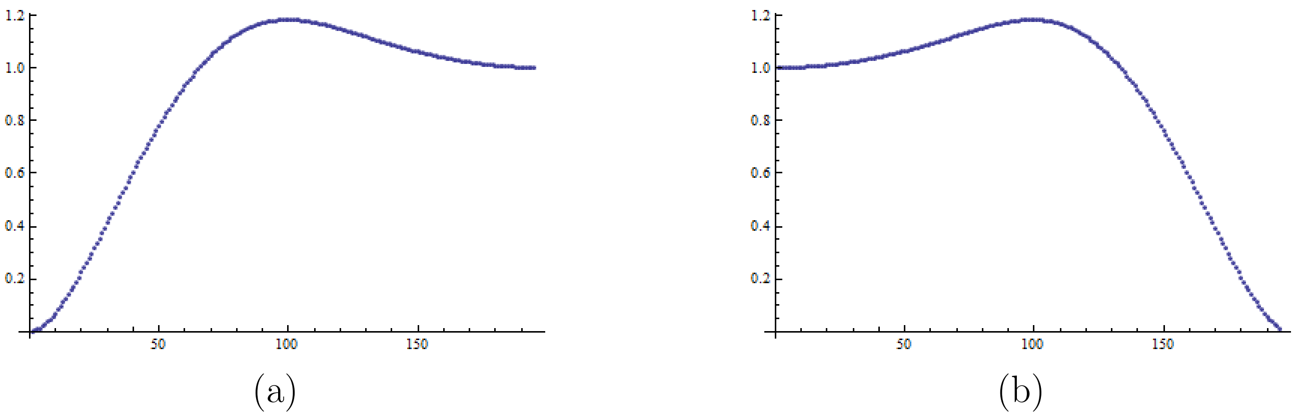


Figure 1. The case $\alpha_1 = \alpha_2 = 1$, $\beta_1 = -\beta_2 = -2$ (supports of μ_1 and μ_2 are symmetric and touching): (a) Function $A(t)$ (b) Function $B(t)$ ($t \in [0,1]$ corresponds to [0,200] on the graph)

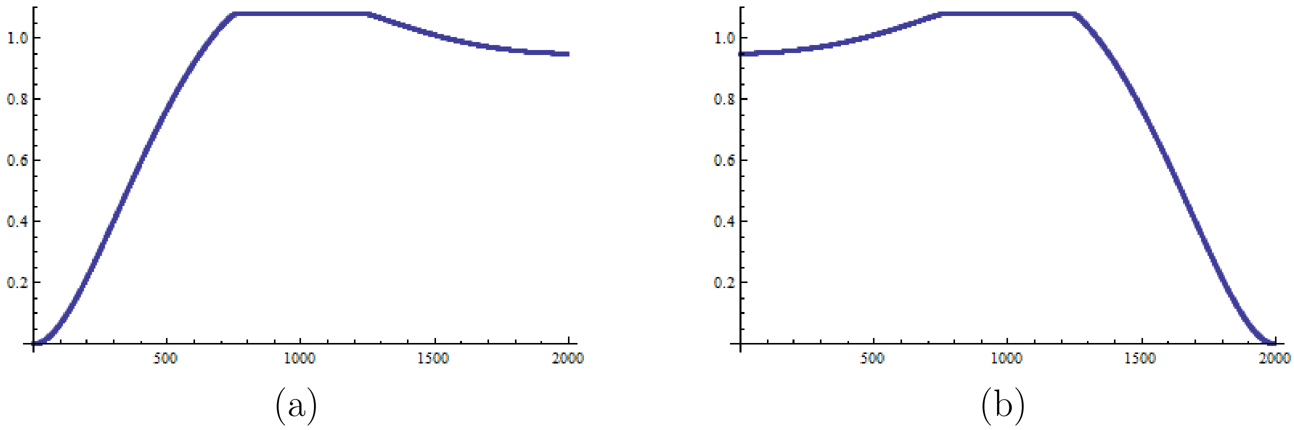


Figure 2. The case $\alpha_1 = \alpha_2 = 0.97$, $\beta_1 = -\beta_2 = 1$ (supports of μ_1 and μ_2 are symmetric but do not touch): (a) Function $A(t)$ (b) Function $B(t)$ ($t \in [0,1]$ corresponds to [0,2000] on the graph, so $t = 1/2$ is at the x-coordinate 1000)

3. Comparing numerics: Angelesco system $d = 2$

The following is a simple but useful observation: the solution of the ODE solved from $t = 0$ depends only on the boundary conditions (2.17), that is only on quantities $(\beta_1 - \beta_2 - 2\alpha_1)^2 - \alpha_2^2$ and α_2^2. So if α_2 and β_2 are fixed (i.e., the support of μ_2), and also $\beta_1 - 2\alpha_1$ is fixed (the left-most edge of the support of μ_1), then the boundary conditions at $t = 0$ are going to be the same.

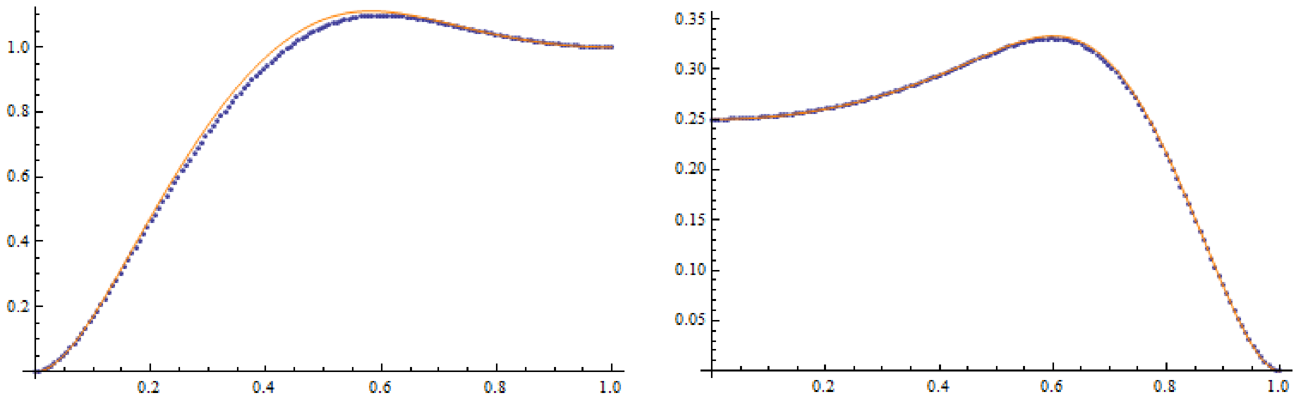
In other words, suppose we have an Angelesco system with $\mathrm{supp}\mu_1 = [f_1, f_2]$, $\mathrm{supp}\mu_2 = [f_3, f_4]$. Then the left parts (before the plateau) of functions $A(t)$, $B(t)$ will coincide with the left parts of functions $\widetilde{A}(t)$, $\widetilde{B}(t)$ for the Angelesco system having $\mathrm{supp}\widetilde{\mu}_1 = [f_1, f_3]$, $\mathrm{supp}\widetilde{\mu}_2 = [f_3, f_4]$ (touching supports, so $\widetilde{A}(t)$, $\widetilde{B}(t)$ have no plateaus).

Similarly, suppose we have an Angelesco system with $\mathrm{supp}\mu_1 = [f_1, f_2]$, $\mathrm{supp}\mu_2 = [f_3, f_4]$. Then the right parts (after the plateau) of functions $A(t)$, $B(t)$ will coincide with the right parts of functions $\widehat{A}(t)$, $\widehat{B}(t)$ for the Angelesco system having $\mathrm{supp}\widehat{\mu}_1 = [f_1, f_2]$, $\mathrm{supp}\widehat{\mu}_2 = [f_2, f_4]$ (touching supports, so $\widehat{A}(t)$, $\widehat{B}(t)$ have no plateaus).

Let us illustrate this with the pictures below. The blue dots will always correspond to the functions $A(t)$ and $B(t)$ approximated via the NNR coefficients: $A(t) \approx a_{n,m}$, $B(t) \approx b_{n,m}$ with $t = \frac{n}{n+m}$ and $n + m \approx 200$ (increasing $n + m$ does not seem to noticeably change the picture). The solid curves will corresponds to the numerical approximation of the solution of the ODE (using Mathematica, not sure if it's Runge–Kutta or something else).

First, let us start with $\beta_1 = -2$, $\alpha_1 = 1$, $\beta_2 = 1$, $\alpha_2 = 0.5$ (so that $\mathrm{supp}\,\mu_1 = [-4,0]$ and $\mathrm{supp}\,\mu_2 = [0,2]$).

In the next three pictures we will modify the left endpoint of $\mathrm{supp}\,\mu_2$ while

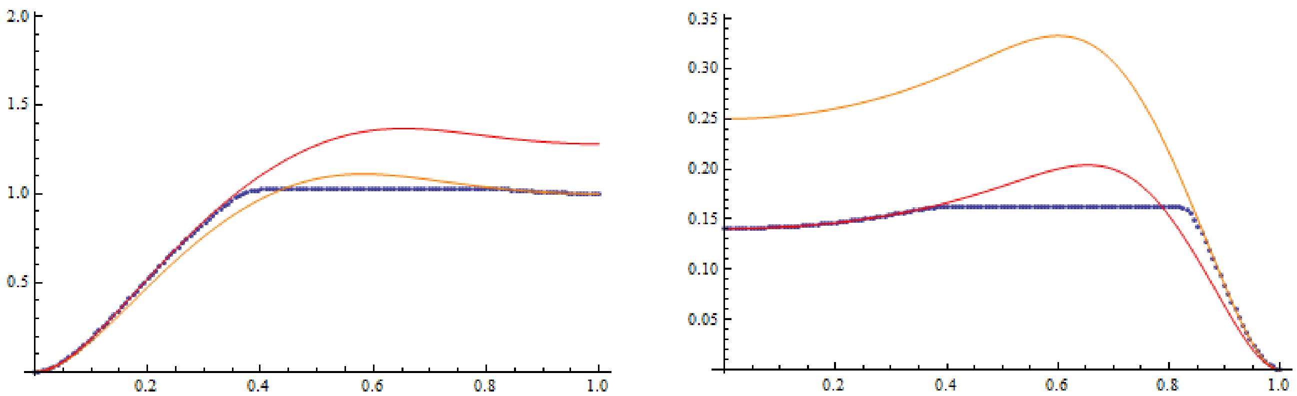


(a) (b)

Figure 3. Functions $A(t)$ and $B(t)$ when $\operatorname{supp}\mu_1 = [-4{,}0]$ and $\operatorname{supp}\mu_2 = [0{,}2]$

keeping the other endpoints fixed.

So let us move to Fig.4: blue dots correspond to the simulations with $\beta_1 = -2$, $\alpha_1 = 1$, $\beta_2 = 1.25$, $\alpha_2 = 0.375$, so that $\operatorname{supp}\mu_1 = [-4{,}0]$ and $\operatorname{supp}\mu_2 = [0.5{,}2]$ (note that $\operatorname{supp}\mu_1$ is the same as before, and $\operatorname{supp}\mu_2$ has the same right endpoint as before too, but now supports do not touch). On the same plot we include the solutions to the ODE for the case $\operatorname{supp}\mu_1 = [-4{,}0]$, $\operatorname{supp}\mu_2 = [0{,}2]$ (orange curve) and for the case $\operatorname{supp}\mu_1 = [-4{,}0.5]$, $\operatorname{supp}\mu_2 = [0.5{,}2]$ (red curve). Notice the orange curve fits correctly around $t = 1$ and red fits correctly around $t = 0$.

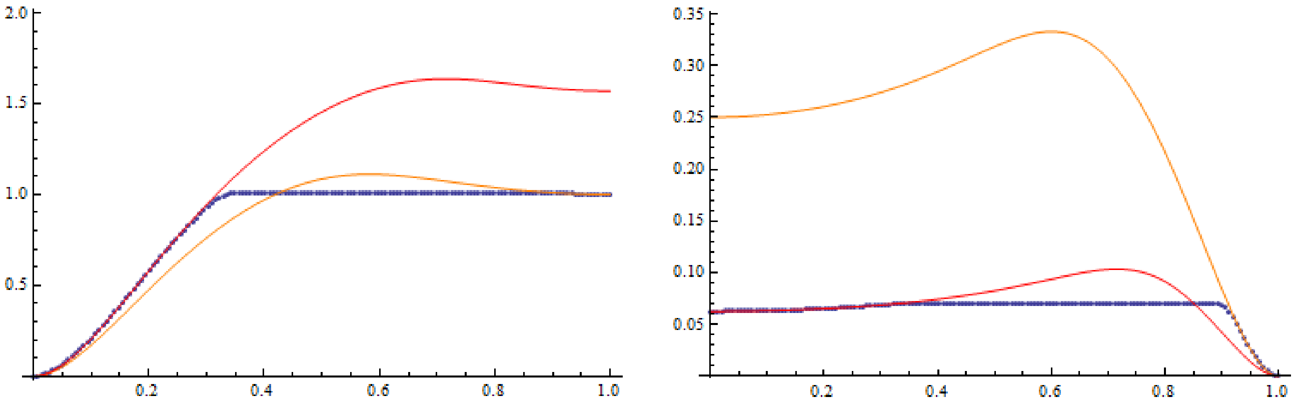


(a) (b)

Figure 4. Blue dots: limits of coefficients when $\operatorname{supp}\mu_1 = [-4{,}0]$, $\operatorname{supp}\mu_2 = [\mathbf{0.5}{,}2]$; Orange curve: ODE when $\operatorname{supp}\mu_1 = [-4{,}0]$, $\operatorname{supp}\mu_2 = [0{,}2]$; Red curve: ODE when $\operatorname{supp}\mu_1 = [-4, \mathbf{0.5}]$, $\operatorname{supp}\mu_2 = [\mathbf{0.5}{,}2]$

Now we move on to Fig.5: blue dots correspond to the simulations with $\beta_1 = -2$, $\alpha_1 = 1$, $\beta_2 = 1.5$, $\alpha_2 = 0.25$, so that $\operatorname{supp}\mu_1 = [-4{,}0]$ and $\operatorname{supp}\mu_2 = [1{,}2]$ ($\operatorname{supp}\mu_1$ and the right endpoint of $\operatorname{supp}\mu_2$ are the same as before). On the same plot we include the solutions to the ODE for the case $\operatorname{supp}\mu_1 = [-4{,}0]$, $\operatorname{supp}\mu_2 = [0{,}2]$ (orange curve) and for the case $\operatorname{supp}\mu_1 = [-4{,}1]$, $\operatorname{supp}\mu_2 = [1{,}2]$

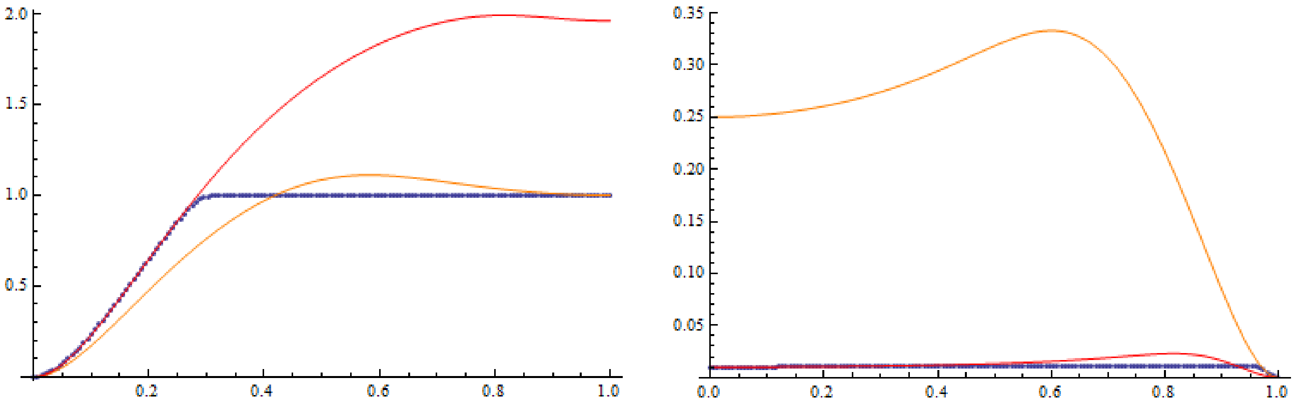
(red curve). Notice the orange curve fits correctly around $t = 1$ and red fits correctly around $t = 0$.



(a) (b)

Figure 5. Blue dots: limits of coefficients when supp $\mu_1 = [-4{,}0]$, supp $\mu_2 = [\mathbf{1}{,}2]$; Orange curve: ODE when supp $\mu_1 = [-4{,}0]$, supp $\mu_2 = [0{,}2]$; Red curve: ODE when supp $\mu_1 = [-4, \mathbf{1}]$, supp $\mu_2 = [\mathbf{1}{,}2]$

Finally we move on to Fig.6: blue dots correspond to the simulations with $\beta_1 = -2$, $\alpha_1 = 1$, $\beta_2 = 1.8$, $\alpha_2 = 0.1$, so that supp $\mu_1 = [-4{,}0]$ and supp $\mu_2 = [1.6{,}2]$ (supp μ_1 and the right endpoint of supp μ_2 are the same as before). On the same plot we include the solutions to the ODE for the case supp $\mu_1 = [-4{,}0]$, supp $\mu_2 = [0{,}2]$ (orange curve) and for the case supp $\mu_1 = [-4{,}1.6]$, supp $\mu_2 = [1.6{,}2]$ (red curve). Notice the orange curve fits correctly around $t = 1$ and red fits correctly around $t = 0$.



(a) (b)

Figure 6. Blue dots: limits of coefficients when supp $\mu_1 = [-4{,}0]$, supp $\mu_2 = [\mathbf{1.6}{,}2]$; Orange curve: ODE when supp $\mu_1 = [-4{,}0]$, supp $\mu_2 = [0{,}2]$; Red curve: ODE when supp $\mu_1 = [-4, \mathbf{1.6}]$, supp $\mu_2 = [\mathbf{1.6}{,}2]$

4. Appendix: parametrization for the touching intervals case

Here we present formulas for the limits of recurrence coefficients (1.12), (1.4) for MOPs with respect to the Angelesco system ($d = 2$) supported on two touched intervals $[-a, 0]$, $a > 0$ and $[0, 1]$. These formulas are based on a parametrization of the end points of the support of zero distribution, see (1.8), (1.10). This parametrization was proposed in [18, 19] and developed in details (for the non-diagonal case) in [17]. We start with this parametrization.

Let $a \in (0, +\infty)$, $b \in (0,1)$ and parameters $s \in (1,2)$, $\tau \in (1, +\infty)$ and $\mathsf{U} := (1,2) \times (1, +\infty)$ is half-strip in $\mathbb{R}^2$. The following smooth functions

$$\mathcal{A}(s, \tau) := R_s(\tau) - 1, \quad \mathcal{B}(s, \tau) := \frac{\mathcal{A}(s, \tau) S(s)}{1 + \mathcal{A}(s, \tau) - S(s)},$$

where

$$R_s(\tau) := \frac{\tau^2(\tau + s - 2)}{(2s - 1)\tau - s}, \quad S(s) := \frac{s(2 - s)^3}{(2s - 1)^3},$$

define a diffeomorphism $(\mathcal{A}, \mathcal{B})$ of U on half - strip $(0, +\infty)_a \times (0,1)_b$. The inverse transformation: $(a, b) \in (0, +\infty)_a \times (0,1)_b \;\to\; (s, \tau) \in \mathsf{U}$ is defined by solution of the equations (the fact of $\exists!$ solutions was proven in [17])

$$\exists\, ! \, s \in (0,1) : \;\; S(s) = \frac{b(1 + a)}{a + b}, \qquad \exists\, ! \, \tau > 1 : \;\; 1 + a = \frac{\tau^2(\tau + s - 2)}{(2s - 1)\tau - s}. \quad (4.1)$$

We also introduce a function $\Theta : \mathsf{U} \to (-1; 1)$

$$\Theta(s, \tau) := (\tau - s) \left(\frac{2 + 2s\tau - s - \tau}{(2s\tau - s - \tau)(s + \tau)(s + \tau - 2)} \right)^{1/2}.$$

Given $\{\mu_1, \mu_2\}$: $\operatorname{supp} \mu_1 = [-a, 0]$ and $\operatorname{supp} \mu_1 = [0, 1]$, we consider extremal problem (1.9) for

$$\tau_1 =: (1 + \theta)/2, \quad \tau_2 =: (1 - \theta)/2, \quad \theta \in (-1, 1).$$

We would like to know answers for the following questions:

1) How to find the value of

$$\theta_a := \min\{\theta \;:\; \operatorname{supp} \omega_1 = [-a,0]\} = \max\{\theta \;:\; \operatorname{supp} \omega_2 = [0,1]\} \;?$$

2) How to find[§] the value of $b_\theta : \operatorname{supp} \omega_2 = [b_\theta,1]$?

The answers from [17] are:

1) Excluding the variable τ from the equations $\mathcal{A}(2, \tau) = a$ and $\Theta(2, \tau) = \theta$, we can get θ_a and a_θ.

$$2) \;\exists\, ! \; (\tilde{s}, \tilde{\tau}) \;:\; \begin{cases} \mathcal{A}(\tilde{s}, \tilde{\tau}) = a \in (0; a_\theta) \\ \Theta(\tilde{s}, \tilde{\tau}) = \theta \in (\theta_a; 1) \end{cases} \quad \Rightarrow \quad b_\theta := \mathcal{B}(\tilde{s}, \tilde{\tau}).$$

[§]For fixed $\theta \in (\theta_a, 1)$

Proposition 1. *Let* $[-a,0],[b,1]$ *be supports* (1.10) *of extremal measures* (1.9) *of the Angelesco system* $\{\mu_1,\mu_2\}$*, and let* $\big(\ s(a,b),\ \tau\big(a,s(a,b)\big)\ \big)$ *be the image of transformation* (4.1)*. Then for limits* (1.12) *of the corresponding NNRR coefficients we have*

$$\alpha_{1,\vec{t}} = C_1\frac{-at^2(\tau-\alpha)}{(\tau-\tau_1)^2(\tau-\tau_2)},\quad \beta_{1,\vec{t}} = C_1\alpha_1 a^2\frac{\tau^3(\tau+s-2)(-2\tau-\tau_1)}{(\tau-\tau_1)^4(\tau-\tau_2)^3}\ . \tag{4.2}$$

$$\textit{where}\quad \tau_1+\tau_2=-(s+\tau-2),\quad \tau_1\tau_2=-\frac{s\tau(s+\tau-2)}{2s\tau-s-\tau},\ \tau_1<\tau_2<\tau\,, \tag{4.3}$$

$$C_1 := \frac{-a\tau_1^2(\tau_1-\alpha)}{(\tau_1-\tau)^2(\tau_1-\tau_2)},\quad \alpha=2-s. \tag{4.4}$$

Proof. The Riemann surface from the Subsection 1.3 can be defined by means of the conformal map of the sphere $\overline{\mathbb{C}}_w \to \mathfrak{R}_{\mathbf{z}}$

$$z(w) := \pi(\mathbf{z}(w)) = \frac{aR_s(w)}{1+a-R_s(w)} = \frac{-aw^2(w-\alpha)}{(w-\tau)(w-\tau_1)(w-\tau_2)},$$

where $\pi:\ \mathfrak{R} \to \overline{\mathbb{C}}$ is the natural projection and for the preimages of $\pi^{-1}(\infty)$: $\tau \to \infty_0, \tau_1 \to \infty_1, \tau_2 \to \infty_2$ we have (4.3) .

Our goal is to define the meromorphic on $\mathfrak{R}$ functions Υ_i and to determine their residues: $\mathbf{z}^2\Upsilon_i(z^{(0)}) = \alpha_{\vec{t},i}(z+\beta_{\vec{c},i}) + \mathcal{O}\big(z^{-1}\big)$ as $z\to\infty$. We have

$$\Upsilon^{(1)}(z(w)) = C_1\frac{w-\tau}{w-\tau_1},\ \ \alpha_{1,\vec{t}} = \Big(z\Upsilon^{(1)}\Big)\Big|_{w=\tau},\ \ \beta_{1,\vec{t}} = \left[z\left(\frac{z}{\alpha_1}\Upsilon^{(1)}-1\right)\right]\Big|_{w=\tau},$$

where $w\in\overline{C}$ and C_1 is defined from the normalization

$$\left(\frac{\Upsilon^{(1)}}{z}\right)\Big|_{w=\tau_1} = 1,\ \ \Rightarrow\ \ C_1 = \left(z(w)\frac{w-\tau_1}{w-\tau}\right)\Big|_{w=\tau_1} = \frac{-a\tau_1^2(\tau_1-\alpha)}{(\tau_1-\tau)^2(\tau_1-\tau_2)}.$$

Thus we get (4.4) and by the same way the expression for $\alpha_{1,\vec{t}}$ in (4.2). Denoting

$$P(w) := \frac{w^2(w-\alpha)}{(w-\tau_1)^2(w-\tau_2)}\ ;\quad \frac{P'}{P}(w) = \frac{2}{w}+\frac{1}{w-\alpha}-\frac{2}{w-\tau_1}-\frac{1}{w-\tau_2}\,,$$

we arrive to

$$\beta_{1,\vec{t}} = C_1\alpha_1 a^2\frac{\tau^2(\tau-\alpha)}{(\tau-\tau_1)(\tau-\tau_2)}\cdot\underbrace{\frac{P(w)-P(\tau)}{w-\tau}\Big|_{w=\tau}}_{=P'(\tau)} = C_1\alpha_1 a^2\frac{\tau^3(\tau-\alpha)(2\tau_2+\tau_1-2\alpha)}{(\tau-\tau_1)^4(\tau-\tau_2)^3}.$$

The Proposition is proved.

References

[1] A. I. Aptekarev, M. Derevyagin, W. Van Assche, Discrete integrable systems generated by Hermite–Pade approximants, Nonlinearity, 29(5):1487–1506, 2016.

[2] A. I. Aptekarev, S. A. Denisov, M. L. Yatsselev, Completely integrable on Z_+^d potentials for electromagnetic Schrodinger operator: rays asymptotics and scattering problem, Keldysh Institute Preprints, 2015, 088, 20 pp. (in Russian) http://www.library.keldysh.ru/preprint.asp?id=2015-88, http://mi.mathnet.ru/ipmp2050

[3] A. I. Aptekarev, Integrable semidiscretization of hyperbolic equations – "computational" dispersion and multidimensional perspective, Keldysh Institute Preprints, 2012, 020, 28 pp. (in Russian) http://www.library.keldysh.ru/preprint.asp?id=2012-20 http://mi.mathnet.ru/ipmp38

[4] A. I. Aptekarev, S. A. Denisov, M. L. Yatsselev, Selfadjoint Jacobi matrices on trees and multiple orthogonal polynomials, arXiv:1806.10531v1.

[5] A. I. Aptekarev, W. Van Assche, M. L. Yatsselev, Hermite-Pade Approximants for a Pair of Cauchy Transforms with Overlapping Symmetric Supports, Communications on Pure and Applied Mathematics, Vol. LXX, 0444—0510, 2017

[6] A.I. Aptekarev, V. Kalyagin, G. Lopez Lagomasino, I.A. Rocha, On the limit behavior of recurrence coefficients for multiple orthogonal polynomials, JAT 139:346–370, 2006.

[7] A.I. Aptekarev, Spectral Problems of High-Order Recurrences, Amer. Math. Soc. Transl., 233 (2014), 43–61

[8] A. I. Aptekarev, V. A. Kalyagin, E. B. Saff, Higher-order three-term recurrences and asymptotics of multiple orthogonal polynomials, Constr. Approx., 30:2 (2009), 175-–223

[9] G. Filipuk, M. Haneczok, W. Van Assche, Computing recurrence coefficients of multiple orthogonal polynomials, Numerical Algorithms, Vol. 70(3):519–543, 2015.

[10] A. A. Gonchar and E. A. Rakhmanov, On the convergence of simultaneous Padé approximants for systems of functions of Markov type, Proc. Steklov Inst. Math. 157 (1983), 31–50.

[11] V. A. Kaliaguine, On operators associated with Angelesco systems, East J. Approx. 1 (1995), no. 2, 157–170.

[12] B. Simon, Szegő's theorem and its descendants: spectral theory for L^2 perturbations of orthogonal polynomials, Princeton University Press, 2011.

[13] W. Van Assche, Chapter 23, Classical and Quantuum Orthogonal Polynomials in One Variable: in (by M.E.H. Ismail), volume 98 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2005. 3, 23

[14] W. Van Assche, Nearest neighbor recurrence relations for multiple orthogonal polynomials, J. of Approx. Theory 163:1427–1448, 2011.

[15] W. Van Assche, Ratio asymptotics for multiple orthogonal polynomials, In Modern trends in constructive function theory, Contemp. Math., 661:73–85, 2016, AMS, Providence, RI.

[16] M. Yattselev, Strong asymptotics of Hermite-Pade approximants for Angelesco systems, Canad. J. Math., 68(5):1159–1200

[17] V.G. Lysov, D.N. Tulyakov, On the Supports of Vector Equilibrium Measures in the Angelesco Problem with Nested Intervals Proc. Steklov Inst. Math., 301:180—196, 2018,

[18] A. I. Aptekarev, V. G. Lysov, D. N. Tulyakov, "Three-sheeted Riemann surfaces of genus 0 with fixed projections of the branch points", Keldysh Institute preprints, 2007, 013, 21 pp. (in Russian) http://www.library.keldysh.ru/preprint.asp?id=2007-13

[19] A. I. Aptekarev, V. A. Kalyagin, V. G. Lysov, D. N. Toulyakov, "Equilibrium of vector potentials and uniformization of the algebraic curves of genus 0", J. Comput. Appl. Math., 233:3 (2009), 602–616