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ИНСТИТУТ ПРИКЛАДНОЙ МАТЕМАТИКИ
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Zero-mean interpolation inequality on the sphere

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Интерполяционные неравенства на сфере для функций с нулевым средним

Доказываются мультипликативные интерполяционные неравенства для вложений пространства Соболева на сфере в критическом случае для функций с нулевым средним, а также аналогичные неравенства для касательных вектор-функций. Соответствующие константы получены в явном виде и с точным ростом по q. Аналогичные неравенства доказаны в одномерном периодическом случае.

Ключевые слова: интерполяционные неравенства, неравенства Либа–Тирринга.

A.A. Ilyin

Zero-mean interpolation inequality on the sphere

We prove multiplicative interpolation inequalities for the imbeddings of the Sobolev space on the sphere in the critical case for zero-mean functions and similar inequalities for tangent vector functions. The corresponding constants are explicitly found with sharp rate of growth with respect to q. Similar inequalities are proved in the one-dimensional periodic case.

Key words: interpolation inequalities, Lieb–Thirring inequalities.

Оглавление

1 Introduction

The following interpolation inequality holds on the sphere $\mathbb{S}^d$:

$$\frac{q-2}{d}\int_{\mathbb{S}^d}|\nabla\varphi|^2d\mu+\int_{\mathbb{S}^d}|\varphi|^2d\mu\geqslant\left(\int_{\mathbb{S}^d}|\varphi|^q d\mu\right)^{2/q}. \tag{1.1}$$

Here $d\mu$ is the normalized Lebesgue measure on $\mathbb{S}^d$:

$$d\mu=\frac{dS}{\frac{2\pi^{d+1}}{\Gamma(\frac{d+1}{2})}},$$

so that $\mu(\mathbb{S}^d)=1$. Next, $q\in[2,\infty)$ for $d=1,2$, and $q\in[2,2d/(d-2)]$ for $d\geqslant 3$. The remarkable fact about (1.1) is that the constant $(q-2)/d$ is sharp and the extremal functions are constants, see [2], [3], [4] and the references therein.

However, in applications (for instance, for the Navier–Stokes equations on the 2D sphere) the functions φ usually play the role of stream function of a divergence free vector function u, $u=\nabla^\perp\varphi$, and therefore without loss of generality φ can be chosen to be orthogonal to constants. In addition, inequalities in multiplicative form on a manifold are not equivalent to interpolation inequalities in the additive form due to the lack of scaling.

The goal of this work is to prove the following multiplicative interpolation inequality for functions on the sphere $\mathbb{S}^2$

$$\|f\|_{L_q(\mathbb{S}^2)}\leqslant\left(\frac{1}{2\pi}\right)^{(q-2)/2q}q^{1/2}\|f\|^{2/q}\|\nabla f\|^{1-2/q},\quad q\in[2,\infty). \tag{1.2}$$

Here $f\in H^1(\mathbb{S}^2)\cap\{\int_{\mathbb{S}^2}f(s)dS=0\}$. (Note that the first factor on the right-hand side is bounded by 1 uniformly in q.)

Our approach based on the Lieb–Thirring inequalities makes it possible to prove similar inequalities in the vector case. Namely, we show that for $u\in H^1(T\mathbb{S}^2)$ with $\operatorname{div}u=0$ it holds

$$\|u\|_{L_q(T\mathbb{S}^2)}\leqslant\left(\frac{1}{2\pi}\right)^{(q-2)/2q}q^{1/2}\|u\|^{2/q}\|\operatorname{rot}u\|^{1-2/q}.$$

Finally, we observe that in the one-dimensional periodic case the rate of growth of the constant in (1.1) is only due to the presence of constants and therefore in the zero-mean case it is natural to consider interpolation inequalities for the critical imbedding $H^{\frac12}(\mathbb{S}^1)\hookrightarrow L_q(\mathbb{S}^1)$. This is done in Section 3.

2 Estimates of Lieb–Thirring constants on $\mathbb{S}^2$

The scalar case

Let Δ be the (scalar) Laplace–Beltrami operator on the 2D sphere. Then

$$-\Delta Y_n^k = \Lambda_n Y_n^k, \quad k = 1, \dots, 2n+1, \quad n = 1, \dots .$$

The eigenvalues $\Lambda_n = n(n+1)$ have multiplicity $2n+1$ and the orthonormal spherical harmonics $Y_n^k(s)$, $k = 1, \dots, 2n+1$, corresponding to the same eigenvalue $n(n+1)$, satisfy the important identity

$$\sum_{k=1}^{2n+1} Y_n^k(s)^2 = \frac{2n+1}{4\pi}. \tag{2.1}$$

We shall also use the following notation

$$-\Delta\varphi_i = \lambda_i\varphi_i, \tag{2.2}$$

where

$$\{\varphi_i\}_{i=1}^{\infty} = \left\{\{Y_n^k\}_{k=1}^{2n+1}\right\}_{n=1}^{\infty}, \qquad \{\lambda_i\}_{i=1}^{\infty} = \{\underbrace{\Lambda_n, \dots, \Lambda_n}_{2n+1 \text{ раз}}\}_{n=1}^{\infty}. \tag{2.3}$$

Let Π be the orthogonal projection onto the subspace of functions orthogonal to constants

$$\Pi\varphi = \varphi - \frac{1}{4\pi}\int_{\mathbb{S}^2}\varphi(s)dS.$$

We consider the Schrödinger operator

$$\Pi(-\Delta + f)\Pi \tag{2.4}$$

with

$$f \in L_p(\mathbb{S}^2), \qquad p > 1.$$

Since the sphere is compact the whole spectrum of the operator (2.4) is discrete and we denote by $\mu_j = \mu_j(f)$ its eigenvalues.

Theorem 2.1. *The following Lieb–Thirring estimate holds for the negative eigenvalues $\mu_j \leqslant 0$:*

$$\sum_{\mu_j \leqslant 0} |\mu_j|^\gamma \leqslant \mathrm{L}_{\mathbb{S}^2}(\gamma) \int_{\mathbb{S}^2} f(s)_-^{1+\gamma} dS. \tag{2.5}$$

Here $\gamma > 0$, $f_- := -\min(f, 0)$ and the Lieb–Thirring constant $\mathrm{L}_\gamma(\mathbb{S}^2)$ satisfies

$$\begin{aligned} \mathrm{L}_{\mathbb{S}^2}(\gamma) &\leqslant \min_{1<k<\gamma+1} F_{\mathbb{S}^2}(\gamma, k), \\ F_{\mathbb{S}^2}(\gamma, k) &= \frac{1}{3\pi}\left(\frac{3(2k-1)}{4k}\right)^k \cdot \frac{\gamma^{\gamma+1} B(\gamma+1-k, k+1)}{(k-1)^k(\gamma+1-k)^{\gamma+1-k}}. \end{aligned} \tag{2.6}$$

In particular,

$$\mathrm{L}_{\mathbb{S}^2}(1) \leqslant \frac{1}{2} \qquad (\mathrm{L}_{\mathbb{S}^2}(1)) \leqslant 0.444\ldots). \tag{2.7}$$

Proof. We set

$$N_r(f) = \#\{\mu_j(f),\ \mu_j(f) \leqslant r\}.$$

Then

$$\sum_{\mu_j \leqslant 0} |\mu_j|^\gamma = \gamma \int_0^\infty r^{\gamma-1} N_{-r}(f) dr. \tag{2.8}$$

Next we use the Birman–Schwinger kernel

$$N_{-r}(f) \leqslant \mathrm{Tr}\Bigl((f + (1-t)r)_-^{1/2} (\Pi(-\Delta + tr)\Pi)^{-1} (f + (1-t)r)_-^{1/2}\Bigr)^k,$$

and the convexity inequality of Lieb and Thirring [9], [1]:

$$\mathrm{Tr}(B^{1/2} C B^{1/2})^k \leqslant \mathrm{Tr}\, B^{k/2} C^k B^{k/2}, \quad B = B^* \geqslant 0,\ C = C^* \geqslant 0.$$

We obtain that

$$N_{-r}(f) \leqslant \mathrm{Tr}(f + (1-t)r)_-^{k/2} (\Pi(-\Delta + tr)\Pi)^{-k} (f + (1-t)r)_-^{k/2}.$$

The above inequalities hold for

$$r > 0, \quad k \geqslant 1, \quad \text{and } t \in [0, 1].$$

We now show that for $k>1$

$$N_{-r}(f)\leqslant\frac{1}{3\pi}\left(\frac{3(2k-1)}{4k}\right)^k(k-1)^{-1}(tr)^{1-k}\int_{\mathbb{S}^2}(f+(1-t)r)_-^k dS. \quad (2.9)$$

In fact, setting $g=(f+(1-t)r)_-$ and using the eigenfunctions in the notation (2.2), (2.3) we obtain

$$\begin{aligned}&\operatorname{Tr} g^{k/2}(\Pi(-\Delta+tr)\Pi)^{-k}g^{k/2}=\operatorname{Tr}(\Pi(-\Delta+tr)\Pi)^{-k}g^k=\\&\int_{\mathbb{S}^2}\sum_{j=1}^\infty(\lambda_j+tr)^{-k}g(s)^k\varphi_j(s)^2dS=\\&\frac{1}{4\pi}\sum_{n=1}^\infty\frac{2n+1}{(n(n+1)+tr)^k}\int_{\mathbb{S}^2}g(s)^kdS,\end{aligned} \quad (2.10)$$

where in the last equality we used (2.1). To estimate $\sum_{n=1}^\infty\frac{2n+1}{(n(n+1)+tr)^k}$ we use the elementary inequality

$$\frac{2n+1}{n(n+1)+tr}\leqslant\frac{2n}{n^2+\frac{2}{3}tr}\quad\text{giving}\quad\frac{2n+1}{(n(n+1)+tr)^k}\leqslant\frac{2n}{(n^2+\frac{2}{3}tr)^k}\ .$$

We set

$$\Phi(x,\rho)=\frac{x}{(x^2+\rho)^k}\,,\quad\rho=\frac{2}{3}tr.$$

The function $\Phi(x,\rho)$ has a global maximum (in x) at

$$x_0=\frac{\rho^{1/2}}{(2k-1)^{1/2}}\quad\text{и}\quad\Phi(x_0,\rho)=\rho^{1/2-k}\,\frac{(2k-1)^{k-1/2}}{(2k)^k}\,.$$

Since $\Phi(x,\rho)$ is decreasing for $x\geqslant x_0$, it follows that

$$\begin{aligned}\sum_{n=1}^\infty\frac{n}{(n^2+\rho)^k}=\sum_{n=1}^\infty\Phi(n,\rho)\leqslant x_0\Phi(x_0,\rho)+\int_{x_0}^\infty\Phi(x,\rho)\,dx=\\ \rho^{1-k}\,\frac{(2k-1)^{k-1}}{(2k)^k}+\rho^{1-k}\,\frac{1}{2(k-1)}\,\frac{(2k-1)^{k-1}}{(2k)^{k-1}}=\rho^{1-k}\frac{1}{k-1}\left(\frac{2k-1}{2k}\right)^k.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{4\pi}\sum_{n=1}^\infty\frac{2n+1}{(n(n+1)+tr)^k}\leqslant\frac{1}{2\pi}\sum_{n=1}^\infty\Phi(n,\rho)\leqslant\\ \frac{1}{2\pi}\rho^{1-k}\frac{1}{k-1}\left(\frac{2k-1}{2k}\right)^k=\frac{1}{k-1}(tr)^{1-k}\cdot\frac{1}{3\pi}\left(\frac{3(2k-1)}{4k}\right)^k.\end{aligned}$$

which proves (2.9).

Using this in (2.8), we obtain for $k \in (1, \gamma+1)$

$$\sum_{\mu_j \leqslant 0} |\mu_j|^\gamma \leqslant \frac{1}{3\pi}\left(\frac{3(2k-1)}{4k}\right)^k \frac{\gamma t^{1-k}}{(k-1)} \int_0^\infty \int_{\mathbb{S}^2} r^{\gamma-k}(f(s)+(1-t)r)_-^k dS dr =$$
$$\frac{1}{3\pi}\left(\frac{3(2k-1)}{4k}\right)^k \frac{\gamma t^{1-k}}{(k-1)} \int_{\mathbb{S}^2} \int_0^\infty r^{\gamma-k}(f(s)+(1-t)r)_-^k dr dS =$$
$$\frac{1}{3\pi}\left(\frac{3(2k-1)}{4k}\right)^k \frac{\gamma t^{1-k}(1-t)^{k-\gamma-1}B(\gamma+1-k, k+1)}{(k-1)} \int_{\mathbb{S}^2} f_-(s)^{\gamma+1} dS =$$
$$F_{\mathbb{S}^2}(\gamma, k) \int_{\mathbb{S}^2} f_-(s)^{\gamma+1} dS,$$

where we have chosen the optimal $t = (k-1)/\gamma$. In the integral with respect to r we set for almost every s

$$r = \frac{1}{1-t} f_-(s)\, \rho.$$

If $f \leqslant 0$ and $f_- = -f$, then $(f+(1-t)r)_- = f_-(\rho-1)_-$ and

$$\int_0^\infty r^{\gamma-k}(f(s)+(1-t)r)_-^k dr = (1-t)^{k-\gamma-1} f_-(x)^{\gamma+1} \int_0^\infty \rho^{\gamma-k}(\rho-1)_-^k d\rho =$$
$$(1-t)^{k-\gamma-1} f_-(s)^{\gamma+1} \int_0^1 \rho^{\gamma-k}(1-\rho)^k d\rho =$$
$$(1-t)^{k-\gamma-1} B(\gamma+1-k, 1+k) f_-(s)^{\gamma+1}.$$

If $f(s) > 0$, then $(f(s)+(1-t)r)_- = 0$, $f_-(s) = 0$ and the above equality formally also holds.

The minimisation with respect to $k \in (1, \gamma+1)$ completes the proof of (2.6). For the proof of (2.7) we observe that

$$\mathrm{L}_{\mathbb{S}^2}(\gamma) \leqslant \min_{1<k<2} F_{\mathbb{S}^2}(1, k) = 0.444\ldots, \quad F_{\mathbb{S}^2}(1, 3/2) = \frac{1}{2}.$$

The proof is complete. □

The dual formulation of this estimate is as follows, see [9].

Theorem 2.2. *Let the family* $\{\varphi_j(s)\}_{j=1}^N \in H^1(\mathbb{S}^2)$ *be orthonormal and* $\int_{\mathbb{S}^2}\varphi_j(s)dS = 0$. *Then the following inequality holds*

$$\left(\int_{\mathbb{S}^2}\rho(s)^{p/(p-1)}dS\right)^{p-1} \leqslant \mathrm{k}_{\mathbb{S}^2}(p)\sum_{j=1}^N\|\nabla\varphi_j\|^2, \tag{2.11}$$

where

$$\rho(s) = \sum_{j=1}^N \varphi_j(s)^2.$$

Here p *satisfies* $1 < p \leqslant 2$, *and the constant* $\mathrm{k}_{\mathbb{S}^2}(p)$ *satisfies*

$$\mathrm{k}_{\mathbb{S}^2}(p) \leqslant \frac{p^p}{(p-1)^{p-1}}\mathrm{L}_{\mathbb{S}^2}(p-1). \tag{2.12}$$

Furthermore, for $p = 2$ *inequality* (2.11) *and estimate* (2.5) *are equivalent and*

$$\mathrm{k}_{\mathbb{S}^2}(2) = 4\mathrm{L}_{\mathbb{S}^2}(1). \tag{2.13}$$

Proof. Let

$$V(s) = \alpha\rho(s)^{1/(p-1)}, \tag{2.14}$$

where $\alpha > 0$ is a parameter. We denote by A the Schrödinger operator:

$$A := \Pi(-\Delta - V)\Pi \tag{2.15}$$

with eigenvalues $\mu_j = \mu_j(V)$. Let further

$$H = L_2(\mathbb{S}^2) \cap \int_{\mathbb{S}^2}\varphi(s)dS = 0.$$

We denote by $\bigwedge^N H$ the N-th exterior product of H. This a Hilbert space with elements that are linear combinations of products $v_1 \wedge \cdots \wedge v_N$. The scalar product of $v_1 \wedge \cdots \wedge v_N$ and $w_1 \wedge \cdots \wedge w_N$ is defined as follows:

$$(w_1 \wedge \cdots \wedge w_N, v_1 \wedge \cdots \wedge v_N) = \det\{(w_i, v_j)\}, \quad 1 \leqslant i, j \leqslant N$$

and is extended to the whole $\bigwedge^N H$ by linearity. We define the operator $A_N : \bigwedge^N H \to \bigwedge^N H$:

$$A_N(v_1 \wedge \cdots \wedge v_N) = (A\,v_1 \wedge \cdots \wedge v_N + \cdots + v_1 \wedge \cdots \wedge A\,v_N).$$

Corresponding to A_N is the quadratic form

$$a_N(v_1 \wedge \cdots \wedge v_N, v_1 \wedge \cdots \wedge v_N) \;=\; (A_N(v_1 \wedge \cdots \wedge v_N), v_1 \wedge \cdots \wedge v_N).$$

If $\varphi_1, \ldots, \varphi_N$ is an orthonormal system, then

$$\begin{gathered} a_N(\varphi_1 \wedge \cdots \wedge \varphi_N, \varphi_1 \wedge \cdots \wedge \varphi_N) = \\ \sum_{j=1}^N \|\nabla \varphi_j\|^2 \,-\, \alpha \sum_{j=1}^N \int_{\mathbb{S}^2} \rho(s)^{1/(p-1)} |\varphi_j(s)|^2 dS = \\ \sum_{j=1}^N \|\nabla \varphi_j\|^2 - \alpha \int_{\mathbb{S}^2} \rho(s)^{p/(p-1)} dS. \end{gathered} \tag{2.16}$$

Let

$$E \;=\; \inf \sigma(A_N).$$

Then in view of (2.5) with (2.14)

$$\begin{gathered} E \;=\; \sum_{j=1}^N \mu_j(V) \geqslant \sum_{\mu_j(V) \leqslant 0} \mu_j(V) \;\geqslant \\ -\mathrm{L}_{\mathbb{S}^2}(\gamma)^{1/\gamma} \left(\int_{\mathbb{S}^2} V(s)^{\gamma+1} dS \right)^{1/\gamma} \;=\; \\ -\mathrm{L}_{\mathbb{S}^2}(\gamma)^{1/\gamma} \alpha^{p/\gamma} \left(\int_{\mathbb{S}^2} \rho(s)^{p/(p-1)} dS \right)^{1/\gamma}, \end{gathered} \tag{2.17}$$

where $\gamma = p - 1 \leqslant 1$ and where we used the elementary inequality

$$\sum |\mu_j| \leqslant \left(\sum |\mu_j|^\gamma \right)^{1/\gamma}.$$

On the other hand, from the variational principle and (2.16)

$$E \;\leqslant\; a_N(\varphi_1 \wedge \cdots \wedge \varphi_N, \varphi_1 \wedge \cdots \wedge \varphi_N) = \sum_{j=1}^N \|\nabla \varphi_j\|^2 - \alpha \int_{\mathbb{S}^2} \rho(s)^{p/(p-1)} dS. \tag{2.18}$$

Combining (2.17) and (2.18) we obtain

$$\alpha \int_{\mathbb{S}^2} \rho(s)^{p/(p-1)} dS - \mathrm{L}_{\mathbb{S}^2}(\gamma)^{1/\gamma} \alpha^{p/\gamma} \left(\int_{\mathbb{S}^2} \rho(s)^{p/(p-1)} dS \right)^{1/\gamma} \;\leqslant\; \sum_{j=1}^N \|\nabla \varphi_j\|^2.$$

Substituting the optimal

$$\alpha = (p/\gamma)^{\gamma/(\gamma-p)} \mathrm{L}_{\mathbb{S}^2}(\gamma)^{1/(\gamma-p)} \left(\int_{\mathbb{S}^2} \rho(s)^{p/(p-1)} dS\right)^{(1-\gamma)/(\gamma-p)},$$

and taking into account that $\gamma = p-1$, we obtain (2.11) with (2.12).

Finally, the equivalence of spectral and integral inequalities with $\gamma = 1$ and $p = 2$ with equality (2.13) for the constants is well known. The proof is complete. □

We are now in position to prove inequality (1.2).

Corollary 2.1. *Let $f \in H^1(\mathbb{S}^2)$ and let $\int_{\mathbb{S}^2} f(s)dS = 0$. Then*

$$\|f\|_{L_q(\mathbb{S}^2)} \leqslant \left(\frac{1}{2\pi}\right)^{(q-2)/2q} q^{1/2} \|f\|^{2/q} \|\nabla f\|^{1-2/q}, \quad q \in [2, \infty). \tag{2.19}$$

Proof. We have to estimate the rate of growth of the constant $\mathrm{k}_{\mathbb{S}^2}(p)$ in (2.12) as $p \to 1$ and set $N = 1$ in (2.11). In view of (2.6), and taking into account that $\gamma = p-1$, $1 < k < p$ and that

$$B(p-k, k+1) < B(p-k, 2) = (p-k)^{-1}(p-k+1)^{-1},$$

we obtain

$$\begin{aligned}
&\mathrm{k}_{\mathbb{S}^2}(p) \leqslant p^p (p-1)^{1-p}\, \mathrm{L}_{\mathbb{S}^2}(p-1) = \\
&\frac{p^p(p-1)}{3\pi} \min_{1<k<p} \left[\left(\frac{3(2k-1)}{4k}\right)^k \frac{B(p-k,k+1)}{(k-1)^k (p-k)^{p-k}}\right] \leqslant \\
&\frac{p^p(p-1)}{3\pi} \min_{1<k<p} \left[\left(\frac{3(2k-1)}{4k}\right)^k \frac{1}{(k-1)^k (p-k)^{p-k+1}(p-k+1)}\right] \leqslant \\
&\frac{p^p(p-1)}{3\pi} [\dots]\big|_{k=\frac{p+1}{2}} = \frac{1}{\pi}\left(\frac{2p}{p-1}\right)^p \left(\frac{3^{p-1}p^{p+1}2^{3-p}}{(p+1)^{p+3}}\right)^{1/2} \leqslant \\
&\frac{1}{2\pi}\left(\frac{2p}{p-1}\right)^p,
\end{aligned} \tag{2.20}$$

where we used that

$$\max_{p\in[1,2]} \frac{3^{p-1}p^{p+1}2^{3-p}}{(p+1)^{p+3}} = \frac{1}{4}.$$

We now set $N=1$ in (2.11) and $\varphi_1=f/\|f\|$. Then (2.11) goes over to

$$\|f\|_{L_{\frac{2p}{p-1}}(\mathbb{S}^2)}\leqslant \mathrm{k}_{\mathbb{S}^2}(p)^{1/2p}\,\|f\|^{1-1/p}\|\nabla f\|^{1/p},$$

which is precisely (2.19), if we set $q=2p/(p-1)\in[4,\infty)$. In the interval $q\in[2,4]$ inequality (2.19) follows from Hölder's inequality. □

Remark 2.1. The exponent $1/2$ of q in (2.19) is sharp as $q\to\infty$.

Remark 2.2. The results described here are certain technical improvements of the corresponding estimates in [6], which, in turn, follow those in [9], see also [13].

Remark 2.3. The series in (2.10) can be estimated without loss of a constant. This was done in [7] where it was shown that for all $\mu\geqslant 0$ and $k=3/2$

$$\mu^{2k-2}\sum_{n=1}^{\infty}\frac{2n+1}{((n(n+1)+\mu^2)^k}<\frac{1}{k-1}.$$

This gives that

$$\mathrm{L}_{\mathbb{S}^2}(1)\leqslant\frac{3}{8}\quad\text{and}\quad \mathrm{k}_{\mathbb{S}^2}(2)=4\mathrm{L}_{\mathbb{S}^2}(1)\leqslant\frac{3}{2}.$$

Finally, using the method of [11], [12] the following improvement was recently obtained in [8]:

$$\mathrm{L}_{\mathbb{S}^2}(1)\leqslant\frac{9}{16\pi}\quad\text{and}\quad \mathrm{k}_{\mathbb{S}^2}(2)=4\mathrm{L}_{\mathbb{S}^2}(1)\leqslant\frac{9}{4\pi},$$

so that in the important particular case $q=4$ (called the Ladyzhenskaya inequality in the context of the Navier–Stokes equations) we obtain

$$\|f\|_{L_4(\mathbb{S}^2)}\leqslant\left(\frac{9}{4\pi}\right)^{1/4}\|f\|^{1/2}\|\nabla f\|^{1/2}.$$

The rate of growth of the constant $\mathrm{L}_{\mathbb{S}^2}(\gamma)$ as $\gamma\to 0$ was not studied in [7], while the results of [8] are essentially restricted to the case $\gamma=1$.

The vector case

In the vector case we have the identity for the gradients of spherical harmonics that substitutes (2.1) (see [6]): for any $s \in \mathbb{S}^2$

$$\sum_{k=1}^{2n+1} |\nabla Y_n^k(s)|^2 = n(n+1)\frac{2n+1}{4\pi}\,. \tag{2.21}$$

This identity is essential for inequalities for vector functions on $\mathbb{S}^2$.

In the vector case we first define the Laplace operator acting on (tangent) vector fields on $\mathbb{S}^2$ as the Laplace–de Rham operator $-d\delta - \delta d$ identifying 1-forms and vectors. Then for a two-dimensional manifold (not necessarily $\mathbb{S}^2$) we have [5]

$$\mathbf{\Delta} u = \nabla \operatorname{div} u - \operatorname{rot}\operatorname{rot} u, \tag{2.22}$$

where the operators $\nabla = \operatorname{grad}$ and div have the conventional meaning. The operator rot of a vector u is a scalar and for a scalar ψ, $\operatorname{rot}\psi$ is a vector:

$$\operatorname{rot} u := \operatorname{div} u^{\perp}, \qquad \operatorname{rot}\psi := \nabla^{\perp}\psi,$$

and in the local frame $u^{\perp} = (u_2, -u_1)$.

Integrating by parts we obtain

$$(-\mathbf{\Delta} u, u)_{L_2(T\mathbb{S}^2)} = \|\operatorname{rot} u\|^2 + \|\operatorname{div} u\|^2. \tag{2.23}$$

The vector Laplacian has a complete in $L_2(T\mathbb{S}^2)$ orthonormal basis of vector eigenfunctions. Using notation (2.3) we have

$$-\mathbf{\Delta} w_j = \lambda_j w_j, \qquad -\mathbf{\Delta} v_j = \lambda_j v_j, \tag{2.24}$$

where

$$w_j = \lambda_j^{-1/2}\nabla^{\perp} y_j, \ \ \operatorname{div} w_j = 0, \qquad v_j = \lambda_j^{-1/2}\nabla y_j, \ \ \operatorname{rot} v_j = 0.$$

Hence, on $\mathbb{S}^2$, corresponding to the eigenvalue $\Lambda_n = n(n+1)$, where $n = 1, 2, \dots$, there are two families of $2n+1$ orthonormal vector-valued eigenfunctions $w_n^k(s)$ and $v_n^k(s)$, where $k = 1, \dots, 2n+1$ and (2.21) gives the following important identities: for any $s \in \mathbb{S}^2$

$$\sum_{k=1}^{2n+1} |w_n^k(s)|^2 = \frac{2n+1}{4\pi}, \qquad \sum_{k=1}^{2n+1} |v_n^k(s)|^2 = \frac{2n+1}{4\pi}. \tag{2.25}$$

We finally observe that since the sphere is simply connected, $-\mathbf{\Delta}$ is strictly positive $-\mathbf{\Delta} \geqslant \Lambda_1 I = 2I$.

Let us consider the vector Schrödinger operator on $\mathbb{S}^2$

$$-\mathbf{\Delta} + f \tag{2.26}$$

and let us denote by $\mu_j = \mu_j(f)$ its eigenvalues.

Theorem 2.3. *The negative eigenvalues $\mu_j \leqslant 0$ of operator (2.26) satisfy*

$$\sum_{\mu_j \leqslant 0} |\mu_j|^\gamma \leqslant 2\mathrm{L}_{\mathbb{S}^2}(\gamma) \int_{\mathbb{S}^2} f(s)_-^{1+\gamma} dS, \tag{2.27}$$

where $\gamma > 0$ and $\mathrm{L}_{\mathbb{S}^2}(\gamma)$ is as in (2.5).

Proof. Once we know from (2.24) that each eigenvalue of the vector Laplacian λ_j is repeated twice and the corresponding vector-valued eigenfunctions satisfy identities (2.25), the proof of the theorem is word for word repetition of that of Theorem 2.1. □

The dual formulation is immediate.

Theorem 2.4. *Let $\{u_j\}_{j=1}^N \in H^1(T\mathbb{S}^2)$ be an orthonormal family of vector fields in $L^2(T\mathbb{S}^2)$. Then for*

$$\rho(s) = \sum_{j=1}^N |u_j(s)|^2$$

it holds

$$\left(\int_{\mathbb{S}^2} \rho(s)^{p/(p-1)} dS\right)^{p-1} \leqslant \mathrm{k}_{\mathbb{S}^2}^{\mathrm{vec}}(p) \sum_{j=1}^N \left(\|\operatorname{rot} u_j\|^2 + \|\operatorname{div} u_j\|^2\right)$$

where $1 < p \leqslant 2$,

$$\mathrm{k}_{\mathbb{S}^2}^{\mathrm{vec}}(p) \leqslant 2\frac{p^p}{(p-1)^{p-1}} \mathrm{L}_{\mathbb{S}^2}(p-1). \tag{2.28}$$

If, in addition, $\operatorname{div} u_j = 0$ (or $\operatorname{rot} u_j = 0$), then

$$\left(\int_{\mathbb{S}^2} \rho(s)^{p/(p-1)} dS\right)^{p-1} \leqslant \mathrm{k}_{\mathbb{S}^2}(p) \cdot \begin{cases} \sum_{j=1}^N \|\operatorname{rot} u_j\|^2, & \operatorname{div} u_j = 0, \\ \sum_{j=1}^N \|\operatorname{div} u_j\|^2, & \operatorname{rot} u_j = 0. \end{cases} \tag{2.29}$$

Setting $N=1$ and using the rate of growth of $\mathrm{k}_{\mathbb{S}^2}(p)$ as $p\to 1$ estimated in (2.20) we obtain the interpolation inequality in the vector case.

Corollary 2.2. *Let* $u\in H^1(T\mathbb{S}^2)$. *Then*

$$\|u\|_{L_q(T\mathbb{S}^2)}\leqslant\left(\frac{1}{\pi}\right)^{(q-2)/2q}q^{1/2}\|u\|^{2/q}\left(\|\operatorname{rot}u\|^2+\|\operatorname{div}u\|^2\right)^{1/2-1/q}.$$

If, in addition, $\operatorname{div}u=0$, *then*

$$\|u\|_{L_q(T\mathbb{S}^2)}\leqslant\left(\frac{1}{2\pi}\right)^{(q-2)/2q}q^{1/2}\|u\|^{2/q}\|\operatorname{rot}u\|^{1-2/q}.$$

Remark 2.4. In the case of the Ladyzhenskaya inequality (that is, when $q=4$) the constant can be substantially improved:

$$\|u\|_{L_4(T\mathbb{S}^2)}\leqslant\left(\frac{9}{4\pi}\right)^{1/4}\|u\|^{1/2}\|\operatorname{rot}u\|^{1/2}.$$

3 Inequalities for periodic functions

In the similar way we can obtain interpolation inequalities of the type $L_q-L_2-L_2$ for periodic functions on $\mathbb{S}^1=[0,2\pi)$ with mean value zero

$$\int_{\mathbb{S}^1}\varphi(x)dx=\int_0^{2\pi}\varphi(x)dx=0$$

lying in the Sobolev space $H^{1/2}(\mathbb{S}^1)$ with norm

$$\|(-\Delta)^{\frac14}\varphi\|^2=\|\varphi\|^2_{\frac12}=\sum_{n=1}^\infty n(a_n^2+b_n^2),$$

where

$$\varphi(x)=\frac{1}{\sqrt{\pi}}\sum_{n=1}^\infty(a_n\sin nx+b_n\cos nx).$$

For this purpose we consider the Schrödinger-type operator

$$\Pi((-\Delta)^{1/2}+f)\Pi,\tag{3.1}$$

where $\Delta=\frac{d^2}{dx^2}$ and

$$\Pi\varphi=\varphi-\frac{1}{2\pi}\int_{\mathbb{S}^1}\varphi(x)dx.$$

Theorem 3.1. *The negative eigenvalues $\mu_j \leqslant 0$ of the operator* (3.1) *satisfy*

$$\sum_{\mu_j\leqslant 0} |\mu_j|^\gamma \leqslant \mathrm{L}_{\mathbb{S}^1}(\gamma) \int_{\mathbb{S}^1} f(x)_-^{1+\gamma} dx. \tag{3.2}$$

Here $\gamma > 0$ and the Lieb–Thirring constant $\mathrm{L}_\gamma(\mathbb{S}^1)$ satisfies

$$\begin{aligned} \mathrm{L}_{\mathbb{S}^1}(\gamma) &\leqslant \min_{1<k<\gamma+1} F_{\mathbb{S}^1}(\gamma, k), \\ F_{\mathbb{S}^1}(\gamma, k) &= \frac{1}{\pi} \frac{\gamma^{\gamma+1} B(\gamma+1-k, k+1)}{(k-1)^k (\gamma+1-k)^{\gamma+1-k}}. \end{aligned} \tag{3.3}$$

In particular,

$$\mathrm{L}_{\mathbb{S}^1}(1) \leqslant F_{\mathbb{S}^1}(1, 3/2) = \frac{3}{2}. \tag{3.4}$$

Proof. The proof goes as that of Theorem 2.1 up to (2.9) instead of which we have

$$N_{-r}(f) \leqslant \frac{1}{\pi} \frac{1}{k-1} (tr)^{1-k} \int_{\mathbb{S}^1} (f + (1-t)r)_-^k dx. \tag{3.5}$$

To prove this, setting as before $g = (f + (1-t)r)_-$ we obtain for $k > 1$

$$\begin{aligned} &\mathrm{Tr}\, g^{k/2} (\Pi((-\Delta)^{1/2} + tr)\Pi)^{-k} g^{k/2} = \mathrm{Tr}(\Pi((-\Delta)^{1/2} + tr)\Pi)^{-k} g^k = \\ &\int_{\mathbb{S}^1} \sum_{j=1}^\infty (\lambda_j + tr)^{-k} g(x)^k \varphi_j(x)^2 dx = \frac{1}{\pi} \sum_{n=1}^\infty \frac{1}{(n+tr)^k} \int_{\mathbb{S}^1} g(x)^k dx \leqslant \\ &\frac{1}{\pi} \int_0^\infty \frac{dy}{(y+tr)^k} \int_{\mathbb{S}^1} g(x)^k dx = \frac{1}{\pi} \frac{1}{k-1} (tr)^{1-k} \int_{\mathbb{S}^1} g(x)^k dx, \end{aligned} \tag{3.6}$$

where the eigenvalues and eigenfunctions of $(-\Delta)^{1/2}$ are

$$\{\lambda_n\}_{n=1}^\infty = \{n, n\}_{n=1}^\infty, \qquad \{\varphi_n(x)\}_{n=1}^\infty = \frac{1}{\sqrt{\pi}} \{\sin nx, \cos nx\}_{n=1}^\infty.$$

We can now complete the proof as in Theorem 2.1:

$$\begin{aligned} &\sum_{\mu_j\leqslant 0} |\mu_j|^\gamma \leqslant \frac{1}{\pi} \frac{\gamma t^{1-k}}{(k-1)} \int_0^\infty \int_{\mathbb{S}^1} r^{\gamma-k} (f(x) + (1-t)r)_-^k dx dr = \\ &\frac{1}{\pi} \frac{\gamma t^{1-k} (1-t)^{k-\gamma-1} B(\gamma+1-k, k+1)}{(k-1)} \int_{\mathbb{S}^1} f_-(x)^{\gamma+1} dx = \\ &F_{\mathbb{S}^1}(\gamma, k) \int_{\mathbb{S}^1} f_-(x)^{\gamma+1} dx, \end{aligned}$$

where the optimal t is as before $t = (k-1)/\gamma$.

The dual formulation is the same as in Theorem 2.2.

Theorem 3.2. *Let the family* $\{\varphi_j(s)\}_{j=1}^N \in H^{\frac{1}{2}}(\mathbb{S}^1)$ *be orthonormal and let* $\int_{\mathbb{S}^1}\varphi_j(x)dx = 0$. *Then for*

$$\rho(x) = \sum_{j=1}^N \varphi_j(x)^2.$$

the following inequality holds

$$\left(\int_{\mathbb{S}^1}\rho(x)^{p/(p-1)}dx\right)^{p-1} \leqslant \mathrm{k}_{\mathbb{S}^1}(p)\sum_{j=1}^N \|(-\Delta)^{\frac{1}{4}}\varphi_j\|^2, \tag{3.7}$$

where $1 < p \leqslant 2$, *and the constant* $\mathrm{k}_{\mathbb{S}^1}(p)$ *satisfies*

$$\mathrm{k}_{\mathbb{S}^1}(p) \leqslant \frac{p^p}{(p-1)^{p-1}}\mathrm{L}_{\mathbb{S}^1}(p-1).$$

Furthermore, for $p = 2$

$$\mathrm{k}_{\mathbb{S}^1}(2) = 4\mathrm{L}_{\mathbb{S}^1}(1).$$

The following result is similar to Corollary 2.1.

Corollary 3.1. *Let* $f \in H^{\frac{1}{2}}(\mathbb{S}^1)$ *and let* $\int_{\mathbb{S}^1} f(x)dx = 0$. *Then*

$$\|f\|_{L_q(\mathbb{S}^1)} \leqslant \left(\frac{2}{\pi}\right)^{(q-2)/2q} q^{1/2}\|f\|^{2/q}\|(-\Delta)^{\frac{1}{4}}f\|^{1-2/q}, \quad q \in [2,\infty). \tag{3.8}$$

Proof. We again estimate the rate of growth of $\mathrm{k}_{\mathbb{S}^1}(p)$ in (2.28) as $p \to 1$. Setting $\gamma = p-1$ so that $1 < k < p$ we obtain

$$\begin{aligned}
&\mathrm{k}_{\mathbb{S}^1}(p) \leqslant p^p(p-1)^{1-p}\,\mathrm{L}_{\mathbb{S}^1}(p-1) = \\
&\frac{p^p(p-1)}{\pi}\min_{1<k<p}\left[\frac{B(p-k,k+1)}{(k-1)^k(p-k)^{p-k}}\right] \leqslant \\
&\frac{p^p(p-1)}{\pi}\min_{1<k<p}\left[\frac{1}{(k-1)^k(p-k)^{p-k+1}(p-k+1)}\right] \leqslant \\
&\frac{p^p(p-1)}{\pi}\left[\dots\right]\big|_{k=\frac{p+1}{2}} = \frac{2}{\pi}\frac{2}{p+1}\left(\frac{2p}{p-1}\right)^p \leqslant \frac{1}{2\pi}\left(\frac{2p}{p-1}\right)^p.
\end{aligned}$$

Setting $N = 1$, $\varphi_1 = f/\|f\|$ in (3.7) and $q = 2p/(p-1) \in [4,\infty)$ completes the proof. ◻

Remark 3.1. The exponent $1/2$ of q in (3.8) is sharp as $q \to \infty$.

Remark 3.2. Arguing as in [8] the value of the constant $\mathrm{k}_{\mathbb{S}^1}(2)$ can be significantly improved:

$$\mathrm{k}_{\mathbb{S}^1}(2) \leqslant \frac{3}{\pi}.$$

so that for $q = 4$ we obtain

$$\|f\|_{L_4(\mathbb{S}^1)} \leqslant \left(\frac{3}{\pi}\right)^{1/4} \|f\|^{1/2} \|(-\Delta)^{\frac{1}{4}} f\|^{1/2},$$

while (3.4) gives

$$\|f\|_{L_4(\mathbb{S}^1)} \leqslant 6^{1/4} \|f\|^{1/2} \|(-\Delta)^{\frac{1}{4}} f\|^{1/2}.$$

□

References

[1] H. Araki, On an inequality of Lieb and Thirring. *Lett. Math. Phys.* **19** (1990), 167–170.

[2] W. Beckner, Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality. *Ann. of Math.* **138**:2 1993, 213–242.

[3] M.F. Bidaut-Veron and L. Veron. Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations. *Invent. Math.* **106** 1991, 489–539.

[4] J. Dolbeault, M.J. Esteban, M. Kowalczyk, and M. Loss. Sharp interpolation inequalities on the sphere: new methods and consequences. *Chin. Ann. Math. Ser. B* **34**:1 (2013), 99–112.

[5] A.A.Ilyin, Navier-Stokes and Euler equations on two-dimensional closed manifolds. *Mat. Sb.* **181**:4 (1990), 521–539; English transl. in *Math. USSR-Sb.* **69**:2 (1991).

[6] A.A. Ilyin, Lieb–Thirring inequalities on the N-sphere and in the plane, and some applications. *Proc. London Math. Soc.* **67**, 159–182 (1993).

[7] A.A. Ilyin, Lieb–Thirring inequalities on some manifolds. *J. Spectr. Theory* **2** (2012), 57–78.

[8] A. Ilyin and A. Laptev, Lieb–Thirring inequalities on the sphere. *Algebra i Analiz*, submitted.

[9] E. Lieb and W. Thirring, Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities, Studies in Mathematical Physics. Essays in honor of Valentine Bargmann, Princeton University Press, Princeton NJ, 269–303 (1976).

[10] S.G. Mikhlin, *Linear partial differential equations.* Vyssh. Shkola, Moscow, 1977; German transl. *Partielle Differentialgleichungen in der mathematischen Physik.* Akademie-Verlag, Berlin, 1978.

[11] M. Rumin, Spectral density and Sobolev inequalities for pure and mixed states. *Geom. Funct. Anal.* **20** (2010), 817–844.

[12] M. Rumin, Balanced distribution-energy inequalities and related entropy bounds. *Duke Math. J.* **160** (2011), 567–597.

[13] R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics,* 2nd ed., Springer-Verlag, New York, 1997.