ISSN 2071-2898 (Print)
ISSN 2071-2901 (Online)

Bobylev A.V., Jin-Cheng Jiang, Potapenko I.F.

On the comparison of Boltzmann and Landau collision integrals

Recommended form of bibliographic references: Bobylev A.V., Jin-Cheng Jiang, Potapenko I.F. On the comparison of Boltzmann and Landau collision integrals // Keldysh Institute Preprints. 2019. No. 30. 14 p. doi:10.20948/prepr-2019-30-e
URL: http://library.keldysh.ru/preprint.asp?id=2019-30&lg=e

О р д е н а Л е н и н а
ИНСТИТУТ ПРИКЛАДНОЙ МАТЕМАТИКИ
имени М.В. Келдыша
Р о с с и й с к о й а к а д е м и и н а у к

A.V. Bobylev, Jin-Cheng Jiang, I.F. Potapenko

On the comparison of Boltzmann and Landau collision integrals

Москва
2019

А.В. Бобылев, Жин-Ченг Янг, И.Ф. Потапенко

О сравнении интегралов столкновений Больцмана и Ландау

В работе получена оценка разницы между интегралами столкновений Больцмана и Ландау для ядра типа дельта-функции. Оценка справедлива для класса гладких и быстро спадающих на бесконечности функций распределения. Это доказано для общего случая многокомпонентной газовой смеси. Результаты могут быть использованы для теоретических и численных задач, относящихся к нелинейному кинетическому уравнению Ландау–Фоккера–Планка.

Ключевые слова: инеграл столкновений Больцмана, кинетическое уравнение Ландау, ядро типа дельта-функции

A.V. Bobylev, Jin-Cheng Jiang, I.F. Potapenko

On the comparison of Boltzmann and Landau collision integrals

The estimate for the difference of the Landau collision integral and the Boltzmann collision integral with delta-kernel is obtained. The estimate holds in a class of smooth and rapidly decreasing at infinity distribution functions. It is proved in the general case of multicomponent gas mixture. Results can be used for theoretical and numerical problems related to the nonlinear Landau–Fokker–Planck kinetic equation.

Key words: Boltzmann collision integral, Landau kinetic equation, delta-kernel

Работа выполнена при поддержке Российского фонда фундаментальных исследований, Грант N 17-51-52007 МНТ_а.

Contents

1 Introduction

The Landau kinetic equation for Coulomb forces was firstly published in 1936 as an asymptotic version of the Boltzmann equation for grazing collisions [1]. Then it was shown by Bogolyubov in 1946 that the Landau equation has a universal nature [2]. It is not directly related to Coulomb forces and Boltzmann equation. The Landau equation can be formally derived from dynamics (via BBGKY-hierarchy) for any weakly interacting system under some assumptions on smoothness of the potential (see e.g. [3] and references therein). Nevertheless the plasma physics remains the main area of applications of the Landau (or, equivalently, Fokker–Planck) equation [4].

Mathematical properties of this equation were studied by many authors (see e.g. [5] for a review). It is remarkable that the existence of classical global solution for the spatially homogeneous Cauchy problem for the Landau equation (in its original version [1], [2]) remains to be an open problem.

There are also many publications on related numerical methods. For example, we can mention some new Monte Carlo methods developed in last two decades (see [3] and references therein). Roughly speaking, these methods are based on the approximation of the Landau collision integral by the Boltzmann collision integral with a special kind of kernels (sometimes called in mathematical works "the scattering cross-section"). In particular, it concerns the so called delta-kernels proposed in [7]. Similar approximation of the Landau equation can be used for its rigorous mathematical study. This approach was briefly discussed in [8].

The present paper can be considered, to some extent, as a continuation of that discussion. Our main aim is to prove some estimates for the order of approximation of the classical Landau collision integral [1] by the Boltzmann collision integral (including the case of delta-kernels). The consideration is done for an arbitrary gas mixture of several components with distinct molecular masses. This is very important for quasi-neutral plasma, where we always have at least two relevant species (electrons and ions).

The main result of the paper is formulated in Proposition 3 at the end of Section 4. It shows that the approximation has the order $O(\varepsilon^{1/3})$ for the classical Landau integral. If we consider a more general collision integral with effective collision frequency proportional to power $5/2 < \gamma \leq 3$ of inverse relative speed, the approximation has the order $O\left(\varepsilon^{p}\right)),\ p = \mathrm{Min}\left(1/2,\ 4/\gamma - 1\right),$ where ε is the standard parameter of the delta-kernel.

We hope to extend this result to a wider class of distribution functions by using ideas of the recent paper [9].

2 Boltzmann equation and Landau approximation

We consider an arbitrary spatially homogeneous mixture of rarefied gases. Let $\{f_i(\mathbf{v},t),\ \ i=1,...,n\}$ be distribution functions of particles with masses $\{m_i,\ i=1,...,n\}$, respectively. The independent variables $\mathbf{v}\in\mathbb{R}^3$ and $t\ge 0$ stand for velocity and time, respectively. Spatial number densities $\{\rho_i(t), i=1,...,n\}$ are given by the integrals

$$\rho_i(t)=\int\limits_{\mathbb{R}^3} d\mathbf{v}\, f_i(\mathbf{v},t),\quad i=1,\ldots,n. \tag{1}$$

Remark 1 *We make a few comments concerning the notation in the main part (Sections 2–4) of the paper. Three-dimensional vectors are denoted by bold characters like* $\mathbf{v}$*, whereas the same ordinary characters denote corresponding absolute values like* $v=|\mathbf{v}|$*. Upper and lower Latin indices are used for any numeration of scalars and vectors like* $\mathbf{v}_i^r$*. We use lower Greek indices just for Cartesian coordinates like* $\mathbf{v}=\{v_\alpha,\ \alpha=1,2,3\}$ *and assume the usual summation rule over repeated Greek indices like* $\mathbf{v}\cdot\mathbf{w}=v_\alpha w_\alpha$*.*

A system of Boltzmann kinetic equations for $f_i(\mathbf{v},t)$ reads

$$\frac{\partial f_i}{\partial t}\ =\ \sum_{j=1}^{n} Q_{ij}(f_i,f_j),\quad i=1,\ldots,n, \tag{2}$$

where

$$\begin{gathered} Q_{ij}(f_i,f_j)=\int\limits_{\mathbb{R}^3\times\mathbb{S}^2} d\mathbf{w}\,d\boldsymbol{\omega}\,g_{ij}\left(u,\frac{\mathbf{u}\cdot\boldsymbol{\omega}}{u}\right)\ [f_i(\mathbf{v}')f_j(\mathbf{w}')-f_i(\mathbf{v})f_j(\mathbf{w})]\,,\\ \mathbf{u}=\mathbf{v}-\mathbf{w},\quad \boldsymbol{\omega}\in\mathbb{S}^2,\qquad \mathbf{v}'=\frac{1}{m_i+m_j}\ (m_i\mathbf{v}+m_j\mathbf{w}+m_j u\,\boldsymbol{\omega})\,,\\ \mathbf{w}'=\frac{1}{m_i+m_j}\ (m_i\mathbf{v}+m_j\mathbf{w}-m_i u\,\boldsymbol{\omega})\,;\qquad i,j=1,\ldots,n. \end{gathered} \tag{3}$$

Functions $g_{ij}(u,\mu)$ are expressed by formulas

$$g_{ij}(u,\mu)=g_{ji}(u,\mu)=u\,\sigma_{ij}(u,\mu),\qquad u=|\mathbf{u}|,\qquad i,j=1,\ldots,n, \tag{4}$$

where $\sigma_{ij}(u,\mu)$ is the differential cross-section (in the center of mass system of colliding particles of sort i and sort j) of scattering at the angle $\theta = \arccos(\mu)$, $|\mu| \le 1$.
Suppose that the distribution functions are three times continuously differentiable and rapidly decreasing with all their derivatives at infinity. Following the original idea of Landau and carrying out the Taylor expansion of the integrands in (3) (with respect to $\mathbf{v}' - \mathbf{v}$ and $\mathbf{w}' - \mathbf{w}$), we obtain a sum of two terms

$$Q_{ij}(f_i, f_j) = Q_{ij}^{(1)}(f_i, f_j) + \Delta_{ij}(f_i, f_j), \tag{5}$$

which can be described in the following way.

The first term corresponds to the Landau collisional integral (for arbitrary $g_{ij}(u,\mu)$ in (3)):

$$\begin{aligned} Q_{ij}^{(1)}(f_i, f_j) = \frac{m_{ij}^2}{2m_i}\frac{\partial}{\partial v_\alpha} \int_{\mathbb{R}^3} d\mathbf{w}\, g_{ij}^{(1)}(u) T_{\alpha\beta}(\mathbf{u}) \times \\ \times \left(\frac{1}{m_i}\frac{\partial}{\partial v_\beta} - \frac{1}{m_j}\frac{\partial}{\partial w_\beta} \right) f_i(\mathbf{v}) f_j(\mathbf{w}), \end{aligned} \tag{6}$$

where the summation over repeated Greek indices $\alpha, \beta = 1, 2, 3$ is assumed,

$$\begin{aligned} \mathbf{u} = \mathbf{v} - \mathbf{w}, \quad m_{ij} = \frac{m_i m_j}{m_i + m_j}, \quad T_{\alpha\beta}(\mathbf{u}) = u^2 \delta_{\alpha\beta} - u_\alpha u_\beta, \\ u = |\mathbf{u}|, \qquad g_{ij}^{(1)}(u) = 2\pi \int_{-1}^{1} d\mu\, g_{ij}(u,\mu)(1-\mu). \end{aligned} \tag{7}$$

Our goal is to estimate the difference

$$\Delta_{ij}(f_i, f_j) = Q(f_i, f_j) - Q^{(1)}(f_i, f_j) \tag{8}$$

in the notation of Eqs. (3), (6),(7). We can consider the case $i = 1, j = 2$ without any loss of generality.Then we can fix a kernel $g_\varepsilon(u,\mu) = g_{12}(u,\mu)$, assuming that $\Delta_{12}(f_1, f_2) \to 0$ for $\varepsilon \to 0$, and present $Q_{12}(f_1, f_2)$ in the form

$$\begin{aligned} Q_{12}(f_1, f_2) = Q_\varepsilon(f, f) = \int_{\mathbb{R}^3\times\mathbb{R}^3} d\mathbf{w}\, d\mathbf{k}\, \frac{\delta\left(\mathbf{k}\cdot\mathbf{u} + k^2/2\right)}{u} \times \\ \times g_\varepsilon\left(u, 1 - \frac{k^2}{2u^2}\right) \left[F(\mathbf{u}+\mathbf{k}, \mathbf{U}) - F(\mathbf{u}, \mathbf{U})\right], \end{aligned} \tag{9}$$

where

$$F(\mathbf{u},\mathbf{U}) = f_1(\mathbf{v})f_2(\mathbf{w}) = f_1\left(\mathbf{U}+\frac{m}{m_2}\mathbf{u}\right) f_2\left(\mathbf{U}-\frac{m}{m_2}\mathbf{u}\right), \\ \mathbf{U} = \frac{m_1\mathbf{v}+m_2\mathbf{w}}{m_1+m_2}, \quad m = \frac{m_1 m_2}{m_1+m_2}, \tag{10}$$

and where $\delta(x)$, $x \in \mathbf{R}$, is the Dirac's delta-function.

This transformation from (3) to (9)–(10) is very convenient for the comparison of the Landau and Boltzmann integrals [10]. It is based on the identity

$$\int_{\mathbb{R}^3} d\mathbf{k}\,\delta\left(\mathbf{k}\cdot\mathbf{u}+\frac{k^2}{2}\right)\psi(\mathbf{k}) = u\int_{\mathbb{S}^2} d\boldsymbol{\omega}\psi(u\boldsymbol{\omega}-\mathbf{u}), \tag{11}$$

where $\psi(\mathbf{k})$ is an arbitrary continuous function.

We consider the difference $F(\mathbf{u}+\mathbf{k}) - F(\mathbf{u})$ in the integral (9), omitting irrelevant argument $\mathbf{U}$. It is, by the integration by parts easy to verify that

$$F(\mathbf{u}+\mathbf{k}) - F(\mathbf{u}) = \left[\left(\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{u}}\right)+\frac{1}{2}\left(\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{u}}\right)^2\right]F(\mathbf{u})+R(\mathbf{u}), \\ R(\mathbf{u}) = \frac{1}{2}\int_0^1 d\theta(1-\theta)^2\left(\frac{\partial}{\partial\theta}\right)^3 F(\mathbf{u}+\mathbf{k}\theta) = \\ = \frac{1}{2}\int_0^1 d\theta(1-\theta)^2\left(\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{u}}\right)^3 F(\mathbf{u}+\mathbf{k}\theta). \tag{12}$$

Substituting this formula into (9), we obtain

$$Q_\varepsilon(f,f) = \int_{\mathbb{R}^3\times\mathbb{R}^3} d\mathbf{w}\,d\mathbf{k}\,\delta\left(\mathbf{k}\cdot\mathbf{u}+\frac{k^2}{2}\right)h_\varepsilon(u,k)\times \\ \times\left[\left(k_i\frac{\partial}{\partial u_i}+\frac{1}{2}k_i k_j\frac{\partial^2}{\partial u_i\partial u_j}\right)F(\mathbf{u},\mathbf{U})+R(\mathbf{u},\mathbf{U})\right], \\ h_\varepsilon(u,k) = \frac{1}{u}g_\varepsilon\left(u,1-\frac{k^2}{2u^2}\right), \tag{13}$$

where the explicit dependence on $\mathbf{U}$ is now indicated. We can compute two integrals on $\mathbf{k}$

$$I_i(\mathbf{u}) = \int\limits_{\mathbb{R}^3} d\mathbf{k}\, \delta\left(\mathbf{k}\cdot\mathbf{u} + k^2/2\right)\ h_\varepsilon(u,k)\ k_i,$$

$$I_{ij}(\mathbf{u}) = \int\limits_{\mathbb{R}^3} d\mathbf{k}\, \delta\left(\mathbf{k}\cdot\mathbf{u} + k^2/2\right)\ h_\varepsilon(u,k)\ k_i k_j, \quad i,j = 1,2,3.$$

Using the standard methods, we get

$$\begin{aligned} I_i(\mathbf{u}) &= -g^{(1)}(u;\varepsilon)\, u_i, \\ I_{ij}(\mathbf{u}) &= g^{(1)}(u;\varepsilon)(u^2\delta_{ij} - u_i u_j) + \frac{1}{2} g^{(2)}(u;\varepsilon)(3u_i u_j - u^2\delta_{ij}); \\ g^{(k)}(u;\varepsilon) &= 2\pi \int\limits_{-1}^{1} d\mu\, g_\varepsilon(u;\mu)\,(1-\mu)^k, \quad k \ge 0. \end{aligned} \tag{14}$$

The equality (13) can be rewritten as

$$\begin{aligned} Q_\varepsilon(f,f) = \ & \frac{1}{2}\int\limits_{\mathbb{R}^3} d\mathbf{w}\ g^{(1)}(u;\varepsilon)\ \left[-2u_i \frac{\partial F(\mathbf{u},\mathbf{U})}{\partial u_i} + \right. \\ & \left. + \left(u^2\delta_{ij} - u_i u_j\right)\ \frac{\partial^2 F(\mathbf{u},\mathbf{U})}{\partial u_i \partial u_j}\right] + \Delta_1(\mathbf{v};\varepsilon) + \Delta_2(\mathbf{v};\varepsilon), \end{aligned} \tag{15}$$

where

$$\begin{aligned} \Delta_1(\mathbf{v};\varepsilon) &= \frac{1}{4}\int\limits_{\mathbb{R}^3} d\mathbf{w}\ g^{(2)}(u;\varepsilon)\ \left(3u_i u_j - u^2\delta_{ij}\right) \frac{\partial^2 F(\mathbf{u},\mathbf{U})}{\partial u_i \partial u_j}, \\ \Delta_2(\mathbf{v};\varepsilon) &= \int\limits_{\mathbb{R}^3} d\mathbf{w} d\mathbf{k}\ \delta\left(\mathbf{k}\cdot\mathbf{u} + k^2/2\right)\ h_\varepsilon(u,k)\ R(\mathbf{u},\mathbf{U}) \end{aligned} \tag{16}$$

in the notation of (10). Here the derivatives on $\mathbf{u}$ under the integral sign are computed for constant $\mathbf{U}$, therefore

$$\frac{\partial}{\partial \mathbf{u}} = m\left(\frac{1}{m_1}\frac{\partial}{\partial \mathbf{v}} - \frac{1}{m_2}\frac{\partial}{\partial \mathbf{w}}\right). \tag{17}$$

For comparison of the first term in the right-hand side of (15) with the Landau integral (6) it is convenient to rewrite (6) as

$$Q_{12}^{(1)}(f_1,f_2) = Q^{(1)}(f_1,f_2) = \frac{1}{2}\int\limits_{\mathbb{R}^3} d\mathbf{w}\,\frac{\partial}{\partial u_i}P_{ij}(\mathbf{u})\,\frac{\partial}{\partial u_j}F(\mathbf{u},\mathbf{U}),$$

$$P_{ij}(\mathbf{u}) = g^{(1)}(u;\varepsilon)\,[u^2\delta_{ij} - u_i u_j]\,,$$

where notations (10) are used. Then the difference of integrals (3) and (6) reads $(i=1, j=2)$

$$\Delta_\varepsilon(\mathbf{v}) = Q_\varepsilon(f_1,f_2) - Q^{(1)}(f_1,f_2) = \Delta_1(\mathbf{v};\varepsilon) + \Delta_2(\mathbf{v};\varepsilon)\,, \tag{18}$$

where $\Delta_{1,2}(\mathbf{v};\varepsilon)$ are given in (16).

In order to be more precise we choose a class of function for which

$$|f_{1,2}^{(s)}(\mathbf{v})| \le Ae^{-\beta m_{1,2}v^2}; \quad s=0,1,2,3; \quad \mathbf{v}\in\mathbb{R}^3\,, \tag{19}$$

where $f_{1,2}^{(s)}(\mathbf{v})$ denote all partial derivatives of order s with respect to $\mathbf{v}$ of functions $f(\mathbf{v}) \equiv f_{1,2}^{(0)}(\mathbf{v})$, and A and β are some positive constants. We note that a consistency (with the BBGKY hierarchy) result for the Landau equation is proved in [3] for a similar class of functions. Then the result of our consideration can be formulated in the following way:

Proposition 1 *We assume that*

(a) *the kernel* $g_{1,2}(u,\mu) = g_\varepsilon(u,\mu)$ *in* (3) *satisfies the conditions*

$$g_\varepsilon(u,\mu)\ge 0, \qquad \int\limits_{-1}^{1} d\mu g_\varepsilon(u,\mu)(1-\mu) \le \frac{c_1 + c_2 u^{2k}}{u^3}\,, \tag{20}$$

with some $k\ge 2$ *and absolute constants* $c_{1,2}>0$*;*

(b) *the functions* $f_{1,2}(v)$ *in* (3) *satisfies conditions* (19).

Then the identity (18)*, where* $Q_\varepsilon = Q_{1,2}$ *and* $Q^{(1)} = Q_{1,2}^{(1)}$*, holds for all* $v\in\mathbb{R}^3$.

The formal derivation of (19) is already done above. It remains to check the convergence of integrals. This will be done in the course of getting estimates in Section 3.

Remark 2 *Of course, the identity* (18) *is valid in a wider class of functions, i.e. inequalities* (19) *are not necessary conditions for the validity of* (18)*. However, the class of functions, satisfying* (19) *is very convenient for applications and estimates.*

3 General estimates

It is easy to check that, under assumptions (19) we obtain

$$\left|F^{(s)}(\mathbf{u},\mathbf{U})\right| \leqslant C\, e^{-\beta(m_1 v^2 + m_2 w^2)}; \quad s = 1,2,3, \tag{21}$$

where $F^{(s)}(\mathbf{u},\mathbf{U})$ denotes all partial derivatives of $F(\mathbf{u},\mathbf{U})$ with respect to $\mathbf{u}$ of order s. That is why

$$|\Delta_1(\mathbf{v};\varepsilon)| \leqslant C \int\limits_{\mathbb{R}^3} d\mathbf{w}\; g^{(2)}(u;\varepsilon)\, u^2\, e^{-\beta(m_1 v^2 + m_2 w^2)}. \tag{22}$$

In the notation of Eqs. (14). Here and below the notation C is used for any constant factor.

Then we consider the term Δ_2 from (16), where (see (12))

$$F(\mathbf{u}+\mathbf{k}) - F(\mathbf{u}) = \left[\left(\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{u}}\right) + \frac{1}{2}\left(\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{u}}\right)^2\right] F(\mathbf{u}) + R(\mathbf{u}),$$

$$R(\mathbf{u}) = \frac{1}{2}\int\limits_0^1 d\theta (1-\theta)^2 \left(\frac{\partial}{\partial\theta}\right)^3 F(\mathbf{u}+\mathbf{k}\theta) =$$

$$= \frac{1}{2}\int\limits_0^1 d\theta (1-\theta)^2 \left(\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{u}}\right)^3 F(\mathbf{u}+\mathbf{k}\theta).$$

Hence,

$$|R(\mathbf{u},\mathbf{U})| \leqslant \frac{Ck^3}{2}\int\limits_0^1 d\theta \exp\left\{-\beta\left[m_1\left(\mathbf{v}+\frac{m}{m_1}\mathbf{k}\theta\right)^2 + m_2\left(\mathbf{w}-\frac{m}{m_2}\mathbf{k}\theta\right)^2\right]\right\},$$

for $F(\mathbf{u}) = F(\mathbf{u},\mathbf{U})$ in the notation of (10). Therefore we obtain from (16)

$$|\Delta_2(\mathbf{v};\varepsilon)| \leqslant \frac{C}{2}\int\limits_{\mathbb{R}^3} d\mathbf{w}\, e^{-\beta(m_1 v^2 + m_2 w^2)}\; J_\varepsilon(u),$$

where

$$J_\varepsilon(u) =$$

$$= \int_0^1 d\theta \int_{\mathbb{R}^3} d\mathbf{k}\, \delta\left(\mathbf{k}\cdot\mathbf{u} + k^2/2\right) h_\varepsilon(u,k)\; k^3\; \exp[-2m\beta(\mathbf{k}\cdot\mathbf{u}\theta + k^2\theta^2/2)] =$$

$$= \int_{\mathbb{R}^3} d\mathbf{k}\, \delta\left(\mathbf{k}\cdot\mathbf{u} + k^2/2\right)\; h_\varepsilon(u,k)\; k^3 \int_0^1 d\theta\; \exp[2m\beta\, k^2\theta(1-\theta)/2] \;\leqslant$$

$$\leqslant \frac{2\pi}{u} \int_0^{2u} dr h_\varepsilon(u,r)\, r^4 \exp(2m\beta r^2/8).$$

After substitution of $h_\varepsilon(u,r)$ from (13) this inequality takes the form

$$J_\varepsilon(u) \leqslant 2^{5/2}\pi \int_{-1}^{1} d\mu\, g_\varepsilon(u,\mu) u^3 e^{2m\beta u^2(1-\mu)/4}\; (1-\mu)^{3/2}.$$

Therefore we obtain

$$|\Delta_2(\mathbf{v},\varepsilon)| \leqslant C \int_{\mathbb{R}^3} d\mathbf{w} \int_{-1}^{1} d\mu\, g_\varepsilon(u,\mu) u^3\; (1-\mu)^{3/2} \times$$
$$\times \exp\left[-2m\beta(v^2+w^2) + 2m\beta\frac{u^2}{4}(1-\mu)\right]. \tag{23}$$

Hence,the following statement is proved.

Proposition 2 *If $f_{1,2}(\mathbf{v})$ satisfy conditions* (19), *then estimates* (22),(23) *are valid for the error terms $\Delta_{1,2}(v;\varepsilon)$ from* (18).

4 Delta-kernels

We consider below a specific class of kernels $g(u;\mu)$ in the Boltzmann collision integrals (3), (9). These kernels are very convenient for approximate solution of the classical Landau equation [1] by both numerical [6] and analytical [8] methods. They have the following form

$$g_\varepsilon(u,\mu) = \frac{1}{2\pi\varepsilon}\delta[1-\mu-2\varepsilon\Phi_\gamma(u)],$$
$$\Phi_\gamma(u) = \mathrm{Min}(u^{-\gamma},\varepsilon^{-\gamma}), \quad 0<\varepsilon<<1, \quad \frac{5}{2}<\gamma\leq 3. \tag{24}$$

Note that the value $\gamma = 3$ corresponds to classical Landau equation [1], other values are included just for generality.

Substituting $g_\varepsilon(u,\mu)$ into (14), we obtain

$$g^{(k)}(u;\varepsilon) = 2\pi \int_{-1}^{1} d\mu g_\varepsilon(u,\mu)(1-\mu)^k = \varepsilon^{k-1} 2^k \Phi_\gamma^k(u), \qquad k \geq 0. \tag{25}$$

Note that $g^{(0)}(u;\varepsilon) = \varepsilon^{-1} = \text{const.}$, i.e. the corresponding collision frequency is constant, like for Maxwell molecules. This is why we chose the function $\Phi_\gamma(u)$ instead of $u^{-\gamma}$ in (24).

Our goal in this section is to evaluate the difference (18) for delta-kernels (24). The estimates are based on Proposition 2. First we use (22),(25) and obtain

$$|\Delta_1(\mathbf{v};\varepsilon| \leq C\,\varepsilon\, e^{-\beta m_1 v^2} I_1(\mathbf{v}), \quad I_1(\mathbf{v}) = \int\limits_{\mathbb{R}^3} d\mathbf{u}\, u^2\, \Phi_\gamma^2(u) = 4\pi\,[I_1^< + I_1^>],$$

where

$$I_1^< = \frac{1}{\varepsilon^2} \int\limits_0^{\varepsilon^{1/\gamma}} du u^4 = \frac{\varepsilon^{5/\gamma - 2}}{5},$$

$$I_1^> = \int\limits_{\varepsilon^{1/\gamma}}^{\infty} du u^{4-2\gamma} = \frac{\gamma}{2\gamma - 5} \varepsilon^{5/\gamma - 2}, \quad \gamma > 5/2.$$

Hence, we get the following estimate:

$$|\Delta_1(v;\varepsilon)| \leq C\, \frac{\gamma}{2\gamma - 5} \varepsilon^{5/\gamma - 1} e^{-\beta m_1 v^2}, \quad \gamma > 2/5, \tag{26}$$

where C is an absolute constant. Then we substitute (25) into (23) and obtain

$$|\Delta_2(v;\varepsilon)| \leq C\,\sqrt{\varepsilon}\, e^{-2m\beta v^2} I_2(\mathbf{v}),$$

$$I_2(\mathbf{v}) = \int\limits_{\mathbb{R}^3} d\mathbf{u} u^3 \Phi^{3/2}(u) \exp[-\beta_1(\mathbf{v} - \mathbf{u})^2 + \varepsilon \beta_1 u^2 \Phi(u)/2],$$

where $\beta_1 = 2m\beta$. Note that the function

$$u^2 \Phi(u) = u^2 \mathrm{Min}(u^{-\gamma}, \varepsilon^{-1}), \quad \gamma > 5/2,$$

has its maximal value at $u = \varepsilon^{1/\gamma}$. Hence,

$$I_2(\mathbf{v}) \leq 4\pi\, e^{m\beta\varepsilon^{2/\gamma}}\, [I_2^< + I_2^>],$$

where

$$I_2^< = \frac{1}{\varepsilon^{3/2}} \int\limits_0^{\varepsilon^{1/\gamma}} du\, u^5 = \frac{1}{6}\varepsilon^{6/\gamma-3/2},$$

$$I_2^> = \int\limits_{\varepsilon^{1/\gamma}}^{\infty} du\, u^{5-3\gamma/2} e^{-\beta_1 (v-u)^2},$$

since $(\mathbf{v}-\mathbf{u})^2 \geq (|\mathbf{v}|-|\mathbf{u}|)^2$. If $5/2 < \gamma \leq 3$, then we obtain the following inequality:

$$|\Delta_1(v;\varepsilon)| + |\Delta_2(v;\varepsilon)| \leq C\left[\varepsilon^{5/\gamma-1} + \sqrt{\varepsilon}\psi(v)\right] \exp(\beta\, \tilde{m}_1 v^2), \tag{27}$$

where

$$\psi(v) = \int\limits_0^{\infty} du\, u^{5-3\gamma/2}\, e^{-2m\beta(v-u)^2}, \quad \tilde{m}_1 = \mathrm{Min}(m_1, 2m), \quad 5/2 < \gamma \leq 3, \tag{28}$$

$m = m_{1,2}$ is given in (10).

Finally we note that the integral $Q^{(1)}(f_1, f_2)$ in (18) with delta-kernel (24) is slightly different from usual Landau integral

$$Q_L(f_1, f_2) = \int\limits_{\mathbb{R}^3} dw \frac{\partial}{\partial u_\alpha}\, Q^{(\gamma)}_{\alpha\beta}(\mathbf{u}) \frac{\partial}{\partial u_\beta} f_1(v) f_(w),$$

$$Q^{(\gamma)}_{\alpha\beta} = u^{-\gamma}\,(u^2\delta_{\alpha\beta} - u_\alpha u_\beta); \quad \alpha, \beta = 1, 2, 3, \tag{29}$$

in the notation of (17). The difference is caused by formula

$$g^{(1)}(u;\varepsilon) = 2\Phi(u) = 2\mathrm{Min}(u^{-\gamma}, 1/\varepsilon),$$

since

$$Q_L(f_1, f_2) - Q^{(1)}(f_1, f_2) = \Delta_0(\mathbf{v};\varepsilon) =$$

$$= \int\limits_{u<\varepsilon^{1/\gamma}} d\mathbf{w} \left(u^{-\gamma} - \frac{1}{\varepsilon}\right)\left[-u_\alpha \frac{\partial}{\partial u_\alpha} + 2(u^2\delta_{\alpha\beta} - u_\alpha u_\beta)\frac{\partial^2}{\partial u_\alpha \partial u_\beta}\right] f_1(\mathbf{v}) f_2(\mathbf{w}).$$

We use assumption (19) and obtain

$$|\Delta_0(\mathbf{v};\varepsilon)| \leq \int\limits_0^{\varepsilon^{1/\gamma}} duu^3(u^{-\gamma} - \varepsilon^{-1})e^{-\beta m_1 v^2} \leq C_1 \varepsilon^{4/\gamma-1} e^{-\beta m_1 v^2}. \qquad (30)$$

Obviously the general inequality reads

$$|Q_\varepsilon(f_1, f_2) - Q_L(f_1, f_2)| \leq |\Delta_0(\mathbf{v};\varepsilon)| + |\Delta_1(\mathbf{v};\varepsilon)| + |\Delta_2(\mathbf{v};\varepsilon)|, \qquad (31)$$

where the Boltzmann collision term $Q_\varepsilon(f_1, f_2)$ is taken with delta-kernel (24), whereas the Landau integral $Q_L(f_1, f_2)$ is given in (29). Collecting the estimates (27), (30) we conclude that the following statement is proved.

Proposition 3 *Let the functions* $f_{1,2}(\mathbf{v})$ *satisfy conditions* (19), *where* $m_{1,2}$ *are corresponding molecular masses. Then the difference* (31) *between the Landau integral*

$$Q_L(f_1, f_2) = \frac{m^2}{m_1} \frac{\partial}{\partial v_\alpha} \int\limits_{\mathbb{R}^3} dw |u|^{-\gamma} \left(|u|^2 \delta_{\alpha\beta} - u_\alpha u_\beta\right) \times$$
$$\times \left(\frac{1}{m_1} \frac{\partial}{\partial v_\beta} - \frac{1}{m_2} \frac{\partial}{\partial w_\beta} \right) f_1(v) f_2(w), \quad 5/2 < \gamma \leq 3,$$

and the Boltzmann integral $Q_\varepsilon(f_1, f_2)$ *with delta-kernel* (24) *satisfies the following estimate for* $5/2 < \gamma \leq 3$

$$|Q_\varepsilon(f_1, f_2) - Q_L(f_1, f_2)| \;\leq\; C \left[\varepsilon^{4/\gamma-1} + \sqrt{\varepsilon}\Psi(v)\right] \exp(-\lambda m_1 v^2),$$
$$\lambda = \mathrm{Min}\left(1, \frac{2m_2}{m_1 + m_2}\right), \quad \Psi(v) = \int\limits_0^\infty duu^{5-3\gamma/2} e^{-\beta(v-u)^2}, \quad v \geq 0,$$

where the constant factor C *depends only on parameters* $\gamma, \beta, m_{1,2}$.

Proposition 3 can be considered as *the main result* of the paper. Note that the integral $\psi(v)$ can be easily estimated as

$$\psi(v) \leq C_1 \left(1 + |v|^{5-3\gamma/2}\right) \quad \text{if} \quad \gamma < 10/3,$$

when C_1 depends only on γ and β.

Acknowledgment. The authors are grateful to Prof. Tai-Ping Liu for fruitful discussions. The support from RFBR for A.V. Bobylev and I.F. Potapenko is appreciated. The research of J-C. Jiang is supported in part by National Science Council Grant NSC 107-2923-M-001-001.

References

[1] Landau L.D. Kinetic equation in case of Coulomb interaction, Phys. Zs. Sov. Union 10 (1936), 154.

[2] Bogolyubov N.N., Problems of a dynamical theory in statistical physics (Moscow, State Technical Press, 1946, in Russian); English translation in Studies in Statistical Mechanics I, edited by J. De Boer, G.E.Uhlenbeck, part A, Amsterdam, North-Holland, 1964.

[3] Bobylev A.V., Pulvirenti M. and Saffirio C., From particle systems to the Landau equation: a consistency result, Commun. Math. Phys. 319 (2013), no.3, 683–702.

[4] Lifshitz E.M., Pitaevskii L.P. Physical kinetics, Butterworth-Helnemann, 1981.

[5] Villani C., A review of mathematical topics in collisional kinetic theory. In : Handbook of Mathematical Fluid Dynamics, Vol.1, edited by S.Friedlander, D.Serre, Amsterdam, North-Holland, 2002, pp. 71–301.

[6] Bobylev A.V. and Potapenko I.F., Monte Carlo methods and their analysis for Coulomb collisions in multicomponent plasma, J. Comput. Phys. 246 (2013), 123–144.

[7] Potapenko I.F., Bobylev A.V and Mossberg E., Deterministic and stochastic methods for nonlinear Landau-Fokker-Planck kinetic equation and application to plasma physics, Transp. Theory Stat. Phys., 37 (2008), 113–170.

[8] Bobylev A.V., Gamba I.M. and Potapenko I.F., On some properties of the Landau kinetic equation, J. Statist. Phys. 160 (2015), 1327–1338.

[9] Jin-Cheng Jiang and Tai-Ping Liu, Boltzmann collision operator for the infinite range potential: a limit problem, Annales de l'Institut Henri Poincare/ Analyse non lineaire (to be published)

[10] A.V. Bobylev, On expansion of Boltzman integral into Landau series, Dokl. Acad. Nauk USSR 225 (1975) 535-538 (in Russian); translation in Sov. Phys. Dokl. 20 (1976) 740–742.