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T. V. Dudnikova

Space–time statistical solutions for the Hamiltonian field–crystal system

Москва — 2020

Дудникова Т.В.

Пространственно-временные статистические решения для гамильтоновой системы “поле – кристалл”

Рассматривается динамика скалярного поля, взаимодействующего с гармоническим кристаллом с n компонентами размерности d, $d,n \geq 1$. Динамика системы является трансляционно-инвариантной относительно дискретной подгруппы $\mathbb{Z}^d$ в $\mathbb{R}^d$. Изучается задача Коши со случайными начальными данными. Предполагается, что начальная мера имеет конечную среднюю плотность энергии, а начальные корреляционные функции трансляционно-инвариантны относительно подгруппы $\mathbb{Z}^d$. Доказывается сходимость пространственно-временны́х статистических решений к гауссовской мере.

Ключевые слова: гармонический кристалл, взаимодействующий со скалярным полем, задача Коши, случайные начальные данные, пространственно-временны́е статистические решения, слабая сходимость мер

Tatiana Vladimirovna Dudnikova

Space–time statistical solutions for the Hamiltonian field–crystal system

We consider the dynamics of a scalar field coupled to a harmonic crystal with n components in dimension d, $d,n \geq 1$. The dynamics of the system is translation-invariant with respect to the discrete subgroup $\mathbb{Z}^d$ of $\mathbb{R}^d$. We study the Cauchy problem with random initial data. We assume that the initial measure has a finite mean energy density and the initial correlation functions are translation invariant with respect to the subgroup $\mathbb{Z}^d$. We prove the convergence of space-time statistical solutions to a Gaussian measure.

Key words: the harmonic crystal coupled to a scalar field, Cauchy problem, random initial data, space-time statistical solutions, weak convergence of measures

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1. Introduction

We study the linear Hamiltonian system consisting of a real scalar field $\psi(x)$ and its momentum $\pi(x)$, $x \in \mathbb{R}^d$, and a "simple lattice" described by the deviations $u(k) \in \mathbb{R}^n$ of the "atoms" and their velocities $v(k) \in \mathbb{R}^n$, $k \in \mathbb{Z}^d$. The Hamiltonian functional of the coupled field-crystal system reads

$$\mathrm{H}(\psi, u, \pi, v) = \mathrm{H_F}(\psi, \pi) + \mathrm{H_L}(u, v) + \mathrm{H_I}(\psi, u),$$

where $\mathrm{H_F}(\psi, \pi)$ ($\mathrm{H_L}(u, v)$) denotes the Hamiltonian for the field (for the crystal, respectively),

$$\mathrm{H_F}(\psi, \pi) := \frac{1}{2} \int \Big(|\nabla\psi(x)|^2 + |\pi(x)|^2 + m_0^2 |\psi(x)|^2 \Big)\, dx,$$

$$\mathrm{H_L}(u, v) := \frac{1}{2} \sum_{k\in\mathbb{Z}^d} \Big(\sum_{j=1}^{d} |u(k+e_j) - u(k)|^2 + \nu_0^2 |u(k)|^2 + |v(k)|^2 \Big),$$

$m_0, \nu_0 > 0$, $e_j \in \mathbb{Z}^d$ stands for the vector with the coordinates $e_j^i := \delta_j^i$. $\mathrm{H_I}(\psi, u)$ denotes the interaction term,

$$\mathrm{H_I}(\psi, u) := \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}^d} R(x-k) \cdot u(k) \psi(x)\, dx,$$

where $R(x)$ is a $\mathbb{R}^n$-valued function. Taking the variational derivatives of $\mathrm{H}(\psi, u, \pi, v)$, we obtain the following system:

$$\left\{ \begin{array}{lcccll} \dot\psi(x,t) & = & \dfrac{\delta \mathrm{H}}{\delta \pi} & = & \pi(x,t), \qquad x \in \mathbb{R}^d, & t \in \mathbb{R}, \\ \dot u(k,t) & = & \dfrac{\partial \mathrm{H}}{\partial v} & = & v(k,t), \qquad k \in \mathbb{Z}^d, & t \in \mathbb{R}, \\ \dot\pi(x,t) & = & -\dfrac{\delta \mathrm{H}}{\delta \psi} & = & (\Delta - m_0^2)\psi(x,t) - \sum\limits_{k'\in\mathbb{Z}^d} u(k',t)\cdot R(x-k'), & \\ \dot v(k,t) & = & -\dfrac{\partial \mathrm{H}}{\partial u} & = & (\Delta_L - \nu_0^2) u(k,t) - \displaystyle\int R(x'-k)\psi(x',t)\, dx'. & \end{array} \right. \tag{1.1}$$

Here Δ_L denotes the discrete Laplace operator on the lattice $\mathbb{Z}^d$,

$$\Delta_L u(k) := \sum_{e, |e|=1} (u(k+e) - u(k)).$$

The system (1.1) can be considered as the description of the motion of electrons in the periodic medium which is generated by the ionic cores. Understanding of this motion is one of the central problem of solid state physics. Now we briefly explain our model. Here $\psi(x,t)$ describes the motion of electron field, $u(k,t)$ is the (small) displacements of the ionic cores from their equilibrium positions. In our approach, we disregard the displacement of the electric and magnetic fields generated by electrons and ions, we neglect the potentials of electrons and the vector potential of ions. For the scalar potential ϕ of ions, $(1/c^2)\ddot{\phi} = \Delta\phi - 4\pi\rho$, where ρ is the density of charge of ions, $\rho \sim \sum_k e\delta(x-k-u(k,t))$. In the static approximation, $(1/c^2)\ddot{\phi} \approx 0$, we obtain

$$\phi(x,t) = \sum_k \frac{e}{|x-k-u(k,t)|} =: r(x-k-u(k,t)) \approx r(x-k) - \nabla r(x-k) \cdot u(k,t),$$

and we substitute $R(x-k) := \nabla r(x-k)$ in the equation of the motion of the electron field. Note that if $n = d$ and $R(x) = -\nabla r(x)$, then the interaction term $\mathrm{H_I}(\psi,u)$ is the linearized Pauli-Fierz approximation of the translation-invariant coupling

$$\sum_k \int r(x-k-u(k))\psi(x)\,dx.$$

A similar model was analyzed by Born and Oppenheimer [1] as a model of a solid state (coupled Maxwell-Schrödinger equations for electrons in the harmonic crystal; see, for instance, [13] and the references therein).

We study the Cauchy problem for system (1.1) with the initial data

$$\left\{ \begin{array}{llll} \psi(x,0) & = \psi_0(x), & \pi(x,0) & = \pi_0(x), & x \in \mathbb{R}^d, \\ u(k,0) & = u_0(k), & v(k,0) & = v_0(k), & k \in \mathbb{Z}^d. \end{array} \right. \tag{1.2}$$

Write

$$\psi^0 := \psi, \quad \psi^1 := \pi, \quad u^0 := u, \quad u^1 := v,$$
$$Y(t) := (Y^0(t), Y^1(t)) \left| \begin{array}{l} Y^0(t) := (\psi^0(x,t), u^0(k,t)) := (\psi(x,t), u(k,t)), \\ Y^1(t) := (\psi^1(x,t), u^1(k,t)) := (\pi(x,t), v(k,t)). \end{array} \right.$$

In other words, $Y^i(\cdot,t)$ are functions defined on the disjoint union $\mathbb{P} := \mathbb{R}^d \cup \mathbb{Z}^d$,

$$Y^i(t) = Y^i(p,t) := \left\{ \begin{array}{ll} \psi^i(x,t), & p = x \in \mathbb{R}^d, \\ u^i(k,t), & p = k \in \mathbb{Z}^d, \end{array} \right. \qquad i = 0{,}1.$$

Then, the system (1.1), (1.2) becomes a dynamical problem of the form

$$\dot{Y}(t) = \mathcal{A}(Y(t)), \qquad t \in \mathbb{R}; \qquad Y(0) = Y_0. \tag{1.3}$$

Here $Y_0 = (\psi_0, u_0, \pi_0, v_0)$ and

$$\mathcal{A} = J\,\nabla\,\mathrm{H}(Y) = \begin{pmatrix} 0 & 1 \\ -\mathcal{H} & 0 \end{pmatrix}, \qquad \mathcal{H} = \begin{pmatrix} -\Delta + m_0^2 & S \\ S^* & -\Delta_L + \nu_0^2 \end{pmatrix},$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \;\; Su(x) = \sum_{k\in\mathbb{Z}^d} R(x-k)u(k), \;\; S^*\psi(k) = \int\limits_{\mathbb{R}^d} R(x-k)\psi(x)\,dx,$$

and $\langle\psi, Su\rangle_{L^2(\mathbb{R}^d)} = \langle S^*\psi, u\rangle_{[l^2(\mathbb{Z}^d)]^n}$, $\psi \in L^2(\mathbb{R}^d)$, $u \in [l^2(\mathbb{Z}^d)]^n$.

We assume that the initial data Y_0 belong to the real phase space $\mathcal{E}$ defined below.

Definition 1.1. *$H^{s,\alpha} = H^{s,\alpha}(\mathbb{R}^d)$, $s \in \mathbb{R}$, $\alpha \in \mathbb{R}$, is the Hilbert space of distributions $\psi \in S'(\mathbb{R}^d)$ with finite norm*

$$\|\psi\|_{s,\alpha} \equiv \|\langle x\rangle^\alpha \Lambda^s \psi\|_{L^2(\mathbb{R}^d)} < \infty,$$

where $\Lambda^s\psi := F^{-1}_{\xi\to x}(\langle\xi\rangle^s\hat{\psi}(\xi))$, $\langle x\rangle := \sqrt{|x|^2+1}$, and $\hat{\psi} := F\psi$ stands for the Fourier transform of a tempered distribution ψ. For $\psi \in D \equiv C_0^\infty(\mathbb{R}^d)$, write $F\psi(\xi) = \displaystyle\int e^{i\xi\cdot x}\psi(x)dx$.

Remark 1.2. *For $s = 0,1,2,\ldots$, the space $H^{s,\alpha}(\mathbb{R}^d)$ is the Hilbert space of real-valued functions $\psi(x)$ with finite norm*

$$\sum_{|\gamma|\le s} \int \langle x\rangle^{2\alpha} |\mathcal{D}^\gamma\psi(x)|^2\,dx < \infty,$$

which is equivalent to $\|\psi\|^2_{s,\alpha}$.

Definition 1.3. *(i) L^α, $\alpha \in \mathbb{R}$, is the Hilbert space of vector-valued functions $u(k) \in \mathbb{R}^n$, $k \in \mathbb{Z}^d$, with finite norm*

$$\|u\|^2_\alpha \equiv \sum_{k\in\mathbb{Z}^d} \langle k\rangle^{2\alpha}|u(k)|^2 < \infty.$$

(ii) $\mathcal{E}^{s,\alpha} := H^{1+s,\alpha}(\mathbb{R}^d) \oplus L^\alpha \oplus H^{s,\alpha}(\mathbb{R}^d) \oplus L^\alpha$ is the Hilbert space of vectors $Y \equiv (\psi, u, \pi, v)$ with finite norm

$$|\!|\!|Y|\!|\!|^2_{s,\alpha} = \|\psi\|^2_{1+s,\alpha} + \|u\|^2_\alpha + \|\pi\|^2_{s,\alpha} + \|v\|^2_\alpha.$$

(iii) The phase space of problem (1.3) is $\mathcal{E} := \mathcal{E}^{0,\alpha}$.

Below we assume that $\alpha < -d/2$.

Using the standard technique of pseudo-differential operators and Sobolev's Theorem (see, e.g., [8]), one can prove that $\mathcal{E}^{0,\alpha} = \mathcal{E} \subset \mathcal{E}^{s,\beta}$ for every $s < 0$ and $\beta < \alpha$, and the embedding is compact.

Definition 1.4. *(i) Write* $\mathfrak{V}^1_\alpha := \mathfrak{V}^1_{\alpha(\psi)} \times \mathfrak{V}^1_{\alpha(u)}$, *where*

$$\mathfrak{V}^1_{\alpha(\psi)} := \left\{\psi(x,t)|\ \psi(x,t) \in C(\mathbb{R}; H^{1,\alpha}(\mathbb{R}^d)),\ \dot\psi(x,t) \in C(\mathbb{R}; H^{0,\alpha}(\mathbb{R}^d))\right\},$$
$$\mathfrak{V}^1_{\alpha(u)} := \left\{u(k,t)|\ u(k,t) \in C^1(\mathbb{R}; L^\alpha)\right\}, \quad \alpha < -d/2.$$

Introduce the seminorms in $\mathfrak{V}^1_\alpha$ *by the rule*

$$|Y^0|^2_{\alpha,1,T} = \max_{|t|\le T}\left[\|\psi(\cdot,t)\|^2_{1,\alpha} + \|\dot\psi(\cdot,t)\|^2_{0,\alpha} + \|u(\cdot,t)\|^2_\alpha + \|\dot u(\cdot,t)\|^2_\alpha\right],\ \ T>0,$$

$Y^0 := (\psi(x,t), u(k,t))$.
(ii) Write $\mathfrak{V}^0_{s,\beta} := \mathfrak{V}^0_{s,\beta(\psi)} \times \mathfrak{V}^0_{\beta(u)}$, *where*

$$\mathfrak{V}^0_{s,\beta(\psi)} := \left\{\psi(x,t) \in L^2_{\rm loc}(\mathbb{R}; H^{1+s,\beta}(\mathbb{R}^d)) \bigcap C(\mathbb{R}; H^{0,\beta}(\mathbb{R}^d))\right\}, \qquad s<0,$$
$$\mathfrak{V}^0_{\beta(u)} := \left\{u(k,t) \in C(\mathbb{R}; L^\beta)\right\}, \qquad \beta < \alpha < -d/2.$$

(iii) Denote by V *the operator* $V : \mathcal{E} \to \mathfrak{V}^1_\alpha$ *such that*

$$V(Y_0) = Y^0(t) \equiv (\psi(x,t), u(k,t)), \tag{1.4}$$

where $(\psi(x,t), u(x,t))$ *is the solution to problem (1.1) with the initial data* $Y_0 = (\psi_0, u_0, \pi_0, v_0)$.

We assume that the initial date Y_0 is a random function. By μ_0 we denote a Borel probability measure on $\mathcal{E}$ giving the distribution of Y_0.

Definition 1.5. *Introduce a Borel probability measure* P *on the space* $\mathfrak{V}^1_\alpha$ *by the rule*

$$P(\omega) = \mu_0(V^{-1}\omega) \quad \textit{for any Borel set } \ \omega \in \mathcal{B}(\mathfrak{V}^1_\alpha).$$

Here and below $\mathcal{B}(X)$ *denotes the* σ*-algebra of Borel sets of a topological space* X*. The measure* P *is called a space-time statistical solution to problem (1.3) corresponding to the initial measure* μ_0*. Denote by* $\{P_\tau, \tau \in \mathbb{R}\}$ *the following family of measures*

$$P_\tau(\omega) = P(S^{-1}_\tau\omega) \quad \textit{for any } \ \omega \in \mathcal{B}(\mathfrak{V}^1_\alpha), \quad \tau \in \mathbb{R}.$$

Here S_τ *denotes the shift operator in time,*

$$S_\tau(Y^0(t)) = Y^0(t+\tau), \qquad \tau \in \mathbb{R}, \qquad Y^0(t) \equiv (\psi(x,t), u(k,t)). \tag{1.5}$$

The main goal of the paper is to prove that the measures P_τ weakly converge as $\tau\to\infty$ to a limit measure on the space $\mathfrak{V}^0_{s,\beta}$, $s<0$, $\beta<\alpha<-d/2$,

$$P_\tau \rightharpoondown P_\infty, \qquad \tau\to\infty. \tag{1.6}$$

This means the convergence of the integrals

$$\int\limits_{\mathfrak{V}^0_{s,\beta}} f(Y^0)P_\tau(dY^0) \to \int\limits_{\mathfrak{V}^0_{s,\beta}} f(Y^0)P_\infty(dY^0) \qquad \text{as} \quad \tau\to\infty$$

for any bounded continuous functional f on $\mathfrak{V}^0_{s,\beta}$. Furthermore, the limit measure P_∞ is a Gaussian measure on the space $\mathfrak{V}^1_\alpha$ supported by the solutions to problem (1.1). Thus, the convergence (1.6) can be considered as an analog of the central limit theorem for a class of solutions to the equations (1.1). The proof of convergence (1.6) is based on the results of [5] and used the technique of the works [10, 16]. Also, we check that the group S_τ is mixing w.r.t. the measure P_∞, i.e., for any $f,g\in L^2(\mathfrak{V}^1_\alpha,P_\infty)$,

$$\lim_{\tau\to\infty}\int\limits_{\mathfrak{V}^1_\alpha} f(S_\tau Y^0)g(Y^0)\,P_\infty(dY^0) = \int\limits_{\mathfrak{V}^1_\alpha} f(Y^0)P_\infty(dY^0)\int\limits_{\mathfrak{V}^1_\alpha} g(Y^0)P_\infty(dY^0). \tag{1.7}$$

In particular, the group S_τ is ergodic w.r.t. the measure P_∞, i.e.,

$$\lim_{T\to\infty}\frac{1}{T}\int\limits_0^T f(S_\tau Y^0)\,d\tau = \int\limits_{\mathfrak{V}^1_\alpha} f(Y^0)\,P_\infty(dY^0) \quad (\mathrm{mod}\,P_\infty).$$

Note that all results remain true for a more general case in which $\mathrm{H_L}$ is the Hamiltonian of the harmonic crystal, i.e.,

$$\mathrm{H_L}(u,v)=\frac{1}{2}\sum_{k\in\mathbb{Z}^d}\Big(\sum_{k'\in\mathbb{Z}^d} u(k)\cdot V(k-k')u(k')+|v(k)|^2\Big),$$

where $V(k)\in\mathbb{R}^n\times\mathbb{R}^n$ and $V(k)$ satisfies the conditions from [4], in particular, $|V(k)|\le Ce^{-\gamma|k|}$ with some $\gamma>0$ and $V^T(-k)=V(k)$ for any $k\in\mathbb{Z}^d$.

For continuous models described by partial differential equations, the behavior of space-time statistical solutions was studied by Komech and Ratanov [10] for wave equations and Ratanov [14] for parabolic equations. For Klein-Gordon equations, the result was obtained in [3]. For infinite harmonic crystals, the time evolution and ergodic properties of their equilibrium states were studied by Lanford, Lebowitz [11] and by van Hemmen [7]. For system (1.1), the long-time behavior of statistical solutions $\mu_t:=[W(t)]^*\mu_0$, where $W(t)$ stands for the solving operator of problem (1.3), was investigated in [5]. In this paper, we extend these results to the *space-time* statistical solutions of problem (1.3).

2. Main results

2.1. Bloch problem. The dynamics of (1.1) is invariant w.r.t. translations in $\mathbb{Z}^d$. Then, we can reduce system (1.1) to the Bloch problem on the torus. We first split $x \in \mathbb{R}^d$ in the form $x = k+y$, $k \in \mathbb{Z}^d$, $y \in K_1^d := [0,1]^d$, and apply the Fourier transform $F_{k\to\theta}$ to the solution $Y(k,t) := \Big(\psi(k+y,t), u(k,t), \pi(k+y,t), v(k,t)\Big)$,

$$\widetilde{Y}(\theta,t) := F_{k\to\theta}Y(k,t) \equiv \sum_{k\in\mathbb{Z}^d} e^{ik\theta}Y(k,t) = (\widetilde{\psi}(\theta,y,t), \widetilde{u}(\theta,t), \widetilde{\pi}(\theta,y,t), \widetilde{v}(\theta,t)),$$

$\theta \in \mathbb{R}^d$, which is a version of the Bloch-Floquet transform. The functions $\tilde{\psi}$, $\tilde{\pi}$ are periodic with respect to θ and quasi-periodic with respect to y, i.e.,

$$\widetilde{\psi}(\theta, y+m, t) = e^{-im\theta}\widetilde{\psi}(\theta,y,t), \quad \widetilde{\pi}(\theta, y+m, t) = e^{-im\theta}\widetilde{\pi}(\theta,y,t), \quad m \in \mathbb{Z}^d.$$

Further, introduce the Zak transform of $Y(\cdot,t)$ (which is also known as Lifshitz-Gelfand-Zak transform, cf [18], [13, p.5]) as

$$\mathcal{Z}Y(\cdot,t) \equiv \widetilde{Y}_\Pi(\theta,t) := (\widetilde{\psi}_\Pi(\theta,y,t), \widetilde{u}(\theta,t), \widetilde{\pi}_\Pi(\theta,y,t), \widetilde{v}(\theta,t)), \qquad (2.1)$$

where $\widetilde{\psi}_\Pi(\theta,y,t) := e^{iy\theta}\widetilde{\psi}(\theta,y,t)$ and $\widetilde{\pi}_\Pi(\theta,y,t) := e^{iy\theta}\widetilde{\pi}(\theta,y,t)$ are periodic functions with respect to y (and quasi-periodic with respect to θ). Denote by $\mathbb{T}_1^d := \mathbb{R}^d/\mathbb{Z}^d$ the real unit d-torus. Set

$$\widetilde{Y}_\Pi(\theta,r,t) \equiv \widetilde{Y}_\Pi(\theta,t) := \begin{cases} (\widetilde{\psi}_\Pi(\theta,y,t), \widetilde{\pi}_\Pi(\theta,y,t)), & r = y \in \mathbb{T}_1^d, \\ (\tilde{u}(\theta,t), \tilde{v}(\theta,t)), & r = 0. \end{cases}$$

Therefore, problem (1.3) is equivalent to the problem on the torus $y \in \mathbb{T}_1^d$ with the parameter $\theta \in K^d \equiv [0,2\pi]^d$,

$$\left\{ \begin{array}{l} \dot{\widetilde{Y}}_\Pi(\theta,t) = \widetilde{\mathcal{A}}(\theta)\widetilde{Y}_\Pi(\theta,t), \quad t \in \mathbb{R} \\ \widetilde{Y}_\Pi(\theta,0) = \widetilde{Y}_{0\Pi}(\theta) \end{array} \right| \qquad \theta \in K^d.$$

Here

$$\widetilde{\mathcal{A}}(\theta) = \begin{pmatrix} 0 & 1 \\ -\widetilde{\mathcal{H}}(\theta) & 0 \end{pmatrix}, \qquad (2.2)$$

and $\widetilde{\mathcal{H}}(\theta) := \mathcal{Z}\mathcal{H}\mathcal{Z}^{-1}$ is the “Schrödinger operator” on the torus $\mathbb{T}_1^d$,

$$\widetilde{\mathcal{H}}(\theta) = \begin{pmatrix} (i\nabla_y + \theta)^2 + m_0^2 & \tilde{S}(\theta) \\ \tilde{S}^*(\theta) & \omega_*^2(\theta) \end{pmatrix},$$

where

$$\omega_*^2(\theta) := 2(1-\cos\theta_1) + \cdots + 2(1-\cos\theta_d) + \nu_0^2,$$

$$\Big(\tilde{S}(\theta)\tilde{u}(\cdot)\Big)(\theta, y) := \tilde{R}_\Pi(\theta, y)\cdot\tilde{u}(\theta),$$

$$\Big(\tilde{S}^*(\theta)\tilde{\psi}_\Pi(\theta,\cdot)\Big)(\theta) := \int_{\mathbb{T}_1^d} \tilde{R}_\Pi(-\theta, y)\tilde{\psi}_\Pi(\theta, y)\, dy,$$

$$\Big\langle \tilde{\psi}_\Pi(\theta,\cdot), (\tilde{S}(\theta)\tilde{u})(\theta,\cdot)\Big\rangle_{L^2(\mathbb{T}_1^d)} = (\tilde{S}^*(\theta)\tilde{\psi}_\Pi)(\theta)\cdot\tilde{u}(\theta),$$

$$\tilde{\psi}_\Pi(\theta,\cdot)\in H^1(\mathbb{T}_1^d),\ \ \tilde{u}(\theta)\in\mathbb{C}^n.$$

2.2. Conditions on the coupled function R. Introduce the space $H_1^s := H^s(\mathbb{T}_1^d)\oplus\mathbb{C}^n$, $s\in\mathbb{R}$, where $H^s(\mathbb{T}_1^d)$ stands for the Sobolev space.

We impose conditions **R1**–**R4** on the coupling function $R(x)\in\mathbb{R}^n$.

R1 $R\in C^\infty(\mathbb{R}^d)$ and $|R(x)|\le \bar{R}\exp(-\varepsilon|x|)$ with some $\varepsilon>0$ and $\bar{R}<\infty$.

R2 The operator $\tilde{\mathcal{H}}(\theta)$ is positive definite for $\theta\in K^d\equiv[0{,}2\pi]^d$. This is equivalent to the uniform bound

$$(X^0, \tilde{\mathcal{H}}(\theta)X^0)\ge\kappa^2\|X^0\|^2_{H_1^1}\quad\text{for } X^0\in H_1^1,\quad \theta\in K^d,$$

where $\kappa>0$ is a constant and $(\cdot,\cdot)$ stands for the inner product in $H_1^0\equiv H^0(\mathbb{T}_1^d)\oplus\mathbb{C}^n$, i.e.,

$$(F,G) = \int\limits_{\mathbb{T}_1^d}\overline{F^1}(y)G^1(y)\,dy + \overline{F^2}\cdot G^2,\quad F=(F^1,F^2),\ G=(G^1,G^2)\in H_1^0.$$

Remark. (i) Condition **R2** ensures that the operator $i\tilde{\mathcal{A}}(\theta)$ is self-adjoint with respect to the energy inner product. This corresponds to the hyperbolicity of problem (1.1).

(ii) Condition **R2** holds, in particular, if the following condition **R2'** holds.

R2'. $\displaystyle\int_{[0,1]^d}\Big|\sum_{k\in\mathbb{Z}^d}R(k+y)\Big|^2 dy < \nu_0^2 m_0^2/2$. Condition **R2'** holds for the functions R satisfying condition **R1** with $\bar{R}\varepsilon^{-d}\ll 1$.

Proposition 2.1. *(see [5]) Let conditions* **R1** *and* **R2** *hold. Then (i) for any $Y_0\in\mathcal{E}$, there exists a unique solution $Y(t)\in C(\mathbb{R},\mathcal{E})$ to the Cauchy problem (1.3). (ii) The operator $W(t): Y_0\mapsto Y(t)$ is continuous in $\mathcal{E}$ for any $t\in\mathbb{R}$,*

$$\sup_{|t|\le T}|\!|\!|W(t)Y_0|\!|\!|_{0,\alpha}\le C(T)|\!|\!|Y_0|\!|\!|_{0,\alpha}\tag{2.3}$$

if α is even and $\alpha\le -2$.

Corollary 2.2. *It follows from (2.3) that for any $Y_0 \in \mathcal{E}$,*

$$|V(Y_0)|_{\alpha,1,T} \le C(T)\|Y_0\|_{0,\alpha}, \quad \forall T > 0.$$

where the operator V is defined in (1.4).

It follows from conditions **R1** and **R2** that, for a fixed $\theta \in K^d$, the operator $\tilde{\mathcal{H}}(\theta)$ is positive definite and self-adjoint in H_1^0 and its spectrum is discrete. Introduce the Hermitian positive-definite operator

$$\Omega(\theta) := \sqrt{\tilde{\mathcal{H}}(\theta)} > 0. \tag{2.4}$$

Denote by $\omega_l(\theta) > 0$ and $\chi_l(\theta, \cdot)$, $l = 1,2,\dots$, the eigenvalues ("Bloch bands") and the orthonormal eigenvectors ("Bloch functions", cf [17]) of the operator $\Omega(\theta)$ in H_1^0, respectively. Note that $\chi_l(\theta, \cdot) \in H_1^\infty := C^\infty(\mathbb{T}_1^d) \oplus \mathbb{C}^n$.

Lemma 2.3. *(see [17, 5]) There exists a closed subset $\mathcal{C}_* \subset K^d$ of zero Lebesgue measure such that the following assertions hold.*

(i) For every point $\Theta \in K^d \setminus \mathcal{C}_*$ and $N \in \mathbb{N}$, there exists a neighborhood $\mathcal{O}(\Theta) \subset K^d \setminus \mathcal{C}_*$ such that each of the functions $\omega_l(\theta)$ and $\chi_l(\theta, \cdot)$, $l = 1, \dots, N$, can be chosen to be real-analytic on $\mathcal{O}(\Theta)$.
(iii) The eigenvalues $\omega_l(\theta)$ have constant multiplicity in $\mathcal{O}(\Theta)$, i.e., one can enumerate them in such a way that for any $\theta \in \mathcal{O}(\Theta)$,

$$\begin{aligned} &\omega_1(\theta) \equiv \ldots \equiv \omega_{r_1}(\theta) < \omega_{r_1+1}(\theta) \equiv \ldots \equiv \omega_{r_2}(\theta) < \ldots, \\ &\omega_{r_\sigma}(\theta) \not\equiv \omega_{r_\nu}(\theta) \quad \text{if} \quad \sigma \neq \nu, \quad r_\sigma, r_\nu \ge 1, \end{aligned}$$

(iii) The spectral decomposition holds,

$$\Omega(\theta) = \sum_{l=1}^{+\infty} \omega_l(\theta) P_l(\theta), \qquad \theta \in \mathcal{O}(\Theta), \tag{2.5}$$

where $P_l(\theta)$ are the orthogonal projectors in H_1^0 onto the linear span of $\chi_l(\theta, \cdot)$, and $P_l(\theta)$ and $\omega_l(\theta)$ depend on $\theta \in \mathcal{O}(\Theta)$ analytically.

Assume that system (1.3) satisfies the next conditions **R3** and **R4**.

R3 For every $\Theta \in K^d \setminus \mathcal{C}_*$, $D_l(\theta) \not\equiv 0$, $l = 1,2,\dots$, where $D_l(\theta) := \det\Big(\dfrac{\partial^2 \omega_l(\theta)}{\partial\theta_i \partial\theta_j}\Big)_{i,j=1}^d$, $\theta \in \mathcal{O}(\Theta)$.

Remark 2.4. *Write $\mathcal{C}_l := \bigcup\limits_{\Theta \in K^d \setminus \mathcal{C}_*} \{\theta \in \mathcal{O}(\Theta) : D_l(\theta) = 0\}$, $l = 1,2,\dots.$ If conditions* **R1** *and* **R2** *hold, then* $\operatorname{mes} \mathcal{C}_l = 0$, $l = 1,2,\dots.$

R4 For each $l \neq l'$, the identities $\omega_l(\theta) \pm \omega_{l'}(\theta) \equiv \text{const}_\pm$, $\theta \in \mathcal{O}(\Theta)$, don't hold with constants $\text{const}_\pm \neq 0$.

For example, conditions **R3** and **R4** hold if $R = 0$.

2.3. Conditions on the initial measure. We assume that the initial data Y_0 in (1.3) is a measurable random function with values in $(\mathcal{E},\, \mathcal{B}(\mathcal{E}))$. Recall that μ_0 is a Borel probability measure on $\mathcal{E}$ which is the distribution of Y_0. Let $\mathbb{E}$ stand for the mathematical expectation w.r.t. this measure.

Definition 2.5. *(i) Write $\mathcal{D} = [D_F \oplus D_L]^2$ with $D_F \equiv C_0^\infty(\mathbb{R}^d)$, and let D_L be the set of vector sequences $u(k) \in \mathbb{R}^n$, $k \in \mathbb{Z}^d$, such that $u(k) = 0$ for $k \in \mathbb{Z}^d$ outside a finite set.*

(ii) For a probability measure μ on $\mathcal{E}$, we denote by $\hat{\mu}$ its characteristic functional (Fourier transform),

$$\hat{\mu}(Z) \equiv \int \exp(i\langle Y, Z\rangle)\, \mu(dY), \qquad Z \in \mathcal{D}.$$

Here $\langle\cdot,\cdot\rangle$ stands for the inner product in $L^2(\mathbb{P}) \otimes \mathbb{R}^N$ with different $N = 1,2,\ldots$,

$$\langle Y, Z\rangle := \sum_{i=0}^{1} \langle Y^i, Z^i\rangle, \qquad Y = (Y^0, Y^1),\ \ Z = (Z^0, Z^1),$$

$$\langle Y^i, Z^i\rangle := \int_{\mathbb{P}} Y^i(p) Z^i(p)\, dp \equiv \int_{\mathbb{R}^d} \psi^i(x)\xi^i(x)\, dx + \sum_{k\in\mathbb{Z}^d} u^i(k)\cdot\chi^i(k),$$

where $Y^i = (\psi^i, u^i)$, $Z^i = (\xi^i, \chi^i)$.

(iii) A measure μ is called Gaussian (of zero mean) if its characteristic functional has the form $\hat{\mu}(Z) = \exp\{-\mathcal{Q}(Z,Z)/2\}$, where $\mathcal{Q}$ is a real-valued nonnegative quadratic form in $\mathcal{D}$.

Definition 2.6. *Denote by $Q_0(p,p') = \Big(Q_0^{ij}(p,p')\Big)_{i,j=0,1}$ the correlation matrix of the measure μ_0, where*

$$Q_0^{ij}(p,p') \equiv \mathbb{E}\Big(Y^i(p) \otimes Y^j(p')\Big), \qquad i,j = 0,1, \qquad p,p' \in \mathbb{P}, \qquad (2.6)$$

where the convergence of the integral in (2.6) is understood in the sense of distributions, i.e., for any $Z_1, Z_2 \in D_F \oplus D_L$,

$$\langle Q_0^{ij}(p,p'), Z_1(p) \otimes Z_2(p')\rangle := \mathbb{E}\langle Y^i(p), Z_1(p)\rangle\langle Y^j(p'), Z_2(p')\rangle.$$

Denote by $\mathcal{Q}_0(Z,Z)$ a real-valued quadratic form on $\mathcal{D}$ with the matrix kernel $Q_0(p,p')$.

We impose conditions **S1–S4** on the initial measure μ_0.

S1. μ_0 has zero mean value, i.e., $\mathbb{E}\,(Y_0(p)) = 0$, $p \in \mathbb{P}$.

S2. The correlation functions $Q_0^{ij}(p,p')$ satisfy the bound

$$|Q_0^{ij}(p,p')| \le h(|p-p'|), \qquad p,p' \in \mathbb{P}, \tag{2.7}$$

where h is a nonnegative bounded function and $r^{d-1}h(r) \in L^1(0,+\infty)$.

S3. The correlation matrix $Q_0(p,p')$, $p,p' \in \mathbb{P}$, is translation invariant w.r.t. the shifts in $\mathbb{Z}^d$, i.e.,

$$Q_0(p+k,p'+k) = Q_0(p,p'), \qquad p,p' \in \mathbb{P}, \quad \text{for any} \quad k \in \mathbb{Z}^d. \tag{2.8}$$

Definition 2.7. *Let $\mathcal{A}$ be an open convex set in $\mathbb{P}$. Denote by $\sigma(\mathcal{A})$ a σ-algebra in $\mathcal{E}$ generated by the linear functionals $Y \mapsto \langle Y,Z\rangle$, where $Z \in \mathcal{D}$ with* $\operatorname{supp} Z \subset \mathcal{A}$. *Introduce the Ibragimov mixing coefficient of the measure μ_0 by the rule*

$$\varphi(r) \equiv \sup_{\substack{\mathcal{A},\mathcal{B} \subset \mathbb{P}:\\ \operatorname{dist}(\mathcal{A},\mathcal{B}) \ge r}} \ \sup_{\substack{A \in \sigma(\mathcal{A}),\, B \in \sigma(\mathcal{B})\\ \mu_0(B) > 0}} \frac{|\mu_0(A\cap B) - \mu_0(A)\mu_0(B)|}{\mu_0(B)}.$$

The measure μ_0 satisfies Ibragimov's strong uniform mixing condition if $\varphi(r) \to 0$ as $r \to \infty$ (cf. [9, Definition 17.2.2]).

S4. The initial mean energy densities are uniformly bounded,

$$e_F(x) := \mathbb{E}(|\nabla\psi_0(x)|^2 + |\psi_0(x)|^2 + |\pi_0(x)|^2) \le \bar e_F < \infty, \quad \text{a.a. } x \in \mathbb{R}^d,$$
$$e_L := \mathbb{E}(|u_0(k)|^2 + |v_0(k)|^2) < \infty, \quad k \in \mathbb{Z}^d.$$

Moreover, μ_0 satisfies Ibragimov's strong uniform mixing condition, and $r^{d-1}\varphi^{1/2}(r) \in L^1(0,+\infty)$.

Remark 2.8. *(i) Condition* **S2** *implies that for any $F,G \in \mathbf{L}^2 := [L^2(\mathbb{P},dp)]^2$, $L^2(\mathbb{P},dp) := L^2(\mathbb{R}^d) \oplus [l^2(\mathbb{Z}^d)]^n$,*

$$|\mathcal{Q}_0(F,G)| \equiv |\langle Q_0(p,p'), F(p)\otimes G(p')\rangle| \le C\|F\|_{\mathbf{L}^2}\|G\|_{\mathbf{L}^2}. \tag{2.9}$$

This follows from the bound (2.7) applying either the Shur test (see, e.g., [12, p.223]) or Young's inequality (see, e.g., [15, Theorem 0.3.1]).

(ii) Conditions **S1** *and* **S4** *imply the bound (2.7) with the function $h(r) = C\max\{\bar e_F, e_L\}\varphi^{1/2}(r)$. This follows from [9, Lemma 17.2.3].*

2.4. The convergence of space–time statistical solutions. Write $\mathcal{D}^0 = D_F \oplus D_L$. Let $[\cdot,\cdot]$ stand for the inner product in $L^2(\mathbb{R};L^2(\mathbb{P};dp))$ (or in its extensions),

$$[F_1,F_2] := \int\limits_{-\infty}^{+\infty} dt \int\limits_{\mathbb{P}} F_1(p,t)F_2(p,t)\,dp$$
$$= \int\limits_{-\infty}^{+\infty} \left(\int\limits_{\mathbb{R}^d} \psi_1(x,t)\psi_2(x,t)dx + \sum_{k\in\mathbb{Z}^d} u_1(k,t)\cdot u_2(k,t) \right) dt,$$

where $F_i \equiv F_i(p,t) \equiv (\psi_i(x,t), u_i(k,t))$, $i=1{,}2$.

Definition 2.9. *Denote by $Q^P_\tau(p_1,p_2,t_1,t_2)$, $p_1,p_2\in\mathbb{P}$, $t_1,t_2\in\mathbb{R}$, the correlation functions of the measures P_τ, $\tau\in\mathbb{R}$, introduced in Definition 1.5. For any $F_1,F_2\in\mathcal{D}^0$, write*

$$\mathcal{Q}^P_\tau(F_1,F_2) := [Q^P_\tau, F_1\otimes F_2] = \int [Y^0,F_1][Y^0,F_2]\,P_\tau(dY^0)$$
$$= \int\limits_{-\infty}^{+\infty} dt_1 \int\limits_{-\infty}^{+\infty} dt_2 \int\limits_{\mathbb{P}} Q^P_\tau(p_1,p_2,t_1,t_2)F_1(p_1,t_1)F_2(p_2,t_2)\,dp, \qquad \tau\in\mathbb{R}.$$

Introduce the adjoint operator V' to the operator V by the rule

$$[VY,F] = \langle Y, V'F\rangle \quad \text{for} \quad Y\in\mathcal{E} \quad \text{and} \quad F\in\mathcal{D}^0.$$

The main result of the paper is the following theorem.

Theorem 2.10. *Let conditions* **R1**–**R4** *be fulfilled. Then the following assertions hold.*

(i) Let conditions **S1** *and* **S2** *be fulfilled. Then the bounds are true:*

$$\sup_{\tau\ge 0} \int |Y^0|^2_{\alpha,1,T} P_\tau(dY^0) \le C(\alpha) < \infty, \qquad \forall T>0, \tag{2.10}$$

where the constant $C(\alpha)$ does not depend on $T>0$.

(ii) Let conditions **S1**–**S3** *be fulfilled. Then the correlation functions of P_τ converge to a limit as $\tau\to\infty$. Moreover, for any $F_1,F_2\in\mathcal{D}^0$,*

$$\mathcal{Q}^P_\tau(F_1,F_2) \to \mathcal{Q}^P_\infty(F_1,F_2) \qquad \textit{as} \quad \tau\to\infty, \tag{2.11}$$

where

$$\mathcal{Q}^P_\infty(F_1,F_2)=\mathcal{Q}_\infty(V'F_1,V'F_2), \tag{2.12}$$

the quadratic form $\mathcal{Q}_\infty$ *is defined in (3.3) below.*

(iii) Let conditions **S1**, **S3**, *and* **S4** *be fulfilled. Then the convergence (1.6) holds. The limit measure* P_∞ *is a Gaussian measure on the space* $\mathfrak{V}^1_\alpha$ *supported by the solutions to problem (1.1).*

(iv) The measure P_∞ *is invariant w.r.t. the shifts in time and the translations in* $\mathbb{Z}^d$*, and (1.7) holds.*

Remark If the initial measure μ_0 is Gaussian, then convergence (1.6) follows from convergence (2.11). In the general case, this doesn't hold. Furthermore, the weak convergence of the measures P_τ doesn't imply, in general, the convergence of their correlation matrices. Therefore, the last fact we prove separately.

3. Proof

3.1. The convergence of statistical solutions. Introduce the statistical solutions μ_t, $t\in\mathbb{R}$, to problem (1.3).

Definition 3.1. *The measure* μ_t *is a Borel probability measure in* $\mathcal{E}$ *giving the distribution of the random solution* $Y(t)$*,*

$$\mu_t(B)=\mu_0(W(-t)B),\quad \forall B\in\mathcal{B}(\mathcal{E}),\quad t\in\mathbb{R}.$$

The correlation functions of the measure μ_t*,* $t\in\mathbb{R}$*, are defined by*

$$Q^{ij}_t(p,p')\equiv\mathbb{E}\Big(Y^i(p,t)\otimes Y^j(p',t)\Big),\quad i,j=0{,}1,\quad p,p'\in\mathbb{P}. \tag{3.1}$$

Here $Y^i(p,t)$ *are the components of the random solution* $Y(t)=(Y^0(\cdot,t),Y^1(\cdot,t))$*. Denote by* $\mathcal{Q}_t(Z,Z)$ *a quadratic form on* $\mathcal{D}$ *with the matrix kernel* $Q_t(p,p')$*,*

$$\mathcal{Q}_t(Z,Z)=\int|\langle Y,Z\rangle|^2\mu_t(dY)=\langle Q_t(p,p'),Z(p)\otimes Z(p')\rangle,\qquad Z\in\mathcal{D}.$$

Since $Y^i(p,t)=(\psi^i(x,t),u^i(k,t))$, we rewrite formula (3.1) as follows:

$$\begin{aligned}Q^{ij}_t(p,p')&=\mathbb{E}[Y^i(p,t)\otimes Y^j(p',t)]\\&=\begin{pmatrix}\mathbb{E}\Big(\psi^i(x,t)\otimes\psi^j(x',t)\Big) & \mathbb{E}\Big(\psi^i(x,t)\otimes u^j(k',t)\Big)\\ \mathbb{E}\Big(u^i(k,t)\otimes\psi^j(x',t)\Big) & \mathbb{E}\Big(u^i(k,t)\otimes u^j(k',t)\Big)\end{pmatrix}\\&\equiv\begin{pmatrix}Q^{\psi^i\psi^j}_t(x,x') & Q^{\psi^iu^j}_t(x,k')\\ Q^{u^i\psi^j}_t(k,x') & Q^{u^iu^j}_t(k,k')\end{pmatrix},\quad i,j=0{,}1,\quad t\in\mathbb{R}.\end{aligned}$$

Let us rewrite the correlation matrices $Q_t^{ij}(p,p')$ using condition **S3**. Introduce the splitting $p = k + r$, where $k \in \mathbb{Z}^d$ and $r \in K_1^d \cup 0$, i.e.,

$$r = \begin{cases} x - [x] \in K_1^d, & \text{if } p = x \in \mathbb{R}^d, \\ 0, & \text{if } p = k \in \mathbb{Z}^d. \end{cases}$$

Since the group $W(t)$ commutes with translations in $\mathbb{Z}^d$, condition **S3** implies that

$$Q_t(k+p, k+p') = Q_t(p,p'), \qquad t \in \mathbb{R}, \qquad k \in \mathbb{Z}^d.$$

Hence,

$$Q_t^{ij}(k+r, k'+r') =: q_t^{ij}(k-k', r, r') \equiv \begin{pmatrix} q_t^{\psi^i\psi^j}(k-k'+r, r') & q_t^{\psi^i u^j}(k-k'+r) \\ q_t^{u^i\psi^j}(k'-k+r') & q_t^{u^i u^j}(k-k') \end{pmatrix}.$$

Using the Zak transform (2.1), introduce the following matrices

$$\tilde Q_t^{ij}(\theta, r, \theta', r') := \mathbb{E}[\tilde Y_\Pi^i(\theta, r, t) \otimes \overline{\tilde Y_\Pi^j(\theta', r', t)}], \quad \theta, \theta' \in K^d, \quad r, r' \in \mathcal{R} \equiv \mathbb{T}_1^d \cup 0.$$

Hence,

$$\tilde Q_t^{ij}(\theta, r, \theta', r') = (2\pi)^d \delta(\theta - \theta') \tilde q_t^{ij}(\theta, r, r'), \quad \theta, \theta' \in K^d, \quad r, r' \in \mathcal{R}, \quad t \in \mathbb{R},$$

where

$$\tilde q_t^{ij}(\theta, r, r') = e^{i(r-r')\theta} \sum_{k \in \mathbb{Z}^d} e^{ik\theta} q_t^{ij}(k, r, r') = \begin{pmatrix} \tilde q_t^{\psi^i\psi^j}(\theta, y, y') & \tilde q_t^{\psi^i u^j}(\theta, y) \\ \overline{\tilde q_t^{u^i\psi^j}}(\theta, y') & \tilde q_t^{u^i u^j}(\theta) \end{pmatrix}. \tag{3.2}$$

Introduce the correlation matrix for the limiting measure μ_∞. For $Z \in \mathcal{D}$, write

$$\begin{aligned} \mathcal{Q}_\infty(Z, Z) &:= \langle Q_\infty(p,p'), Z(p) \otimes Z(p') \rangle \\ &= (2\pi)^{-d} \int_{K^d} \left(\tilde q_\infty(\theta), \tilde Z_\Pi(\theta, \cdot) \otimes \overline{\tilde Z_\Pi(\theta, \cdot)} \right) d\theta, \end{aligned} \tag{3.3}$$

where $\tilde q_\infty(\theta)$ is the operator-valued function given by the rule

$$\tilde q_\infty(\theta) := \sum_{l=1}^{+\infty} P_l(\theta) \frac{1}{2} \begin{pmatrix} \tilde q_0^{00}(\theta) + \tilde{\mathcal{H}}^{-1}(\theta) \tilde q_0^{11}(\theta) & \tilde q_0^{01}(\theta) - \tilde q_0^{10}(\theta) \\ \tilde q_0^{10}(\theta) - \tilde q_0^{01}(\theta) & \tilde{\mathcal{H}}(\theta) \tilde q_0^{00}(\theta) + \tilde q_0^{11}(\theta) \end{pmatrix} P_l(\theta), \tag{3.4}$$

for $\theta \in K^d \backslash \mathcal{C}_*$. Here the symbol $\tilde q_0^{ij}(\theta) = \mathrm{Op}\Big((\tilde q_0^{ij}(\theta, r, r') \Big)$ stands for the integral operator with the integral kernel $\tilde q_0^{ij}(\theta, r, r')$, $r, r' \in \mathcal{R} = \mathbb{T}_1^d \cup \{0\}$ (see formula (3.2) with $t = 0$), and $P_l(\theta)$ is the spectral projection operator introduced in (2.5).

Theorem 3.2. *(see [5]) Let conditions* **R1**–**R4** *hold. Then the following assertions are valid.*

(i) Let conditions **S1** *and* **S2** *hold. Then*

$$\sup_{t\in\mathbb{R}} \int \|Y\|^2_{0,\alpha}\, \mu_t(dY) \le C < \infty. \tag{3.5}$$

(ii) Let conditions **S1**–**S3** *hold. Then the correlation functions of the measures μ_t converge to a limit. For any $Z_1, Z_2 \in \mathcal{D}$,*

$$\mathcal{Q}_t(Z_1, Z_2) \to \mathcal{Q}_\infty(Z_1, Z_2), \qquad t \to \infty,$$

where the quadratic form $\mathcal{Q}_\infty$ is defined in (3.3).

(iii) The measures μ_t weakly converge to a limiting measure μ_∞ on the space $\mathcal{E}^{s,\beta}$ with any $s < 0$ and $\beta < \alpha < -d/2$. The measure μ_∞ is Gaussian in $\mathcal{E} \equiv \mathcal{E}^{0,\alpha}$, its characteristic functional is of a form

$$\hat{\mu}_\infty(Z) = \exp\{-\mathcal{Q}_\infty(Z,Z)/2\}, \quad Z \in \mathcal{D}.$$

(iv) The measure μ_∞ is time stationary, i.e., $[W(t)]^*\mu_\infty = \mu_\infty$, $t \in \mathbb{R}$.

Proof of Theorem 2.10 (i) At first, note that

$$P_\tau(\omega) = \mu_\tau(V^{-1}\omega) \quad \text{for any} \;\; \omega \in \mathcal{B}(\mathfrak{V}^1_\alpha) \;\; \text{and} \;\; \tau > 0, \tag{3.6}$$

where μ_τ is defined in Definition 3.1 and the operator V in (1.4). To prove the bound (2.10), we apply (3.6) and obtain

$$\begin{aligned}\int |Y^0|^2_{\alpha,1,T} P_\tau(dY^0) &= \int |VY|^2_{\alpha,1,T}\, \mu_\tau(dY) = \sup_{|s|\le T} \int \|W(s)Y\|^2_{0,\alpha}\, \mu_\tau(dY)\\ &= \sup_{|s|\le T} \int \|Y\|^2_{0,\alpha}\, \mu_{s+\tau}(dY) \le \sup_{t\in\mathbb{R}} \mathbb{E}\|W(t)Y\|^2_{0,\alpha} \le C < \infty\end{aligned}$$

by the bound (3.5). The assertion (i) is proved.

(ii) Let $Y^0(t) \equiv (\psi(x,t), u(k,t))$ be a solution to problem (1.1) with the initial data Y_0. Then, for any $F \in \mathcal{D}^0$,

$$[Y^0, F] = [VY_0, F] = \langle Y_0, V'F\rangle, \tag{3.7}$$

where V' is the adjoint operator to the operator V. Using (3.6) and (3.7) gives the following equality for any $F_1, F_2 \in \mathcal{D}^0$:

$$\begin{aligned}\mathcal{Q}^P_\tau(F_1,F_2) &= \int [VY, F_1][VY, F_2]\, \mu_\tau(dY) = \int \langle Y, V'F_1\rangle\langle Y, V'F_2\rangle\, \mu_\tau(dY)\\ &= \langle Q_\tau(p,p'), V'F_1(p) \otimes V'F_2(p')\rangle \equiv \mathcal{Q}_\tau(V'F_1, V'F_2).\end{aligned}$$

Then, the convergence (2.11) follows from the following four facts:

(a) the quadratic form $\mathcal{Q}_\tau(Z,Z)$ converges to a limit for any $Z\in\mathcal{D}$ (see Theorem 3.2 (ii));

(b) $\mathcal{D}$ is dense in $\mathcal{L}:=L^2(\mathbb{R}^d)\oplus[\ell^2(\mathbb{Z}^d)]^n\oplus H^1(\mathbb{R}^d)\oplus[\ell^2(\mathbb{Z}^d)]^n$ (evidently);

(c) the quadratic forms $\mathcal{Q}_\tau(Z,Z)$, $\tau\in\mathbb{R}$, are equicontinuous in $\mathcal{L}$;

(d) $V'F\in\mathcal{L}$ for any $F\in\mathcal{D}^0$.

To prove fact (c) we introduce the operator $W'(t)$ which is adjoint to the operator $W(t)$:

$$\langle W(t)Y,Z\rangle=\langle Y,W'(t)Z\rangle,\qquad Y\in\mathcal{E},\qquad Z\in\mathcal{D}.$$

Therefore, the equicontinuity of the quadratic forms $\mathcal{Q}_\tau$ follows from bound (2.9). Indeed,

$$|\mathcal{Q}_\tau(Z,Z)|=|\mathcal{Q}_0(W'(\tau)Z,W'(\tau)Z)|\le C\|W'(\tau)Z\|^2_{\mathbf{L}^2}\le C_1\|Z\|^2_{\mathcal{L}}\,, \tag{3.8}$$

where the constant C_1 doesn't depend on $\tau\in\mathbb{R}$. The last inequality in (3.8) is proved in [5, formula (5.4)].

To check fact (d) we write the operator V' using the operator $W'(t)$:

$$V'F=\int\limits_{-\infty}^{+\infty}W'(t)\vec{F}\,dt,\quad \text{where}\quad \vec{F}:=(F,0),\quad F\in\mathcal{D}^0. \tag{3.9}$$

By [5, formulas (5.4) and (7.5)], we have $\|W'(t)Z\|_{\mathcal{L}}\le C\|Z\|_{\mathcal{L}}$. Hence, for $F\in\mathcal{D}^0$,

$$\|V'F\|_{\mathcal{L}}\le\int\limits_{-\infty}^{+\infty}\|W'(t)\vec{F}(\cdot,t)\|_{\mathcal{L}}\,dt\le C\int\limits_{-\infty}^{+\infty}\|F(\cdot,t)\|_{L^2(\mathbb{P})}\,dt\le C_2<\infty.$$

The assertion (ii) of Theorem 2.10 is proved.

(iii) To establish the weak convergence of the measures P_τ on the space $\mathfrak{V}^0_{s,\beta}$ it is enough to prove the following two assertions (A1) and (A2):

(A1) *The family of measures* $\{P_\tau,\tau\in\mathbb{R}\}$ *is weakly compact in* $\mathfrak{V}^0_{s,\beta}$;

(A2) *The characteristic functionals of* P_τ *converge to a limit as* $\tau\to\infty$.

The first (second) assertion provides the existence (resp., uniqueness) of the limit measures P_∞.

The bound (2.10) and the Prokhorov theorem (see, e.g., [6]) imply assertion (A1). This can be proved using the technique of [16, Theorem XII.5.2] and the Dubinskii embedding theorems (see, e.g., [2] or [16, Theorem IV.4.1]).

To prove assertion (A2) we apply (3.6) and (3.7), and obtain for any $F\in\mathcal{D}^0$

$$\hat{P}_\tau(F):=\int e^{i[Y^0,F]}P_\tau(dY^0)=\int e^{i\langle Y,V'F\rangle}\mu_\tau(dY)=:\hat{\mu}_\tau(V'F).$$

Then, convergence of $\hat{P}_\tau(F)$ to a limit as $\tau \to \infty$ follows from the following facts:

(a') $\hat{\mu}_\tau(Z)$ converges to a limit as $\tau \to \infty$ for any $Z \in \mathcal{D}$ (Theorem 3.2 (iii));
(b') $\mathcal{D}$ is dense in $\mathcal{L}$ (evidently);
(c') the characteristic functionals $\hat{\mu}_\tau(Z)$, $\tau \in \mathbb{R}$, are equicontinuous in $\mathcal{L}$;
(d') $V'F \in \mathcal{L}$ for any $F \in \mathcal{D}^0$ (this is proved above).

It remains to check the equicontinuity of $\hat{\mu}_\tau(Z)$, $\tau \in \mathbb{R}$. Indeed, by the Cauchy-Schwartz inequality and (3.8), one obtains

$$\begin{aligned}|\hat{\mu}_\tau(Z_1) - \hat{\mu}_\tau(Z_2)| &= \Big| \int \left(e^{i\langle Y,Z_1\rangle} - e^{i\langle Y,Z_2\rangle}\right) \mu_\tau(dY)\Big| \le \int \Big|e^{i\langle Y,Z_1-Z_2\rangle} - 1\Big| \, \mu_\tau(dY)\\ &\le \int |\langle Y, Z_1 - Z_2\rangle| \, \mu_\tau(dY) \le \sqrt{\int |\langle Y, Z_1 - Z_2\rangle|^2 \, \mu_\tau(dY)}\\ &= \sqrt{\mathcal{Q}_\tau(Z_1 - Z_2, Z_1 - Z_2)} \le C\|Z_1 - Z_2\|_{\mathcal{L}}.\end{aligned}$$

Assertion (A2) and then, item (iii) of Theorem 2.10 is proved. The invariance of the measure P_∞ w.r.t. the shifts in time follows from convergence (1.6). ■

Remark. Now we simplify formula (2.12) using (3.4). Denote by $G_t^{ij}(\theta)$, $i,j, = 0,1$, the entries of the matrix-valued operator $G_t(\theta)$ defined by the rule

$$G_t(\theta) := e^{\tilde{\mathcal{A}}(\theta)t} = \begin{pmatrix} \cos\Omega(\theta)t & \sin\Omega(\theta)t\,\Omega^{-1}(\theta)\\ -\Omega(\theta)\sin\Omega(\theta)t & \cos\Omega(\theta)t\end{pmatrix}, \quad \theta \in K^d,$$

where the functions $\tilde{\mathcal{A}}(\theta)$ and $\Omega(\theta)$ are introduced in (2.2) and (2.4), respectively. Therefore, by (3.9),

$$(\widetilde{V'F})_\Pi(\theta, r) = \int\limits_{-\infty}^{+\infty} e^{\tilde{\mathcal{A}}^T(\theta)t}\vec{\widetilde{F}}_\Pi(\theta, r, t)\,dt = \int\limits_{-\infty}^{+\infty} \left(G_t^{00}(\theta), G_t^{01}(\theta)\right) \widetilde{F}_\Pi(\theta, r, t)\,dt.$$

Hence, by (2.12) and (3.3),

$$\begin{aligned}\mathcal{Q}^P_\infty(F_1, F_2) &= (2\pi)^{-d}\int\limits_{K^d} \left(\tilde{q}_\infty(\theta), (\widetilde{V'F_1})_\Pi(\theta,\cdot) \otimes \overline{(\widetilde{V'F_2})_\Pi(\theta,\cdot)}\right) d\theta\\ &= (2\pi)^{-d}\int\limits_{-\infty}^{+\infty} dt_1 \int\limits_{-\infty}^{+\infty} dt_2 \int\limits_{K^d} \sum_{i,j=0}^{1} \left(G^{0i}_{t_1}(\theta)\tilde{q}^{ij}_\infty(\theta)G^{0j}_{t_2}(\theta), \widetilde{F_{1\Pi}}(\theta,\cdot) \otimes \overline{\widetilde{F_{2\Pi}}}(\theta,\cdot)\right) d\theta.\end{aligned}$$

Hence, due to (3.4), the correlation function of the measure P_∞ has the form

$$Q^P_\infty(k{+}r, k'{+}r', t_1, t_2) =: q^P_\infty(k{-}k', r, r', t_1{-}t_2), \;\; k, k' \in \mathbb{Z}^d, \;\; r, r' \in \mathcal{R}, \;\; t_1, t_2 \in \mathbb{R},$$

where $\tilde{q}^P_\infty(\theta,r,r',t) := \cos\Omega(\theta)t\,\tilde{q}^{00}_\infty(\theta,r,r') - \sin\Omega(\theta)t\,\Omega^{-1}(\theta)\,\tilde{q}^{01}_\infty(\theta,r,r')$, $\tilde{q}^{ij}_\infty$ are entries of the matrix $\tilde{q}_\infty$ defined in (3.4).

Now we verify the mixing property (1.7) for the limit measure P_∞. Since P_∞ is Gaussian with zero mean value, it is enough to prove that for any $F_1, F_2 \in \mathcal{D}^0$,

$$I_\tau := \mathbb{E}_\infty\left([S_\tau Y^0, F_1][Y^0, F_2]\right) \to 0 \qquad \text{as } \ \tau\to\infty, \tag{3.10}$$

where $\mathbb{E}_\infty$ denotes the integral w.r.t. the measure P_∞, S_τ is defined in (1.5).

Indeed, using (3.6), (3.7) and (3.3), we obtain

$$\begin{aligned} I_\tau &= (2\pi)^{-d}\int\limits_{K^d}\left(\hat{q}_\infty(\theta), (\widetilde{V'S_\tau^{-1}F_1})_\Pi(\theta,\cdot)\otimes\overline{(\widetilde{V'F_2})_\Pi}(\theta,\cdot)\right)\,d\theta \\ &= \sum_{\pm,l}(2\pi)^{-d}\int\limits_{K^d\setminus\mathcal{C}_*} e^{\pm i\omega_l(\theta)\tau}c^l_\pm(\theta)\,d\theta, \end{aligned} \tag{3.11}$$

where

$$c^l_\pm(\theta) := \frac{1}{2}\int\limits_{-\infty}^{+\infty}dt_1\int\limits_{-\infty}^{+\infty}e^{\pm i\omega_l(\theta)(t_1-t_2)}\Big(P_l(\theta)M_l(\theta)P_l(\theta), \widetilde{F_{1\Pi}}(\theta,\cdot,t_1)\otimes\overline{\widetilde{F_{2\Pi}}}(\theta,\cdot,t_2)\Big)dt_2.$$

Here $M_l(\theta) := \left[\tilde{q}^{00}_\infty(\theta) \pm i\omega_l^{-1}(\theta)\tilde{q}^{01}_\infty(\theta)\right]$, $\omega_l(\theta)$ and $P_l(\theta)$ are introduced in Lemma 2.3. The oscillatory integrals in (3.11) vanish by the Lebesgue-Riemann theorem, because $c^l_\pm \in L^1(K^d)$. This can be proved using the technique from [5, Sec. 7].

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