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# Hyperbolic volume of 3-d manifolds, A-polynomials, numerical hypothesis testing



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**ИНСТИТУТ ПРИКЛАДНОЙ МАТЕМАТИКИ**
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# Hyperbolic volume of 3-$d$ manifolds, $A$-polynomials, numerical testing of hypothesis

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Мы продолжаем наше знакомство со связями гиперболического объёма дополнения узла в трехмерной сфере с топологическими инвариантами узла. В этот раз мы уделим внимание $A(M, L)$ параметризации аффинного многообразия с каспом, порожденным узлом (т.н. $A$-многочленам). Затем, используя известные выражения $A$-многочленов для ряда узлов, мы приведем результаты численной проверки гипотез об асимптотике решений $q$-разностных уравнений, связанных с гиперболическим объемом этих узлов.

***Ключевые слова:*** узлы, фундаментальная группа дополнения узла, $\mathrm{SL}_2$-представление, $A$-многочлен, WKB-асимптотика, $q$-разностное уравнение, Гипотеза Объёма.

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We continue our study of the connections between the hyperbolic volume of the complement of a knot in the three dimensional sphere with topological invariants of this knot. This time we pay attention to $A(M, L)$ parametrization for the affine variety with casp, produced by a knot (so-called $A$-polynomials). Then, using the known expressions of $A$-polynomials for number of knots we present results of the numerical tests for the conjectures on asymptotics of solutions of $q$-difference equations connected with the hyperbolic volume of these knots.

***Key words:*** knots, fundamental group of the complement of a knot, $\mathrm{SL}_2$-representation, $A$-polynomials, WKB-asymptotics, $q$-difference equation, Volume Conjecture.

# 1. Introduction

In this preprint, we continue our earlier (see [1, 2, 3]) acquaintance with the connections of the hyperbolic volume $\mathrm{vol}(\mathbb{S}^3 \setminus K)$ of the complement of the knot $K$ in the three-dimensional sphere $\mathbb{S}^3$ with the topological invariants of the knot $K$. These connections between the *geometric* characteristic: $\mathrm{vol}(\mathbb{S}^3 \setminus K)$ and the *combinatorial, algebraic* characteristics of the knot $K$ are established using *analytical* methods: asymptotic WKB analysis of $q$-difference equations for $q$-hypergeometric functions.

This time (see Section **2**) we will pay attention to well-known *algebraic* concepts: the fundamental group of the knot $\pi_1(K)$, its $\mathrm{SL}_2(\mathbb{S}^3\setminus K, \mathbb{C})$ representation, and $A(M, L)$ parametrization of $\mathrm{SL}_2$ as an affine variety with cusp generated by the knot $K$ (the so-called $A$-polynomial).

Then (see Section **3**), using the known $A$-polynomials for some knots $K$, we present the results of a numerical test of hypotheses about the asymptotics of solutions of $q$-difference equations associated with $\mathrm{vol}(\mathbb{S}^3 \setminus K)$.

Now, in continuation of the introduction, we will dwell on the formulation of the problem and on the main points of the *analytical* approach to this problem.

**1.1. Knot invariants and $q$-difference equations.** The famous Kashaev conjecture (see [4, 5]) relates the hyperbolic volume $\mathrm{vol}(\mathbb{S}^3\setminus K)$ to some topological invariant of the knot $K$. Later (see [6, Theorem 4.9]) this Kashaev invariant was expressed in terms of a classical invariant: the Jones polynomial $J(q)$ and its *colors* by $n$-dimensional representations of the quantum group, the so-called *colored* Jones polynomials (see [7, 8]):

$$\{J_n(q)\}_{n=1}^{\infty}, \qquad J_1(q) = 1, \quad J_2(q) = J(q), \quad \ldots \; . \tag{1.1}$$

In turn, Garofalidis and Le (see [9]) proved that the colored Jones polynomials $\{J_n(q)\}_{n=1}^{\infty}$ are $q$-hypergeometric functions and, therefore,[1] are particular solutions of the $q$-difference equations:

$$\sum_{j=0}^{d} A_j(q^n, q) Q_{n+j}(q) = 0, \tag{1.2}$$

satisfying the initial conditions from (1.1). That is, assuming in (1.2)

$$Q_n(q) := J_n(q), \qquad n = 1, \ldots, d\,, \tag{1.3}$$

we obtain the entire sequence (1.1). We will use this property as the definition of colored Jones polynomials.

[1] like the classical orthogonal polynomials and their generalizations, the jointly orthogonal polynomials with respect to classical weights satisfying $d$-term recurrence relations

Example. For the *eight*-knot $4_1$ the initial conditions are:

$$J_1(q) = 1, \qquad J_2(q) = q^2 - q + 1 - q^{-1} + q^{-2}$$

and coefficients:

$$a_0(q) := \frac{q^{1-n}(1+q^{n-1})(1-q^{2n-1})}{1-q^n}, \qquad a_2(q) := -\frac{(1-q^{n-2})(1-q^{2n-1})}{(1-q^n)(1-q^{2n-3})},$$
$$a_1(q) := \frac{q^{2-2n}(1-q^{n-1})^2(1+q^{n-1})(1+q^{4n-4}-q^{n-1}-q^{2n-3}-q^{2n-1}-q^{3n-3})}{(1-q^n)(1-q^{2n-3})}$$

using recurrence relations[2]

$$J_n(q) = \sum_{j=1}^{2} a_j(q) J_{n-j}(q) + a_0(q), \quad n \geq 3, \tag{1.4}$$

define the entire sequence of colored Jones polynomials.

**1.2. Volume Hypothesis – Limiting Regimes.** Kashaev's conjecture (**VC**) on the volume of hyperbolic knots $K$ states:

$$2\pi \lim_{N\to\infty} \frac{1}{N} \ln |J_N(q = e^{2\pi i/N})| = \mathrm{vol}(\mathbb{S}^3 \setminus K). \tag{1.5}$$

Let's look at the left-hand side. There is a specific knot $K$, and for it the initial colored Jones polynomials (1.1) and the coefficients of the recurrence relations (1.2) are known. A large number $N$ is fixed, and using (1.3), (1.2) the polynomials $\{J_n(q)\}_{n=1}^N$ are successively calculated. Then in the polynomial $J_N(q)$ the substitution $q = e^{2\pi i/N}$ is done and the resulting quantity, depending only on $N$, grows exponentially with increasing $N$, and the rate of its growth is proportional to the hyperbolic volume $\mathbb{S}^3 \setminus K$.

Thus, we have two connected limits: $N \to \infty$ and $q = e^{2\pi i/N} \to 1$, along the arc $\mathbb{S}^1$ in the upper half-plane $\mathbb{C}_+$, which motivates us to consider the original $q$-difference equation (1.2) in the following double-scale regime:

$$\sum_{j=0}^{d} A_j(q^n, q) Q_{n+j}(q) = 0, \qquad \begin{cases} N \to \infty, \\ \dfrac{n}{N} \to t \in [0,1], \quad q^n \to e^{2\pi i t} =: z \in \mathbb{S}^1, \end{cases} \tag{1.6}$$

which in the limit gives the algebraic function $\lambda(z)$ (*spectral curve*):

$$P(z,\lambda) := \sum_{j=0}^{d} A_j(z,1)\lambda^j \;=\; 0, \quad \to \quad \lambda(z) := \{\lambda_m(z)\}_{m=1}^d. \tag{1.7}$$

[2]the inhomogeneous form of the equation (1.4) can easily be reduced to the homogeneous form (1.2) with $d = 3$

**1.3. VKB - asymptotics of the general solution of the $q$-difference equation.** On closed arcs $I := \{z\} \subseteq \mathbb{S}^1$, where the roots of the polynomial (1.7) *are separated*:

$$\exists\, \delta > 0 \,:\, \{|\lambda_m(z) - \lambda_k(z)|\} \geq \delta, \qquad \forall\, z \in I \quad \text{and} \quad m \neq k, \tag{1.8}$$

there are approaches (see [12, 1, 2, 3] ) for obtaining formal (and asymptotic) expansions in the small parameter $1/N$ for the general (*fundamental*) solution of the $q$-difference equation (1.2). If we write the recurrence relations (1.2) in matrix form: $\overrightarrow{Q}_{n+1} = \mathcal{A}_n \overrightarrow{Q}_n$, where $d \times d$ matrix $\mathcal{A}_n$ is formed by the coefficients $\{A_j(q^n, q)\}_{j=0}^d$ of (1.2), and the vector $\overrightarrow{Q}_n$ denotes

$$\overrightarrow{Q}_n \equiv \overrightarrow{Q}_n(q^n, q) := (Q_n, Q_{n+1}, \ldots, Q_{n+d-1})^T, \quad n \in \mathbb{N}, \tag{1.9}$$

then the limiting matrix $\mathcal{A} := \mathcal{A}(q^n, q)|_{q=1, q^n=z}$ has *characteristic* polynomial $P(z, \lambda)$ and eigenvalues $\{\lambda_m(z)\}_{m=1}^d$, the same as in (1.7).

According to the general theorems (for details[3] see [1, 3]) on the expansion of the fundamental solution (1.9) of the $q$-difference equation (1.2) in the zones of separated eigenvalues (1.8), the following asymptotics holds:

$$\overrightarrow{Q}_n(q^n, q)|_{q=e^{2\pi i/N}, q^n=z} = \sum_{j=1}^{d} c_j\, e^{N\varphi_{-1}^{(j)}(z)} \left( \overrightarrow{F}_j(z) + O\left(\frac{1}{N}\right)\right), \quad \text{где} \tag{1.10}$$

$$\overrightarrow{F}_j(z) := e^{\varphi_0^{(j)}(z)}(1, \lambda_j(z), \ldots, \lambda_j^{d-1}(z))^T, \quad \varphi_{-1}^{(j)}(e^{2\pi i t}) = \int_{t_1}^{t} \ln(\lambda_j(e^{2\pi i\tau}))\, d\tau.$$

Assuming the maximizing (integral above) $j := 1$, for a particular solution $Q_n$ in *general position* (i.e. all $c_j \neq 0$) in the zone (1.8) started at $z = 1$ we have:

$$\lim_{N\to\infty,\, n/N \to t} \frac{\ln |Q_n(e^{2\pi i/N})|}{N} = \operatorname{Re} \varphi_{-1}^{(1)}(e^{2\pi i t}) = \int_0^t \ln |\lambda_1(e^{2\pi i\tau})|\, d\tau. \tag{1.11}$$

Thus, if for some knot $K$: **1)** the limiting eigenvalues are *separate* on the whole $\mathbb{S}^1$, and **2)** the particular solution $\{J_n(2\pi i/N)\}_{n=1}^N$ grows exponentially as $n := [tN], t \in (0, 1)$, then (see also [12]) for the left-hand side in (1.5) we have:

$$2\pi \lim_{N\to\infty} \frac{\ln |J_N(e^{2\pi i/N})|}{N} = 2\pi \int_0^1 \ln |\lambda_1(e^{2\pi i\tau})|\, d\tau \overset{(1.5)}{=} \operatorname{vol}(\mathbb{S}^3 \setminus K). \tag{1.12}$$

**1.4. VKB - asymptotics of particular solutions.** Turning to the conditions for the existence of the limit on the left-hand side of (1.12), we note:

All known to us spectral curves $\lambda(z)$ have branch points on $\mathbb{S}^1$, i.e. cases **1)** - continuation of the zone (1.8) to the entire circle - were not observed[4].

[3] below, in Appendix 1, we present the main points and statements

[4] by the end of the work on the preprint we encountered case **1)**, see below point 3.10

In this connection, an important problem arises of rigorously justifying the possibility of continuing asymptotics of the form (1.11) for particular solutions from the zone $I$ - the separation of the seigenvalues (1.8) to the zone $\tilde{I}$ - the holomorphic continuation of the branch[5] $\lambda_1$, regardless of the presence of branch points of other branches in the continuation zone $\tilde{I}$. This problem is more complicated than the problem of the asymptotic expansion in $I$ of the fundamental solution, and its general solution is unknown to us. In the particular case of 3- and 4-term (ordinary, not "$q$-") recurrence relations, it was solved in [13].

Before continuing to discuss the conditions for the existence of a limit for the colored Jones polynomials on the left-hand side of (1.12), let us dwell on the second equality on the right-hand side of (1.12):

$$2\pi \int_0^1 \ln|\lambda_1(e^{2\pi i\tau})|\, d\tau \overset{?}{=} \mathrm{vol}(\mathbb{S}^3 \setminus K). \tag{1.13}$$

In [3] we numerically investigated the fulfillment of the equality (1.13) on the simplest knots $4_1, 5_2$. To our surprise, we found (with a large number of significant digits coinciding) that the integral on the left-hand side of (1.13) is twice as large as the known volumes on the right:

$$2\pi \int_0^1 \ln|\lambda_1(e^{2\pi i\tau})|\, d\tau \approx \mathrm{vol}(\mathbb{S}^3 \setminus K) \ * \ 2 \ (!) \tag{1.14}$$

This fact, of course, was known before, see [15].

Now let us return to the condition **2)** of the existence of a limit on the left-hand side of (1.12) for a particular solution of the equation (1.2), (1.3) (which assumed exponential growth of $\{J_n(e^{\frac{2\pi i}{N}})\}_{n=1}^N$ for $n := [tN]$, $t \in (0,1)$ and $N \to \infty$). Here we note that from the well-known relation, see [6], for $q := e^{\frac{2\pi i}{N}}$ we have

$$J_n(q) = \overline{J_{N-n}(q)}, \ 1 \leqslant n < N \quad \Rightarrow \quad J_1(q) = J_{N-1}(q) = 1, \tag{1.15}$$

and moreover, for the knot $4_1$ it is easy to prove[6], and for a number of other knots it is possible to verify numerically that for all $n < N$ the colored Jones polynomials (for $q := e^{\frac{2\pi i}{N}}$) are bounded and at the same time for $n = N$ they grow exponentially, in accordance with the Kashaev conjecture (1.5):

$$|J_n(e^{\frac{2\pi i}{N}})| \lesssim C, \ \ 1 \leqslant n < N \qquad \lim_{N\to\infty} \frac{1}{N} \ln|J_N(e^{\frac{2\pi i}{N}})| = \frac{\mathrm{vol}(\mathbb{S}^3 \setminus K)}{2\pi}. \tag{1.16}$$

Thus we see that the particular solution (1.1) defined by the Cauchy problem (1.3) for the $q$-difference equation (1.2) (i.e., the colored Jones polynomials) cannot directly express the value of the limit on the left-hand side of the volume hypothesis (1.12) using WKB analysis (1.11).

[5]the branch defining the growth rate of this particular solution in the zone $I$

[6]see Appendix 2 below for the estimate of $\{J_n(e^{\frac{2\pi i}{N}})\}$ for *twisted* knots

At the same time, we note that the particular solution $\{J_n(e^{\frac{2\pi i}{N}})\}_{n=1}^N$ has in the limit regime (1.6) a very interesting local singularity (1.16). The study of this singularity and finding the volume in connection with it seems to be a very important task, even in the simplest particular case of the knot $4_1$.

Still, let's return to the WKB asymptotics (1.12) and try to "make it work"to find the limit on the left side of (1.12) and understand the conflict (1.13) vs. (1.14).

Let us imagine for $N \gg 1$ *continualization* of a particular solution (1.1):

$$\{J_n(e^{\frac{2\pi i}{N}})\}_{n=1}^N, \;\; n := [tN], \;\; t \in [0,1] \quad \longrightarrow \quad \{J_{[tN]}(e^{\frac{2\pi i}{N}})\}_{t\in[0,1]}.$$

We can imagine a function $J_{[tN]}$ continuous in $t \in [0,1]$, oscillating with high frequency, with local (bounded as $N$ grows) "minima" at points $t_n := n/N$, $n = 1,...,N-1$ and with alternating local "maxima", growing exponentially with $N$, at least in the left neighborhood of the maximum of the function $J_{[tN]}$ at $t = 1$. Now, if we were to impose a small *perturbation* on the function, shifting the neighborhoods of the local "maxima" to the *discretization* points $t_n$, then the exponential growth rate at $t = 1$ could be calculated using the WKB integral (1.11) along the growth zone of the perturbed particular solution.

An example of such *perturbation* could be the following trick. Let us denote

$$f_n(q) := (1-q^n)J_n(q) \quad \text{then} \quad J_n(q) = \frac{J_n(q)(1-q^n)}{(1-q^n)} = \frac{f_n(q)}{(1-q^n)}. \qquad (1.17)$$

Obviously, the right-hand side here has uncertainty at $q \to e^{2\pi i/N}$, $n = N$, and therefore, by L'Hôpital's rule, we have

$$J_N(q = e^{2\pi i/N}) = -\left.\frac{f_n'(q)}{nq^{n-1}}\right|_{n=N,q=e^{2\pi i/N}}, \quad \text{where} \;\; f_n'(q) := \frac{d}{dq}f_n(q). \qquad (1.18)$$

The following statement is true (For the proof, see [3, Lemma 2]):

*The spectral curves for $J_n$ and for $f_n'$ coincide. Besides,*

$$\lim_{N\to\infty}\frac{1}{N}\ln|f_N'(q = e^{2\pi i/N})| = \lim_{N\to\infty}\frac{1}{N}\ln|J_N(q = e^{2\pi i/N})|. \qquad (1.19)$$

Let us illustrate the validity of this proposition using the example of knot $4_1$. Let us transform the recurrence relations (1.4) to the form:

$$f_n = \tilde{a}_1(n,q)f_{n-1} + \tilde{a}_2(n,q)f_{n-2} + \tilde{a}_0(n,q), \quad n \geq 3, \qquad (1.20)$$
$$f_1 := 1-q, \quad f_2 := (1-q^2)(q^2-q+1-q^{-1}+q^{-2}) = -q^4+q^3-q^{-1}+q^{-2},$$

where

$$\tilde{a}_j(n,q) := \frac{a_j(n,q)(1-q^n)}{1-q^{n-j}}, \quad j = 0,1,2. \qquad (1.21)$$

Further, for the derivatives $f_n'$ these recurrences take the form:

$$\begin{aligned} &f_n' = \tilde{a}_1(n,q) f_{n-1}' + \tilde{a}_2(n,q) f_{n-2}' + \tilde{A}_0(n,q), \quad n \geq 3, \\ &f_1' := -1, \quad f_2' := -4q^3 + 3q^2 + q^{-2} - 2q^{-3}, \end{aligned} \tag{1.22}$$

where $\tilde{a}_j$, $j = 1{,}2$, are defined in (1.21),

$$\tilde{A}_0(n,q) := \frac{d}{dq}\tilde{a}_0 + f_{n-1}\frac{d}{dq}\tilde{a}_1 + f_{n-2}\frac{d}{dq}\tilde{a}_2.$$

As we see, the homogeneous parts of both recurrence relations (1.20) and (1.22) coincide, and it is easy to verify that in the limit regime (1.6) their spectral curves coincide with the spectral curve of the original recurrence relation (1.2) - (1.4).

Thus, in order to obtain a formula for the limit in (1.19) in the form of the WKB integral (1.12), it is necessary to check whether the terms of the sequence $\{f_n'(q)|_{q=e^{2\pi i/N}}\}_{n=1}^N$ exhibit exponential growth as $N \to \infty$ and $n/N =: t \in [\tilde{t}, 1], \tilde{t} < 1$. We have not yet been able to strictly prove the corresponding lower bounds, but numerical calculations for the knots $4_1$ and $5_2$ give a "positive" answer to this question for some $\tilde{t} \in (1/2, 1)$. Thus, based on these numerical calculations, in [3] we proposed a hypothesis about representing the limit on the left-hand side of (1.5) as a WKB integral:

$$2\pi \lim_{N\to\infty} \frac{\ln|J_N(e^{2\pi i/N})|}{N} = 2\pi \int_{\tilde{t}}^1 \ln|\lambda_1(e^{2\pi i\tau})|\, d\tau \tag{1.23}$$

where $\tilde{t} \in (1/2, 1)$ defines the branch point $\tilde{z} := e^{2\pi i\tilde{t}}$ of $\lambda(z)$ closest to $(-1)$ on $\mathbb{S}^1$, whose branch $\lambda_1 : (\exists, !)$ is holomorphic and $|\lambda_1| > 1$ on the arc $(\tilde{z}, e^{2\pi i}] \subset \mathbb{S}^1$. Since the $\lambda(z)$ known to us[7], corresponding to the knots, had for all their branches $\{\lambda_j(z)\}$ the unit modulus on the arc $(\overline{\tilde{z}},\, \tilde{z}) \ni (-1)$, then the integral in (1.23) can be extended (without changing the value) to the entire lower semicircle, and also, having defined $(\exists, !)$ the holomorphic branch $\lambda_1^* : |\lambda_1^*| > 1$ on the arc $[1, \overline{\tilde{z}}) \subset \mathbb{S}^1$, extend to the upper semicircle:

$$\int_0^{1/2} \ln|\lambda_1^*(e^{2\pi i\tau})|\, d\tau = \int_{1/2}^1 \ln|\lambda_1(e^{2\pi i\tau})|\, d\tau = \int_{\tilde{t}}^1 \ln|\lambda_1(e^{2\pi i\tau})|\, d\tau, \tag{1.24}$$

which agrees with (1.14).

[7] at the time of publication of [3]

## 2. Obtaining $A-$polynomials

Along with the polynomial $P(z, \lambda)$, which characterizes the spectral curve (2.7) and is related to the (analytic) left-hand side of the volume hypothesis (1.5), of interest are the so-called $A(M, L)$-polynomials introduced in [16] and parametrizing the affine variety for the $SL_2(\mathbb{S}^3 \setminus K, \mathbb{C})$ representation of the fundamental group of the knot $\pi_1(K)$, where $M$ is the *latitude* and $L$ is the *longitude* of the small torus (cusp) enclosing the knot $K$ (i.e. the $A(M, L)$-polynomial is related to the right-hand (algebraic-geometric) side (1.5)).

This interest is due to the fact that, according to the **AJ** *conjecture*[8] of Garofalidis [10, p.297] both of these polynomials coincide after the cancellation of $A(M, L) := A(M, L)/\tilde{A}(M)$ of some power of $L$ and a factor $\tilde{A}$ depending only on $M$, and the identification $z \equiv M^2, \lambda \equiv L \Rightarrow :$

$$P(z, \lambda) \equiv A(M, L). \tag{2.1}$$

Moreover, if we denote by $V$ the volume of the manifold defined by the parametrization $A(M, L) = 0$, then the following formula is known [11, C.D. Hodgson, 1986]:

$$dV = -2(\log|L|\, d\,(\mathsf{arg}M) \;-\; \log|M|\, d\,(\mathsf{arg}L)). \tag{2.2}$$

If, using (2.1), we pass to the variables $z$, $\lambda$, then we obtain

$$dV = -\log|\lambda|\, d\,(\mathsf{arg}z) \;+\; \log|z|\, d\,(\mathsf{arg}\lambda),$$

while on the boundary torus $z := \exp(it)$ we have

$$dV = -\log|\lambda|\, dt. \tag{2.3}$$

Thus, the volume formula (2.2) in variables (2.3) demonstrates a direct connection of the WKB integrals (1.12), (1.23), (1.24) with the geometric right-hand side of the volume hypothesis (1.5).

This section is of a methodological nature. In it, using known sources, starting from the simplest knots, we will sequentially construct their fundamental group $\pi_1(K)$, $\mathrm{SL}_2(\mathbb{S}^3 \setminus K, \mathbb{C})$-representations and obtain an explicit form of the corresponding $A-$polynomials.

**2.1. Fundamental group $\pi_1(K)$ of knot $K$.** For the definition of $\pi_1(M, x_0)$ for a manifold $M$ with respect to a point $x_0 \in M$, see [17, p. 534].

We are interested in manifolds $M := \mathbb{S}^3 \setminus K$, where $x_0 = \infty \in \mathbb{S}^3$, and $K$ is a knot. Using the example of knot $K := 3_1$, following [17, p. 655], we will construct generators of this group and obtain relations connecting them.

[8] $AJ-$ the initial letters $A-$ of the polynomial and the Jones polynomials $J$

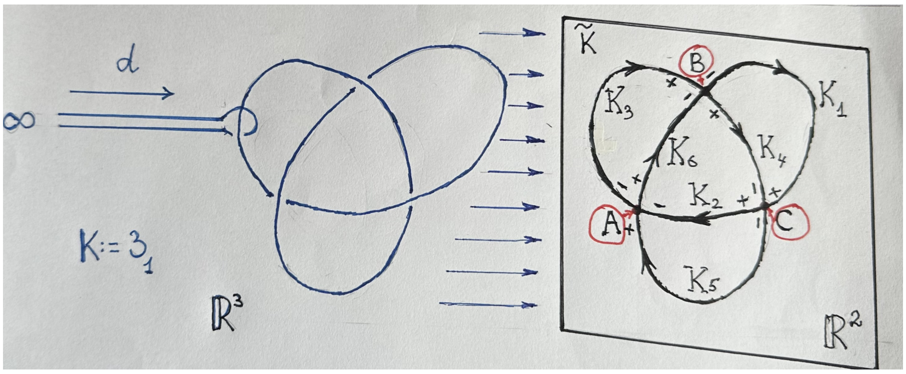


*Рис. 1.* Basic paths for the group $\pi_1(K)$ for "trefoil" $3_1$

Consider $\widetilde{K}$ - the projection of $K$ onto $\mathbb{R}^2$ (the direction of "general position" $d$ – the projection is chosen so that $\widetilde{K}$ is a planar graph with vertices $\{A, B, \ldots\}$ and edges $\{\mathrm{K}_1, \mathrm{K}_2, \ldots\}$, and exactly four edges converge at each vertex, their ends are labeled $+$ or $-$ depending on the location of the projected section of the knot $K$: $(+)$ – "above" or $(-)$ – "below". The direction and numbering of the edges is induced by the fixed direction of traversal of $K \subset \mathbb{S}^3$. For example, for $K := 3_1$ the graph has the form, see Fig. 1:

$$\begin{array}{llll} K = \{A, B, C\} \sqcup \{\mathrm{K}_1, \ldots, \mathrm{K}_6\}: & & \mathrm{K}_1 := [B_{(-)}C_{(+)}], & \mathrm{K}_2 := [C_{(+)}A_{(-)}], \\ \mathrm{K}_3 := [A_{(-)}B_{(+)}], & \mathrm{K}_4 := [B_{(+)}C_{(-)}], & \mathrm{K}_5 := [C_{(-)}A_{(+)}], & \mathrm{K}_6 := [A_{(+)}B_{(-)}]. \end{array}$$

*Generators (basic)* of the pathes of the group $\pi_1$ (we denote $a_j \in \pi_1$) are defined as follows: path $a_j$ starts from point $\infty \in \mathbb{S}^3$ in direction $d$, reaches point $K$, corresponding to the middle of edge $\mathrm{K}_j$, goes around $K$ and returns back to point $\infty$. We obtain *relations* connecting the generators of the group $\pi_1$. Let 4 edges converge at a vertex: $\mathrm{K}_{j_1}, \ldots, \mathrm{K}_{j_K}$, and in one pair, say in $(j_1, j_2)$, the ends of the edges have the same sign, and in the other – $(j_3, j_4)$ – the sign of the ends is opposite to the sign of $(j_1, j_2)$. If $(j_1, j_2)$ has the sign $(+)$, then we obviously have:

$$a_{j_1} = a_{j_2}, \quad j_2 = j_1 + 1, \tag{2.4}$$

in this case the pair $(j_3, j_4)$ has the sign $(-)$, and for the generators we obtain

$$a_{j_4} = a_{j_1}^{-1} a_{j_3} a_{j_1}. \tag{2.5}$$

The set of relations (2.4), (2.5) generates all relations in the group $\pi_1(\mathbb{S}^3 \setminus K, \infty)$. For the trefoil $K = 3_1$ we have:

$$\begin{array}{llll} B \to a_3 = a_4 =: x, & a_1 = a_3^{-1} a_6 a_3 & \Rightarrow & y = x^{-1} w^{-1} x, \\ C \to a_1 = a_2 =: y, & a_5 = a_1^{-1} a_4 a_1 & \Rightarrow & w = y^{-1} x^{-1} y, \\ A \to a_5 = a_6 =: w^{-1}, & a_3 = a_5^{-1} a_2 a_5 & \Rightarrow & x = w\, y\, w^{-1}. \end{array}$$

Thus, the representation of the fundamental group will be:

$$\pi_1(\mathbb{S}^3 \setminus 3_1, \infty) = \langle x, y : \; xw = wy, \; w = y^{-1}x^{-1}y \rangle. \tag{2.6}$$

We note a wide class of knots (the so-called two-bridge knots, see [18]), for which the fundamental group is generated by two generators, and its representation has the form (2.6):

$$\pi_1(\mathbb{S}^3 \setminus K, \infty) = \langle x, y : \; xw = wy, \; w = W(x, y, x^{-1}, y^{-1}) \rangle, \tag{2.7}$$

where $W$ are some words with the letters $(x, y, x^{-1}, y^{-1})$. For example, for the simplest hyperbolic knots $4_1$ (the "eight" knot) and $5_2$, see Fig. 2, the fundamental group (2.7) is defined by the words:

$$4_1 \to W_{4_1} := y^{-1}\, x\, y\, x^{-1}, \tag{2.8}$$

and

$$5_2 \to W_{5_2} := y\, x^{-1}\, y\, x\, y^{-1}\, x, \tag{2.9}$$

and also, see Fig. 2, the so-called twisted knots $K_p$ have

$$K_p \to W_{K_p} := (x\, y^{-1}\, x^{-1} y)^p. \tag{2.10}$$

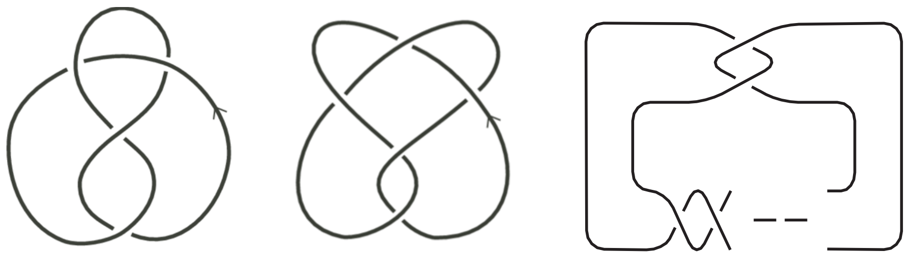

*Рис. 2.* $K = 4_1$, $K := 5_2$ and $K := K_p$ - twisted knots

**2.2. Representation $\rho : \pi_1(\mathbb{S}^3 \setminus K, \infty) \to \mathrm{SL}_2(\mathbb{C})$ and the algorithm for constructing $A$–polynomial.** In [16] for an oriented three-dimensional manifold $X \subset \mathrm{H}^3$, whose boundary can be enclosed inside a torus, i.e. $\partial X \subset \mathbb{T}^2$ (for example, $X = \mathbb{S}^2 \setminus K$), using the representation $\rho$ of the fundamental group $\pi_1(X)$ as a group $\mathrm{SL}(2, \mathbb{C}) = \{U\}$ – unimodular ($\det U = 1$) matrices $2 \times 2$, an isometric to the hyperbolic metric $\mathrm{H}^3$ parametrization of $(m, l)$ in the form of an algebraic curve $A(m, l) = 0$ is proposed. In this case, the variable $m$ in the polynomial $A$ has even degrees and corresponds to the points of the circle of the meridional section of the boundary torus $M$ – *latitude*, and $l$ – to the points of the cycle $L$ transversal to the meridian (intersecting it at one point) – *longitude*.

The general idea of this approach is the following. Cycles $M$ and $L$ in the manifold $X$ are identified as products of generators of the group $\pi_1(X)$. Then, for these generators, their representations from $\mathrm{SL}_2(\mathbb{C})$ are determined so that the elements of the corresponding matrices contain coordinates $m \in M$ and $l \in L$. Finally, considering the required number of non-trivial representations corresponding to trivial cycles, all undefined parameters except $m$ and $l$ are eliminated, leaving a polynomial dependence $A(m,l)=0$.

We describe this procedure for the two-bridge knots given above, see (2.7). Following [16], we turn to (2.7) and assume

$$M := x \quad \text{and} \quad L := x^n\, w\, w^*,$$

where the word $w^*$ is a mirror permutation of the word $w$, and $n$ is chosen so that the sum of the exponents of $L$ is zero. Defining the representation of generators

$$\rho(M) := \rho(x) := \begin{bmatrix} m & 1 \\ 0 & m^{-1} \end{bmatrix}, \quad \rho(y) := \begin{bmatrix} m & 0 \\ t & m^{-1} \end{bmatrix}. \tag{2.11}$$

Next, substituting into (2.7) a specific word $w$, corresponding to the knot under consideration, we generate a matrix

$$P(m,t) := \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} := \rho(xw) - \rho(wy) = \rho(x)\rho(w) - \rho(w)\rho(y), \tag{2.12}$$

which for hyperbolic two-bridge knots must have a zero main diagonal

$$p_{11} = p_{22} = 0. \tag{2.13}$$

Let us denote the Laurent polynomial in $m$:

$$p(m,t) := p_{12},$$

generate matrix

$$Q(m,t) := \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} := \rho(L) = \rho(x^n)\rho(w)\rho(w^*),$$

and denote

$$q(m,t) := q_{11}. \tag{2.14}$$

Finally, from the system of equations

$$\begin{cases} p(m,t) = 0 \\ q(m,l) = l \end{cases} \tag{2.15}$$

we exclude $t$ (by calculating the resultant of the polynomials $m^r p$ and $m^s(q-l)$ with respect to $t$) and obtain the desired polynomial $A(m,l)$.

Let us illustrate this procedure on specific knots (2.6), (2.7), (2.8), (2.9).

**2.3.** $A(m,l)$ **is a polynomial for the** $3_1$ **trefoil knot.** Substituting into (2.6) the representations (2.11) for $y^{-1}, x^{-1}, y$, we obtain $\mathrm{SL}_2(\mathbb{C})$ the representation for $w$:

$$\rho(w) = \rho(y^{-1}x^{-1}y) = \rho(y^{-1})\cdot\rho(x^{-1})\cdot\rho(y) = \begin{bmatrix} m+tm^{-1} & m^{-2} \\ -tm^2+(1-t)t & (1-t)m^{-1} \end{bmatrix}.$$

Next, according to the procedure described above, we find the representation $\rho(xw)-\rho(wy)$ - matrix $P(m,t)$. Here we have a surprise[9] – condition (2.13) is not satisfied, but, although $p_{22}=0$, we have

$$p_{11} = -(m^4+m^2t-m^2+1)+m^{-2} \neq 0. \tag{2.16}$$

After thinking a little about the fundamental group (2.6), we have

$$y = w^{-1}xw = (y^{-1}xy)x(y^{-1}x^{-1}y) = (y^{-1}xyx^{-1})x(xy^{-1}x^{-1}y),$$

thus the groups (2.6) and the fundamental group of the twisted knot $K_1$, see (2.10), coincide

$$\langle x, y : x\widetilde{w} = \widetilde{w}y,\ \widetilde{w} = xy^{-1}x^{-1}y\rangle = \pi_1(\mathbb{S}^3\setminus K_1,\infty), \tag{2.17}$$

as well as the knots that generate them[10] $3_1$ and $K_1$. Now, applying the above procedure from [16] to the group (2.17), we obtain the following expressions for the matrices $\rho(\widetilde{w})$ and $\rho(\widetilde{w})^*$:

$$\begin{bmatrix} (t-1)^2+tm^2 & (t-1)m^{-1}+m \\ (t-1)tm^{-1}+tm & tm^{-2}+1 \end{bmatrix}, \begin{bmatrix} tm^2+1 & (t-1)m+m^{-1} \\ (t-1)tm+tm^{-1} & (t-1)^2+tm^{-2} \end{bmatrix}.$$

For the matrix $P(m,t)$, as expected, we have (2.13): $p_{11}=p_{22}=0$, and for the Laurent polynomial $p_{12}$ we obtain

$$p(m,t) := p_{12} = (m^4+m^2t-m^2+1)m^{-2}. \tag{2.18}$$

Having calculated the matrix $Q(m,t) := \rho(\widetilde{w}\widetilde{w}^*) = \rho(\widetilde{w})\rho(\widetilde{w}^*)$, we have for the element $q_{11}$:

$$q(m,t) := q_{11} = (m^6t^2+m^4t^3-m^4t^2+m^4t+m^2t^3-m^2t^2+m^2+t^2-t)m^2.$$

Thus, the system (2.15) is obtained. From the first equation we immediately express $t$:

$$p(m,t_0)=0 \quad\Longrightarrow\quad t_0 = 1-m^2-m^{-2}. \tag{2.19}$$

Substituting the obtained $t$ into the second equation, we obtain (after all reductions):

$$q(m,t)|_{t=t_0} = l \quad\Longrightarrow\quad A(m,l) = -(m^6+l). \tag{2.20}$$

[9]Knot $3_1$– is not hyperbolic, so (2.13) is not guaranteed

[10]which can be easily verified by comparing Fig. 1 and $K_n$, $n=1$ in Fig. 2

**2.4.** $A(m,l)$ **polynomial for the figure-eight knot** $4_1$**.** Substituting into (2.8) the representations (2.11) for $x, y$ and their inverses, we obtain the $\mathrm{SL}_2(\mathbb{C})$ representation for $w$ and $w^*$:

$$\rho(w) = \rho(y^{-1})\rho(x)\rho(y)\rho(x^{-1}) = \begin{bmatrix} (t+m^2)m^{-2} & (1-t-m^2)m^{-1} \\ (t(1-t)-tm^2)m^{-1} & (1-t)^2+tm^2 \end{bmatrix}.$$

$$\rho(w^*) = \begin{bmatrix} ((1-t)^2\cdot m^2+t)m^{-2} & ((1-t)m^2-1)m^{-1} \\ ((1-t)m^2t-t)m^{-1} & m^2t+1 \end{bmatrix}.$$

We calculate the matrix $P(m,t) := \rho(xw) - \rho(wy)$. The test $p_{11} = p_{22} = 0$ is passed, and for the Laurent polynomial $p_{12}$ we have:

$$p(m,t) := (m^4t - m^4 + m^2t^2 - 3m^2t + 3m^2 + t - 1)m^{-2}.$$

Calculating the matrix $Q(m,t) := \rho(ww^*)$, we have for the element $q_{11}$:

$$q(m,t) := q_{11} = (m^6t^2 - m^6t + m^4t^2 - m^2t^3 + m^4 - m^2t^2 + m^2t + t^2)m^{-4}.$$

Finally, we get rid of $t$ in the system (2.15) by calculating the resultant of the polynomials $m^2p$ and $m^4q$ with respect to $t$. We obtain the desired polynomial

$$A_{4_1}(m,l) := -lm^8 + lm^6 + l^2m^4 + 2lm^4 + m^4 + lm^2 - l. \tag{2.21}$$

**2.5.** $A(m,l)$ **is a polynomial for knot** $5_2$**.** As before, starting from (2.9), we obtain a $\mathrm{SL}_2(\mathbb{C})$ representation for $w$ and $w^*$:

$$\rho(w) := \begin{bmatrix} m^2t^2 - 2m^2t + m^2 + t & \dfrac{(t-1)(m^2t - 2m^2 + 1)}{m} \\ \dfrac{t(m^2t^2 - 3m^2t + 2m^2 + t - 1)}{m} & \dfrac{m^2t^3 - 4m^2t^2 + 4m^2t + t^2 - 2t + 1}{m^2} \end{bmatrix},$$

$$\rho(w^*) := \begin{bmatrix} m^2t^2 - 2m^2t + t^3 + m^2 - 4t^2 + 4t & \dfrac{m^2t - m^2 + t^2 - 3t + 2}{m} \\ \dfrac{(t-1)t(m^2 + t - 2)}{m} & \dfrac{m^2t + t^2 - 2t + 1}{m^2} \end{bmatrix}.$$

We calculate the matrix $P(m,t) := \rho(xw) - \rho(wy)$, and for $p_{12}$ we have:

$$p(m,t) := \frac{m^4t^2 - 3m^4t + m^2t^3 + 2m^4 - 5m^2t^2 + 8m^2t - 3m^2 + t^2 - 3t + 2}{m^2}.$$

Next, for the element $q_{11}$ of the matrix $Q(m,t) := \rho(ww^*)$, we get $q(m,t) :=$

$$-\,(m^{12}\,t^5 - 5\,m^{12}\,t^4 + m^{10}\,t^6 + 9\,m^{12}\,t^3 - 5\,m^{10}\,t^5 - 7\,m^{12}\,t^2 + 9\,m^{10}\,t^4 + 2\,m^8\,t^6 \\ + 2\,m^{12}\,t - 6\,m^{10}\,t^3 - 11\,m^8\,t^5 + 23\,m^8\,t^4 + 2\,m^6\,t^6 + m^{10}\,t - 21\,m^8\,t^3 - 10\,m^6\,t^5 \\ + 7\,m^8\,t^2 + 17\,m^6\,t^4 + 2\,m^4\,t^6 - 8\,m^6\,t^3 - 12\,m^4\,t^5 - 6\,m^6\,t^2 + 28\,m^4\,t^4 + m^2\,t^6 \\ + 6\,m^6\,t - 31\,m^4\,t^3 - 6\,m^2\,t^5 - m^6 + 16\,m^4\,t^2 + 14\,m^2\,t^4 - 3\,m^4\,t - 16\,m^2\,t^3 + t^5 \\ + 10\,m^2\,t^2 - 6\,t^4 - 4\,m^2\,t + 13\,t^3 - 12\,t^2 + 4\,t)/m^6.$$

Calculating the resultant $(m^2 \cdot p, m^6(q-l), t)$, we obtain:

$$A_{5_2}(m,l) := l^3\,m^{14} - l^2\,(m^{14} + 2\,l\,m^{12} + 2\,m^{10} - m^6 + m^4) + l\,(m^{10} - m^8 + 2\,m^4 + 2\,m^2 - 1) + 1.$$

**2.6.** $A(m,l)$ **is a polynomial for the knot** $7_4$**.** Using *MAPLE* we have calculated $A$-polynomials for other knots as well. For example, for knot $7_4$ we know the representation (2.7) of the fundamental group, where

$$7_4 \to W_{7_4} := y\,x^{-1}\,y\;\;x\,y^{-1}\,x\,y^{-1}\,x\,y^{-1}\,x\,y^{-1}\;\;x\,y^{-1}\,x.$$

From it, as before, we arrive at the resultant $(p,(q-l),t)$, which turns out to be factorized: $A_{7_4}(m,l) := A_{7_4}(m,l)^{(1)}\,A_{7_4}(m,l)^{(2)}$, where

$$\begin{aligned} A^{(1)}_{7_4}(m,l) :=& l^3\,m^{14} + (-2\,m^{14} + 6\,m^{12} + 2\,m^{10} - 7\,m^8 + 2\,m^6 + 3\,m^4 - 2\,m^2 + 1)\,l^2 \\ & + (m^{14} - 2\,m^{12} + 3\,m^{10} + 2\,m^8 - 7\,m^6 + 2\,m^4 + 6\,m^2 - 2)\,l \;+\; 1. \\ A^{(2)}_{7_4}(m,l) :=& (l^2\,m^8 + (-m^8 + m^6 + 2\,m^4 + m^2 - 1)\,l \;+\; 1)^2. \end{aligned} \tag{2.22}$$

**2.7. A note on the AJ - conjecture.** This conjecture has been proved for the knots $3_1, 4_1, 7_4$ (cf. Garoufalidis [10] and Koutschan–Garoufalidis [32]); for torus knots (cf. Hikami [33], Tran [34]), for some classes of two-bridge knots, including all twist knots and pretzels (cf. Le [35], [36], [37]).

The explicit form of the $A$−polynomials for various knots can be found in [40, 39].

# 3. Roots of $P(z,\lambda)$ at $|z| = 1$ and WKB integrals

This section is the central one in our preprint. In it we present new material: for various knots with known $A$−polynomials[11] and volumes, we analyze the branches $\{\lambda_j(z)\}$ for $|z| = 1$ of the algebraic function (1.7) and integrate them numerically to test our conjecture (1.23), (1.24) about $(\exists, !)$ of the holomorphic branch $\lambda_1 : |\lambda_1| > 1$ on the arc of the upper semicircle $\mathbb{S}^1_+$:

$$2\pi \lim_{N\to\infty} \frac{\ln|J_N(e^{2\pi i/N})|}{N} = 2\pi \int_0^{\mathbf{(1/2)}} \ln|\lambda_1(e^{2\pi i\tau})|\,d\tau \overset{(1.5)}{=} \mathrm{vol}(\mathbb{S}^3 \setminus K). \tag{3.1}$$

[11] assuming the **AJ**-hypothesis

**3.1. Knot $6_1$: branches $\{\lambda_j\}$ and volume.** As we have already noted, we take the explicit form of the characteristic polynomials (1.7) from the known $A(M,L)$-polynomials, setting $z \equiv M^2, \lambda \equiv L$ in (2.1). In view of the interest in (3.1), we study for $|z|{=}1$ the branches $\{\lambda_j(z)\}_{j=1}^d$ of the algebraic function $\lambda(z)$ defined by (1.7). Let us note the general properties[12] of the roots of colored Jones polynomials (in $\lambda$) of the polynomial $P(z,\lambda)$ for $|z|{=}1$:

**1)** $\lambda(z) = \overline{\lambda(\bar z)}$, i.e. the sets $\{\lambda_j(z)\}_{j=1}^d$ and $\{\overline{\lambda_j(\bar z)}\}_{j=1}^d$ coincide;

**2)** $\forall j \;\; \exists\, k \neq j: \quad |\lambda_j(z)\lambda_k(z)| = 1, \quad j,k = 1,...d, \;\; |z| = 1.$

For the knot $6_1$ we have:

$$P(z,\lambda) := \lambda^4 z^4 + \lambda^3(-2z^6+3z^5+3z^4+z-1) + \lambda(-z^8+z^7+3z^4+3z^3-2z^2) + \lambda^2(z^8-3z^7-z^6+3z^5+6z^4+3z^3-z^2-3z+1) + z^4 \;=\; 0, \qquad (3.2)$$

and $\quad P(-1,\lambda) = (\lambda-1)^4, \qquad P(1,\lambda) = (\lambda+1)^4.$

The analysis of branches $\{\lambda_j(z)\}_{j=1}^4$ is enough to do on the $\mathbb{S}^1_+$-upper semicircle. The discriminant in (3.2) is equal to $D(z) := z^6\,(z-1)^{12}(z+1)^{12}\Delta$, where $\Delta :=$

$$5z^{12}-32z^{11}+56z^{10}-118z^9+124z^8+32z^7+123z^6+32z^5+124z^4-118z^3+56z^2-32z+5$$

Of all the zeros of $\Delta$ on the upper semicircle there are two branch points $\lambda(z)$: $z_1 := -0.84.. + i\,0.53.. =: \boldsymbol{e}^{it_1}, t_1 = 2.57..,\;\; z_2 := 0.052.. + i\,0.99.. =: \boldsymbol{e}^{it_2}, t_2 =$ 1.51.. We have $\lambda_j(-1) = 1$, $j = 1{,}2{,}3{,}4$. Moving from $z = \boldsymbol{e}^{i\pi}$ clockwise, we calculate the roots of the polynomial (3.2) and obtain that $|\lambda_j(z)| = 1$, $j = 1{,}2{,}3{,}4$, on the arc $\{z = \boldsymbol{e}^{it}, t \in [\pi, t_1]\}$. At point $z_1$ we have: $\lambda_1(z_1) = \lambda_4(z_1)$, and $|\lambda_1(z)| > |\lambda_2(z)| = 1 = |\lambda_3(z)| > |\lambda_4(z)|$ for points $z = \boldsymbol{e}^{it}, t \in (t_1, t_2]$.

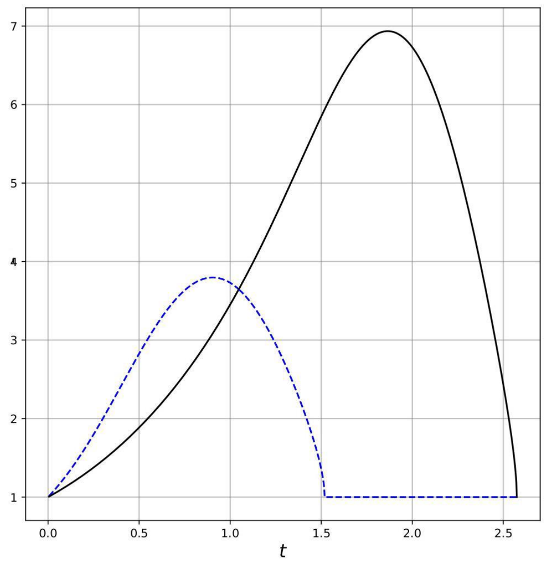


*Рис. 3.* $|\lambda_1(z)|, |\lambda_2(z)| \geq 1$ at $z = \boldsymbol{e}^{it}, t = 0,\ldots,t_1$, for knot $6_1$

[12]follow from the general properties of the roots of colored Jones polynomials

At point $z_2$ the branches $\lambda_2 = \lambda_3$ coincide, and at the output (clockwise) from $z_2$ we have: $|\lambda_1(z)| > |\lambda_2(z)| > 1 > |\lambda_3(z)| > |\lambda_4(z)|, z = \boldsymbol{e}^{it}, t \in (t_2, t^*]$. Note that point $t^* = 1.047...$ on arc $(t_2, 0)$, in which $|\lambda_1(z)| = |\lambda_2(z)| > 1 > |\lambda_3(z)| = |\lambda_4(z)|$ and then at $z = \boldsymbol{e}^{it}, t \in (t^*, 0)$ the order of branches is rebuilt according to the magnitude of the modulus: $|\lambda_2(z)| > |\lambda_1(z)| > 1 > |\lambda_4(z)| > |\lambda_3(z)|$. Finally, for $z = 1$, $\lambda_j(1) = -1$, $j = 1,2,3,4$.

Note that on the upper semicircle we have $|\lambda_1(z)||\lambda_4(z)| = 1$ and $|\lambda_3(z)||\lambda_2(z)| = 1$. It is also obvious that on the lower semicircle (going counterclockwise from p.$z = -1$) there will be the same structure of branches of the algebraic function $\lambda(z)$.

Let us proceed to the calculation of the integral[13] in (3.1). The integral sum gives:

$$\int_0^{t_1} \ln|\lambda_1(e^{it})|\,dt \approx \sum_{k=0}^{99} \ln\left|\lambda_1(\boldsymbol{e}^{ikt_1/100})\right| \frac{t_1}{100} = 3.161001...$$

Known volume value for knot $6_1$:

$$\mathrm{vol}(\mathbb{S}^3 \setminus 6_1) = 3.16396322...$$

A. B. Batkhin developed a special numerical method for identifying branches and calculating integrals (in the neighborhoods of branching points), which allows obtaining results with a fairly high accuracy (see in [2] the calculations of the integral (3.1) for knot $5_2$). For knot $6_1$ the numerical value of the integral is:

$$\int_0^{\pi} \ln|\lambda_1(e^{it})|\,dt \approx 3.163963228883...$$

**3.2. Knot $7_2$: branches $\{\lambda_j\}$ and volume.** We continue the numerical verification of the hypothesis (3.1) for various knots. We present the characteristic polynomial (1.7) that defines the spectral curve $\lambda(z)$ for knot $7_2$:

$$\begin{aligned} P(z,\lambda) :=& \lambda^5 + \lambda^4 a_4(z) + \lambda^3 a_3(z) + \lambda^2 a_2(z) + \lambda a_1(z) + z^{11} = 0, \qquad (3.3)\\ a_1 :=& z^4 - z^5 + 3z^9 + 4z^{10} - 2z^{11};\\ a_2 :=& -2z^2 + 5z^3 + z^4 - 4z^5 + 6z^7 + 5z^8 + 2z^9 - 4z^{10} + z^{11};\\ a_3 :=& 1 - 4z + 2z^2 + 5z^3 + 6z^4 - 4z^6 + z^7 + 5z^8 - 2z^9;\\ a_4 :=& -2 + 4z + 3z^2 - z^6 + z^7. \end{aligned}$$

Note: $\quad P(-1,\lambda) = (\lambda-1)^4, \qquad P(1,\lambda) = (\lambda+1)^4.$

[13] since $|\lambda| = 1$ on $(t_1, \pi)$, then the upper limit in the integral can be replaced by $\pi$

The discriminant of the polynomial $P(z,\lambda)$ is $D(z) := z^{16}\,(z^2-1)^{20}\Delta$, where

$$\begin{aligned}\Delta := {} & 20z^{16} - 192z^{15} + 581z^{14} - 882z^{13} + 1649z^{12} - 2214z^{11} + \\ & 1146z^{10} + 730z^9 + 2733z^8 + 730z^7 + 1146z^6 - 2214z^5 + \\ & 1649z^4 - 882z^3 + 581z^2 - 192z + 20. \end{aligned} \tag{3.4}$$

Of the 16 zeros of $\Delta$ on the circle - 8, on $\mathbb{S}^1_+$ lie two branch points of $\lambda(z)$: $z_1 := -0.89.. + i\,0.45.. =: \boldsymbol{e}^{it_1}, t_1 = 2.67..,\ \ z_2 := -0.28.. + i\,0.96.. =: \boldsymbol{e}^{it_2}, t_2 = 1.85..$

The branches $\{\lambda_j(z)\}_{j=1}^5$ of the spectral curve (3.3) for the knot $7_2$ on the circle $\mathbb{S}^1$ behave in the same way as the branches $\lambda(z)$ for $6_1$. One exception: an additional branch has appeared (denote it by $\lambda_3(z)$), which is holomorphic and equal in absolute value to one on the whole $\mathbb{S}^1$. Moving along $\mathbb{S}^1_+$ from $z = \boldsymbol{e}^{i\pi}$ clockwise, we have:

$$\begin{array}{lr}
\lambda_j(-1) = 1, \quad |\lambda_j(z)| = 1, \quad j = 1{,}2{,}3{,}4{,}5, & z = \boldsymbol{e}^{it}, \quad t \in [\pi, t_1]; \\
\lambda_1(z_1) = \lambda_5(z_1), \quad |\lambda_1(z)| > 1 = |\lambda_j(z)| > |\lambda_5(z)|, \quad j = 2{,}3{,}4, & t \in (t_1, t_2]; \\
\lambda_2(z_2) = \lambda_4(z_2), \quad |\lambda_1(z)| > |\lambda_2(z)| > 1 = |\lambda_3(z)| > |\lambda_4(z)| > |\lambda_5(z)|, & t \in (t_2, t^*]; \\
|\lambda_1(z^*)| = |\lambda_2(z^*)| > 1 = |\lambda_3(z^*)| > |\lambda_4(z^*)| = |\lambda_5(z^*)|, & z^* = \boldsymbol{e}^{it^*}, \quad t^* = 1.45..; \\
|\lambda_2(z)| > |\lambda_1(z)| > 1 = |\lambda_3(z)| > |\lambda_5(z)| > |\lambda_4(z)|, & t \in (t^*, 0]; \\
\lambda_j(1) = -1, \quad j = 1{,}2{,}3{,}4{,}5. &
\end{array}$$

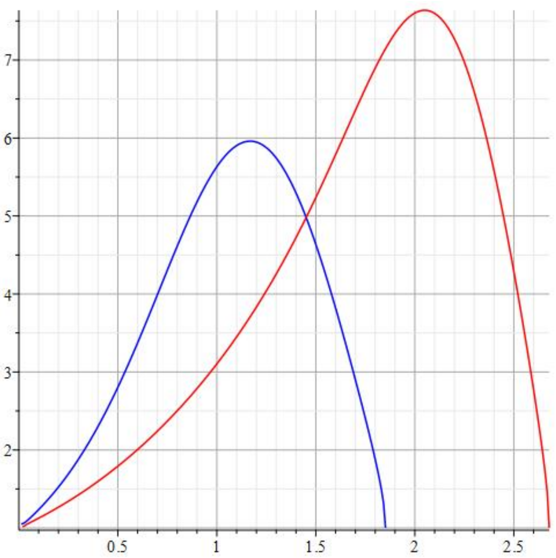


*Рис. 4.* $|\lambda_1(z)|, |\lambda_2(z)| \geq 1$ at $z = \boldsymbol{e}^{it}, t = 0, \ldots, t_1$, for the knot $7_2$

Known volume value for knot $7_2$: $\mathrm{vol}(\mathbb{S}^3 \setminus 7_2) = 3.331744232...$

For knot $7_2$ the numerical value of the integral is:

$$\int_0^{\pi} \ln|\lambda_1(e^{it})|\,dt \approx 3.3317442316411...$$

**3.3. Knot $7_4$: branches $\{\lambda_j\}$ and volume.** This knot has a peculiarity – its characteristic polynomial is factorized[14] $P(z,\lambda) = P^{(1)}(z,\lambda)P^{(2)}(z,\lambda)$: $\mathsf{deg}_\lambda[P^{(1)}(z,\lambda)] = 3$, $\mathsf{deg}_z[P^{(1)}] = 7$, $\mathsf{deg}_\lambda[P^{(2)}(z,\lambda)] = 2$, $\mathsf{deg}_z[P^{(2)}] = 4$. This circumstance facilitates the analysis of the structure of the 5 branches of the spectral curve $\lambda(z)$, since the branches of the factors can be considered independently.

The discriminant of $P^{(1)}(z,\lambda)$ is equal to $D_1(z) := z\,(z^2-1)^6\Delta_1$, where

$$\Delta_1 := (4z^6 + 12z^4 + 27z^3 + 12z^2 + 4)(2z^4 - 5z^3 + 8z^2 - 5z + 2)^2,$$

and $P^{(2)}(z,\lambda)$ has discriminant $D_2(z) := (z^2-1)^2\Delta_2$, $\Delta_2 := (z^2+z+1)(z^2-3z+1)$. On $\mathbb{S}^1_+$, $\Delta_1$ has a single zero $z_1 := -0.94.. + i\,.032.. =: \boldsymbol{e}^{it_1}, t_1 = 2.81...$ There, on $\mathbb{S}^1_+$ and $\Delta_2$ there is only one zero $z_2 := -\frac{1}{2} + i\,\frac{\sqrt{3}}{2} =: \boldsymbol{e}^{it_2}, t_2 = \frac{2\pi}{3}$.

Denote by $\{\lambda_{2k-1}(z)\}_{k=1}^3$ the branches of the curve $\lambda^{(1)}(z)$ defined by the equation $P^{(1)}(z,\lambda) = 0$, and by $\{\lambda_{2k}(z)\}_{k=1}^2$ the branches of $\lambda^{(2)}$: $P^{(2)}(z,\lambda) = 0$.

Moving clockwise along $\mathbb{S}^1_+$ from $z = \boldsymbol{e}^{i\pi}$, we have for $\lambda^{(1)}(z)$:

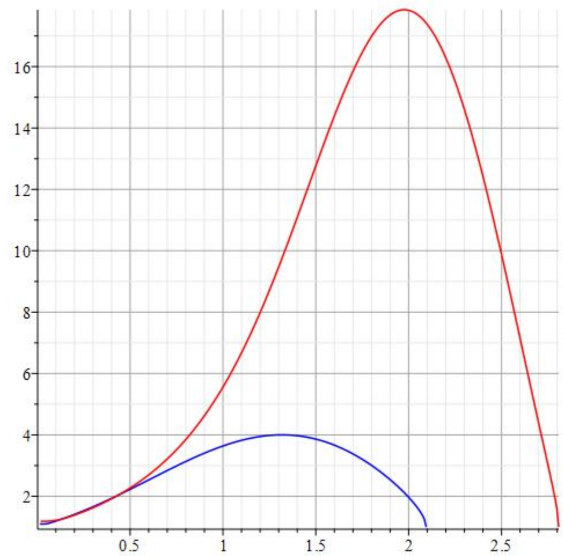


*Рис. 5.* $|\lambda_1(z)|, |\lambda_2(z)| \geq 1$ at $z = \boldsymbol{e}^{it}, t = 0, \ldots, t_1$, for knot $7_4$

$$\begin{cases} \lambda_j(-1) = 1, \quad j = 1,3,5, \quad |\lambda_j(z)| = 1, \quad z = \boldsymbol{e}^{it}, \quad t \in [\pi, t_1], \quad \lambda_1(z_1) = \lambda_5(z_1); \\ |\lambda_1(z)| > 1 = |\lambda_3(z)| > |\lambda_5(z)|, \quad z = \boldsymbol{e}^{it}, \quad t \in (t_1, 0), \quad \lambda_j(1) = -1, \ j = 1,3,5; \end{cases}$$

Similarly, for $\lambda^{(2)}(z)$:

$$\begin{cases} \lambda_j(-1) = 1, \ j = 2,4, \quad |\lambda_j(z)| = 1, \quad z = \boldsymbol{e}^{it}, \ t \in [\pi, t_2], \quad \lambda_2(z_2) = \lambda_4(z_2); \\ |\lambda_2(z)| > 1 > |\lambda_4(z)|, \quad z = \boldsymbol{e}^{it}, \ t \in (t_2, 0), \quad \lambda_j(1) = -1, \ j = 2,4. \end{cases}$$

---

[14]see (2.22), following [32] we omit the square of $A^{(2)}_{7_4}$

Note that in the neighborhood of the point $z = 1$, $\exists z^* = \boldsymbol{e}^{it^*}$ we have:
$|\lambda_1(z)| > |\lambda_2(z)|$ for $0 < t \in (t^*, 0)$, and $|\lambda_1(z) < |\lambda_2(z)|$ for $\pi > t \in (\pi, t^*)$.

Known volume value for the knot $7_4$: $\mathrm{vol}(\mathbb{S}^3 \setminus 7_4) = 5.13794120....$
For the knot $7_4$ the numerical value of the integral is:

$$\int_0^\pi \ln|\lambda_1(e^{it})|\, dt \approx 5.137941201873417769...$$

**3.4. Knot $7_5$: branches $\{\lambda_j\}$ and volume.** Starting from this point, we will not give in the main text[15] the explicit form of the polynomials $P(z, \lambda)$ that define the spectral curve $\lambda(z)$ in (1.7). For the knot $7_5$, we only note that

$$\mathsf{deg}_\lambda[P(z,\lambda)] = 8,\ \mathsf{deg}_z[P(z,\lambda)] = 34, \qquad P(\pm 1, \lambda) = (\lambda \pm 1)^8.$$

The discriminant $P(z, \lambda)$ has degree 50, not counting the zeros of high even multiplicity at the points $\pm 1, 0$. However, only three branch points $z_j := \boldsymbol{e}^{it_j}, j{=}1,...,3$ and one more point $z_0 := \boldsymbol{e}^{it_0}$ : fall on the upper semicircle $\mathbb{S}^1_+$.
$t_0 := 2.237035759..; t_1 := 2.848733829..; t_2 := 2.233540134..; t_3 := 2.190746731..,$ two branches intersect holomorphically, and in the neighborhood of the point $z_0$ their modulus is equal to 1.

All branches $\{\lambda_j(z)\}_{j=1}^8$ of the spectral curve of knot $7_5$ on the circle $\mathbb{S}^1$ in the neighborhood of point $\boldsymbol{e}^{i\pi}$ have a modulus equal to 1, and two branches preserve this property throughout $\mathbb{S}^1$, while in the neighborhood of point 1 the other three branches have moduli greater than 1, and the three remaining ones have moduli equal to the inverse values of the previous moduli. Moving along $\mathbb{S}^1_+$ from $z = \boldsymbol{e}^{i\pi}$

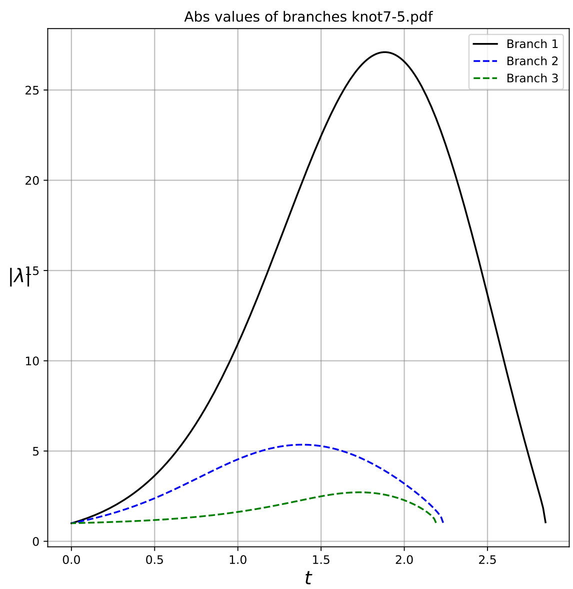


*Рис. 6.* $|\lambda_j(z)| \geq 1,\ j = 1,2,3$, at $z = \boldsymbol{e}^{it}, t \in (0, \pi)$, for the knot $7_5$

[15] see **http://katlas.math.toronto.edu/wiki/Data:7_5/A-polynomial**

clockwise, we select branches modulo greater than one. At the point $z_1$, the only such branch branches, denoted by $\lambda_1(z)$, the remaining branches remain equal to (or less than) one in modulo). This branch will preserve the maximum modulus on the path from $z_1$ to 1, on this path it is holomorphic (does not branch) and forms the answer. The other two branches $\lambda_2(z)$ and $\lambda_3(z)$, branching at points $z_2$ and $z_3$, respectively, and having a modulus greater than 1 on the holomorphy regions from $z_2$ to 1 and from $z_3$ to 1, preserve the ordering of the modules on these regions: $|\lambda_1(z)| > |\lambda_2(z)| > |\lambda_3(z)| > 1$.

Known volume value for the knot $7_5$: $\mathrm{vol}(\mathbb{S}^3 \setminus 7_5) = 6.443537381.....$

For the knot $7_5$ the numerical value of the integral is:

$$\int_0^\pi \ln|\lambda_1(e^{it})|\, dt \approx 6.4435373808505754761...$$

**3.5. Knot $7_6$: branches $\{\lambda_j\}$ and volume.** For the polynomials $P(z,\lambda)$[16] that define in (1.7) the spectral curve $\lambda(z)$ of the knot $7_6$ we have

$\mathsf{deg}_\lambda[P(z,\lambda)] = 9$, $\mathsf{deg}_z[P(z,\lambda)] = 27$, $\qquad P(\pm 1,\lambda) = (\lambda \pm 1)^9$.

The discriminant $P(z,\lambda)$ has degree 84, not counting the zeros of high even multiplicity at the points $\pm 1, 0$. However, only four branch points $z_j := \boldsymbol{e}^{it_j}$, $j$=1,...,4 fall on the upper semicircle $\mathbb{S}^1_+$:

$t_1 := 2.880078732..; t_2 := 2.321655981..; t_3 := 2.305603863.., t_4 := 1.827810801;$ and at two more points $z_0j := \boldsymbol{e}^{it_0j}$ : $\quad t_{01} := 2.3814402100..; t_{02} := 2.3464745745..$, intersect holomorphically two branches with equal 1 moduli.

All branches $\{\lambda_j(z)\}_{j=1}^9$ of the spectral curve of knot $7_6$ on the circle $\mathbb{S}^1$ in the neighborhood of point $\boldsymbol{e}^{i\pi}$ have modulus $= 1$, and one branch preserves this property on the whole $\mathbb{S}^1$, and in the neighborhood of point 1 the other four branches have moduli $> 1$ , and the four remaining moduli are equal to the inverse values of the previous moduli, i.e. $<1$.

Moving along $\mathbb{S}^1_+$ from $z = \boldsymbol{e}^{i\pi}$ clockwise, we select branches whose absolute value is greater than one. At the point $z_1$, the only such branch branches, denoted by $\lambda_1(z)$, the remaining branches remain equal (or $<$) to one absolute value. This branch will preserve its maximum absolute value on the path from $z_1$ to 1, on this path it is holomorphic (does not branch) and forms the answer. The other two branches $\lambda_2(z)$ and $\lambda_3(z)$, branching at points $z_2$ and $z_3$, respectively, and having modulus $> 1$ on the holomorphy sections from $z_2$ to 1 and from $z_3$ to 1, preserve the ordering of the modules on these sections: $|\lambda_1(z)| > |\lambda_2(z)| > |\lambda_3(z)| > 1$.

Finally, the branch $\lambda_4(z)$, branching at point $z_4$, having modulus $> 1$ on the holomorphy region from $z_4$ to 1, increases its modulus as it moves (clockwise), so

[16]see **http://katlas.math.toronto.edu/wiki/Data:7_6/A-polynomial**

that first $|\lambda_1(z)| > |\lambda_2(z)| > |\lambda_4(z)| > |\lambda_3(z)| > 1$, and then $|\lambda_1(z)| > |\lambda_4(z)| > |\lambda_2(z)| > |\lambda_3(z)| > 1$, see Fig.7.

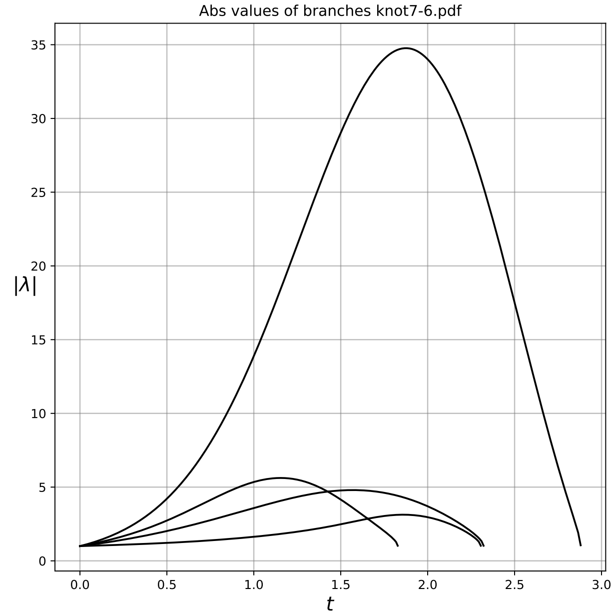


*Рис. 7.* $|\lambda_j(z)| \geq 1$, $j = 1,...,4$, at $z = \boldsymbol{e}^{it}, t \in (0, \pi)$, for the knot $7_6$

Known volume value for the knot $7_6$: $\mathrm{vol}(\mathbb{S}^3 \setminus 7_6) = 7.084925954...$

For the knot $7_6$ the numerical value of the integral is:

$$\int_0^\pi \ln|\lambda_1(e^{it})|\, dt \approx 7.08492595351096864 84...$$

**3.6. Knot $7_7$: branches $\{\lambda_j\}$ and volume.** This knot, like $7_4$, has a factorization of the characteristic polynomial [17] $P(z,\lambda) = P^{(1)}(z,\lambda)P^{(2)}(z,\lambda)$: $\mathsf{deg}_\lambda[P^{(1)}(z,\lambda)] = 4$, $\mathsf{deg}_z[P^{(1)}] = 14$, $\mathsf{deg}_\lambda[P^{(2)}(z,\lambda)] = 3$,$\mathsf{deg}_z[P^{(2)}] = 5$. This circumstance facilitates the analysis of the 7 branches of the spectral curve $\lambda(z)$, since the branches $\lambda^{(j)}(z), j = 1{,}2$ of the factors $P$ can be considered independently.

The discriminant of the polynomial $P^{(1)}(z,\lambda)$ is $D_1(z) := z^{16}(z^2-1)^{12}\Delta_1$, where $\Delta_1 := (16z^8 - 68z^7 + 44z^6 + 120z^5 + 33z^4 + 120z^3 + 44z^2 - 68z + 16)$ $(2z^{10} - 22z^9 + 91z^8 - 176z^7 + 163z^6 - 108z^5 + 163z^4 - 176z^3 + 91z^2 - 22z + 2)^2$; and $P^{(2)}(z,\lambda)$ has the discriminant $D_2(z) := (z^2-1)^6\Delta_2$, where

$$\Delta_2 := z^8 - 6z^7 + 11z^6 - 12z^5 - 11z^4 - 12z^3 + 11z^2 - 6z + 1.$$

On $\mathbb{S}^1_+$, $\Delta_1$ has two single zeros (branch points) at points: $\boldsymbol{e}^{it_1^{(1)}}, \boldsymbol{e}^{it_2^{(1)}}$ and one doble zero at point $\boldsymbol{e}^{it_0^{(1)}}$. $\Delta_2$ on $\mathbb{S}^1_+$ has a single zero at point $\boldsymbol{e}^{it_1^{(2)}}$: $t_1^{(1)} := 2.905300...$, $t_1^{(2)} := 2.407169...$, $t_2^{(1)} := 1.535100...$, $t_0^{(1)} := 2.216967...$ .

[17] We have $P(\pm1,\lambda) = (\lambda\pm1)^7$; $A$-polynomial in http://katlas.math.toronto.edu/wiki/Data:7_7/A-polynomial

At point $\boldsymbol{e}^{it_0^{(1)}}$ two branches of curve $\lambda^{(1)}(z)$ intersect holomorphically with equal 1 moduli, and also at each of points $\boldsymbol{e}^{it_{01}^{(12)}}, t_{01}^{(12)} := 2.408663..$ and $\boldsymbol{e}^{it_{02}^{(12)}}, t_{02}^{(12)} := 1.700625..$ the branch $\lambda^{(2)}(z)$ and the branch $\lambda^{(1)}(z)$ with equal moduli are intersected.

All branches $\{\lambda_j^{(1)}(z)\}_{j=1}^4$ and $\{\lambda_j^{(2)}(z)\}_{j=1}^3$ of the spectral curve of knot $7_7$ on the circle $\mathbb{S}^1$ in the neighborhood of point $\boldsymbol{e}^{i\pi}$ have modulus $= 1$, and one branch of the curve $\lambda^{(2)}(z)$ preserves this property on the whole $\mathbb{S}^1$, and in the neighborhood of point 1 on $\mathbb{S}^1_+$ the other branch $\lambda^{(2)}(z)$ and two branches $\lambda^{(1)}(z)$ have moduli $>1$, and the three remaining branches $\lambda(z)$ modules are equal to the reciprocals of the previous modules.

Moving along $\mathbb{S}^1_+$ from $z = \boldsymbol{e}^{i\pi}$ clockwise, we select branches whose absolute value is greater than one. At the point $z_1 := \boldsymbol{e}^{it_1^{(1)}}$, the only such branch branches, denoted by $\lambda_1(z) := \lambda_1^{(1)}(z)$, the remaining branches remain equal (or $<$) to one absolute value. This branch will preserve its maximum absolute value on the path from $z_1$ to 1, on this path it is holomorphic and forms the answer.

Let us fix two other branches $\lambda_2(z) := \lambda_1^{(2)}(z)$ and $\lambda_3(z) := \lambda_2^{(1)}(z)$, branching at points $z_2 := \boldsymbol{e}^{it_1^{(2)}}$ and $z_3 := \boldsymbol{e}^{it_2^{(1)}}$, respectively, and having modulus $> 1$ on the holomorphy regions from $z_2$ to 1 and from $z_3$ to 1. Moreover, their modulus there is less than the modulus of the branch $\lambda_1(z)$. We add that on the arc $\mathbb{S}^1_+$ from $z_3$ to 1 $\exists z_* \ : \ |\lambda_2(z)| > |\lambda_3(z)|$ from $z_3$ to $z_*$, but $|\lambda_2(z)| < |\lambda_3(z)|$ from $z_*$ to 1.

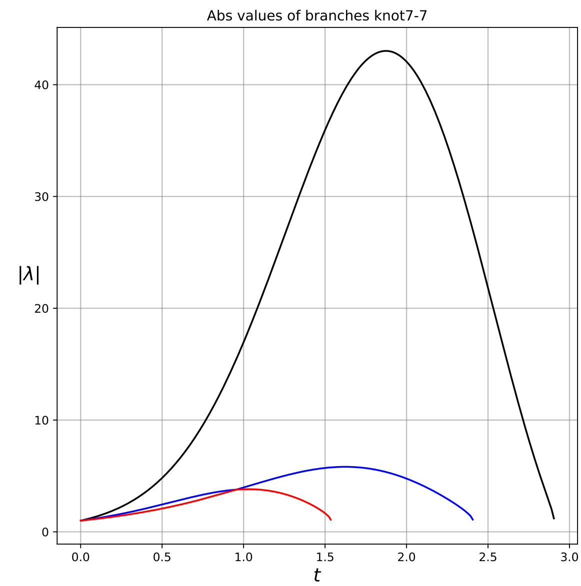


*Рис. 8.* $|\lambda_j(z)| \geq 1$, $j = 1{,}2{,}3$, at $z = \boldsymbol{e}^{it}, t \in (0, \pi)$, for the knot $7_7$

Known volume value for the knot $7_7$: $\mathrm{vol}(\mathbb{S}^3 \setminus 7_7) = 7.643375172...$
For the knot $7_7$ the numerical value of the integral is:

$$\int_0^{\pi} \ln|\lambda_1(e^{it})|\, dt \approx 7.6433751723599555...$$

**3.7. WKB integrals and Mahler measures.** Before moving on to the final series of knots considered, we note that the connection between WKB integrals in (1.12) and special functions expressing hyperbolic volumes was noted in D. Boyd' papers, see [14, 15], devoted to Mahler measures. *The logarithmic Mahler measure of the polynomial* $P(x_1, ..., x_n)$ is called

$$m(P) := \int_0^1 \cdots \int_0^1 \log |P(e(t_1), ..., e(t_n))| \, dt_1 \cdots dt_n, \quad e(t) := \exp(2\pi i t).$$

Accordingly, simply *the Mahler measure of a polynomial* is $\exp(m(P))$.

For a polynomial of two variables $P(z, \lambda)$ the Mahler measure is $m(P) :=$

$$\int_0^1 \int_0^1 \log |P(e(t_1), e(t_2))| \, dt_2 dt_1 = \int_0^1 \left( \int_0^1 \log \prod_{j=1}^n |(e(t_2)) - \lambda_j(e(t_1))| dt_2 \right) dt_1.$$

Applying Jensen's formula to the inner integral,

$$m(P) := \int_0^1 \sum_{j=1}^n \left( \int_0^1 \log |e(t_2) - \lambda_j(e(t_1))| dt_2 \right) dt_1 = \sum_{j=1}^n \int_0^1 \log^+ |\lambda_j(\boldsymbol{e}^{2\pi i t_1})| dt_1,$$

we obtain the sum of the integrals (1.12) over all branches $\lambda_k(z) : |\lambda_k| \geq 1, \ z \in \mathbb{S}^1$.

In the paper by D. Boyd [15] (as polynomials of two variables) $A(M, L)-$ are considered. Examples of three knots are given: $k5_{15}, k5_7$ and $10_{125}$, for which the integrals of the logarithms of the moduli of the branches of the algebraic functions $A(M, L) = 0$ are calculated.[18]. In the next three sections, we will also analyze these examples under the assumption that the **AJ**-hypothesis is valid (2.1).

**3.8. Knot $10_{125}$: branches $\{\lambda_j\}$ and volume.** This knot was considered in [15, Example 3]. Besides the number $10_{125}$ in the Rolfsen classification, it is also called the $k6_{20}$ knot and the $K(-2, 3, -5)$ pretzel. For the polynomials $P(z, \lambda)$[19], defining in (1.7) the spectral curve $\lambda(z)$ of the $10_{125}$ knot, we have $\mathsf{deg}_\lambda[P(z, \lambda)] = 9$, $\mathsf{deg}_z[P] = 27$, $P(1, \lambda) = (\lambda - 1)^2(\lambda + 1)^7$, $P(-1, \lambda) = (\lambda^2 + 1)^2(\lambda - 1)^5$.
The discriminant $P(z, \lambda)$, not counting the zeros of high even multiplicity at the points $\pm 1, 0$, has 28 single zeros, 32 double zeros, and quadruple zeros at the roots of the equation $(Z^4 + 1)^4 = 0$. Moreover, on the upper semicircle $\mathbb{S}_+ = 1$, in addition to half of the mentioned quadruple zeros and three double zeros at the points of the holomorphic intersection of the branches $z_{0j} := \boldsymbol{e}^{it_{0j}}$ : $t_{01} :=$ 2.6551684954..; $t_{02} :=$ 1.4920790567.., $t_{03} :=$ 1.0826837985.., five branch points

[18]Unfortunately, in [15] neither an explicit form nor a clear reference to where the $A(M, L)$ polynomials for these knots were taken from are given.

[19]see **http://katlas.math.toronto.edu/wiki/Data:10_125/A-polynomial**

(single zeros) are encountered: $z_j := \boldsymbol{e}^{it_j},\ j$=1,...,5 : $t_1 := 3.126927230..; t_2 := 2.347421331..; t_3 := 2.342125309.., t_4 := 1.094478093..; t_5 := 0.328860611..$ .

In the neighborhood of point $\boldsymbol{e}^{i\pi}$ (more precisely, on the arc $[t_1, 2\pi - t_1]$ of the circle $\mathbb{S}^1$) all $\{\lambda_j(z)\}_{j=1}^{9}$ have modulus = 1, and one branch [20] preserves this property on the whole $\mathbb{S}^1$, and from point 1 there emerge three branches with moduli > 1, three with moduli equal to the inverse values of the previous moduli, i.e. <1, and the three remaining ones have moduli =1.

We move along $\mathbb{S}^1_+$ from $z = \boldsymbol{e}^{i\pi}$ clockwise. At branch point $z_1$, two branches with modulus = 1 bifurcate into branches $\lambda_1 : |\lambda_1| > 1$[21] and $\lambda_9 : |\lambda_9||\lambda_1| = 1$. Similarly, at branch points $z_j, j$ = 2,3,4, branches $\lambda_j : |\lambda_j| > 1, j$ = 2,3,4 are formed, and their partners: $\lambda_j : |\lambda_j| < 1, j$ = 8,7,6. Recall that there remains one branch $\lambda_5 : |\lambda_5| = 1$ on the entire $\mathbb{S}^1$. Thus, all branches $\{\lambda_j(z)\}_{j=1}^{9}$ on $\mathbb{S}^1_+$ are fixed.[22] Note the proximity of points $t_2, t_3$ and $t_{03}, t_4$.

The most interesting thing happens on the arc $[t_4, 0]$. First, at the point $t_{03}$ there is a holomorphic intersection of the branch $\lambda_3 : |\lambda_3(\boldsymbol{e}^{it})| > 1,\ t \in (t_4, t_{03})$ with its partner $\lambda_7 : |\lambda_7| = |\lambda_3|^{-1}$. Thus, on the arc $(t_{03}, t_5)$ we have $|\lambda_j| > 1,\ j$ = 1,2,4,7. At point $t_5$ $\lambda_7$ branches with its partner $\lambda_3$ and then on the arc $(t_5, 0]$ we have $|\lambda_7| = |\lambda_3| = 1$, and for the remaining branches $|\lambda_2| > |\lambda_1| > |\lambda_4| > 1$. In Fig. 9.-2) we can trace the change in the order of the modules $|\lambda_j| > 1,\ j$ = 1,2,4,7.

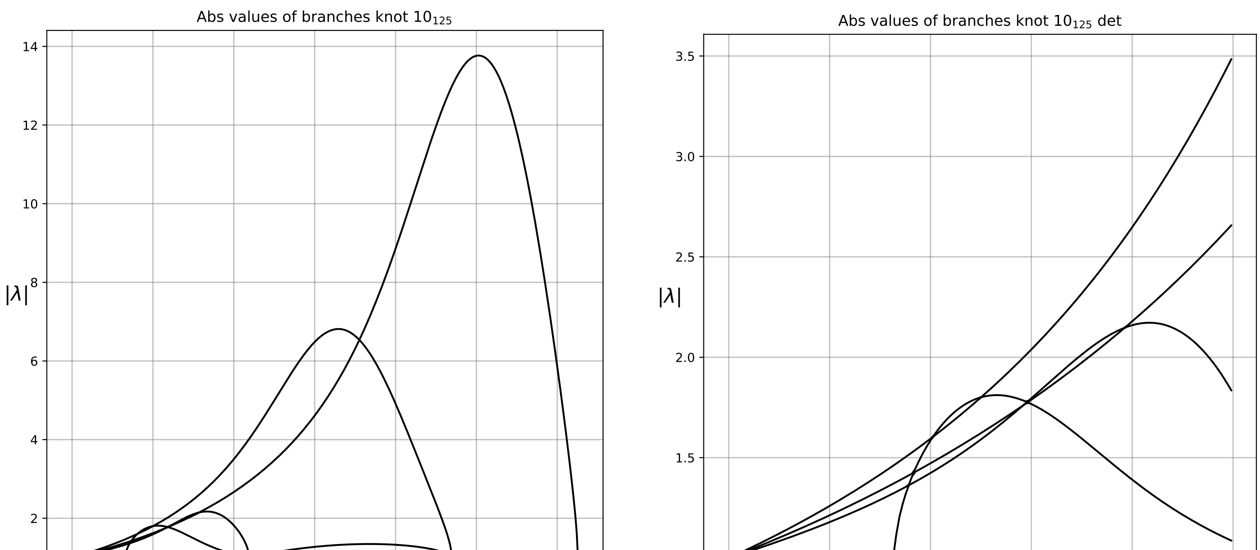


*Рис. 9.* 1) $|\lambda_j(z)| \geq 1,\ j = 1,...,5$ at $z = \boldsymbol{e}^{it}, t \in (0, \pi)$ for knot $10_{125}$
2) scaling: $t \in (0, 1)$; $|\lambda_4| = |\lambda_1| = |\lambda_7|$ at point $t^*$ =0.59...

[20] we denote it by $\lambda_5$

[21] this branch will preserve holomorphy on the arc $[t_1, 0]$ and form the answer.

[22] Note that on the considered section $[t_3, t_4]$ a change occurs: $|\lambda_1| > |\lambda_2|$ changes to $|\lambda_1| < |\lambda_2|$.

Known volume value for the knot $10_{125}$: vol($\mathbb{S}^3 \setminus 10_{125}$) =4.611961375....

For the knot $10_{125}$ the numerical value of the integral is:

$$\int_0^{\pi} \ln|\lambda_1(e^{it})|\,dt \approx 4.61196.$$

Comparing these results with [15], we note the discrepancy between the values of $\mathsf{deg}_z[P]$ and the disagreement with the statement in [15] that $\lambda_1, \lambda_2, \lambda_4 \in \mathcal{H}(\mathbb{S}^1)$.

**3.9. Knot $k5_{15}$: branches $\{\lambda_j\}$ and volume.** This knot was considered in [15, Example 1]. For $P(z,\lambda)^{23}$, defining in (1.7) the spectral curve $\lambda(z)$ of the knot $k5_{15}$, we have $\mathsf{deg}_\lambda[P(z,\lambda)] = 16$, $\mathsf{deg}_z[P] = 291$, $P(1,\lambda) = (\lambda-1)^7(\lambda+1)^9$, $P(-1,\lambda) = (\lambda^{14}+8\lambda^{13}+39\lambda^{12}+24\lambda^{11}+5\lambda^{10}-40\lambda^9+19\lambda^8+16\lambda^7+19\lambda^6-40\lambda^5+5\lambda^4+24\lambda^3+39\lambda^2+8\lambda+1)(\lambda-1)^2$. Moreover, among the roots of $P(-1,\lambda)$ there are $\lambda_j : |\lambda_j(-1)| = 5.80909...$, $j =$ 1,2, and $|\lambda_j(-1)| = 0.17214...$, $j =$ 15,16, the rest have $|\lambda_j(-1)| = 1$, $j = 3, ..., 14$.

The discriminant $P(z,\lambda)$, not counting the zeros of high even multiplicity at the points $1, 0$, has 26 single zeros, and 1 double zero at $-1$. Moreover, in addition to the double zero at $\boldsymbol{e}^{i\pi}$ - the point of holomorphic intersection of branches equal in modulus to one, one branch point (single zero) falls on the upper semicircle $\mathbb{S}^1_+$: $z_3 := \boldsymbol{e}^{it_3} : t_3 := 3.1098279565..$ .

We move along $\mathbb{S}^1_+$ from $z = \boldsymbol{e}^{i\pi}$ clockwise. Recall that at this point two holomorphic branches are conjugate. Fix $\lambda_1$ with increasing modulus, $\lambda_2$ with decreasing modulus. At the branch point $z_3$ two branches with modulus $= 1$ branch into branches $\lambda_3 : |\lambda_3| > 1$ and $\lambda_{14} : |\lambda_{14}||\lambda_3| = 1$, which are holomorphic at $(t_3, 0]$.

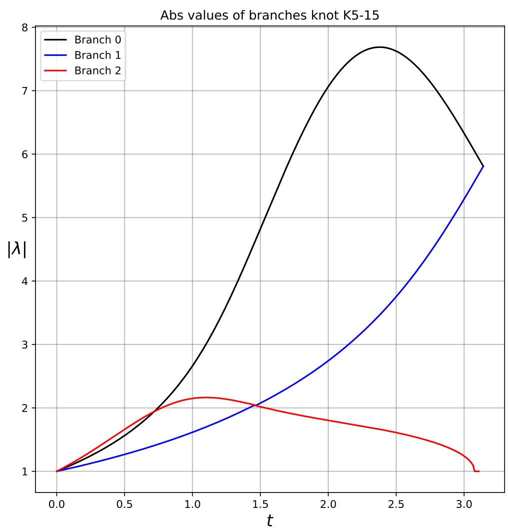


*Рис. 10.* $|\lambda_j(z)| \geq 1$, $j =$ 1,2,3, at $z = \boldsymbol{e}^{it}, t \in (0,\pi)$, for the knot $k5_{15}$

[23] $A$-polynomials for the knot $k5_{15}$ will be given in the Appendix, see section 4.3

Thus, for the first time we have faced an example of a knot whose spectral curve $\lambda(e^{it})$ has branches that are holomorphic on the entire circle and not equal in absolute value to one: $\lambda_j(e^{it}) \in \mathcal{H}(\mathbb{S}^1)$, $|\lambda_j(e^{it})| > 1$, $j = 1{,}2$. Moreover, $\lambda_1(z) = \overline{\lambda_2(\bar{z})}$. There are also 12 branches $\{\lambda_j\}$ that are holomorphic on the entire circle: $|\lambda_j| = 1$, $j = 4, \ldots, 13$. The remaining two branches $\lambda_3, \lambda_{14} \in \mathcal{H}(\mathbb{S}^1) \setminus \{z_3, \bar{z}_3\}$. Here also $\lambda_3(z) = \overline{\lambda_{14}(\bar{z})}$, where $|\lambda_3(e^{it})| > 1 > |\lambda_{14}(e^{it})|$ on the arc $t \in (t_3, 0, 2\pi - t_3)$ and $|\lambda_3(e^{it})| = 1 = |\lambda_{14}(e^{it})|$ on the arc $t \in (2\pi - t_3, 2\pi, t_3)$.

Known volume value for the knot $k5_{15}$: $\mathrm{vol}(\mathbb{S}^3 \setminus k5_{15})$ =4.1885842865...

The numerical value of the integrals fot $\lambda_j, j = 1{,}2{,}3$ of the knot $k5_{15}$ are

$$I_1 := \int_0^\pi \ln|\lambda_1(e^{it})|\, dt \approx 4.239491778, \quad I_2 := \int_0^\pi \ln|\lambda_2(e^{it})|\, dt \approx 2.538974570,$$
$$I_3 := \int_0^\pi \ln|\lambda_3(e^{it})|\, dt \approx 1.649603.$$

**3.10. Knot $k5_7$: branches $\{\lambda_j\}$ and volume.** This knot was considered in [15, Example 2]. For the polynomials $P(z, \lambda)$[24], defining in (1.7) the spectral curve $\lambda(z)$ of the knot $k5_{15}$, we have $\deg_\lambda[P(z,\lambda)] = 17$, $\deg_z[P] = 325$,

$$P(1, \lambda) = (\lambda - 1)^8(\lambda\ + 1)^9, \quad P(-1, \lambda) = L(\lambda)(\lambda - 1),\ \deg[L(\lambda)] = 16.$$

Moreover, the roots of $P(-1, \lambda)$ are $\lambda_j : |\lambda_j(-1)| = 3.900505..., j = 1{,}2$ and $|\lambda_j(-1)| = 3.2043057..., j = 3{,}4$, as well as their partners $\lambda_{17-j},\ j = 1, 2, 3, 4 :$ $|\lambda_{17-j}(-1)| = |\lambda_j(-1)|^{-1}$, while the rest have $|\lambda_j(-1)| = 1, j = 5, \ldots, 13$.

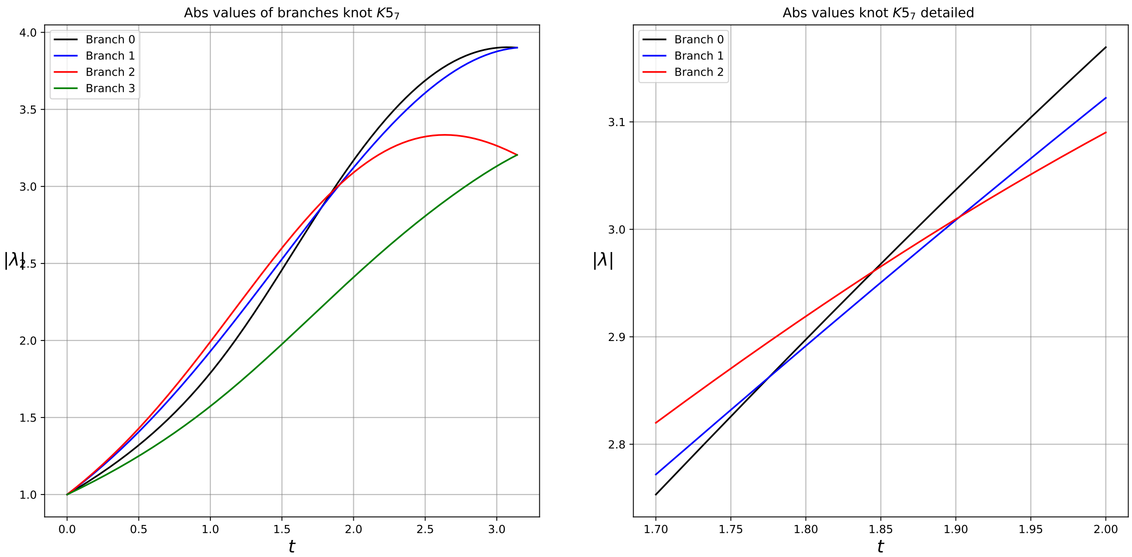


*Рис. 11.* 1) $|\lambda_j(z)| \geq 1$, $j = 1, \ldots, 4$ at $z = \boldsymbol{e}^{it}, t \in (0, \pi)$ for the knot $k5_7$
2) scaling: $t \in (1.7, 2.0)$

[24] $A$ - polynomials for the knot $k5_{15}$ will be given in the Appendix, see section 4.3

The discriminant $P(z,\lambda)$, not counting the zeros of high even multiplicity at the points $1, 0$, has 28 single zeros and 8 double zeros. At the same time, except for 2 double zeros at $\boldsymbol{e}^{it_0}, t_0 := \pm 2.46535...$ - the points of holomorphic intersection of branches equal in modulus to one, no zeros fall on the circle $\mathbb{S}^1$.

We move along $\mathbb{S}^1_+$ from $z = \boldsymbol{e}^{i\pi}$ clockwise. Recall that at this point two $k = 1,2$ pairs of holomorphic branches with modulus $> 1$ : $\lambda_{2k-1}(-1) = \overline{\lambda_{2k}(-1)}$ are conjugate: (we fix $\lambda_{2k-1}$ a branch with increasing modulus, $\lambda_{2k}$ with decreasing modulus). Thus, their partners are also fixed: $\lambda_j \in \mathcal{H}(\mathbb{S}^1)$ with modulus $< 1$. Fixing the remaining branches (with modulus $= 1$) is not important for us. Thus, for the knot $k5_7$ all branches of the spectral curve $\lambda(z) = \{\lambda(z)\}_{j=1}^{17}$ are holomorphic on $\mathbb{S}^1$. We are interested in the branches $\{\lambda(z)\}_{j=1}^{4}$ modulo $> 1$ on the entire arc $\mathbb{S}^1 \setminus \{1\}$. Let us mark the points $\boldsymbol{e}^{it_{j,k}} \in \mathbb{S}^1_+$ where the moduli of the branches $\lambda_j$ and $\lambda_k$ coincide: $t_{1,2} \approx 1.7752568285336$, $t_{1,3} \approx 1.84493232546964$, $t_{2,3} \approx 1.9025733709708$. On $\mathbb{S}^1_+$ we have the following order of modules for branches $|\lambda_j| > 1$ (see Fig.11-2):

$|\lambda_1| > |\lambda_2| > |\lambda_3| > |\lambda_4|$ на $(t_{2,3}, 0)$; $|\lambda_1| > |\lambda_3| > |\lambda_2| > |\lambda_4|$ на $(t_{1,3}, t_{2,3})$;
$|\lambda_3| > |\lambda_1| > |\lambda_2| > |\lambda_4|$ на $(t_{1,2}, t_{1,3})$; $|\lambda_3| > |\lambda_2| > |\lambda_1| > |\lambda_4|$ на $(\pi, t_{1,3})$.

For branches $\lambda_j, j = 1,..,4$ of the knot $k5_7$, the numerical value of the integrals in $\mathbb{S}^1_+$ are:

$$I_1 := \int_0^\pi \ln|\lambda_1(e^{it})|\,dt \approx 2.652510600, \quad I_2 := \int_0^\pi \ln|\lambda_2(e^{it})|\,dt \approx 2.717740319,$$
$$I_3 := \int_0^\pi \ln|\lambda_3(e^{it})|\,dt \approx 2.644672858, \quad I_4 := \int_0^\pi \ln|\lambda_4(e^{it})|\,dt \approx 2.088765966.$$

Known volume value for the knot $k5_7$: $\mathrm{vol}(\mathbb{S}^3 \setminus k5_7) = 4.0545040273...$

**3.11. Knots $k5_7, k5_{15}$: comparison with [15].** We must to admit the obvious: for the last two knots the values of the integrals $I_j$ calculated by us are in no way similar to the known values of the volumes taken by us from [7]. Moreover, in [15] similar integrals (designated there by $Vj$) are calculated and coincide with the known volumes with many signs. For the knot $k5_7$ in [15] the following is given: [V1; V2] = [4.054504027..; 1.315746892..] [V3; V4] = [2.436059319..; 2.297379506..]. However, if we take integrals over the entire domain of holomorphy, then

$I_1+I_2 \approx 2.652510600+2.717740319=4.733438824=4.054504027+1.315746892\approx \mathrm{V1}+\mathrm{V2}$
$I_3+I_4 \approx 2.644672858+2.088765966=5.370250919=2.436059319+2.297379506\approx \mathrm{V3}+\mathrm{V4}$

For the knot $k5_{15}$ in [15] it is given: [V1; V2]= [4.188584286..; 2.589882062..][25] Similarly, we have:

$I_1+I_2 \approx 4.239491778+2.538974570=6.778466348=[4.188584286+2.589882062\approx \mathrm{V1}+\mathrm{V2}$

[25] and V3 = 1.64960971..

# 4. Appendixes

**4.1. WKB basis for solutions of q-recurrences (details).** Expansions of fundamental solutions of difference equations (recurrence relations) – are a classical section of asymptotic analysis, founded by the works of Poincaré, Perron, and Birkhoff. Among modern studies, we highlight the work by O. Costin—R. Costin [25] and the above-mentioned article by S. Garofalidis and D. Geronimo [12], from which we cite:

«This subject is classical and has been reinvented over the past hundred years by several groups, often unaware of each others results. ... Our results are hardly new and are contained or can be obtained by minor modifications from results of Costin–Costin or from work of Birkhoff and collaborators ... ».

Our «group» (with D.N. Tulyakov) developed its own modification of the approach to this problem (see [26] – [30] , [13], [31]), which was subsequently (with «supervision» by S. Garofalidis and with the active participation of T. Dudnikova), adapted to $q$-recurrence relations (see [1]–[3]).

For recurrence relations (1.2) written in matrix form[26]:

$$\overrightarrow{Q}_{n+1} = \mathcal{A}_n \overrightarrow{Q}_n, \tag{4.1}$$

expansions of basic solutions in overlapping zones are sought: in *zones of separated eigenvalues* matrices $\mathcal{A}_n$, see (1.8), and in *in zones of convergence of some eigenvalues.* Matching bases in overlapping zones allows obtaining global representations of particular solutions.

In the zone of separated eigenvalues, the main technical point of the approach is to find a diagonalizing transformation («diagonalizer») of $V_n$ such that the matrix $V_{n+1}^{-1}\mathcal{A}_n V_n$ is close to a diagonal matrix

$$D_n := \operatorname{diag}\,[V_{n+1}^{-1}\mathcal{A}_n V_n] \;\approx\; V_{n+1}^{-1}\mathcal{A}_n V_n\,. \tag{4.2}$$

Formally, the basis vectors are the columns of the matrix

$$B_n := V_n \prod_{k=k_0}^{n-1} D_k \;=:\; V_n\,\Pi_n. \tag{4.3}$$

Действительно,

$$\mathcal{A}_n B_n = V_{n+1} V_{n+1}^{-1}\mathcal{A}_n V_n\,\Pi_n = V_{n+1}\,\Pi_{n+1} = B_{n+1}\,.$$

Thus, the problem of constructing an asymptotic basis of solutions (1.10) is reduced to finding expansions of the «diagonalizers» $V_n$ (the main problem) and the product $\Pi^{(n)}$ of diagonal operators (the solution follows from the main one).

As a result, in this zone we have

[26] where $d\times d$ matrix $\mathcal{A}_n$ is formed by the coefficients $\{A_j(q^n,q)\}_{j=0}^{d}$ from (1.2), and $\overrightarrow{Q}_n$ is introduced in (1.9).

**Theorem 4.1.** *Let q-difference equations* (1.2), (1.9) *have in the limit scale* (1.6) *the spectral curve* (1.7). *Then for the basis* $B_n := V_n \operatorname{diag}\left\{\pi_n^{(j)}\right\}_{j=1}^{d}$ *of their general solutions the following statements are true.*

*(i) In the zone of separated eigenvalues there are formal decompositions:*

$$\begin{cases} \mathbf{V}(N,t) := V_n(q^n,q)|_{n=Nt,q=e^{2\pi it}} = \mathbf{V}_0(t) + \dfrac{1}{N}\mathbf{V}_1(t) + \cdots \\ \pi_n^{(j)} = \exp\left\{ \sum\limits_{i=-1}^{\infty} \dfrac{\varphi_i^{(j)}(z)}{N^i} \right\}, \qquad z = e^{2\pi it}. \end{cases} \tag{4.4}$$

*(ii) The elements of the matrices* $\mathbf{V}_0(t), \mathbf{V}_1(t), \dots$ *are algebraic functions, and*

$$\mathbf{V}_0(t) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_1(z) & \lambda_2(z) & \dots & \lambda_d(z) \\ \vdots & \vdots & & \vdots \\ \lambda_1^{d-1}(z) & \lambda_2^{d-1}(z) & \dots & \lambda_d^{d-1}(z) \end{pmatrix}, \qquad z = e^{2\pi it}, \quad \det\mathbf{V}_0(t) \neq 0. \tag{4.5}$$

*(iii) The elements of the diagonal matrix* $\operatorname{diag}\{\varphi_i^{(j)}\}_{j=1}^{d}$ *are Abelian integrals, and*

$$\frac{d}{dt}\varphi_{-1}^{(j)}(e^{2\pi it}) = \ln \lambda_j(e^{2\pi it}).$$

Note again that the expansion coefficients obtained here for $V_n$ – are algebraic functions, and for $\varphi_i^{(j)}$ – are Abelian integrals. Using additional information about the asymptotics of the coefficients $A_j(q^n,q)$ of the recurrence relations (1.2) or (1.9), one can prove that the formal series (4.4) turn out to be asymptotic.

Let us note a detail concerning the coefficients $\{c_j\}$ in the (1.10) expansion in the basis of fundamental solutions (4.3). Generally speaking, these constants can change when passing inside the zone $I$ - separated by eigenvalues points, where the order of their (s.v.) moduli changes. That is, let $I = \sqcup_p I_p :\ \exists$ a permutation $\sigma_p$ of the set $\{1,\dots,d\} : \left|\lambda_{\sigma_p(1)}(z)\right| \geq \left|\lambda_{\sigma_p(2)}(z)\right| \geq \cdots \geq \left|\lambda_{\sigma_p(d)}(z)\right|, \quad \forall z \in I_p$. Then $c_j := c_j^p,\ z \in I_p$.

In the zone of close eigenvalues other approaches are used (see [26, 27, 28]). Here an additional small parameter (closeness of eigenvalues) as it tends to zero transforms the difference problem (4.1) into a differential one, and the obtained coefficients of the asymptotic expansion for $V_n$ are already solutions of the equations: hypergeometric, Bessel, Airy, Painlevé.

**4.2. On the growth of particular solutions: estimates for twisted knots.** Let us return to Section 1.4 and recall that numerical calculations forknots $4_1$, $5_2$ (see in [3]) showed that, despite the demonstration of exponential growth[27] for $J_N(\boldsymbol{e}^{2\pi i/N})$ as $N \to \infty$, no corresponding growth was observed for $J_n(\boldsymbol{e}^{2\pi i/N})$ in the two-scale regime (1.11): $N \to \infty,\ n/N \to t \in \mathcal{K} \Subset (0,1)$.
This made it impossible to use WKB analysis to prove that the growth rate is equal to the integral in (1.12). Moreover, the integral in (1.12) for a particular solution in general position turned out to be twice as large as the corresponding volume. However, the perturbation of the sequence $\{J_n(q)\}$ of the form $\{f_n'(q)\}$ : $f_n(q) := (1-q^n)J_n(q)$, see (1.17), (1.18), which has the same spectral curve and asymptotics for $n = N,\ q := \boldsymbol{e}^{2\pi i/N},\ N \to \infty$, and demonstrates (in numerical calculations) exponential growth in the regime $N \to \infty, n/N \to t \in \mathcal{K} \Subset (\mathbf{1/2}, 1)$, made it possible to use WKB$-$analysis for $t \in$ (1/2,1) and the hope of halving the integral in (1.12).

Thus, for a rigorous justification of (1.23), (1.24) (even for specific knots) we need to be able to prove for the sequence

$$\{f_n'(q)\} \quad \text{in the limit regime} \quad N \to \infty,\ \ n/N \to t \in (0,1)\ \ : \tag{4.6}$$

a) absence of exponential growth and decay when $t \in (0, 1/2)$;
b) presence of $\tilde{t} \in$ [1/2,1) : exponential growth takes place $\forall t \in \mathcal{K} \Subset (\tilde{t}, 1)$.

The following statement, proved by T. Dudnikova, contains a positive answer to part of point a) for all twisted knots.

**Lemma 4.1.** *Let $K$ be a twisted knot $K_p$, $p \in \mathbb{N}$, and*

$$n_p(N) := N \cdot \Big[\frac{1}{2\pi} \arccos\Big(\frac{2p-1}{2p}\Big)\Big], \tag{4.7}$$

*where $[a]$ is the integer part of a number $a$.*

*Then, for $q = e^{2\pi i/N}$ u $n = 1, \dots, n_p(N)$, the following bounds hold,*

$$|J_n(q)| \le n \quad \textit{and} \quad |f_n'(q)| \le C_p' n^2, \tag{4.8}$$

*where $C_p' := 1 + 2C_p + 2\sqrt{4p-1}$, $C_p := (9p-1)/(4p)$.*

[27] with the index $\approx Vol(K)/(2\pi)$, according to the hypothesis proved for these knots

**4.3. Explicit form of $A$-polynomials for the knots $k5_{15}, k5_7$.** One of the common knot classifications is Rolfsen's classification, which is based on the number of crossings in the projection of the knot onto the sphere. For example, knot $3_1$ is the first (and only) in the series of knots with three crossings, and knot $10_{125}$ is in 125th place in the list of knots with 10 crossings. Another classification (census manifolds) is based on the number of regular tetrahedra into which the complement of this knot in $\mathbb{S}^3$ is partitioned. For example, the manifold $\mathbb{S}^3 \setminus 10_{125}$ is partitioned into 6 tetrahedra and ranks 20th (among 6 tetrahedral manifolds) in terms of hyperbolic volume, so knot $10_{125}$ has number $k6_{20}$ in this classification. It happens that a knot, the complement of which consists of a small number of tetrahedra, has a very large number of intersections in its projection and is therefore not described in Rolfsen's classification. Such knots include the knots $k5_{15}$ and $k5_7$.

For the knots of the Rolfsen classification, the exact and numerical value of the volume of their complement, as well as the explicit form of the $A$-polynomial, can be easily found on the site [39]. For the knots of the classification (census manifolds), in particular $k5_{15}, k5_7$, the corresponding volumes can be found in the classical article [7], and the explicit form of the $A$-polynomials lies elsewhere: on the site [40]. Due to the discrepancy between our numerical calculations of the integrals (1.24) and the exact values from [7], we present below the expressions of the $A$-polynomials from [40] that we used, in the hope of resolving the discrepancy that has arisen.

$$A_{k5_{15}} := (M^{570})(-1*M^{12}) + (L^1*M^{534})(1*M^{10}-5*M^{12}+2*M^{14}+1*M^{16}-1*M^{18}) + (L^2*M^{498})*(1*M^8+3*M^{10}-5*M^{12}+12*M^{14}-5*M^{16}) + (L^3*M^{462})*(-1*M^6+6*M^8-6*M^{10}+21*M^{10}-9*M^{14}+3*M^{16}) + (L^4*M^{426})*(-3*M^6+4*M^8-7*M^{10}-12*M^{12}+1*M^{14}+3*M^{16}) + (L^5*M^{390})*(-4*M^6+1*M^8-25*M^{10}-12*M^{12}-5*M^{14}+3*M^{16}) + (L^6*M^{354})*(2*M^6-11*M^8+50*M^{10}-34*M^{12}+7*M^{14}) + (L^7*M^{318})*(3*M^4-8*M^6+38*M^8+19*M^{10}+25*M^{12}-7*M^{14}) + (L^8*M^{282})*(6*M^4-9*M^6-18*M^8+18*M^{10}+9*M^{12}-6*M^{14}) + (L^9*M^{246})*(7*M^4-25*M^6-19*M^8-38*M^{10}+8*M^{12}-3*M^{14}) + (L^{10}*M^{210})*(-7*M^4+34*M^6-50*M^8+11*M^{10}-2*M^{12}) + (L^{11}*M^{174})*(-3*M^2+5*M^4+12*M^6+25*M^8-1*M^{10}+4*M^{12}) + (L^{12}*M^{138})*(-3*M^2-1*M^4+12*M^6+7*M^8-4*M^{10}+3*M^{12}) + (L^{13}*M^{102})*(-3*M^2+9*M^4-21*M^6+6*M^8-6*M^{10}+1*M^{12}) + (L^{14}*M^{66})*(5*M^2-12*M^4+5*M^6-3*M^8-1*M^{10}) + (L^{15}*M^{30})*(1-1*M^2-2*M^4+5*M^6-1*M^8) + (L^{16}*M^{(-6)})*(1*M^6);$$

$$A_{k5_7} := (M^{(-8)})*(1*M^8) + (L^1*M^{30})*(1*M^8) + (L^2*M^{68})*(-2*M^4+6*M^6-12*M^8) + (L^3*M^{106})*(-3*M^4+8*M^6-12*M^8-2*M^{10}+1*M^{12}) + (L^4*M^{144})*(1-8*M^2+28*M^4-47*M^6+54*M^8-4*M^{10}+5*M^{12}-1*M^{14}) + (L^5*M^{182})*(-1*M^2+10*M^4-13*M^6+5*M^8+35*M^{10}-7*M^{12}-1*M^{14}) + (L^6*M^{220})*(4*M^4-15*M^6-40*M^{10}-6*M^{12}+1*M^{14}) + (L^7*M^{258})*(1*M^2-6*M^4-4*M^6+5*M^8-35*M^{10}-32*M^{12}+16*M^{14}-1*M^{16}) + (L^8*M^{296})*(-1*M^4-15*M^6+65*M^8-10*M^{10}+30*M^{12}+1*M^{14}) + (L^9*M^{334})*(1*M^6+30*M^8-10*M^{10}+65*M^{12}-15*M^{14}-1*M^{16}) + (L^{10}*M^{372})*(-1*M^4+16*M^6-32*M^8-35*M^{10}+5*M^{12}-4*M^{14}-6*M^{16}+1*M^{18}) + (L^{11}*M^{410})*(1*M^6-6*M^8-40*M^{10}-15*M^{14}+4*M^{16}) + (L^{12}*M^{448})*(-1*M^6-7*M^8+35*M^{10}+5*M^{12}-13*M^{14}+10*M^{16}-1*M^{18}) + (L^{13}*M^{486})*(-1*M^6+5*M^8-4*M^{10}+54*M^{12}-47*M^{14}+28*M^{16}-8*M^{18}+1*M^{20}) + (L^{14}*M^{524})*(1*M^8-2*M^{10}-12*M^{12}+8*M^{14}-3*M^{16}) + (L^{15}*M^{562})*(-12*M^{12}+6*M^{14}-2*M^{16}) + (L^{16}*M^{600})*(1*M^{12}) + (L^{17}*M^{638})*(1*M^{12})$$

Let us recall connection (2.1) between the polynomials $P(z, \lambda)$ and $A(M, L)$:

$$z \equiv M^2, \lambda \equiv L : \quad \Rightarrow \quad P(z, \lambda) \equiv A(M, L).$$

Also note that the $A(M, L)$ polynomials for $10_{125}$ that we used in section 3.8 are the same as those given in both [39] and [40].

**4.4. Concluding remarks.** Finally, we note that

**1)** Our goal is to find the limit on the left-hand side of (1.5) **VC** (volume hypothesis), based on the fact that the $q$-polynomials $J_n(q)$ are a solution of the Cauchy problem (i.e. a particular solution) of the homogeneous $q$-difference equation (1.2).

**2)** As an approach to the goal, we consider the WKB asymptotics of the fundamental solutions of this equation, the leading terms of which have exponential growth (decrease) with exponents of the form (1.10), determined by the integrals of the logarithms of the moduli of the branches $\lambda(z)$ – the spectral curve (1.7).

**3)** However, the left-hand side in **VC** - (1.5) for all knots $K$ is the polynomials $J_n(q)|_{q=e^{2\pi i/N}}$ as $n \to N$ (from the left) are bounded, due to the symmetry property (1.15), while $J_N(q)|_{q=e^{2\pi i/N}}$ grow exponentially, due to **VC**. Therefore, directly, exponentially growing asymptotics of the form (1.10), (1.11) are not applicable.

However, in (1.17) - (1.19) particular solutions (1.2) are proposed that do not have (1.15) symmetry and, therefore, have the possibility of exponential growth for $J_n(q)|_{q=e^{2\pi i/N}}$ as $n \to N$ (on the left) and, as $n = N$, achieve a growth rate equal to the limit on the left-hand side of **VC** - (1.5).

**4)** The latter circumstance led to the hypothesis that $\exists!$ branch $\lambda_1(z)$ of the spectral curve $\lambda(z)$:

$$\lim_{N\to\infty} 2\pi \frac{1}{N} \ln |J_N(e^{\frac{2\pi i}{N}})| = 2\pi \int_{\mathbf{1/2}}^{1} \ln |\lambda_1(e^{2\pi i\tau})| \, d\tau \tag{4.9}$$

**5)** In this preprint we do not set ourselves the task of testing the hypothesis (4.9), but simply look at how (under the assumption of the validity of the **AJ** - hypothesis for the knot $K$) the right-hand side in (4.9) relates to the right-hand side of **VC** - (1.5), i.e., with the hyperbolic volume $\mathbb{S}^3 \setminus K$.

**6)** In section 4 we perform a numerical analysis of the behavior of the branches of the spectral curve $\lambda(z)$ and present the values of the (for a number of knots $K$) on the right-hand side of (4.9). We note the coincidence (within the accuracy of the numerical method we use) of these values with the known values with volumes $\mathbb{S}^3 \setminus K$ for the knots $6_1, 7_2, 7_4$,$7_5$,$7_6$,$7_7$ and $10_{125}$.

**7)** We are confident that our calculations for knots $k5_{15}, k5_7$ adequately reflect the input data, but we do not think that the mismatch of the values being tested can lead to counterexamples to the **AJ** or **VC** hypotheses. Rather, we are talking about a mismatch of the $A$-polynomials we used with the known volumes $\mathrm{vol}(\mathbb{S}^3 \setminus K)$ for $K := k5_{15}, k5_7$.

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