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Asymptotic Relative Dynamics for Spacecraft on Close Hyperbolic Trajectories

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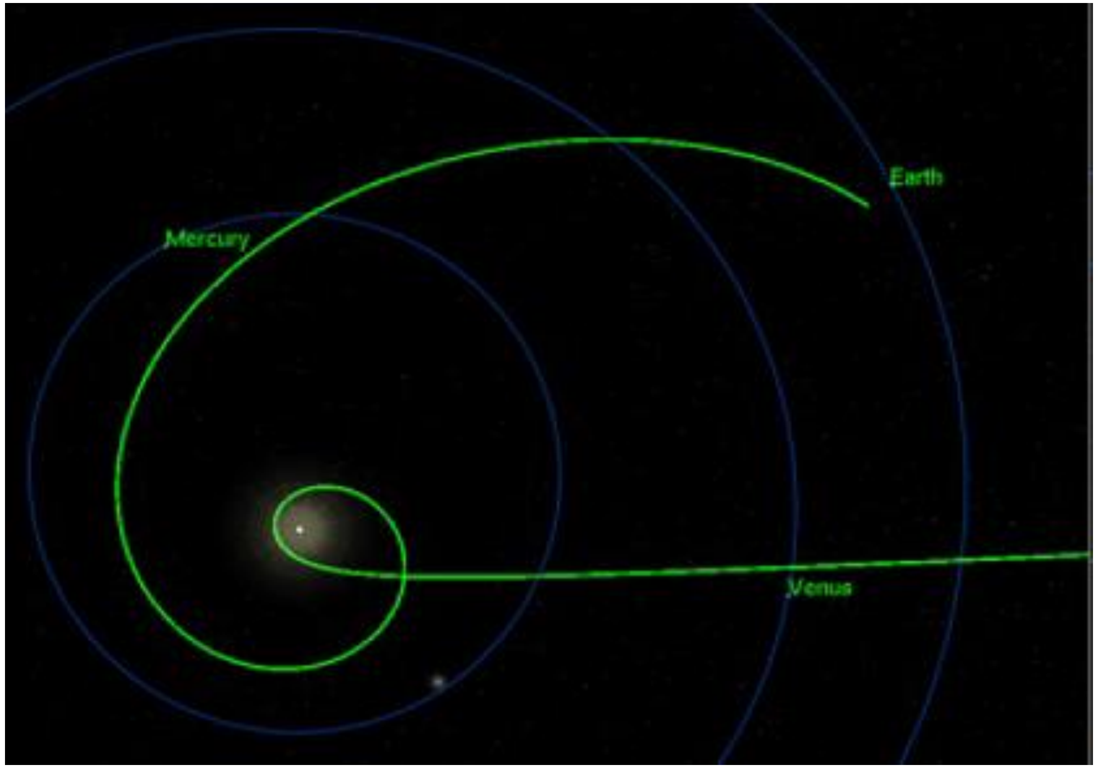
MOTIVATION: HYPERBOLIC TRAJECTORIES IN REAL MISSIONS

Past missions

- Pioneer 10 & 11
- Voyager 1 & 2
- New Horizons

Future mission concepts

- Interstellar Probe mission
- Solar Gravitational Lens' Focus mission
- "Sundiver" concept



Possible trajectory of the SGLF mission implementing the "Sundiver" concept

Smallsats and their formations have great potential for deep space exploration

Birth of smallsats	Regular launches of single smallsats	Smallsats formations	Single deep space smallsats	**Deep space smallsat formations**
1990-2000	2000-2010	2010-2020	2020-2030	2030-…..

CIRCULAR vs ELLIPTIC vs HYPERBOLIC RELATIVE MOTION

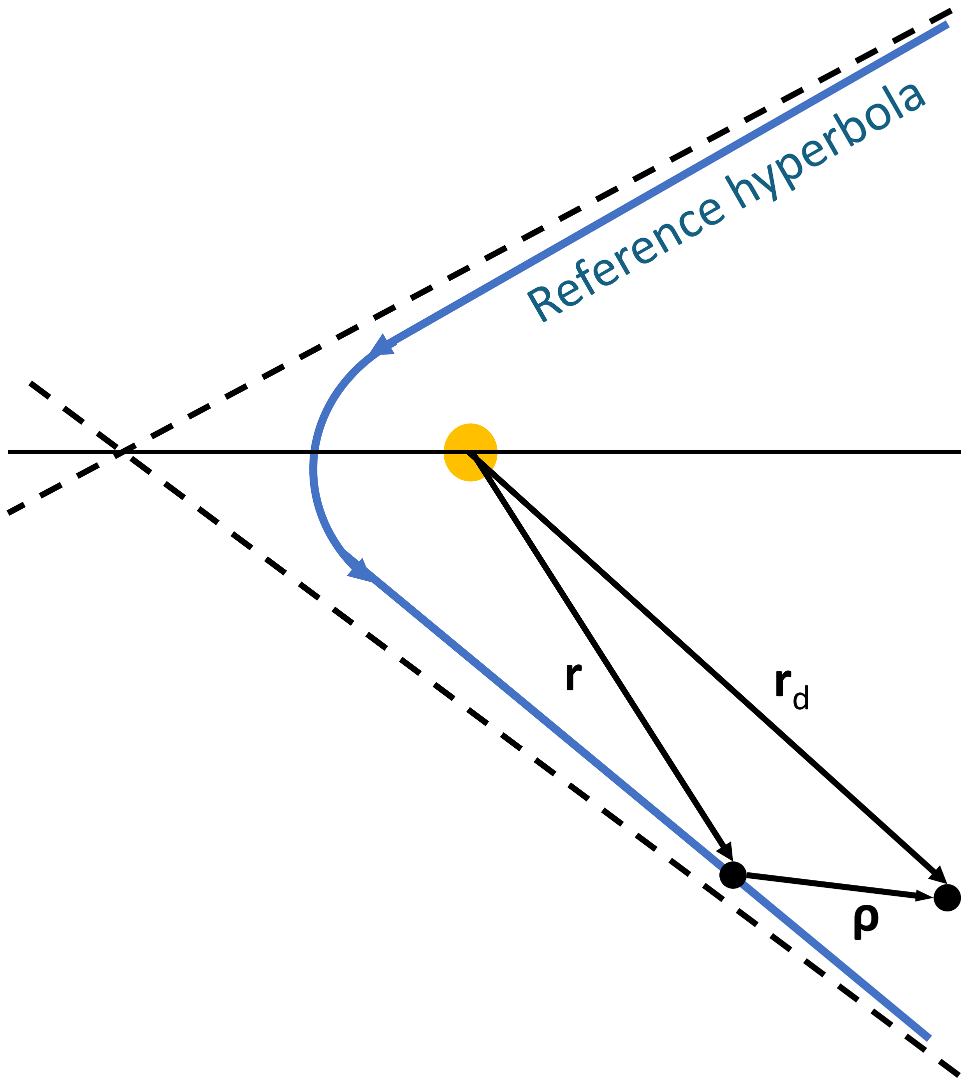
Circular reference orbit
- Hill-Clohessy-Wiltshire (HCW) linearized equations
- Simple time-explicit solution

Elliptic reference orbit
- Tschauner-Hempel linearized equations
- Simple true-anomaly-explicit solution

Hyperbolic reference orbit
- General 2BP linearized equations
- **Simple solution is to be obtained**

GOALS OF STUDY

1. To obtain a *practical description* of a relative motion in close hyperbolas
2. To find out *what types* of a hyperbolic relative motion are possible
3. To demonstrate *how to design* a formation in close hyperbolic trajectories



$\mathbf{r}$ – chief's radius-vector

$\mathbf{r}_d$ – deputy's radius-vector

$\boldsymbol{\rho} = \mathbf{r}_d - \mathbf{r}$ – the relative position vector

Each spacecraft moves along a hyperbola with the Sun at the focus; spacecraft do not interact

$$\ddot{\mathbf{r}} = -\frac{\mu \mathbf{r}}{r^3} \qquad \ddot{\mathbf{r}}_\mathrm{d} = -\frac{\mu \mathbf{r}_\mathrm{d}}{r_\mathrm{d}^3} \qquad \ddot{\boldsymbol{\rho}} = \ddot{\mathbf{r}}_\mathrm{d} - \ddot{\mathbf{r}}$$

Linearization under the assumption $\frac{\rho}{r} \ll 1$

$$\ddot{\boldsymbol{\rho}} = -\frac{\mu}{r^3}\left(\boldsymbol{\rho} - \frac{3(\mathbf{r} \cdot \boldsymbol{\rho})\mathbf{r}}{r^2}\right)$$

ASYMPTOTIC COORDINATE SYSTEM AND δ VARIABLE

Assume we know the reference hyperbola $(\mathrm{a}, \mathrm{e}, \mathrm{i}, \Omega, \omega, \tau)$

We define the Asymptotic Coordinate System (ACS) as following

- The origin – the attractive center (the Sun)
- $\mathbf{e}_1$ – along the outgoing asymptote
- $\mathbf{e}_2$ – complement to the right-handed basis
- $\mathbf{e}_3$ – along the orbital angular momentum

We introduce a new angle δ

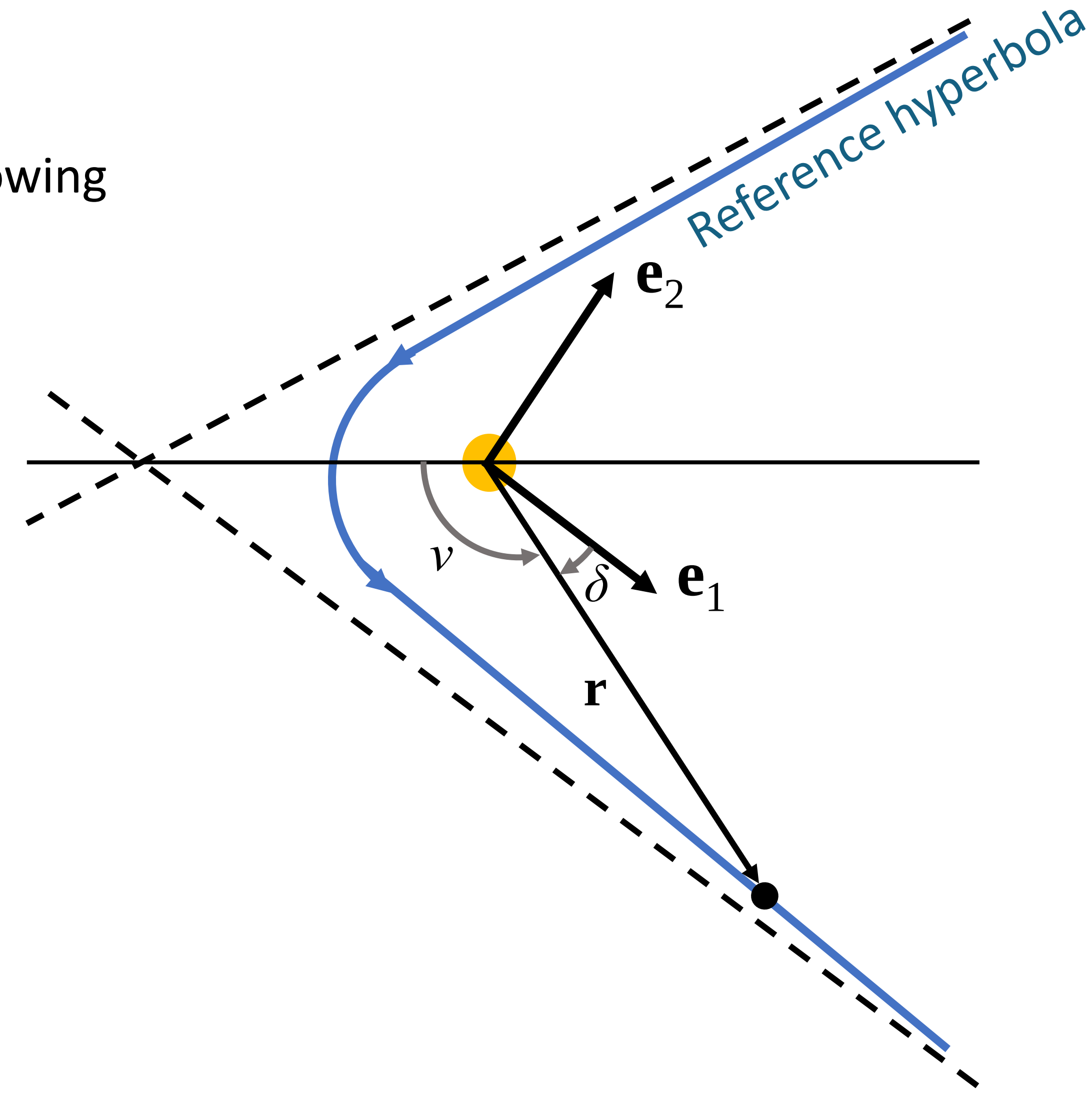
- ν – the chief's true anomaly

$$\nu \in (-\nu_{max}, \nu_{max}),\ \nu_{max} = \arccos\left(\frac{-1}{\mathrm{e}}\right)$$

- $\delta \coloneqq \nu_{max} - \nu$

$$\delta \in (0, 2\nu_{max}),\ \dot{\delta} < 0,$$

$$\delta \xrightarrow[t\to+\infty]{} 0,\ \delta = O\left(\frac{1}{r}\right)$$



SOLUTION OF THE LINEARIZED SYSTEM

Linearized system

$\dot{\mathbf{x}} = \boldsymbol{A}(\mathrm{t})\,\mathbf{x}$

$\mathbf{x} := \begin{pmatrix} \boldsymbol{\rho} \\ \dot{\boldsymbol{\rho}} \end{pmatrix}, \Delta\mathrm{t} := \mathrm{t} - \mathrm{t}_0$

Solution

$\mathbf{x}(\mathrm{t}) = \boldsymbol{U}(\mathrm{t}, \mathrm{t}_0)\boldsymbol{\xi}$ → $\boldsymbol{\xi}$ is a vector of constants

Relates to the *initial conditions* via
$\boldsymbol{\xi} = \boldsymbol{U}^{-1}(\mathrm{t}_0, \mathrm{t}_0)\mathbf{x}(\mathrm{t}_0)$

We designate components
$\boldsymbol{\xi} = (\alpha_{-1} \quad \alpha_0 \quad \beta_{-1} \quad \beta_0 \quad \gamma_{-1} \quad \gamma_0)^{\mathrm{T}}$

Note: $\boldsymbol{\xi}$ has units of distance (km)

$\boldsymbol{U}(\mathrm{t}, \mathrm{t}_0)$ is a *fundamental matrix* [Reynolds, 2022]:

$\boldsymbol{U}(\mathrm{t}, \mathrm{t}_0) = \boldsymbol{U}(\mathbf{r}(\mathrm{t}), \mathbf{v}(\mathrm{t}), \Delta\mathrm{t})$

$$\mathbf{r} = \frac{\mathrm{a}(e^2 - 1)}{1 - \cos\delta + \eta \sin\delta}\begin{pmatrix} \cos\delta \\ -\sin\delta \\ 0 \end{pmatrix}$$

$$\boldsymbol{U}\big(\mathbf{r}(\delta), \mathbf{v}(\delta), \Delta\mathrm{t}(\delta, \delta_0)\big)$$

$$\mathbf{v} = \frac{\mathrm{v}_\infty}{\eta}\begin{pmatrix} \eta + \sin\delta \\ \cos\delta - 1 \\ 0 \end{pmatrix}$$

We use δ instead of time

Here $\eta := \sqrt{e^2 - 1}$

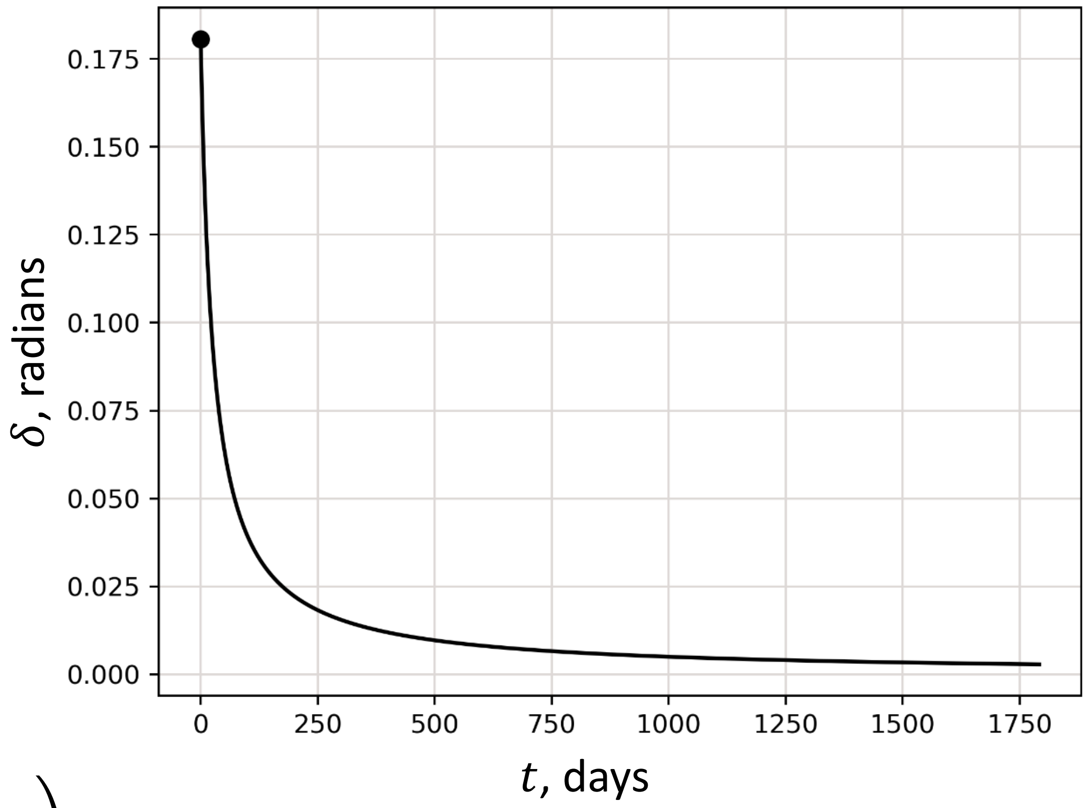
$\boldsymbol{U}\big(\mathbf{r}(\delta), \mathbf{v}(\delta), \Delta\mathrm{t}(\delta, \delta_0)\big)$ Let us exploit the natural asymptotic behavior: $\delta \to 0+$

$$\mathbf{r} = \frac{\mu\eta}{\mathrm{v}_\infty^2}\begin{pmatrix} \frac{1}{\delta} - \frac{1}{2\eta} + \left(-\frac{1}{3} + \frac{1}{4\eta^2}\right)\delta + O(\delta^2) \\ -1 + \frac{\delta}{2\eta} + O(\delta^2) \\ 0 \end{pmatrix}$$

$$\mathbf{v} = \mathrm{v}_\infty \begin{pmatrix} 1 + \frac{\delta}{\eta} + O(\delta^3) \\ -\frac{\delta^2}{2\eta} + O(\delta^3) \\ 0 \end{pmatrix}$$

$$\Delta\mathrm{t}\,(\delta, \delta_0) = \frac{\mu}{\mathrm{v}_\infty^3}\left(\frac{\eta}{\delta} + \left(\frac{1}{2} - \mathrm{M}_0\right) - \ln\frac{2\eta}{\delta \mathrm{e}} - \frac{4\eta^2 + 9}{12\eta}\delta + O(\delta^2)\right)$$

$\mathrm{M}_0 \coloneqq \mathrm{e}\,\sinh \mathrm{H}(\delta_0) - \mathrm{H}(\delta_0)$, H is the *hyperbolic anomaly*



Angle δ as a function of time

δ-EXPLICIT AND TIME-EXPLICIT ASYMPTOTIC SOLUTIONS

$$x(\delta) = \frac{\alpha_{-1}}{2\delta\eta} - \frac{3\alpha_{-1}}{2\eta^2}\ln\frac{2\eta}{\delta e} + \alpha_0 - \alpha_{-1}\left(\frac{6M_0 - 11}{4\eta^2}\right) + O(\delta\ln\delta),$$

$$y(\delta) = \frac{\beta_{-1}}{\delta} + \beta_0 + \frac{\alpha_{-1}}{\eta} - \left(\frac{\beta_{-1}}{3} + \frac{\beta_0}{2\eta} + \frac{5\alpha_{-1}}{4\eta^2}\right)\delta + o(\delta),$$

$$z(\delta) = \frac{\gamma_{-1}}{\delta} + \gamma_0 - \left(\frac{\gamma_{-1}}{3} + \frac{\gamma_0}{2\eta}\right)\delta + o(\delta).$$

<u>Note</u>:

$\alpha_{-1}, \beta_{-1}, \gamma_{-1}$ are first met with δ^{-1}

$\alpha_0, \beta_0, \gamma_0$ are first met with δ^0

δ-explicit asymptotic solution

$$\delta = \eta\frac{1}{\tau_\pi} - \eta\frac{\ln\tau_\pi}{\tau_\pi^2} + \eta\left(\frac{1}{2} + \ln\frac{e}{2}\right)\frac{1}{\tau_\pi^2} + o\left(\frac{1}{\tau_\pi^2}\right), \quad \tau_\pi \to +\infty, \quad \text{where } \tau_\pi := \frac{\mathrm{v}_\infty^3}{\mu}(t - \tau)$$

$$x(\tau_\pi) = \frac{\alpha_{-1}}{2\eta^2}\tau_\pi - \frac{\alpha_{-1}}{\eta^2}\ln\left(\frac{2}{e}\tau_\pi\right) + \alpha_0 + \alpha_{-1}\left(\frac{5}{2\eta^2} - \frac{3M_0}{2\eta^2}\right) + O\left(\frac{\ln\tau_\pi}{\tau_\pi}\right),$$

$$y(\tau_\pi) = \frac{\beta_{-1}}{\eta}\tau_\pi + \frac{\beta_{-1}}{\eta}\ln\left(\frac{2}{e}\tau_\pi\right) + \beta_0 + \frac{2\alpha_{-1} - \beta_{-1}}{2\eta} + O\left(\frac{\ln\tau_\pi}{\tau_\pi}\right),$$

$$z(\tau_\pi) = \frac{\gamma_{-1}}{\eta}\tau_\pi + \frac{\gamma_{-1}}{\eta}\ln\left(\frac{2}{e}\tau_\pi\right) + \gamma_0 - \frac{\gamma_{-1}}{2\eta} + O\left(\frac{\ln\tau_\pi}{\tau_\pi}\right).$$

Time-explicit asymptotic solution

POSSIBLE TYPES OF HYPERBOLIC RELATIVE MOTION

1. The relative motion is bounded if and only if
   $\alpha_{-1} = \beta_{-1} = \gamma_{-1} = 0$,
   otherwise it is unbounded

2. If the motion is bounded, the relative trajectory is generally a line segment
   (if $\alpha_0, \beta_0, \gamma_0 \neq 0$, then $(x, y, z)^T$ tends to $(\alpha_0, \beta_0, \gamma_0)^T$)
   - If at least one of $\alpha_0, \beta_0, \gamma_0$ is zero, the geometry is more complicated (for example, it can be a segment of parabola in certain 2D projections)

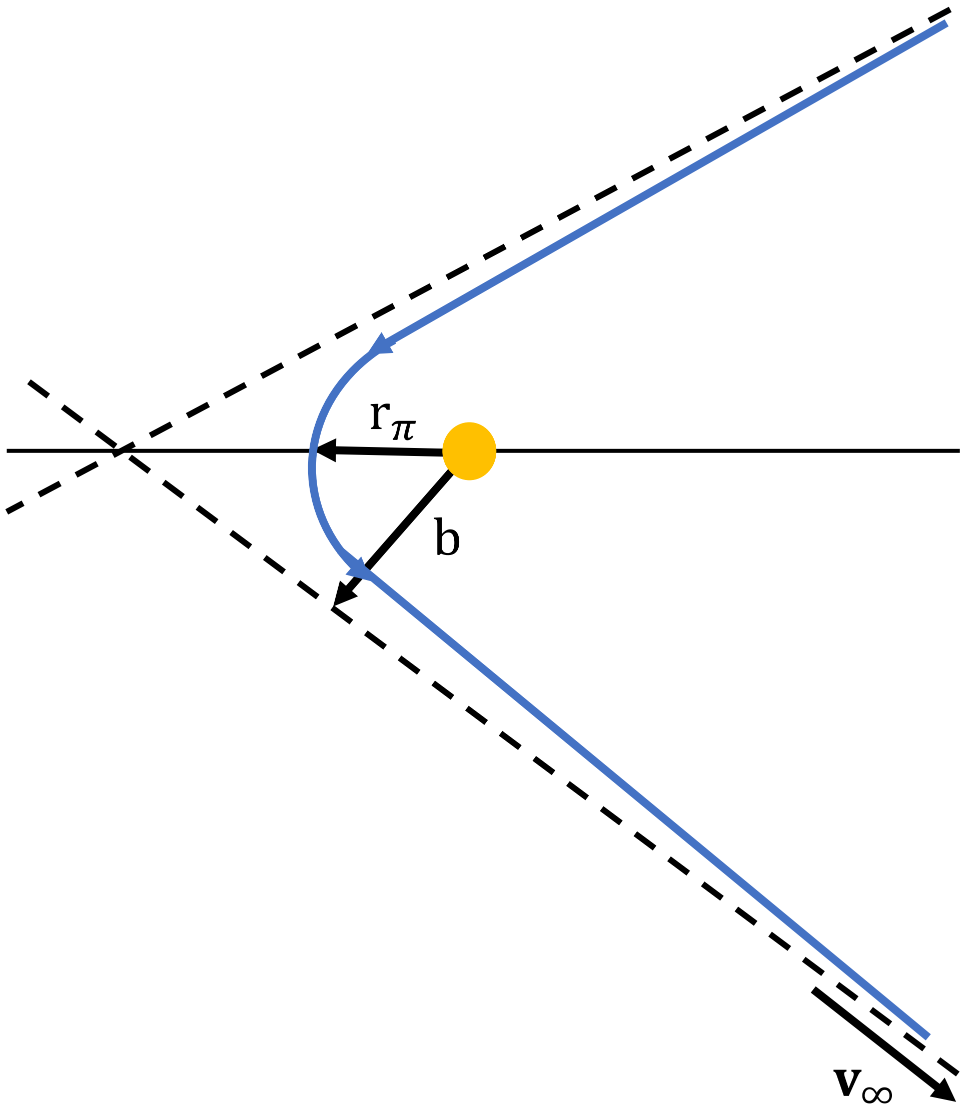
3. If the motion is unbounded, the relative trajectory is generally an infinite ray
   ($x = \frac{\alpha_{-1}}{2\eta^2}\tau_\pi + o(\tau_\pi),\ y = \frac{\beta_{-1}}{\eta}\tau_\pi + o(\tau_\pi),\ z = \frac{\gamma_{-1}}{\eta}\tau_\pi + o(\tau_\pi)$)
   - If at least one of $\alpha_{-1}, \beta_{-1}, \gamma_{-1}$ is zero, the geometry is more complicated (for example, it can be a hyperbola in certain 2D projections)

Note:
These conditions are equal to three *linear equations* on the initial state $\mathbf{x}(\mathrm{t}_0)$ (initial relative velocity ≈ 0)

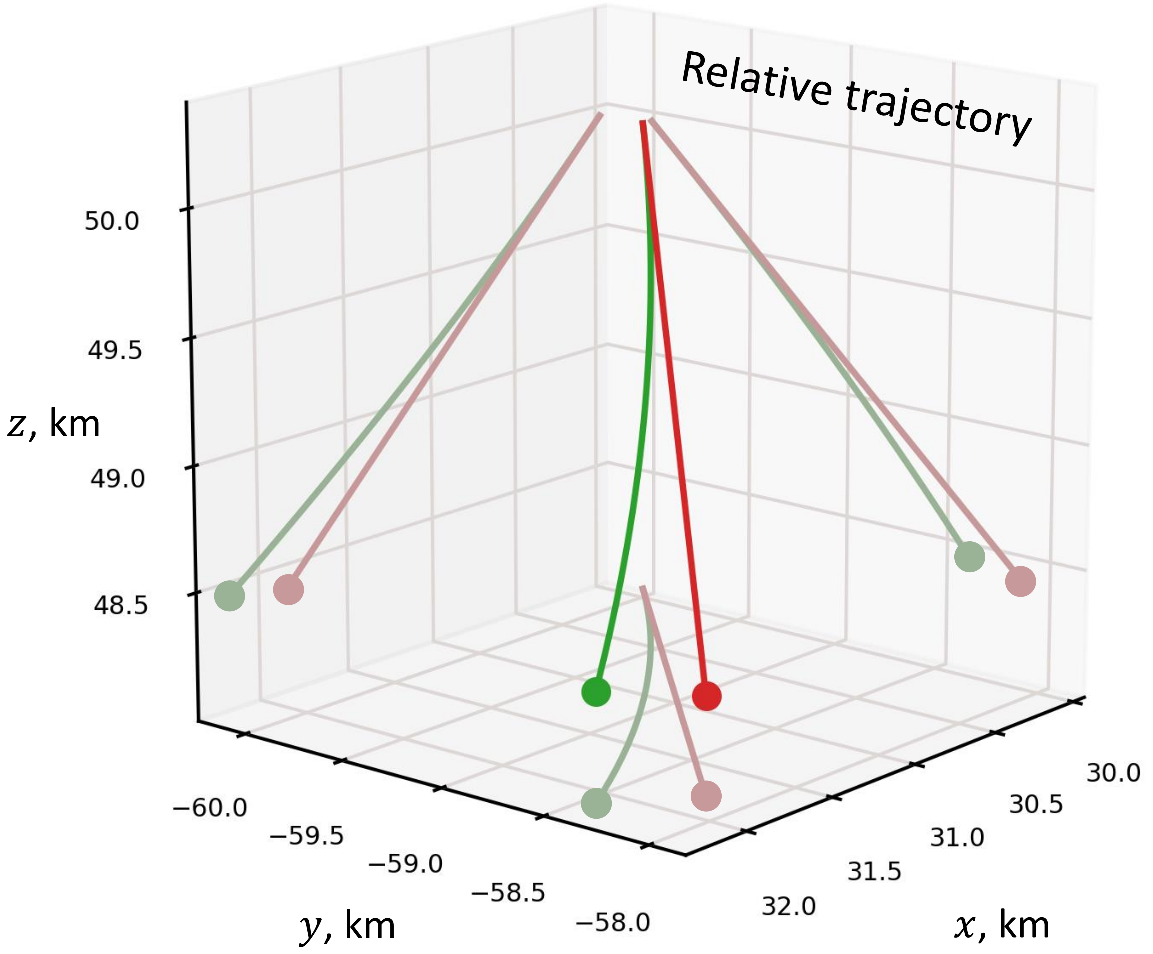
The hyperbolic case is much poorer in types of relative motion than the elliptic/circular case

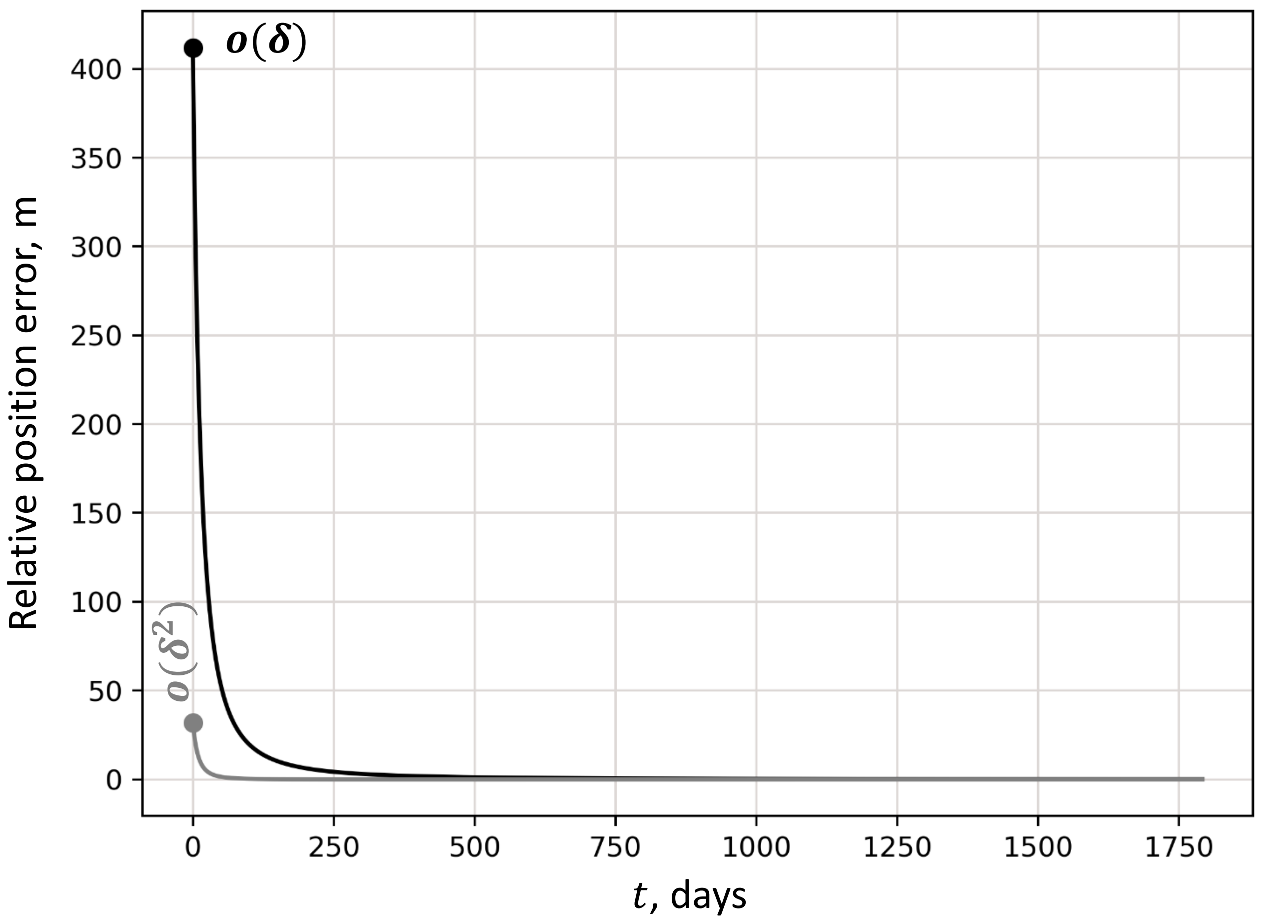
THE REFERENCE TRAJECTORY

- Heliocentric
- Hyperbolic excess velocity $\mathrm{v}_\infty = 20\,\frac{\mathrm{AU}}{\mathrm{year}} \approx 95\,\frac{\mathrm{km}}{\mathrm{s}}$
- Eccentricity $\mathrm{e} \approx 3.03$
- Impact parameter $\mathrm{b} = 0.28\ \mathrm{AU}$
- Pericenter distance $\mathrm{r}_\pi = 0.20\ \mathrm{AU}$
- Semi-major axis $a = 0.10$ AU
- Initial chief's position $r_0 = 1.52\ \mathrm{AU}\ \ (\delta_0 \approx 0.1807)$
- Final chief's position $r_f = 100\ \mathrm{AU}\ \ (\delta_f \approx 0.0028)$



EXAMPLE: BOUNDED MOTION





Error $|\boldsymbol{\rho}_{exact} - \boldsymbol{\rho}_{approx}|$ between exact (2BP) and first/second-order approximate solutions

Unperturbed 2-body problem solution

First-order asymptotic solution

$\alpha_{-1} = \beta_{-1} = \gamma_{-1} = 0$

$\alpha_0 = 30$ km

$\beta_0 = -60$ km

$\gamma_0 = 50$ km

$$x(\delta) = \alpha_0\left(1 + \frac{\delta}{\eta}\right),\ \ y(\delta) = \beta_0\left(1 - \frac{\delta}{2\eta}\right),\ \ z(\delta) = \gamma_0\left(1 - \frac{\delta}{2\eta}\right)$$

EXAMPLE: REGULAR TETRAHEDRON FORMATION

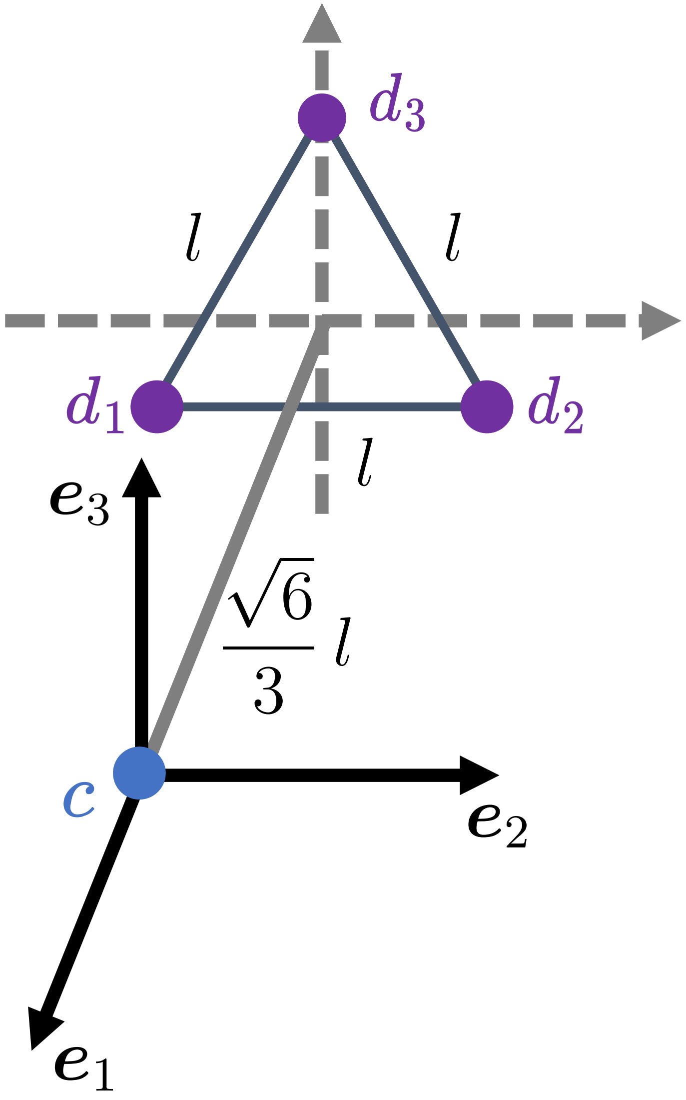
Since the motion is bounded, for all deputies
$\alpha_{-1} = \beta_{-1} = \gamma_{-1} = 0$

Recalling that $(x, y, z)^T$ tends to $(\alpha_0, \beta_0, \gamma_0)^T$,
it is easy to find

$$\boldsymbol{\xi}_1 = \begin{pmatrix} 0 & -\frac{\sqrt{6}}{3}l & 0 & -\frac{1}{2}l & 0 & -\frac{1}{2\sqrt{3}}l \end{pmatrix}^T$$

$$\boldsymbol{\xi}_2 = \begin{pmatrix} 0 & -\frac{\sqrt{6}}{3}l & 0 & \frac{1}{2}l & 0 & -\frac{1}{2\sqrt{3}}l \end{pmatrix}^T$$

$$\boldsymbol{\xi}_3 = \begin{pmatrix} 0 & -\frac{\sqrt{6}}{3}l & 0 & 0 & 0 & \frac{1}{\sqrt{3}}l \end{pmatrix}^T$$

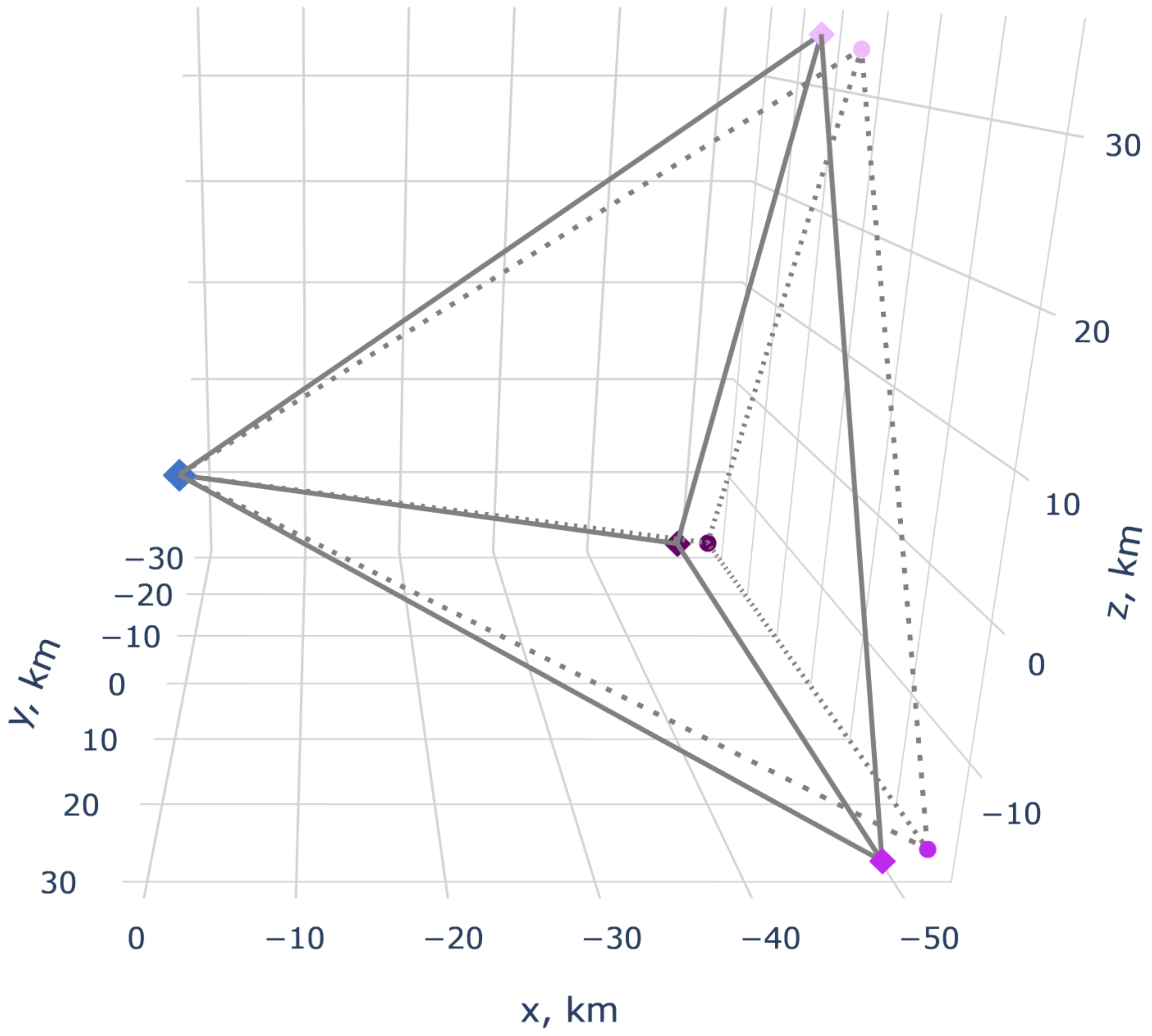


Convert into initial relative position & velocity

$$\boldsymbol{\rho}_1(\mathrm{t}_0) = l\ [\mathrm{km}] \cdot (-0.87\ \mathrm{km} \quad -0.48\ \mathrm{km} \quad -0.28\ \mathrm{km}\ )^T$$

$$\dot{\boldsymbol{\rho}}_1(t_0) = l\ [\mathrm{km}] \cdot \left(0.019\ \frac{\mathrm{mm}}{\mathrm{s}} \quad -0.010\ \frac{\mathrm{mm}}{\mathrm{s}} \quad -0.004\ \frac{\mathrm{mm}}{\mathrm{s}}\right)^T$$

EXAMPLE: REGULAR TETRAHEDRON FORMATION SIMULATION



Evolution of a tetrahedron with $l = 60$ km
from the epoch at $\mathrm{r} = 1.52$ AU
to the epoch at $\mathrm{r} = 100$ AU

Dashed lines – tetrahedron at the initial epoch
Solid lines – tetrahedron at the final epoch

Simulation was done numerically
in the <u>unperturbed 2-body problem</u>
with initial conditions calculated analytically

During all the flight time, the tetrahedron
was almost regular with the required size

No control or correction was applied!

CONCLUSION

- A new asymptotic representation for describing relative motion between spacecraft in hyperbolic trajectories is developed
  - Time-explicit formulae are obtained

- Based on the representation, possible types of relative motion are classified
  - Hyperbolic case turns out to be quite poor in terms of types compared to elliptic/circular case

- A way to design a spacecraft formation using the representation is proposed and demonstrated by the regular tetrahedron example

- <u>Areas of application</u>:
  control & navigation problems solution;
  formation deployment and maintenance;